Chapter 12
Order in the Reals

You've seen numbers ever since you've been in school, and you know a lot about them. It is possible to give them a careful mathematical foundation. In fact, it's possible to construct the natural numbers (and you can do so in Project 29.3). Then, if you try to introduce operations like addition and subtraction, you'll find that you are missing something: the negative numbers. So you look at the integers, and try again. Now, trying to introduce multiplication and division, you'll find you are missing something again: multiplicative inverses. So you look at the rational numbers, and you'll find you are missing something yet again. That brings you to the real numbers. Our ultimate goal will be to discuss what's missing in $\mathbb{Q}$, and to show you why $\mathbb{R}$ has what's missing. This is known as completeness of $\mathbb{R}$. In this and the next chapter, we'll show you some wonderful applications of completeness.

What do we mean by this property "completeness," that $\mathbb{R}$ has, but $\mathbb{Q}$ doesn't? If we take a stroll along the real number line, we can walk right up to any real number and it will be there waiting for us. In contrast to this, if we walk along the real line, this time stepping on rational numbers only, we might walk right up to where we'd expect $\sqrt{2}$ to be, but it will be out having lunch with other irrationals. We'll make this precise by the end of the chapter. We remind you that in this text we are assuming that the real numbers satisfy the algebraic and order properties listed in the Appendix on pages 363–364. We now turn to the important terminology that we need.

A nonempty subset A of $\mathbb{R}$ is **bounded above** if there is a real number M such that $x \leq M$ for all $x \in A$. We call a real number M satisfying $x \leq M$ for all $x \in A$ an **upper bound** of A. The nonempty subset A of $\mathbb{R}$ is **bounded below** if there is a real number m such that $m \leq x$ for all $x \in A$. We call a real number m satisfying $m \leq x$ for all $x \in A$ a **lower bound** of A. We say a nonempty set is **bounded** if it is bounded above and below. For example, the open interval $(0, 1)$ is bounded above, since every $x \in (0, 1)$ satisfies $x \leq 1$. The number 1 is an upper bound, and so is the real number 1.5. In fact, every number greater than or equal to 1 is an upper bound. Similarly, since $x \geq 0$ for all $x \in (0, 1)$, the set $(0, 1)$ is bounded below and 0 is an example of a lower bound of the set. Since $(0, 1)$ is bounded above and below, it is an example of a bounded set.

Exercise 12.1. For each of the following sets of real numbers, decide whether it is bounded above, bounded below, and (consequently) whether or not it is bounded. If the set is bounded above, give three different examples of upper bounds in $\mathbb{R}$. If the set is bounded below, give three different examples of lower bounds in $\mathbb{R}$. Use your intuition; we'll prove things rigorously later. The sets are:

(a) $\{x \in \mathbb{R} : x^2 \leq 5\}$;
(b) $\{x \in \mathbb{R} : x^3 < 5\}$;
(c) $\{x \in \mathbb{N} : x \leq 5\}$;
(d) $\{x \in \mathbb{Q} : x^2 < 2\}$.   ○

Sometimes a subset of $\mathbb{R}$ that is bounded above contains a largest element, and we give this element a special name: a maximum. The real number M is a **maximum** of the set A, if $M \in A$ and $x \leq M$ for all $x \in A$. We will write $M = \max A$ for a maximum of the set A. Note that a maximum is an upper bound that lies in the set A. Likewise, a real number m is a **minimum** of the set A, if $m \in A$ and $m \leq x$ for all x in A. We will write $m = \min A$ to denote a minimum of the set A. Again, notice that a minimum is a lower bound that lies in the set A.

It should now be clear that if a set A has a maximum, then A must be bounded above, and if the set has a minimum, then A must be bounded below. What about the converse?

Example 12.2. Give an example of a bounded set that has neither a maximum nor a minimum.

We claim that the set $(0,2)$ is bounded and has neither a maximum nor a minimum.

Proof. For each $x \in (0,2)$, we know that $0 < x < 2$. Therefore 0 is a lower bound of the set and 2 is an upper bound. Thus, $(0,2)$ is bounded. To see that it has no maximum, suppose to the contrary that s is a maximum of the set $(0,2)$. Then, by definition of maximum, s must be in the set, so $0 < s < 2$. But (as you can check) $0 < s < (2+s)/2 < 2$, and therefore $(2+s)/2$ is in the set $(0,2)$ and larger than s, a contradiction. In a similar fashion, you can check that there is no minimum.   □

It turns out that there is an upper bound that can help us when we don't have a maximum (called the supremum), and a lower bound that can help us when we don't have a minimum (called the infimum). We define these below.

Let A be a nonempty set of real numbers that is bounded above. Then a real number U is said to be a **supremum** of A or **least upper bound** of A if

(i) $a \leq U$ for all $a \in A$, and
(ii) if $M \in \mathbb{R}$ satisfies $a \leq M$ for all $a \in A$, then $U \leq M$.

Note that (i) says that U is an upper bound, while (ii) says that U is least among all upper bounds. While the phrase "least upper bound" is more descriptive, most authors prefer the term "supremum."

The following lemma tells us that the supremum, when it exists, is unique.

Lemma 12.3. *If a nonempty subset of* $\mathbb{R}$ *has a supremum, then the supremum is unique.*

We first try to understand the problem. Let's call our set S. We are not asking whether or not a supremum of S exists. What we are trying to do is to show that there cannot exist two different real numbers a and b, such that both a and b fulfill the properties of supremum of S. (We've seen examples of sets with more than one upper bound. Maybe there are sets with more than one least upper bound.)

So we turn to devising a plan. Let's suppose that there are two such numbers a and b, and try to show that they must be equal.

Proof. Let S be a nonempty subset of $\mathbb{R}$. Suppose a and b are two real numbers that satisfy properties (i) and (ii) in the definition of supremum. Then a is an upper bound. Since b is a supremum, property (ii) implies that $b \leq a$. On the other hand, since b is an upper bound and a is a supremum, property (ii) implies that $a \leq b$. Thus $a = b$, and we conclude that there is at most one supremum.   □

From here on in, we will refer to "the" supremum, and we will denote the supremum of a nonempty set A by $\sup A$.

The last proof was your first proof of uniqueness. This particular proof is fairly standard. You'll frequently be able to prove uniqueness by supposing that you have two such objects, and showing that they must be equal.

Exercise 12.4. Return to Exercise 12.1. Use your intuition to decide which of the sets have a supremum in $\mathbb{R}$. For the sets that you decide have a supremum, find a real number that you believe is the supremum. At this point, you may use your intuition to find an answer. We will ask you to prove that your answer is correct later.   ○

So we have defined two notions, supremum and maximum. What are the differences and what are the similarities? A close look at the definition shows that the maximum of a set A must be in A, while the supremum of A need not. On the other hand, we also have the following.

Exercise 12.5. Let A be a nonempty subset of real numbers that is bounded above. Show that if A has a maximum M, then M is the supremum of A. Conclude that a maximum of a set is unique and thus we can speak of "the" maximum of a set A—if it exists.   ○

In the following exercise you will define and investigate the infimum (or greatest lower bound) of a set. This is an important exercise, and we will refer to it frequently.

Exercise 12.6. Let A be a nonempty subset of $\mathbb{R}$ that is bounded below.

(a) Define **infimum** (or **greatest lower bound**) of the set A.
(b) Do what you always do when confronted with a new definition: Find examples and nonexamples.   ○

The infimum of a set A, denoted $\inf A$, is also unique (Problem 12.17) and hence we can speak of "the" infimum. If a set A has a minimum, then this minimum is the infimum of the set. Thus, $\min A$ is also unique, if it exists.

Some students find the words supremum and infimum difficult to remember. But once you get used to it, these words will sound like what they are: If the supremum is in the set, it's the maximum. If it's not in the set, the supremum (as suggested by the word "superior") lies above the set. Similarly, if the infimum is in the set, it's the minimum. If it's not in the set, then the infimum (as suggested by the word "inferior") lies below the set.

The next example shows how to prove rigorously that a particular number is the infimum (or supremum) of a set. Remember that to show ℓ is the infimum, we must show that it is a lower bound, and that if y is another lower bound, then $y \le \ell$. We will actually show the contrapositive of this last assertion: if $y > \ell$, then y is not a lower bound.

Example 12.7. Show that $\inf(3,4] = 3$.

Proof. We note that $3 \le x$ for all $x \in (3,4]$. Therefore, 3 is a lower bound.

To see that it is the infimum, we will show that nothing larger can be a lower bound. To this end, let y be chosen so that $3 < y$. If $y > 4$, then y is not a lower bound of $(3,4]$. If $y \le 4$, then $(3+y)/2$ is a real number such that $3 < (3+y)/2 < y \le 4$. Therefore $(3+y)/2 \in (3,4]$ and $(3+y)/2 < y$. Thus, y is not a lower bound of $(3,4]$. Hence, 3 is the infimum of $(3,4]$. □

There is one point in the proof above that is very important and, unfortunately, very easy to overlook. When we checked that "nothing larger than 3 can be a lower bound," we chose $y > 3$ and showed that $(3+y)/2 < y$. This, alone, will not convince someone that y is not a lower bound of $(3,4]$; if $(3+y)/2$ is not in the set $(3,4]$, it won't help us at all. That's why we also checked that $(3+y)/2 \in (3,4]$. Our point is this: to show y_0 is *not* a lower bound of a set S, you must find something in the set S that is smaller than y_0. (Keep this in mind when you solve the problems; in particular, when you solve Problem 12.1.)

Example 12.7 provides us with an example of a bounded set that has no minimum, but does have an infimum.

Exercise 12.8. Return to Exercise 12.4 and assume for now that the real numbers $\sqrt{5}$ and $5^{1/3}$ are irrational. (We will discuss this assumption in the next chapter.) If the set has an infimum, say what you think the infimum is. Then give an argument that shows that your answers for supremum (from Exercise 12.4) and infimum for the first three sets are correct.

We saw in Example 12.2 that there are sets that are bounded, but have no maximum or minimum. Since a maximum is a supremum, it may seem that there exist sets that are bounded above but have no supremum in $\mathbb{R}$. It turns out that in $\mathbb{R}$ this is not the case; the real numbers are constructed to guarantee the existence of a supremum of every bounded nonempty set. This will not be proved; in a way it is an agreement. The technical term for such a statement is axiom.

The completeness axiom of the reals. *Every nonempty subset of real numbers that is bounded above has a supremum.*

Exercise 12.9. State a version of the completeness axiom of $\mathbb{R}$ replacing the word "supremum" by the word "infimum." What conditions, if any, must be placed on the set? Prove that the two versions are equivalent. (Once you have your version of the completeness axiom, you may use Problem 12.7 to complete this exercise.)   ○

Here's an extremely useful consequence of the work we have built up in this chapter:

Theorem 12.10 (Archimedean property of $\mathbb{R}$). *Let a and b be two positive real numbers. Then there exists a positive integer n such that $a < nb$.*

Proof. Suppose that this is not true; that is, suppose that there are two positive real numbers, a and b, such that $a \geq nb$ for all $n \in \mathbb{N}$. This means that a/b is an upper bound of $\mathbb{N}$. Therefore, by the completeness axiom of $\mathbb{R}$, it follows that $\mathbb{N}$ has a supremum, which we will call u. Now consider $u - 1$. Since this is less than the supremum u, it can't be an upper bound of $\mathbb{N}$. So there exists $m \in \mathbb{N}$ with $m > u - 1$. Therefore, $m + 1 \in \mathbb{N}$ and $m + 1 > u$. Since no element of $\mathbb{N}$ can be greater than the upper bound u, this is a contradiction.   □

When we have a result that follows from a theorem that we just proved, we call it a corollary. You will show, in Problem 12.14, that the following is indeed a corollary of the Archimedean property.

Corollary 12.11. *For every real number a, there is an integer n such that $a < n$.*

We now turn to the well-ordering principle of $\mathbb{N}$, which is concerned with a fundamental property of the natural numbers. There is another important principle, called the principle of mathematical induction, which we will introduce in Chapter 18 and Project 29.3. If you work the project, you will learn that induction is one of the five Peano axioms that can be used to construct the natural numbers. For now, we will state and use the well-ordering principle without proof. In Chapter 18, we will show that the well-ordering principle of $\mathbb{N}$ and the principle of mathematical induction are equivalent.

Well-ordering principle of the natural numbers. *Every nonempty subset of the natural numbers contains a minimum.*

As a consequence of the well-ordering principle, we obtain an interesting theorem about where the rationals "live." The next theorem suggests that they can really fill up space! The curious thing about them, which we will return to in Chapter 23, is that there really aren't that many of them.

Theorem 12.12. *Let a and b be two real numbers satisfying $a < b$. Then there is a rational number c such that $a < c < b$.*

The proof of this result will be much easier to follow if you understand the basic idea. It's this: if the difference between a and b were greater than one, then there would have to be an integer m with $a < m < b$ and we would be done. Of course, the difference does not have to be greater than one, but we can sort of force it to be: Look at $b - a$ and multiply by an integer n so that $n(b - a) > 1$. Now the difference between nb and na is greater than one, so there has to be an integer m between them (but this needs proof). So we will prove that there exists an integer m with $na < m < nb$. Divide by n to obtain the desired rational number, m/n.

Proof. As you will show in Problem 12.21, we may assume without loss of generality that $a > 0$. By Theorem 12.10 there is an integer n such that $n(b - a) > 1$. Thus,

$$nb > 1 + na. \tag{12.1}$$

Now consider the subset A of $\mathbb{N}$ defined by $A = \{r \in \mathbb{N} : na < r\}$. By Corollary 12.11, A is nonempty. The well-ordering principle implies that A has a minimum, which we call m. Thus $m \in A$, and from the definition of A we see that $na < m$; in other words, $a < m/n$. Let c be the rational number m/n. Then we have the lower inequality, $a < c$, and we are halfway there. For the upper inequality, note that $m - 1$ is not in the set A (what would happen if it were?) so $na \geq m - 1$. So, putting this together with equation 12.1 we get

$$nb > 1 + na \geq 1 + (m - 1).$$

So $nb > m$, and $b > m/n$. Now $c = m/n$ is a rational number between a and b, and this completes the proof. $\square$

Definitions

Definition 12.1. A nonempty subset A of $\mathbb{R}$ is **bounded above**, if there is a real number M such that $x \leq M$ for all $x \in A$. We say a nonempty subset A of $\mathbb{R}$ is **bounded below**, if there is a real number m such that $m \leq x$ for all $x \in A$.

Definition 12.2. A nonempty subset of $\mathbb{R}$ is **bounded** if it is bounded above and bounded below.

Definition 12.3. Let A be a nonempty subset of $\mathbb{R}$. A real number M that satisfies $x \leq M$ for all $x \in A$ is called an **upper bound** of A. We call a real number m that satisfies $m \leq x$ for all $x \in A$ a **lower bound** of A.

Definition 12.4. The real number M is the **maximum** of the subset A of $\mathbb{R}$, written $M = \max A$, if $M \in A$ and $x \leq M$ for all $x \in A$.

Definition 12.5. The real number m is the **minimum** of the subset A of $\mathbb{R}$, written $m = \min A$, if $m \in A$ and $m \leq x$ for all x in A.

Definition 12.6. Let A be a nonempty set of real numbers that is bounded above. Then a real number U is said to be the **supremum** of A or **least upper bound** of A, written $U = \sup A$, if

(i) $a \leq U$ for all $a \in A$, and
(ii) if $M \in \mathbb{R}$ satisfies $a \leq M$ for all $a \in A$, then $U \leq M$.

Definition 12.7. Let A be a nonempty set of real numbers that is bounded below. Then a real number ℓ is said to be the **infimum** of A or **greatest lower bound** of A, written $\ell = \inf A$, if

(i) $\ell \leq b$ for all $b \in A$, and
(ii) if $m \in \mathbb{R}$ satisfies $m \leq b$ for all $b \in A$, then $m \leq \ell$.

Solutions to Exercises

Solution (12.1). We include brief answers to each part here.

(a) This set is bounded, and therefore bounded above and below. Some possible upper bounds are $\sqrt{5}$, 3, and 121. Some possible lower bounds are $-\sqrt{5}, -10$, and -2π.
(b) This set is bounded above, and it is not bounded below. Some possible upper bounds are $5^{1/3}$, 10, and 21.3.
(c) This set is bounded, and therefore bounded above and below. Some possible upper bounds are 5, 121, and 1000. Some possible lower bounds are 0, -3, and -12.
(d) This set is bounded above and below, and therefore bounded. Every real number greater than or equal to $\sqrt{2}$ will work as an upper bound, and every real number less than or equal to $-\sqrt{2}$ will work as a lower bound.

Solution (12.4). Our intuition tells us that we have $\sup\{x \in \mathbb{R} : x^2 \leq 5\} = \sqrt{5}$, $\sup\{x \in \mathbb{R} : x^3 < 5\} = 5^{1/3}$, $\sup\{x \in \mathbb{N} : x \leq 5\} = 5$, and $\sup\{x \in \mathbb{Q} : x^2 < 2\} = \sqrt{2}$.

Solution (12.5). Let $M = \max A$. Then $a \leq M$ for all $a \in A$ and property (i) of the definition of supremum is fulfilled. Now suppose that K is a real number satisfying $a \leq K$ for all $a \in A$. Since M is in A we have in particular that $M \leq K$. Thus, property (ii) holds and $M = \sup A$.

By Lemma 12.3, the supremum of A is unique. Since $\max A$ is the supremum of A, it is also unique.

Solution (12.6).

(a) Let A be a nonempty set of real numbers that is bounded below. A real number m is the infimum (or greatest lower bound) of A if

(i) $a \geq m$ for all $a \in A$, and
(ii) if y is a real number satisfying $a \geq y$ for all $a \in A$, then $m \geq y$.

(b) Let $A = \{x \in \mathbb{R} : x > 10\}$. Then A is bounded below (e.g., by 0), nonempty (e.g., $20 \in A$), and $\inf A = 10$. This can be seen as follows.

  (i) $a \geq 10$ for all $a \in A$ by the definition of A.
  (ii) If y is a real number satisfying $a \geq y$ for all $a \in A$ and $y > 10$, then $10 < (y + 10)/2 < y$. Thus $(y + 10)/2 \in A$ and $(y + 10)/2 \not\geq y$. This is a contradiction and it follows that $y \leq 10$.

These two conditions imply that $\inf A = 10$.
Using the set A above, $0 \neq \inf A$ and $20 \neq \inf A$, giving two nonexamples.
We do not seem to be able to find a nonempty set that is bounded below and does not have an infimum. That no such set exists will follow from Exercise 12.9 below.

Solution (12.8).
  In this problem we assume that there is a real number, x, with the property that $x^2 = 5$. Similarly, we assume that there is a real number y with the property that $y^3 = 5$. A rigorous argument for the existence of x will be given in Theorem 13.2.
  First, let $S = \{x \in \mathbb{R} : x^2 \leq 5\}$. We claim that $m = -\sqrt{5} = \inf S$. If $x \in S$, then $x^2 \leq 5 = m^2$. Thus, $|x| \leq |m|$. Since $m < 0$, we have $m \leq |x| \leq -m$. So we conclude that $m \leq x$. Thus m is a lower bound of S. If v is any other lower bound of S, then $v \leq -\sqrt{5}$ because $-\sqrt{5} \in S$. Thus $v^2 \geq 5 = m^2$ and hence $v \leq m$. This establishes the claim. (This illustrates Exercise 12.5. We could also have used the result of this exercise and we will, from here on.) An entirely similar proof shows that $\sup S = \sqrt{5}$.
  For the second set, $T = \{x \in \mathbb{R} : x^3 < 5\}$, note that T is not bounded below. Hence it has no infimum. In Exercise 12.4 we claimed that $\sup T = 5^{1/3}$. Set $M = 5^{1/3}$. If $x \in T$, then $x^3 < 5 = M^3$. Hence $x < M$, showing that M is an upper bound. Suppose that U is an upper bound of T and that $U < M$. Then $U < (U + M)/2 < M$. Thus $((U + M)/2)^3 < M^3 = 5$. Hence $(U + M)/2 \in T$. This shows that U cannot be an upper bound and implies that $M = \sup T$.
  Finally, the third set is simply $W = \{0, 1, 2, 3, 4, 5\}$. We see that 0 is the minimum and 5 is the maximum. From Exercise 12.5, we conclude that the infimum is 0 and the supremum is 5.
  We note here that the fourth set is more complicated because of the requirement that x be a member of $\mathbb{Q}$.

Solution (12.9). Completeness axiom of $\mathbb{R}$; infimum version. *Every nonempty subset of real numbers that is bounded below has an infimum.*
  To show the equivalence of the two versions we first assume that every nonempty set of real numbers that is bounded above has a supremum. Let S be a nonempty subset of the reals that is bounded below. Define $T = \{x \in \mathbb{R} : -x \in S\}$. By the result of Problem 12.7, we conclude that T is bounded above. By the completeness axiom of $\mathbb{R}$, the set T has a supremum. Again using the result of Problem 12.7, we conclude that the set S has an infimum. This establishes the infimum version of the axiom.
  A similar proof establishes the fact that the infimum version of the completeness axiom implies the supremum version.

Problems

Problem 12.1. A student solved the following problem: Let $S = \mathbb{Q} \cap (0,4)$. Show that $\sup S = 4$. Below is the student's solution. Criticize it.

Not a proof (Student Solution). First we show that 4 is an upper bound. Let $x \in S$. Then $0 < x < 4$, so 4 is clearly an upper bound. Suppose to the contrary that 4 is not the supremum. Then there exists an upper bound u with $u < 4$. But $u < (u+4)/2$, and we have shown that u is not an upper bound of S. This shows that 4 must be the supremum.   ⁇

Problem 12.2. Consider the sets below. For each one, decide whether the set is bounded above. If it is, give the supremum in $\mathbb{R}$. Then decide whether or not the set is bounded below. If it is, give the infimum. Finally, decide whether or not the supremum is a maximum, and whether or not the infimum is a minimum:

(a) The closed interval $[0,4]$;
(b) The open interval $(0,4)$;
(c) The natural numbers $\mathbb{N}$;
(d) The set $[0, \sqrt{2}] \cap \mathbb{Q}$.

Problem 12.3. Consider the interval $(1,4)$ in $\mathbb{R}$. Show in detail

(a) that 4 is the supremum, and
(b) that 1.1 is not a lower bound.

Problem 12.4. Let $X = \mathscr{P}(\mathbb{N}) \setminus \{\emptyset\}$. For $R, S \in X$ we define $R \sim S$ if and only if $\min R = \min S$.

(a) Give an example of two different elements of X that are related to each other with respect to the relation $\sim$ defined above.
(b) Prove that $\sim$ as defined above is an equivalence relation on X.

Problem 12.5. Show that $\inf\{1/n : n \in \mathbb{Z}^+\} = 0$.

Problem 12.6. Show that $\sup\{1 - 1/n : n \in \mathbb{Z}^+\} = 1$.

Problem[#] **12.7.** Let $S \subseteq \mathbb{R}$ and define $T = \{x \in \mathbb{R} : -x \in S\}$.

(a) Prove that if S is nonempty and bounded below, then T is bounded above. Further, prove that if $\sup T$ exists, then $\inf S$ exists and $\inf S = -\sup T$.
(b) Prove that if S is nonempty and bounded above, then T is bounded below. Further, prove that if $\inf T$ exists, then $\sup S$ exists and $\sup S = -\inf T$.

Problem[#] **12.8.** Let S be a nonempty subset of $\mathbb{R}$. Prove that S is bounded if and only if there exists $M \in \mathbb{R}$ such that $|x| \leq M$ for all $x \in S$.

Problem[#] **12.9.** Let S be a nonempty bounded subset of $\mathbb{R}$. Show that $\inf S \leq \sup S$. Under what conditions on S would you have $\inf S = \sup S$?

Problem 12.10. Let S be a nonempty bounded subset of $\mathbb{R}$ and let u be a real number such that $u < \sup S$. Show that there exists $s \in S$ such that $u < s$.

Problem 12.11. Let S and T be nonempty bounded subsets of $\mathbb{R}$.

   (a) Show that $\sup(S \cup T) \geq \sup S$, and $\sup(S \cup T) \geq \sup T$.
   (b) Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.
   (c) Try to state the results of (a) and (b) in English, without using mathematical symbols.

Problem 12.12. Let $x \in \mathbb{R}$ and let S be a nonempty subset of $\mathbb{R}$ that is bounded above. We define a new set, $x + S$, by $x + S = \{x + s : s \in S\}$.

   (a) Prove that $x + S$ is bounded above.
   (b) Prove that $x + \sup S$ is an upper bound of $x + S$. Using this result, conclude that $\sup(x + S) \leq x + \sup S$.
   (c) Prove that $x + \sup S = \sup(x + S)$.

Problem 12.13. Let A and B denote nonempty bounded subsets of $\mathbb{R}$ and define $A \oplus B = \{x + y : x \in A, y \in B\}$.

   (a) Prove that $A \oplus B$ is also a nonempty bounded subset of $\mathbb{R}$.
   (b) Let $A = [1, 3]$ and $B = [2, 4]$. Find $A \oplus B$, $\sup A$, $\sup B$, and $\sup(A \oplus B)$.
   (c) Show that $\sup(A \oplus B) \leq \sup A + \sup B$.
   (d) Show that $\sup A + \sup B \leq \sup(A \oplus B)$.

Problem# 12.14. Prove Corollary 12.11.

Problem 12.15. Find the supremum and infimum of the set $(0, 2) \cap \mathbb{Q}$ and justify your answer. (If you haven't worked Problem 12.1, you might want to work it before solving this problem.)

Problem 12.16. Let ε be a positive real number. Prove that for every real number a, there exists a rational number b (depending on a) such that $|a - b| < \varepsilon$.

Problem# 12.17. Prove that if a subset A of the reals has an infimum, then the infimum is unique.

Problem 12.18. Prove that every nonempty bounded subset of $\mathbb{N}$ has a maximum.

Problem 12.19. Show that there does not exist a real number b such that $a \leq b$ for all $a \in \mathbb{R}$.

Problem 12.20. Let $a \in \mathbb{Q}$, $a \neq 0$, and $b \in \mathbb{R} \setminus \mathbb{Q}$. Prove the following:

   (a) $a + b \in \mathbb{R} \setminus \mathbb{Q}$;
   (b) $ab \in \mathbb{R} \setminus \mathbb{Q}$;
   (c) $1/b \in \mathbb{R} \setminus \mathbb{Q}$.

Problem[#] **12.21.** Suppose that we have established Theorem 12.10 for the case when $a > 0$; that is, suppose we know that if a and b are real numbers and $0 < a < b$, then there exists a rational number r with $a < r < b$. Show that it follows that for all real numbers x and y with $x < y$, there is a rational number s with $x < s < y$. (This is what we meant by "we may assume without loss of generality that $a > 0$." We suggest you add an appropriate real number M to x and y to reduce your problem to the case $0 < a < b$.)