Chapter 11
Partitions

It is sometimes helpful to split a nonempty set into disjoint smaller pieces. For example, we might have reason to split the integers into positive integers, negative integers, and the set containing zero alone. We often split the real numbers into rational numbers and irrational numbers, or we might want to break $\mathbb{R}^2$ down into distinct vertical lines. All of these are examples of partitioning a space.

Though we have an intuitive feel for what a partition is, we need a precise definition. We turn to that now. A **partition of a nonempty set** X is a collection $\mathscr{A}$ of subsets of X that satisfies the following three conditions.

(i) Every set $A \in \mathscr{A}$ is nonempty,
(ii) $\bigcup_{A \in \mathscr{A}} A = X$, and
(iii) for all $A, B \in \mathscr{A}$, if $A \cap B \neq \emptyset$, then $A = B$.

Looking back at the example of the integers that we discussed earlier, we see that the collection $\mathscr{A} = \{\mathbb{Z}^+, \mathbb{Z}^-, \{0\}\}$ is a partition of $\mathbb{Z}$.

Figure 11.1 provides a diagram of a partition of $X = \{a,b,c,d,e,f\}$ into sets A_1, A_2, and A_3, defined by

$$A_1 = \{a,b\}, \ A_2 = \{c,d,e\}, \text{ and } A_3 = \{f\}.$$

While it is often clear that the sets in the collection are nonempty, you should still check. Condition (ii) says that every element of X is in at least one of the sets in the collection. It's a sort of existence statement: for each element x of X, there exists a set A in $\mathscr{A}$ of which x is a member. The third condition is a fancy way of saying that two of the sets in the partition are disjoint or they are equal. This is a sort of uniqueness statement: if x belongs to two sets A and B, then the sets must be equal.

We hope the examples below will clarify these concepts.

Example 11.1. For each $x \in \mathbb{R}$, define $A_x = \{y \in \mathbb{R} : |x| = |y|\}$. We will show that $\{A_x : x \in \mathbb{R}\}$ forms a partition of $\mathbb{R}$.

Before we begin, note that our proposed partition is an indexed collection of sets, and that in this case it is possible that two different indices give rise to the same set.

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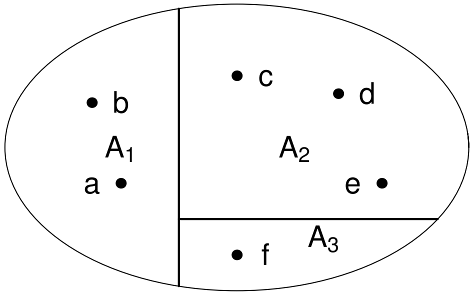

Fig. 11.1 A partition of $\{a,b,c,d,e,f\}$

For example, $A_1 = \{1,-1\}$ and $A_{-1} = \{-1,1\}$, so $A_1 \cap A_{-1} \neq \emptyset$. That will not be a problem though, and it does illustrate why we stated condition (iii) the way we did. It happens that $A_1 \cap A_{-1} \neq \emptyset$, but $A_1 = A_{-1}$, as required.

Proof. We note that all the sets A_α are nonempty. So there are two things left to show, namely, conditions (ii) and (iii) of the definition of partition. We begin by showing that (ii) holds; that is, $\bigcup_{x\in\mathbb{R}} A_x = \mathbb{R}$. First, $\bigcup_{x\in\mathbb{R}} A_x \subseteq \mathbb{R}$ since each $A_x \subseteq \mathbb{R}$. We show the reverse containment using an element-chasing argument. Let $y \in \mathbb{R}$. Then $|y| = |y|$, so $y \in A_y$. Since $y \in A_x$ for some x (namely, $x = y$), we may conclude that $y \in \bigcup_{x\in\mathbb{R}} A_x$. Thus $\bigcup_{x\in\mathbb{R}} A_x = \mathbb{R}$.

Next, suppose that $A_x \cap A_y \neq \emptyset$. Then we must show that $A_x = A_y$. By our assumption, there exists $z \in \mathbb{R}$ such that $z \in A_x \cap A_y$. Therefore, $|x| = |z|$ and $|y| = |z|$. In particular, $|x| = |y|$. So,

$$A_x = \{w \in \mathbb{R} : |x| = |w|\} = \{w \in \mathbb{R} : |y| = |w|\} = A_y. \qquad \square$$

Well, we showed the two sets A_x and A_y are equal with nary an element-chasing argument in sight. What happened? We certainly could have started with an element from one side and showed it was in the other, switched sides, repeated what we did, and then concluded we were done. But this is somewhat cumbersome and doesn't show us what is really going on. So from now on, even though we can use element-chasing, we are going to use whatever produces the most elegant or enlightening proof.

Exercise 11.2. For each $n \in \mathbb{N}$, let $A_n = [-n,n]$. Show that the collection $\mathscr{A} = \{A_n : n \in \mathbb{N}\}$ does not form a partition of $\mathbb{R}$. However, if we define $B_n = [n,n+1)$, then the collection $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ does partition $\mathbb{R}$. $\bigcirc$

In Example 11.1 and Exercise 11.2, the third condition may have reminded you of transitivity. If so, then it may not surprise you to learn that there is a connection between equivalence relations and partitions. As we have already seen, every equivalence relation on a set X gives rise to equivalence classes in a natural way. These equivalence classes are sets and as we will see, these sets partition our set X.

Conversely, a collection of sets that partitions a set X gives rise to an equivalence relation on X. How? Well, we say two elements in X are related if they belong to the same set of the partition. We shall now show that this relation is an equivalence relation. We can shorten our proof of this theorem, if we first prove something less ambitious. A helpful result that is used to prove a theorem is called a lemma. Lemmas are sometimes of independent interest.

Lemma 11.3. *Let X be a nonempty set and let $\sim$ be an equivalence relation on X. For two arbitrary elements x and y in X, if $E_x \cap E_y \neq \emptyset$, then $E_x = E_y$.*

The very first thing we should probably ask ourselves before beginning our proof is: What is E_x? If we don't know, we can't understand the proof. So, before reading the proof, we recall the definition:

$$E_x = \{z \in X : x \sim z\}.$$

Now the proof should be easy.

Proof. Let $z \in E_x$. Hence $x \sim z$. Since we assume that $E_x \cap E_y \neq \emptyset$, we may choose $w \in E_x \cap E_y$. Thus $w \in E_x$, and therefore $x \sim w$. Similarly, $w \in E_y$ and therefore $y \sim w$. By symmetry, $w \sim x$. So $y \sim w, w \sim x$, and $x \sim z$. By transitivity, $y \sim z$. Thus, $z \in E_y$, and we may conclude that $E_x \subseteq E_y$.

Exactly the same argument ($\star$) shows that $E_y \subseteq E_x$. Hence $E_x = E_y$. $\square$

One comment on the proof above: When we use the words "exactly the same argument," as in ($\star$), that means nothing would be changed except (possibly) the symbols. If you use words to that effect (like "similarly" or "exactly as above"), make sure that what you say is true. Now that we have our lemma, we turn to the proof of our main theorem.

Theorem 11.4. *Let $\sim$ be an equivalence relation on a nonempty set X. Then the indexed collection of equivalence classes $\{E_x : x \in X\}$ is a partition of X. Furthermore, if $\mathscr{A}$ is a partition of a nonempty set X and for $x, y \in X$ we define $x \sim y$ if and only if $x, y \in A$ for some $A \in \mathscr{A}$, then $\sim$ is an equivalence relation on X.*

Before beginning the proof, let's reflect on what we need to do. For the first assertion ("$\{E_x : x \in X\}$ is a partition") we need to show that the sets are nonempty, and satisfy conditions (ii) and (iii) in the definition of partition.

What do we expect to use? Our assumptions, of course. We are assuming $\sim$ is an equivalence relation, so we should use the fact that $\sim$ is reflexive, symmetric, and transitive. But that's only one direction—this would show that an equivalence relation gives rise to equivalence classes and these, in turn, form a partition of our set.

For the other direction, we want to show that if we have a relation defined by a partition $\mathscr{A}$, then the relation is an equivalence relation. So that means we must show that $\sim$ is reflexive, symmetric, and transitive. How will we do that? Well, probably the first thing to do is to make sure we know what $\sim$ is. Remember, $x \sim y$ if and only if there exists $A \in \mathscr{A}$ such that $x, y \in A$. Now, finally, we may begin.

Proof. First we'll show that given an equivalence relation on X, the indexed collection of sets $\{E_x : x \in X\}$ forms a partition of X. We first show that each E_x is nonempty. Since the relation is reflexive, $x \sim x$ for each $x \in X$. Thus $x \in E_x$ for each $x \in X$, and $E_x \neq \emptyset$.

Now we need to check condition (ii): that $\bigcup_{y \in X} E_y = X$. If $x \in X$, then we have just seen that $x \in E_x$. This shows that $x \in \bigcup_{y \in X} E_y$. Thus $X \subseteq \bigcup_{y \in X} E_y$. Since the opposite inclusion follows from the fact that $E_y \subseteq X$ for each $y \in X$, we know that $X = \bigcup_{y \in X} E_y$. Thus, condition (ii) holds.

To show that condition (iii) holds, suppose that for $x, y \in X$, we have $E_x \cap E_y \neq \emptyset$. By Lemma 11.3, we conclude that $E_x = E_y$, and condition (iii) holds. Thus, the set of equivalence classes $\{E_x : x \in X\}$ satisfies conditions (i), (ii), and (iii) and therefore forms a partition of X.

To prove the converse, suppose that $\mathscr{A}$ is a partition of X. By condition (ii), $X = \bigcup_{A \in \mathscr{A}} A$. Thus, for $x \in X$, there exists $A \in \mathscr{A}$ such that $x \in A$. Since x is in the same set as itself, $x \sim x$. Since x was arbitrary, $\sim$ is reflexive.

Suppose now that $x, y \in X$ and $x \sim y$. Then there exists $A \in \mathscr{A}$ such that $x, y \in A$. But if $x, y \in A$, then $y, x \in A$. Consequently, $y \sim x$. Therefore $\sim$ is symmetric.

Finally, suppose that $x, y, z \in X$ where $x \sim y$ and $y \sim z$. We must show that $x \sim z$. By the definition of $\sim$ we see that there exists $A \in \mathscr{A}$ such that $x, y \in A$, and there exists $B \in \mathscr{A}$ such that $y, z \in B$. Therefore, $A \cap B \neq \emptyset$. By property (iii) of partitions, $A = B$. Thus $x, z \in A$. Therefore, $x \sim z$, as desired. We conclude that the partition gives rise to an equivalence relation, since $\sim$ is symmetric, transitive, and reflexive. □

We will illustrate this connection between partitions and equivalence relations with two very simple examples.

Example 11.5. (a) Consider the set $X = \{1,2,3,4,5\}$, then the collection of sets $\mathscr{A} = \{\{1,2\},\{3\},\{4,5\}\}$ is a partition of X. Describe the corresponding equivalence relation.

We have exactly the following relations (and no others):

$$1 \sim 1, 2 \sim 2, 3 \sim 3, 4 \sim 4, 5 \sim 5, 1 \sim 2, 2 \sim 1, 4 \sim 5, 5 \sim 4$$

By Theorem 11.4, this relation is an equivalence relation.

(b) We consider the set $Y = \{1,2,3\}$ and define an equivalence relation on $\mathscr{P}(Y)$ by $A \sim B$ if and only if the number of elements of A is equal to the number of elements of B, for $A, B \in \mathscr{P}(Y)$.

This is clearly an equivalence relation and Theorem 11.4 shows that the following collection of sets, $\mathscr{B}$, is a partition of $\mathscr{P}(Y)$:

$$\mathscr{B} = \{\{\emptyset\},\{\{1\},\{2\},\{3\}\},\{\{1,2\},\{1,3\},\{2,3\}\},\{\{1,2,3\}\}\}.$$ ○

Theorem 11.4 shows how one can obtain a partition from an equivalence relation and vice versa. Now suppose we start with a partition $\mathscr{A}_1$ and use the theorem to get a corresponding equivalence relation. Then we use that equivalence relation to obtain a second partition $\mathscr{A}_2$. What's the relation between $\mathscr{A}_1$ and $\mathscr{A}_2$? And what about turning this procedure around: Say we start with an equivalence relation R_1, and we use it to determine a partition. Then we use this partition to obtain an equivalence relation R_2. What's the relation between R_1 and R_2? You're probably thinking that in both cases you will end up where you started. If you work Problems 11.14 and 11.15 you will find out that your intuition is correct: $\mathscr{A}_1$ and $\mathscr{A}_2$ will be the same, and so will R_1 and R_2. We say that Theorem 11.4 provides a one-to-one correspondence between the equivalence relations on a nonempty set X and the partitions of the set X.

Exercise 11.6. For $r \in \mathbb{R}$, let $A_r = \{(x,y) \in \mathbb{R}^2 : x+y=r\}$. Show that $\{A_r : r \in \mathbb{R}\}$ is a partition of $\mathbb{R}^2$. Then describe the equivalence relation and equivalence classes associated with this partition. ○

Definition

Definition 11.1. A **partition of a nonempty set** X is a collection $\mathscr{A}$ of subsets of X that satisfies the following three conditions.

(i) Every set $A \in \mathscr{A}$ is nonempty,
(ii) $\bigcup_{A \in \mathscr{A}} A = X$, and
(iii) for all $A, B \in \mathscr{A}$, if $A \cap B \neq \emptyset$, then $A = B$.

Solutions to Exercises

Solution (11.2). The collection $\mathscr{A}$ does not partition $\mathbb{R}$ because condition (iii) is not satisfied: $A_1 \cap A_2 \neq \emptyset$, but $A_1 \neq A_2$. The collection $\mathscr{B}$ does partition $\mathbb{R}$: For each $n \in \mathbb{Z}$, the set B_n is nonempty, the union of the sets satisfies

$$\bigcup_{B \in \mathscr{B}} B = \bigcup_{n \in \mathbb{Z}} B_n = \bigcup_{n \in \mathbb{Z}} [n, n+1) = \mathbb{R},$$

and if $B_n, B_m \in \mathscr{B}$ with $B_n \cap B_m \neq \emptyset$, then $[n,n+1) \cap [m,m+1) \neq \emptyset$. Since m and n are integers, the intervals $[n,n+1)$ and $[m,m+1)$ are either equal or disjoint. We conclude that $[n,n+1) = [m,m+1)$; in other words, $B_n = B_m$.

Solution (11.6). Note that for $r \in \mathbb{R}$, the ordered pair $(0,r)$ satisfies the condition $0+r=r$. Thus $(0,r) \in A_r$ and A_r is nonempty. Since it is clear that $\bigcup_{r \in \mathbb{R}} A_r \subseteq \mathbb{R}^2$, we check the reverse inclusion. If $(u,v) \in \mathbb{R}^2$, then $s = u+v \in \mathbb{R}$ and consequently $(u,v) \in A_s$. Thus $(u,v) \in \bigcup_{r \in \mathbb{R}} A_r$, and $\bigcup_{r \in \mathbb{R}} A_r = \mathbb{R}^2$, completing the proof of condition (ii) in the definition of partition. Finally, suppose that $A_r \cap A_s \neq \emptyset$. Then there exists $(u,v) \in A_r \cap A_s$. By the definition of A_r and A_s this means that $r = u+v = s$. Thus, $A_r = A_s$, as desired.

We associate an equivalence relation on $\mathbb{R}^2$ as follows. For $(x,y),(u,v) \in \mathbb{R}^2$, we will say $(x,y) \sim (u,v)$ if and only if $x+y = u+v$. By our work above and Theorem 11.4, this is an equivalence relation on $\mathbb{R}^2$. The equivalence classes are the lines with slope -1.

In the two exercises in this chapter, the third condition (of partition) is satisfied because the indices (n and m in Exercise 11.2, and r and s in Exercise 11.6) are equal. Though this can happen, Example 11.1 shows that the two sets can be equal without the indices being equal. Condition (iii) in the definition of partition requires that we show that the two sets are equal—not the two indices.

Problems

Problem 11.1. For each of the relations in Problem 10.2 that you determined to be equivalence relations, describe the partition associated with it.

Problem 11.2. Determine whether or not the following are equivalence relations on $\mathbb{R}^2$. If they are, describe the partition associated with each:

(a) $(x,y) \sim (w,z)$ if and only if $y = w$;
(b) $(x,y) \sim (w,z)$ if and only if $x^2 = w^2$;
(c) $(x,y) \sim (w,z)$ if and only if $xw = yz$.

Problem 11.3. None of the following partitions $\mathbb{R}$. Say precisely why the collection of sets fails to be a partition of $\mathbb{R}$.

(a) $\{\{x \in \mathbb{R} : |x| = r\} : r \in \mathbb{R}\}$;
(b) $\{\{x \in \mathbb{R} : n < |x| \leq n+1\} : n \in \mathbb{N}\}$;
(c) $\{\mathbb{R} \setminus \mathbb{R}^+, \mathbb{R} \setminus \mathbb{R}^-\}$.

Problem 11.4. (a) For each $r \in \mathbb{R}$, let $A_r = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = r\}$. Is this a partition of $\mathbb{R}^3$? If so, give a geometric description of the partitioning sets.
(b) For each $r \in \mathbb{R}$, let $A_r = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = r^2\}$. Is this a partition of $\mathbb{R}^3$? If so, give a geometric description of the partitioning sets.

Problem 11.5. (a) Let $A = \{1,2,\ldots,10\}$. Describe a partition of A that gives rise to five distinct partitioning sets.
(b) Describe a partition of $\mathbb{Z}$ that gives rise to five distinct partitioning sets.
(c) Can you describe a partition of $\mathbb{R}$ that gives rise to five distinct partitioning sets?

Problem 11.6. (a) Suppose that we partition $\mathbb{R}^3$ into horizontal planes. What equivalence relation is associated with this partition?

(b) Suppose that we partition $\mathbb{R}^3$ into concentric spheres, centered at $(0,0,0)$. What equivalence relation is associated with this partition?

Problem 11.7. Suppose that we look at the set X containing all circles in the plane. Define an equivalence relation on this set of circles by $c \sim d$ if and only if the circles c and d have the same center. Describe the partition associated with this equivalence relation.

Problem 11.8. Consider the set P of polynomials with real coefficients. Decide whether or not each of the following collection of sets determines a partition of P. If you decide that it does determine a partition, show it carefully. If you decide that it does not determine a partition, list the part(s) of the definition that is (are) not satisfied and justify your claim with an example. (See Problem 10.13 for more information about polynomials.)

(a) For $m \in \mathbb{N}$, let A_m denote the set of polynomials of degree m. The collection of sets is $\{A_m : m \in \mathbb{N}\}$.
(b) For $c \in \mathbb{R}$, let A_c denote the set of polynomials p such that $p(0) = c$. The collection of sets is $\{A_c : c \in \mathbb{R}\}$.
(c) For a polynomial q, let A_q denote the set of all polynomials p such that q is a factor of p; that is, there is a polynomial r such that $p = qr$. The collection of sets is $\{A_q : q \in P\}$.
(d) For $c \in \mathbb{R}$, let A_c denote the set of polynomials p such that $p(c) = 0$. The collection of sets is $\{A_c : c \in \mathbb{R}\}$.

Problem 11.9. For two nonempty disjoint sets, I and J, let $\{A_\alpha : \alpha \in I\}$ be a partition of $\mathbb{R}^+$ and $\{A_\alpha : \alpha \in J\}$ be a partition of $\mathbb{R}^- \cup \{0\}$. Prove that $\{A_\alpha : \alpha \in I \cup J\}$ is a partition of $\mathbb{R}$.

Problem 11.10. Let X be a nonempty set and $\{A_\alpha : \alpha \in I\}$ be a partition of X.

(a) Let B be a subset of X such that $A_\alpha \cap B \neq \emptyset$ for every $\alpha \in I$. Is $\{A_\alpha \cap B : \alpha \in I\}$ a partition of B? Prove it or give a counterexample.
(b) Suppose further that $A_\alpha \neq X$ for every $\alpha \in I$. Is $\{X \setminus A_\alpha : \alpha \in I\}$ a partition of X? Prove it or show that it is not a partition. (Make sure you consider each of the following cases: the partition $\{A_\alpha : \alpha \in I\}$ has zero, one, two, or at least three elements.)

Problem 11.11. Recall that for an integer n, the symbol $3|n$ means that there exists $m \in \mathbb{Z}$ such that $n = 3m$. For each integer i, where $i = 0,1,2$, we define the set $A_i = \{x \in \mathbb{Z} : 3|(x-i)\}$. Show that $\{A_i : i = 0,1,2\}$ is a partition of $\mathbb{Z}$.

Problem 11.12. Let $A = \{x \in \mathbb{R} : x > 0\}$ and $B = \{x \in \mathbb{R} : x \leq 0\}$. Describe the equivalence relation on $\mathbb{R}$ that is associated with the partition $\{A,B\}$ of $\mathbb{R}$.

Problem 11.13. For each $r \in \mathbb{R}$ define a subset of $\mathbb{R}^3$ by

$$A_r = \{(x,y,z) : y^2 + z^2 = r^2\}.$$

(a) Prove that $\{A_r : r \in \mathbb{R}\}$ is a partition of $\mathbb{R}^3$.
(b) Describe the equivalence relation associated with this partition geometrically.

Problem 11.14. Let X be a nonempty set with an equivalence relation $\sim$ on it. According to Theorem 11.4, this equivalence relation gives rise to a partition on X which in turn defines a new equivalence relation $\bar{\sim}$ on X. Prove that $\sim$ and $\bar{\sim}$ are the same equivalence relation on X.

Problem 11.15. Let $\mathscr{A}$ be a partition of a nonempty set X. By Theorem 11.4, this partition gives rise to an equivalence relation on X and this equivalence relation, in turn, defines a new partition $\mathscr{B}$ of X. Prove that $\mathscr{A} = \mathscr{B}$.

Problem 11.16. Let $X = \{x \in \mathbb{Z}^+ : x \leq 100\}$; that is, X is the set of all integers from 1 to 100. For each $Y \in \mathscr{P}(X)$ we define $A_Y = \{Z \in \mathscr{P}(X) : Y$ and Z have the same number of elements$\}$.

(a) Prove that $\{A_Y : Y \in \mathscr{P}(X)\}$ partitions $\mathscr{P}(X)$.
(b) Let $\sim$ denote the equivalence relation on $\mathscr{P}(X)$ that is associated with this partition (according to Theorem 11.4). If possible, find A, B, and C such that

1. $A \sim \{1,2,3\}$ and $A \neq \{1,2,3\}$
2. $B \not\sim \{7,8,9\}$
3. $C \sim X$ and $C \neq X$.

Problem 11.17. Let the collection $\mathscr{S} = \{A_\alpha : \alpha \in I\}$ be a partition of a nonempty set A and $\mathscr{T} = \{B_\beta : \beta \in J\}$ be a partition of a nonempty set B. We define the following collection of sets $\mathscr{W} = \{A_\alpha \times B_\beta : \alpha \in I$ and $\beta \in J\}$.

(a) Give a "small" example of this situation using $A = B = \mathbb{R}$ and two different partitions $\mathscr{S}$ and $\mathscr{T}$, each having exactly two sets. (Specify $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{W}$.)
(b) Does the collection $\mathscr{W}$ for your example above partition the plane $\mathbb{R} \times \mathbb{R}$? (No proof needed for this part, just state your answer.)
(c) Prove that with $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{W}$ as defined at the beginning of this problem, $\mathscr{W}$ is a partition of $A \times B$.

Problem 11.18. Let X be a nonempty set and $\mathscr{A}$ a partition of X. Is $\{\mathscr{P}(A) : A \in \mathscr{A}\}$ a partition of $\mathscr{P}(X)$? If it is, prove it. If it isn't, give a simple example of sets X and $\mathscr{A}$ such that $\{\mathscr{P}(A) : A \in \mathscr{A}\}$ is not a partition of $\mathscr{P}(X)$.

Problem 11.19. Consider the set consisting of all three-letter "words" that you can make with the two letters O and T, so that

$$S = \{OOO, TOO, OTO, OOT, OTT, TOT, TTO, TTT\}.$$

(a) Define an equivalence relation on the set S of "words" by saying that $x \sim y$ for $x, y \in S$ if it is possible to rearrange the letters in x to obtain y. Show that this is an equivalence relation and find the partition $\mathscr{A}$ associated with $\sim$.

(b) Now define $x \sim_1 y$ for $x, y \in S$ if and only if there is a set $A \in \mathscr{A}$ such that $x, y \in A$. Is this an equivalence relation? If so, find the associated equivalence classes.

(c) Is your answer to part (b) what you expected? Why or why not? (If a theorem applies, state it.)

Problem 11.20. Do there exist sets X for which $\mathscr{P}(X)$ is a partition of X? If so, give an example of such a set X. If there do not exist such sets, prove that $\mathscr{P}(X)$ is not a partition of X for all sets X.

Problem 11.21. For $k \in \mathbb{Z}$, we define $A_k = \{x \in \mathbb{Z} : x = 5\ell + k \text{ for some } \ell \in \mathbb{Z}\}$.

(a) Prove that $\{A_k : k \in \mathbb{Z}\}$ partitions $\mathbb{Z}$.

(b) We denote by $\sim$ the equivalence relation on $\mathbb{Z}$ that is obtained from the partition of part (a). Give as simple a description of $\sim$ as possible; that is, given two elements x and y in $\mathbb{Z}$, find a simple condition "$C(x, y)$" on x and y so that $x \sim y$ if and only if "$C(x, y)$" holds.

Tips on Putting It All Together

> In my own writing, I average about five pages a day. Unfortunately, they're all the same page.—Michael Alley [5, p. 246]

Now we will build upon the foundations we have created.

- In each section, work through the definitions. (Check "Tips on Definitions.") If you don't know the definitions, you cannot get started. So the first step is to make sure that you have *mastered* them.
- Next, learn and understand all theorems. You don't have to memorize their number, of course, but you should know by name each theorem that has a name. Make sure you can restate every theorem in the text correctly.
- If you are asked to prove something, look for a proof or theorem that reminds you of your problem. Read it over.
- If your problem is too difficult, try a simpler one first.
- Whenever you claim something is true, say why (at least to yourself, if it is minor, and to the reader, if it is major). Is it a definition? a theorem? Are the techniques the same as in a proof everyone has already seen? If you are writing up your homework, tell the grader which theorem (now you should give a number) or what definition you are using.
- If you can check your solution, do so. Is your answer reasonable?
- Does your theorem make sense? Does it agree with other theorems in the text? (It's supposed to agree, of course!) Does your proof use everything you were given?
- Your first draft is precisely that. No one should have to read someone else's first draft. Work out the solution, write it up, put it away, read it again, and rewrite it.