Chapter 5
Proof Techniques

> The fact that Adolphe Sax invented the saxophone hardly proves that William Tell invented the telephone.—Markus M. Ronner (Swiss journalist)[1]

In this chapter, we introduce you to some of the most common proof techniques. The three methods we will examine in this section are:

- direct proof (just get started and keep going),
- proof by contradiction (show that the negation of the statement you wish to prove implies the impossible), and
- proof in cases (which may be used when conditions dictate that different situations occur).

There are many more. For example, another proof technique that you may be familiar with from the study of calculus is the method of exhaustion, such as computing area or volume calculations by "filling up the object" with a sequence of more familiar smaller sets. Sometimes these techniques are used in combination. Some other methods, such as proof of existence and uniqueness of an object or proof by induction, will appear in subsequent chapters.

The first example is a direct proof. We want to show that "If A, then B is true." So we do it in our most direct manner: We start with A and keep going until we get to B. Before getting started, we make sure we know the meaning of every word in the implication and we try to make sure that the implication is true.

Theorem 5.1. *If a, b, and c are integers such that a divides b and a divides c, then a divides $b+c$.*

"Understanding the problem." Okay, before we get started, let's identify the hypothesis and conclusion. What are they? The hypothesis is a, b, and c are integers such that a divides b and a divides c. We get to start with that. What does a divides b mean? We say that a nonzero integer a **divides** b if there is an integer n such that $b = an$. There is even a standard symbol for this, namely, $a|b$. Since we have already

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[1] The translation is ours.

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defined everything here, we understand the problem and we feel confident—raring to go, in fact. What's the conclusion we need to come to? The conclusion is a divides $b+c$, and we know what this means because we understand "divides."

"Devising a plan." So we know that, in the notation we used above, $b = am$ and $c = an$ where m and n are both integers. We need to show that a divides $b+c$, or that there is an integer j with $b+c = aj$. Looking at what we were given and what the desired conclusion is should suggest the plan.

Proof. Since a, b, and c are integers such that $a|b$ and $a|c$, we know that there exist integers m and n such that $b = am$ and $c = an$. Therefore, $b+c = am+an = a(m+n)$. Since $m+n$ is an integer and $a \neq 0$, we conclude that $a|(b+c)$. $\square$

"Looking back." Let's admire this proof for a minute. It's so lovely. There are complete sentences, periods, and all symbols are carefully defined. We say where we are starting; that is, what the assumption is, and we end by saying what the conclusion is. Just in case the reader hasn't noticed, though, we indicate that we are done by adding the little box, $\square$. Other people use Q.E.D. *(quod erat demonstrandum* which is Latin for *which was to be demonstrated).* Your proofs should be just as appealing as the one above.

What follows is an example of a proof by contradiction, sometimes referred to as *reductio ad absurdum*. The idea of such a proof is that we suppose that what we wish to conclude is false and show that something really silly happens (hence the absurdum). Below is an example of this idea that goes back to the Pythagoreans. This is one of two proofs presented by G. H. Hardy in his famous book *A Mathematician's Apology* [45], as an example of a beautiful proof. (The first proof in Hardy's text is in the problems. If you haven't read his book, it is another one that we highly recommend.)

Theorem 5.2. *The number $\sqrt{2}$ is not rational.*

"Understanding the problem." Before we begin, we make sure that we know what all the words mean, what we are assuming, and what we are trying to prove. A rational number is a number of the form p/q where p and q are integers, and q is nonzero. So we need to show that $\sqrt{2}$ is not of this form; that is, there are no integers p and q (with q nonzero) such that $\sqrt{2} = p/q$. That may seem like a tall order, since it seems to mean we have to look through all possible integers! This leads directly to:

"Devising a plan." Perhaps it would be easiest to assume $\sqrt{2} = p/q$ (with p and q integers and $q \neq 0$) and see what, if anything, happens. This is precisely the idea behind proof by contradiction.

Proof. Suppose, to the contrary, that $\sqrt{2}$ is rational. Then there exist integers p and q (with q nonzero) such that $\sqrt{2} = p/q$. We may assume that p and q have no common factor, for if they did, we would simplify and begin again. Now, we have that $\sqrt{2}q = p$. Squaring both sides, we obtain $2q^2 = p^2$. Thus p^2 is even. Since p^2 is even, we know from Problem 3.2 that p must be even. Therefore, $p = 2m$ for some integer m. This means that $2q^2 = 4m^2$. Dividing, we see that $q^2 = 2m^2$. But this means that q^2 is even. Again we know from Problem 3.2 that q is even. So p and q have a common factor 2, which is completely absurd, since we assumed they had no common factor. Therefore our assumption that $\sqrt{2}$ is rational must be wrong and we have completed the proof of the theorem. □

"Looking back." Note that we slipped in a reference to Problem 3.2. If we hadn't, you would have read "Since p^2 is even, p must be even." Your reaction to this could have been "Oh yeah, we did that already." That's fine. But you could also have stopped, tried to think about why it is true, tried to prove it, and so on. That's fine too, in some sense, but you don't want to re-prove everything we have already done. So if the writer tells the reader why something is true, it saves the reader valuable time. Or, you could also have skipped right over it, never worrying about why it is true. That's not fine. You need to understand each sentence in a proof!

Knowing how to split a proof into cases, which we will refer to as a "proof in cases," is something that will be extremely useful too. Here is an example of something defined in cases. Once we understand this definition, we'll prove something using it.

For a real number x, the **absolute value** of x is defined in cases by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} .$$

Is this what you were expecting the definition to be? If not, let's make sure it agrees with what you were expecting. If $x = 3$, then $x \geq 0$, and we conclude that $|3| = 3$. If $x = -3$, then $x < 0$, and we conclude that $|-3| = -(-3) = 3$. If you feel comfortable with this definition, you are ready to move on to the theorem. If not, work out a few more examples and then move on.

Theorem 5.3. *Let x and y be real numbers. Then $|xy| = |x||y|$.*

We made sure that we understood the definition of absolute value before proceeding to the theorem, so we understand the problem. Let's think about devising a plan.

"Devising a plan." Absolute value was defined in cases, and therefore $|xy|$ depends on whether $xy \geq 0$ or $xy < 0$. The first, $xy \geq 0$, is actually two cases again: $xy > 0$ or $xy = 0$. What are the possibilities? Well, $xy > 0$ would mean that both $x > 0$ and $y > 0$, or both $x < 0$ and $y < 0$. The case $xy = 0$ would mean that $x = 0$ or $y = 0$. The final possibility, $xy < 0$, would mean that one of the two, x or y, is negative and the other is positive. It seems that we have four cases to consider: both x and y positive, both negative, at least one of the numbers is zero, and one of the two numbers negative while the other is positive.

Proof. First, suppose that $x > 0$ and $y > 0$. Then $xy > 0$ and we have $|xy| = xy$, $|x| = x$, and $|y| = y$. Therefore,

$$|xy| = xy = |x||y| ,$$

and we have established the result in this case.

Second, suppose that $x < 0$ and $y < 0$. Then $xy > 0$ and we have $|xy| = xy$, $|x| = -x$, and $|y| = -y$. Therefore,

$$|xy| = xy = (-x)(-y) = |x||y| \,,$$

and we have the result for this case as well.

Third, suppose that $x = 0$ or $y = 0$. Then $xy = 0$ and we have $|xy| = 0$. Also, $|x| = 0$ or $|y| = 0$. Therefore,

$$|xy| = 0 = |x||y| \,,$$

establishing the result in this case too.

For our final case, suppose that one number is positive and the other is negative. Thus, we may assume that $x < 0$ and $y > 0$. Then $xy < 0$ and we have $|xy| = -(xy)$, $|x| = -x$, and $|y| = y$. Therefore,

$$|xy| = -(xy) = (-x)y = |x||y| \,.$$

We have now established the result for all four possible cases and we may conclude that $|xy| = |x||y|$ for all real numbers x and y. □

Once again, look at the form of the proof. There are four cases and we tell the reader which case we are discussing before we discuss it. We can conclude something in each case, but it isn't until we cover all four possible cases that we can write "we may conclude that $|xy| = |x||y|$ for all real numbers x and y."

It will also be helpful to know how to show something is not true. A statement whose truth is anticipated, but for which we have no proof yet is called a conjecture. There are many different ways that one might arrive at a conjecture. It can be due to the intuition or insight of a great mathematician, or it can be a generalization of observations gleaned from many examples. The latter has become more common in recent years, in part due to the capabilities of powerful calculators and computers. Once we find a proof, the conjecture turns into a theorem. One of the most famous examples in recent history is a proof by Andrew Wiles. In 1995, Wiles turned Fermat's last conjecture into Fermat's last theorem, [111]. Fermat's last theorem was a conjecture for over 350 years. (Watch the excellent Nova episode "The Proof" for the full story on the history of Fermat's last theorem, [10].)

Another recent and major achievement was the proof of the 100-year-old Poincaré conjecture by Grigori Perelman. In 2002 and 2003, Perelman made three papers available on the Internet and, by 2006, experts were convinced that Perelman's papers provided a positive answer to the Poincaré conjecture. Such an achievement was well worthy of the Fields Medal, one of the top two honors a mathematician can receive and often considered the equivalent of the Nobel Prize (with the other award being the Abel Prize). Perelman declined the Fields Medal. Because the Poincaré conjecture is one of the seven millennium problems for which the Clay Mathematics Institute has offered a one million dollar award for the solution, Perelman was offered this award. As expected, he declined this prize, too. There is a biography of Grigori Perelman and the culture of Russian mathematicians in the 20th century,

[33]. The Clay Institute has a website with all seven problems listed, and some of them are formulated as conjectures. (See [19].)

It's important to note that just because you believe something might be true, doesn't mean that it necessarily is true. Sometimes you will find that a conjecture someone else has made (or even one that you have made) is, in fact, false. In these cases, you need to find an example of something that satisfies the hypotheses of your conjecture, but not the conclusion. An example is the following conjecture of Pierre de Fermat—one of the very few of his conjectures that turned out to be wrong.

Consider numbers of the form $2^{2^m}+1$, where m is a natural number. The first number, $2^{2^0}+1=3$, is prime. The second, $2^{2^1}+1=5$, is also prime, as are the third, fourth, and fifth numbers. In fact, Fermat conjectured that if m is a nonnegative integer, then $2^{2^m}+1$ is prime. In 1732, the Swiss mathematician Leonhard Euler showed that this was false by showing that the sixth number in this list, $2^{2^5}+1$ $= 4294967297$, can be factored. In fact, our calculator tells us that

$$2^{2^5}+1 = 641 \cdot 6700417\,.$$

Thus Fermat's conjecture is false.

An example that shows that a statement is false is called a counterexample. You only need one to show something is false!

We end this chapter by discussing one final statement form that appears frequently in mathematics, and this is a statement of the form "*A* if and only if *B*." This is a really convenient way of saying two things: *A* if *B* and *B* if *A*. In other words, it's two statements in one! We have already seen examples of this; for example, in Problem 3.2 (c) you summarized a result as

An integer x is odd if and only if x^2 is odd.

We proved this statement in two installments. First we showed the implication of Theorem 3.3: *For all real numbers x, if x^2 is odd, then x is odd.* We proved the converse in Problem 3.2 (a): *For all real numbers x, if x is odd, then x^2 is odd.* This approach, proving the necessity and sufficiency, is what you will do each and every time. So whenever you have to prove an "if and only if" statement you should rejoice: You get to give two proofs for one problem. To make your proof clear and easy to follow, it's a good idea to break the proof into two parts, indicate clearly which part you are proving, and indicate clearly when you have finished proving each part. Here's an example.

Example 5.4. Show that for a real number x, we get $-2 \le x < 1$ if and only if $(2x+1)/(x-1) \le 1$.

"Devising a plan." We have to prove an equivalence, so we have two separate problems, each of which is an implication. We will first show that for all real numbers x, if $-2 \le x$ and $x < 1$, then $(2x+1)/(x-1) \le 1$. A direct proof should work here. Then we will prove the converse: For all real numbers x, if $(2x+1)/(x-1) \le 1$, then $-2 \le x$ and $x < 1$. We will find it convenient to multiply the inequality by

$x-1$. Since we don't know whether $x-1$ is positive or negative, we will need to consider two cases. Consequently, we expect this proof to be a proof in cases.

Proof. We first assume that $-2 \leq x$ and $x < 1$. Adding $x+1$ to the first inequality, $-2 \leq x$, we get $x-1 \leq 2x+1$. Subtracting 1 from the inequality $x < 1$ leads to $x-1 < 0$. Therefore, when we divide by $x-1$, this "flips the sign." As a result we obtain $(2x+1)/(x-1) \leq 1$, as required.

For the converse we assume that $(\star)(2x+1)/(x-1) \leq 1$. Note first that this implies that $x \neq 1$. Now our proof will be in cases, $x-1 > 0$ and $x-1 < 0$. So first suppose that $x-1 > 0$. Multiplying $(\star)$ by $x-1$, we get $2x+1 \leq x-1$. This, and our assumption that $x-1 > 0$, implies $0 < x-1 \leq -3$. This is a contradiction and shows that this case is impossible. Now consider the remaining case, namely, $x-1 < 0$, or equivalently, $x < 1$. As before, this implies that $2x+1 \geq x-1$. This, in turn, implies that $x \geq -2$. We conclude that $-2 \leq x < 1$, completing the proof of the converse.

Since we have proved both the statement "$-2 \leq x < 1$ implies $(2x+1)/(x-1) \leq 1$," and its converse, we have completed the proof of the result. □

"Looking back." Reviewing the structure of the proof above, we note that we have proved an implication and its converse. We use certain phrases for the benefit of the reader; for example, when we say "We will first show that...," this is to let the reader know what's coming. Of course, the astute reader would figure it out, eventually, without being told. But most people like to be told where they are going before they get to their destination, and the same is true in mathematics. After indicating clearly what our hypothesis is, we work until we obtain the desired conclusion and we say when we have reached our conclusion. There are, sometimes, "if and only if" proofs in which each step of the proof is reversible, and some authors will work both directions simultaneously to save time and space. Even in this case, it is less confusing for the reader if you prove one direction and, after carefully checking that all steps are reversible, say something on the order of "since all the steps above are reversible, the converse follows." ○

It's very convenient, when taking notes or writing quickly, to follow the lead of many mathematicians and write "iff" for "if and only if." While it can be a real time saver, we don't recommend it in formal writing.

Definitions

Definition 5.1. A nonzero integer a **divides** an integer b if there is an integer n such that $b = an$. We write this as $a|b$.

Definition 5.2. For a real number x, the **absolute value** of x is defined to be

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} .$$

Problems

Problem 5.1. Below is the other proof Hardy chose to present ([45, pp. 92–94]). This theorem and its proof were known to Euclid, and appear in the *Elements* IX 20, [47]. Can you read and understand this proof? Read the whole thing. Underline anything you don't understand the first time. Reread it slower this time. Underline anything you can't figure out. You may need to spend 10 minutes on each sentence; you may not. Then write the general idea of the proof in "street talk." A bright, interested twelve-year-old should be able to follow your outline of the proof.

Before you begin, make sure you understand what will be assumed and what we will try to do. Make sure you know what all the words mean. "Infinite" has not yet been defined; prime number has.

Theorem 5.5. *There are infinitely many prime numbers.*

Proof. To prove this statement, suppose, to the contrary, that there are finitely many primes. Then we may write these finitely many primes in ascending order as

$$2, 3, 5, \ldots, N ,$$

where N is the largest prime. Now consider the number M defined by

$$M = (2 \cdot 3 \cdot 5 \cdot \cdots \cdot N) + 1 .$$

If M is prime, then M is a prime that is larger than the largest prime N. Therefore, we must conclude that M is not prime, and so it is divisible by some prime number, P. However, P must appear in the list of primes

$$2, 3, 5, \ldots, N ,$$

which we gave earlier. But when we divide M by P, we obtain a remainder of 1. Therefore, P cannot be a factor of M, and we have contradicted our assumption that there are finitely many primes. Thus, there exist infinitely many primes. □

Problem 5.2. Prove that if n is an integer, then $4n^2 + 4n + 8$ is an even integer. What kind of proof did you use?

Problem 5.3. Prove that if n is an integer, then $n^2 + 3n + 2$ is an even integer. What method of proof did you use?

Problem 5.4. In this problem, we outline a proof of the following theorem:

Theorem 5.6. *Let x and y be real numbers. If $xy > 1/2$, then $x^2 + y^2 > 1$.*

Your mission is to fill in the gaps and blanks, leaving no detail omitted.

Proof. The proof will proceed by (*insert name of proof technique or description of proof strategy here*). So suppose that $x^2 + y^2 \leq 1$. Now we know that $(x - y)^2 \geq 0$. (*Insert missing steps of proof here*.) Therefore $xy \leq 1/2$, and the proof is complete. □

Problem 5.5. In this problem, we outline a proof of the following theorem:

Theorem 5.7. *If n and m are nonzero integers, then* $n^2 - m^2 \neq 1$

Your mission is to fill in the gaps and blanks, leaving no detail omitted.

Proof. The proof will proceed by (*insert name of proof technique or description of proof strategy here*). So suppose that $n^2 - m^2 = 1$. Then $(n-m)(n+m) = 1$. Since (*insert relevant reason here*), we conclude that both factors, $(n-m)$ and $(n+m)$, are equal. Now we cannot have $n - m = n + m$, because (*insert reason here*). Therefore (*insert concluding sentence*). □

Problem 5.6. Consider the two statements about real numbers x and y.

Statement 1. If the product, xy, is not a rational number, then x or y must be an irrational number.

Statement 2. If x is a rational number and y is an irrational number, then $x + y$ is irrational.

(a) What method(s) of proof would you use to prove Statement 1? Statement 2? Why?
(b) Prove Statement 1 in the most efficient way possible. Do you need to modify your answer to the previous part of the problem? Explain.
(c) Prove Statement 2 in the most efficient way possible. Do you need to modify your answer to the previous part of the problem? Explain.

Your first proof of these statements may not be the best one. Run through the various techniques until you come up with the nicest proof of each statement!

Problem 5.7. Provide counterexamples to each of the following.

(a) Every odd number is prime.
(b) Every prime number is odd.
(c) For every real number x, we have $x^2 > 0$.
(d) For every real number $x \neq 0$, we have $1/x > 0$.
(e) Every function $f : \mathbb{R} \to \mathbb{R}$ is linear (of the form $mx + b$).

Problem 5.8. Define two sets, A and B, by

$$A = \{x \in \mathbb{Z} : x = 2n \text{ for some } n \in \mathbb{Z}\} \text{ and}$$
$$B = \{x \in \mathbb{Z} : x = 2m + 1 \text{ for some } m \in \mathbb{Z}\}.$$

(a) Using these definitions, give a rigorous proof that A and B have no element in common. Make sure you write out all details.
(b) What type of proof did you use in part (a)?

Problem 5.9. Let n be an integer. Prove that if n^2 is divisible by 3, then n is divisible by 3.

Problem 5.10. Show that $\sqrt{3}$ is not rational. (You may want to use the result of Problem 5.9 to work this problem.)

Problem 5.11. Prove that $\sqrt{2}+\sqrt{3}$ is not a rational number.

Problem 5.12. Prove that the square of an integer cannot be of the form $3k+2$, where k is an integer.

Problem 5.13. Prove that $\sin^2 x \leq |\sin x|$ for all $x \in \mathbb{R}$.

Problem# 5.14. Let x be a real number.

(a) Prove that $-|x| \leq x \leq |x|$.
(b) Let $a \geq 0$. Prove that $|x| \leq a$ if and only if $-a \leq x \leq a$.
(c) Use parts (a) and (b) to prove the theorem below.

Theorem 5.8 (The triangle inequality). *Let x and y be real numbers. Then*

$$|x+y| \leq |x|+|y| \ .$$

Problem 5.15. Prove the lower triangle inequality: Let x and y be real numbers. Then

$$||x|-|y|| \leq |x-y| \ .$$

There are many ways to do this, but we would like to firmly suggest using the triangle inequality (see Theorem 5.8 above).

Problem 5.16. Prove that for real numbers z and w,

$$|(1+z)(1+w)-1| \leq (1+|z|)(1+|w|)-1 \ .$$

Problem 5.17. Prove or refute the following conjecture. There are no positive integers x and y such that $x^2-y^2=10$.

Problem 5.18. Prove or refute the following conjecture. There are no positive integers x and y such that $x^2-3xy+2y^2=10$.

Problem 5.19. Prove that for all real numbers x, we have $x \leq -5$ if and only if $1 \leq (2x+3)/(x-2) \leq 2$.

Problem 5.20. Find all points in the xy-plane that lie on the surface

$$4 = 5(x-3)^2+3(y-\pi)^2+2(z+2)^2.$$

Write up your solution carefully. What method of proof did you use?

Problem 5.21. Let n be an integer. Prove that if $n^2-(n-2)^2$ is not divisible by 8, then n is even.

Problem 5.22. Let $n \in \mathbb{Z}^+$, $a_0, \ldots, a_n \in \mathbb{R}$, and $a_n \neq 0$. Prove that the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ can have at most n different roots. (Some remarks are in order here. To work this problem, you must understand it. Recall that $c \in \mathbb{R}$ is a root of a polynomial p if $p(c) = 0$. In order to solve the problem, you also need to recall that if c is a root of p, then $x - c$ is a factor of p.)

Problem 5.23. Let $a_1, \ldots, a_{10}$ be real numbers and $y = (x - a_1)\cdots(x - a_{10})$ be the equation of a curve in the plane $\mathbb{R}^2$. Prove that this curve has a horizontal tangent at $x = a_1$ if and only if $a_1 = a_j$ for some j in $\{2, 3, \ldots, 10\}$. (Note: This problem requires knowledge of calculus.)

Problem 5.24. Consider the following statement.

$$\forall x, (x \in \mathbb{Z}^+ \to \exists y, \exists z,$$
$$((y \in \mathbb{Q}) \wedge (z \in \mathbb{Q}) \wedge (yz \neq 0) \wedge (x^2 = y^2 + z^2))).$$

(a) Change this symbolic statement to an English sentence.
(b) Prove the statement you found in (a).

Tips on Definitions

> "When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean—neither more nor less."—Lewis Carroll, [17, p. 163]

In your previous courses, you may or may not have had to memorize definitions. Now it becomes essential that you memorize them, understand them, and investigate them before venturing on to use them. Here are some suggestions on how to do these things.

- The first step is to make sure you know the definition. This does not mean that you highlight it with a marker and read it over a few times. It means that you, first of all, understand it, and, second of all, memorize it. You must know whether the quantifiers are "for all" or "there exist," you must know what order they come in, you must watch the order on implications, and you must be sure that what you write is correct. Every single itty bitty detail must be correct or chances are that your definition is wrong.
- It's very difficult to memorize something you don't understand. So once you see a definition (in bold black print in this book) write it down and think about what it means.
- Give many examples, until you feel that you know what an example looks like.
- Negate the definition and try to find nonexamples (that show when things won't satisfy the definition).

- Go back and see if you can write out the definition without looking at it. Wait a few hours and do that again. If anything is out of place, ask yourself if it matters. If it does, repeat the appropriate steps here.
- Definitions are often stated as implications. This leads students to ask if the definition is an equivalence. The answer is "yes." Consider the following definition: "An integer m is even if there exists an integer n such that $m = 2n$." Since this is how we defined "even," we also mean that "if m is even, then there exists an integer n such that $m = 2n$."
- Note that you either have defined something correctly or not; you can't "get close" to a definition unless you get it right. For example, suppose you are asked to define *cat*. Let's say your definition is "A carnivorous mammal, domesticated since early times." Fair enough, it seems. Now suppose that working with this definition you try to purchase a mail-order cat and you receive a large St. Bernard. While this might upset you, it shouldn't surprise you. Your definition was close enough that you didn't receive a mail-order minivan, for example, but it was also wrong. In the same way, mathematicians will be upset if your definition includes things it shouldn't. Be very careful.

Some teachers and students find it helpful to make definition notebooks. In such a notebook, you will do all the steps above as often as necessary. We heartily recommend such an approach.

In this text, we summarize the definitions at the end of each chapter. This is intended to *help* you with a definition notebook; it is not meant to replace the notebook. Working the following additional problems will help you appreciate the value of a good definition.

Problem 5.25. Consider an object that all of you know well: A *car*.

(a) Define a *car*. (You may not say, "A car is an automobile." Why are we ruling this out?)
(b) Give an example.
(c) Give a few nonexamples. Your nonexample should be close to the definition of "car" but not close enough to be correct.

Problem 5.26. Definition. We will call a natural number an *s-difference* if it is the difference of the squares of two natural numbers.

(a) Give three different examples of s-differences.
(b) Write the definition symbolically. What should the universe be?
(c) Give two nonexamples from the universe that you chose in (b).

Problem 5.27. See Problem 5.26 for the definition of an s-difference.

Definition. We will call an s-difference *simply even*, if it is even but not a multiple of 4.

(a) Give an example of a simply even s-difference.
(b) Give two nonexamples.

(c) Is this a useful definition? Why or why not?

Problem 5.28. At the beginning of Chapter 4 we noted that *set* is an undefined expression. Some texts define *set* to be "a collection of objects." Why is this "definition" not satisfactory?

Problem 5.29. For the first three parts of this problem, you will need to use your knowledge of calculus.

(a) Consider the following: A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if it is continuous. Is this a definition of *differentiable*? Why or why not?
(b) Consider the following: A *sequence of real numbers* is a list of real numbers. Is this a definition of a *sequence of real numbers*? Why or why not?
(c) We found the following "definition" of *tangent line* online: "A line that touches a curve at a point without crossing over." What's wrong with this definition? (Say as much as you can!)
(d) Is this a definition of *perfect square*? A real number is a *perfect square* if there exists n such that $x = n^2$.