Chapter 10
Relations

In the last chapter we introduced relations. We will now look at three useful properties of relations.

Recall that "S is a relation on a set X" is one way of saying that S is a subset of $X \times X$, and therefore the elements of S are ordered pairs, (x,y). Many authors write $x \sim y$ rather than $(x,y) \in S$. Sometimes we will write $x \sim y$ and other times we will write $(x,y) \in S$, and this is exactly the same thing. So why do it? Because sometimes one notation is more convenient than the other. Use the next exercise to familiarize yourself with both notations.

Exercise 10.1. Let $S = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x > y\}$.

   (a) Sketch the set S.
   (b) With this relation is $1 \sim 2$?
   (c) With this relation is $3.5 \sim 2$?   ○

   At this point, we investigate one more relation that will be familiar to you.

Exercise 10.2. Let X be a nonempty set. We consider the usual set inclusion $\subset$ on the set $\mathscr{P}(X)$.

   (a) Using the ordered pair notation and calling this relation R, describe the set R.
   (b) Give an example of a set X and two sets $A, B \in \mathscr{P}(X)$ such that $A \subset B$.
   (c) Give an example of a set X and two different sets $A, B \in \mathscr{P}(X)$ such that $A \not\subset B$.
   (d) If $A, B \in \mathscr{P}(X)$ with $A \neq B$, can you conclude that $A \subset B$ or $B \subset A$?   ○

You will learn more about the relation in Exercise 10.2 above in Chapter 13. We now turn to another important type of relation.

A relation on a set X is said to be **reflexive** if $x \sim x$ for all $x \in X$. The relation is **symmetric** if for all $x, y \in X$, whenever $x \sim y$, then $y \sim x$. Finally, the relation is **transitive** if for all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$. If a relation is reflexive, symmetric, and transitive, then the relation is said to be an **equivalence relation**.

For the remainder of this chapter, we will look at examples and nonexamples of equivalence relations. There's something to note before we begin. To show that a relation is reflexive, we show that $x \sim x$ for all $x \in X$. But to show it is symmetric, we must choose two arbitrary elements of X, suppose that $x \sim y$, and then show that $y \sim x$. If we don't use the fact that $x \sim y$, we probably haven't done it correctly. Finally, to show that a relation is transitive, we must choose three arbitrary elements, suppose that $x \sim y$ and $y \sim z$, and then show that $x \sim z$. Remember to use your assumptions to show that a relation is symmetric or transitive.

Example 10.3. Define a relation on the real numbers $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Show that this relation is an equivalence relation.

Before we begin to show that this is an equivalence relation, we will do appropriate things to understand this definition. Here are a few examples of pairs that satisfy the relation:

$$3 \sim 4,\ 0 \sim -2384,\ 7 \sim 7,\ \pi \sim \pi + 7,\ -3.7 \sim 4.3.$$

On the other hand, the following pairs do not satisfy the relation:

$$3 \nsim 3.5,\ 0 \nsim \pi,\ -3.7 \nsim 3.7.$$

If you have a sense of what the relation does, you are ready to move on to the proof. (If you don't have a sense of what is happening, look for more examples and nonexamples.)

Proof. To show that this relation is reflexive, let $x \in \mathbb{R}$. Then $x - x = 0$. Since $0 \in \mathbb{Z}$, we see that $x - x \in \mathbb{Z}$. Therefore, $x \sim x$ for all $x \in \mathbb{R}$ and $\sim$ is reflexive.

To show that this relation is symmetric, let $x, y \in \mathbb{R}$. If $x \sim y$, then $x - y \in \mathbb{Z}$. But $y - x = -(x - y) \in \mathbb{Z}$, and therefore $y \sim x$. Hence this relation is symmetric.

To show that this relation is transitive, let $x, y, z \in \mathbb{R}$. If $x \sim y$ and $y \sim z$, then $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$. Now the sum of two integers is an integer and therefore $x - z = (x - y) + (y - z) \in \mathbb{Z}$. In other words, $x \sim z$. Thus $x \sim z$ is transitive.   □

Exercise 10.4. Let R be a relation on a set X. Write the definition of each of the following in symbolic notation: R is reflexive, symmetric, and transitive on X. Once you have completed this task, negate each of your (symbolic) definitions.   ○

In Example 10.3 it's interesting to try to describe, in words, the set of numbers that are related to 0, $1/2$, π, and x. As is the case with our examples and nonexamples appearing above, we hope that describing these sets will help us to more fully understand this relation.

For 0, we look for $\{x \in \mathbb{R} : x \sim 0\} = \{x \in \mathbb{R} : x - 0 \in \mathbb{Z}\}$. Thus, the set of elements related to 0 is just $\mathbb{Z}$.

For $1/2$, we look for $\{x \in \mathbb{R} : x \sim 1/2\} = \{x \in \mathbb{R} : x - 1/2 \in \mathbb{Z}\}$. Thus, the set of all elements related to $1/2$ is the set $\{1/2 + k : k \in \mathbb{Z}\}$.

For π, we look for $\{x \in \mathbb{R} : x - \pi \in \mathbb{Z}\}$. Thus, the set of all elements related to π is the set $\{\pi + k : k \in \mathbb{Z}\}$.

In general it appears that for every $x \in \mathbb{R}$, the set of all elements related to x is the set $\{x + k : k \in \mathbb{Z}\}$.

Once we have an equivalence relation on a set X, we define the **equivalence class** of an element $x \in X$ to be the set E_x where $E_x = \{y \in X : x \sim y\}$. It might help to think of equivalence classes as houses. "Being in the household of" is the equivalence relation on the people in the town and the houses form the equivalence classes. Saying the relation is reflexive is the same as saying each person lives in the same house as herself. The fact that the relation is symmetric implies that if Barbara lives in the same house as Bob, then Bob lives in the same house as Barbara. Finally, the fact that the relation is transitive implies that if Louis lives in the same house as Lois and Lois lives in the same house as Lilly, then Louis and Lilly live in the same house. If David, Esther, Florian, and Gregg all live in the house and David is your best friend, you'll probably call the house David's house. If Esther is your best buddy, though, you might call it Esther's house. No matter what you call the house, it's the same house and the same four people live there. And these are the main things to remember: equivalence classes are nonempty, x is always in the same class as x, if x is in the same class as y, then y is in the same class as x, and if x is in the same class as y and y is in the same class as z, then x and z are in the same class. When x and y are in the class, you might call the class E_x or you might call it E_y; either way, it will be the same class, as we will show more precisely below. It's time for a little practice with this new definition.

Using the equivalence class notation, we see that the sets of points related to 0, $1/2$, π, and x following Example 10.3 were actually descriptions of E_0, $E_{1/2}$, E_π, and E_x.

Exercise 10.5. Write each relation below using set notation. Then decide whether or not the following relations are reflexive, symmetric, or transitive. If they are all three, prove it and describe the equivalence classes. If they are not, give a particular example to show why the property fails to hold.

 (a) Define a relation on $\mathbb{Z}$ by $x \sim y$ if and only if $x = -y$.
 (b) Define a relation on $\mathbb{Z}$ by $x \sim y$ if and only if $x - y$ is even.
 (c) Define a relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by $(x, y) \sim (w, z)$ if and only if $xz = yw$. $\bigcirc$

Example 10.6. We look at $R = \{(1,1), (2,2), (3,3), (1,3)\}$. This is a relation on the set $X = \{1, 2, 3\}$. We can represent this relation with a diagram. Such a diagram is obtained using a point for each element of X, and these will be the vertices of the diagram. If $(x, y) \in R$, then we will sketch an arrow from x to y. These arrows are the directed edges of the diagram (that is, they are edges with an indicated direction). Figure 10.1 shows the diagram of our relation R. (Such a diagram is usually called a directed graph or a digraph in graph theory. Since we will soon use the term graph for something else, we will call such a picture a diagram.) $\bigcirc$
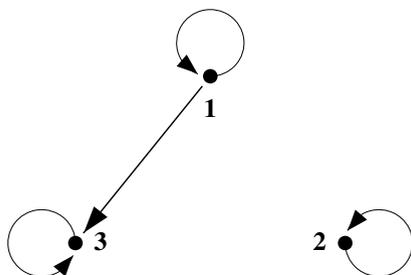
Fig. 10.1 Diagram of a relation on $\{1,2,3\}$

Exercise 10.7. Give a detailed description of a diagram that came from an equivalence relation; that is, give precise conditions that a diagram must satisfy in order to fulfill each of the conditions an equivalence relation must satisfy. How can you identify the equivalence classes in the diagram?   ○

Definitions

Definition 10.1. A relation $\sim$ on a set X is **reflexive** if $x \sim x$ for all $x \in X$.

Definition 10.2. A relation $\sim$ on a set X is **symmetric** if for all $x, y \in X$, whenever $x \sim y$, then $y \sim x$.

Definition 10.3. A relation $\sim$ on a set X is **transitive** if for all $x, y, z \in X$, whenever $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 10.4. A relation on a set X is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Definition 10.5. Given an equivalence relation $\sim$ on a set X, the **equivalence class** of $x \in X$ is the set $E_x = \{y \in X : x \sim y\}$.

Solutions to Exercises

Solution (10.1). The set S is represented by the shaded area of Figure 10.2.
   Using the defined relation, $(1, 2) \notin S$ but $(3.5, 2) \in S$. Thus, the answer to (b) is no, and the answer to (c) is yes.

Solution (10.2).

  (a) The relation is $R = \{(A, B) \in \mathscr{P}(X) \times \mathscr{P}(X) : A \subset B\}$.
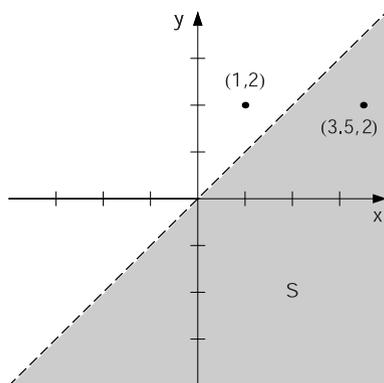
Fig. 10.2 $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x > y\}$

(b) For our example, we may take X to be any nonempty set. Then $\emptyset$ and X are different, both are in $\mathscr{P}(X)$, and $\emptyset \subset X$.

(c) Using the same sets as above, we have $X \not\subset \emptyset$.

(d) In general, we cannot conclude from $A, B \in \mathscr{P}(X)$ with $A \neq B$ that $A \subset B$ or $B \subset A$. To see this, let $\{a,b\} \subseteq X$ with $a \neq b$. Then $\{a\}, \{b\} \in \mathscr{P}(X)$ and $\{a\} \neq \{b\}$. But, $\{a\} \not\subset \{b\}$ and $\{b\} \not\subset \{a\}$. However, the conclusion does hold if X contains only one element. In that case $\mathscr{P}(X) = \{\emptyset, X\}$ and $\emptyset \subset X$.

Solution (10.4). Since R is a relation on the set X, we have $R \subseteq X \times X$. The universe for all variables below is the set X. Then,

R is reflexive, if $\forall x, (x,x) \in R$.

R is symmetric, if $\forall x, \forall y, ((x,y) \in R \rightarrow (y,x) \in R)$.

R is transitive, if $\forall x, \forall y, \forall z, (((x,y) \in R \wedge (y,z) \in R) \rightarrow (x,z) \in R)$.

R is not reflexive, if $\exists x, (x,x) \notin R$.

R is not symmetric, if $\exists x, \exists y, ((x,y) \in R \wedge (y,x) \notin R)$.

R is not transitive, if $\exists x, \exists y, \exists z, ((x,y) \in R \wedge (y,z) \in R \wedge (x,z) \notin R)$.

Solution (10.5).

(a) This relation is neither reflexive nor transitive (but it is symmetric). It is not reflexive because, for example, $1 \neq -1$ and therefore $1 \not\sim 1$. It is not transitive because $1 \sim -1$ and $-1 \sim 1$, but $1 \not\sim 1$.

(b) This relation is an equivalence relation. To see this, let $x \in \mathbb{Z}$. Then $x - x = 0$ and therefore $x \sim x$, which shows that $\sim$ is reflexive. To show symmetry, let $x, y \in \mathbb{Z}$. If $x \sim y$, then $x - y$ is even. Since $y - x = -(x - y)$, it follows that $y - x$ is even. Therefore $y \sim x$, and $\sim$ is symmetric. For transitivity, let $x, y, z \in \mathbb{Z}$. If $x \sim y$ and $y \sim z$, then $x - y$ and $y - z$ are both even; in other words, there exist integers m and n such that $x - y = 2m$ and $y - z = 2n$. Now, $x - z = (x - y) + (y - z) = 2m + 2n = 2(m + n)$, and $m + n \in \mathbb{Z}$. Thus $x - z$ is even. Therefore $x \sim z$, and $\sim$ is transitive. We conclude that the relation is an equivalence relation.

What are the equivalence classes? You can check that if x is even, then the equivalence class corresponding to x is the set of even numbers. If x is odd, then the equivalence class corresponding to x is the set of odd numbers.

(c) This relation is an equivalence relation.

If $(x,y) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, then $xy = yx$. Therefore, $(x,y) \sim (x,y)$. Thus, the relation is reflexive.

If $(x,y), (w,z) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ and $(x,y) \sim (w,z)$, then $xz = yw$. This implies that $wy = zx$. Thus, $(w,z) \sim (x,y)$. This shows that the relation is symmetric.

Suppose $(x,y), (u,v)$, and (w,z) are elements of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ such that $(x,y) \sim (u,v)$ and $(u,v) \sim (w,z)$. By definition, y, v, and z are all nonzero, $xv = yu$, and $uz = vw$. We need to show that $xz = yw$. Multiplying both sides of the equation $xv = yu$ by z and both sides of the equation $uz = vw$ by y, we obtain the two equations $xvz = yuz$ and $uzy = vwy$. Therefore $xvz = vwy$. Now v is nonzero, so we may cancel to obtain $xz = yw$, which shows that $(x,y) \sim (w,z)$. Therefore $\sim$ is transitive.

Finally, $E_{(x,y)} = \{(w,z) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) : xz = yw\}$. If we think of (x,y) as the rational number x/y, then $(w,z) \sim (x,y)$ if and only if $x/y = w/z$. So this relation is a way of identifying all fractions with the same value.

Solution (10.7). A relation is reflexive if and only if its diagram has an arrow from each vertex to itself. It is symmetric if and only if whenever we have an arrow between two different vertices, then there is also an arrow in the other direction between these two vertices. Finally, a relation is transitive if and only if whenever an arrow goes from a vertex x to a vertex y and a second arrow from vertex y to a vertex z, then there is an arrow directly from x to z. The relation is an equivalence relation on the set of the vertices of the diagram, if the diagram satisfies the three conditions outlined above.

Two vertices are in the same equivalence class if and only if there is an arrow between them. Thus two vertices are in different equivalence classes if and only if there is no arrow between them. In a diagram representing an equivalence relation, we call a subset of the vertices with the property that any two vertices of the subset are connected by arrows a component of the diagram. With this language, the equivalence classes are the components of the diagram.

Problems

Problem 10.1. We define several relations on $\mathbb{R}^2$ below. Prove that each is an equivalence relation and give a geometric description of the equivalence classes.

(a) We define $(x,y) \sim (w,z)$ if and only if $x + y = w + z$.
(b) We define $(x,y) \sim (w,z)$ if and only if $x^2 + y^2 = w^2 + z^2$.
(c) We define $(x,y) \sim (w,z)$ if and only if $3x + y = 3w + z$.
(d) We define $(x,y) \sim (w,z)$ if and only if $x = w$.

Problem 10.2. Decide whether or not the following relations are reflexive, symmetric, or transitive. If a property holds, prove that it does. If a property does not hold, prove that it does not hold. If the relation is an equivalence relation, give the equivalence class of a general point $x \in X$.

(a) On $\mathbb{R}$, we define $x \sim y$ if and only if $x < y$.
(b) On $\mathbb{R}$, we define $x \sim y$ if and only if $x \leq y$.
(c) On $\mathbb{Z}$, we define $x \sim y$ if and only if $x - y$ is divisible by 3.
(d) If X is a nonempty set, define a relation on $\mathscr{P}(X)$ by $A \sim B$ if and only if $A \subseteq B$.
(e) If X is a nonempty set, define a relation on $\mathscr{P}(X)$ by $A \sim B$ if and only if $A \setminus B \neq \emptyset$.
(f) On $\mathbb{Z}$, we define $x \sim y$ if and only if $|x| = |y|$.
(g) On $\mathbb{Z}^+$, we define $x \sim y$ if and only if there exists a rational number m such that $x = y^m$.

Problem 10.3. Define three relations on $\mathbb{R}$ by $x \sim y$ if and only if there exists $n \in \mathbb{Z}$ such that

(a) $x, y \in [n, n+1]$;
(b) $x, y \in [n, n+1)$;
(c) $x, y \in [n, n+2)$.

In each case, determine whether we have defined an equivalence relation or not. Give reasons for your answers.

Problem 10.4. If $a, b \in \mathbb{C}$, say that $a \sim b$ if and only if $a^k = b^k$ for some positive integer k. Prove that this is an equivalence relation.

Problem 10.5. We define a relation $\sim$ on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ if and only if $y_2 - y_1 \in 2\mathbb{Z}$ (see Example 6.1 for the definition of $2\mathbb{Z}$). Show that the relation $\sim$ is an equivalence relation and describe the equivalence class of the point $(0, 1)$.

Problem 10.6. Define an equivalence relation on $\mathbb{Z}$ that has exactly three equivalence classes.

Problem 10.7. Let $X = \{1, 2, 3, 4, 5\}$.

(a) If possible, define a relation on X that is an equivalence relation.
(b) If possible, define a relation on X that is reflexive, but neither symmetric nor transitive.
(c) If possible, define a relation on X that is symmetric, but neither reflexive nor transitive.
(d) If possible, define a relation on X that is transitive, but neither reflexive nor symmetric.

Problem 10.8. Define a relation $\sim$ on $\mathbb{R}$ as follows: For $x, y \in \mathbb{R}$, we say $x \sim y$ if and only if $x^2 - y^2 \in \mathbb{Z}$.

(a) Prove that $\sim$ as defined above is an equivalence relation on $\mathbb{R}$.
(b) Give five different real numbers that are in the equivalence class $E_{\sqrt{2}}$.

Problem 10.9. Define a relation $\sim$ on $\mathbb{R}^2$ as follows: For $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, we say that $(x_1, x_2) \sim (y_1, y_2)$ if and only if both $x_1 - y_1$ and $x_2 - y_2$ are even integers. Is this relation an equivalence relation? Why or why not?

Problem# 10.10. Let X be a nonempty set with an equivalence relation $\sim$ on it. Prove that for all elements x and y in X, the equality $E_x = E_y$ holds if and only if $x \sim y$.

Problem 10.11. What, if anything, is wrong with the following argument?
   We claim that if a relation on a set X is symmetric and transitive, then it is reflexive. Here's a proof of this claim:

Proof. Let $x \in X$. Let $y \in X$ with $x \sim y$. By symmetry we have $y \sim x$. We now use transitivity to conclude that $x \sim x$.   ⁇

Problem 10.12. Give an example of a relation on $\mathbb{Z} \times \mathbb{Z}$ that is not transitive, but is reflexive and symmetric.

Problem 10.13. Recall that a **polynomial** p over $\mathbb{R}$ is an expression of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$ where each $a_j \in \mathbb{R}$ and $n \in \mathbb{N}$. The largest integer j such that $a_j \neq 0$ is the **degree** of p. We define the degree of the constant polynomial $p = 0$ to be $-\infty$. (A polynomial over $\mathbb{R}$ defines a function $p : \mathbb{R} \to \mathbb{R}$.)

(a) Define a relation on the set of polynomials by $p \sim q$ if and only if $p(0) = q(0)$. Is this an equivalence relation? If so, what is the equivalence class of the polynomial given by $p(x) = x$?
(b) Define a relation on the set of polynomials by $p \sim q$ if and only if the degree of p is the same as the degree of q. Is this an equivalence relation? If so, what is E_r if $r(x) = 3x + 5$?
(c) Define a relation on the set of polynomials by $p \sim q$ if and only if the degree of p is less than or equal to the degree of q. Is this an equivalence relation? If so, what is E_r, where $r(x) = x^2$?

Problem 10.14. Figure 10.3 shows the diagram of a relation on $X = \{a, b, c, d, e, f\}$. Is this the diagram of an equivalence relation on X? Give reasons for your answer. If it is an equivalence relation, find all equivalence classes of the relation.

Problem 10.15. We define a relation on a subset X of $\mathbb{Z}$ as follows. For $x, y \in X$, $x \sim y$ if and only if there is a prime number p such that $p|x$ and $p|y$. For each of the two choices of X below, draw a diagram of the relation. From the diagram determine whether or not the relation on X is an equivalence relation. If it is an equivalence relation, use the diagram to find all equivalence classes. How many are there?

(a) $X = \{x \in \mathbb{Z} : 1 \leq x \leq 10\}$;
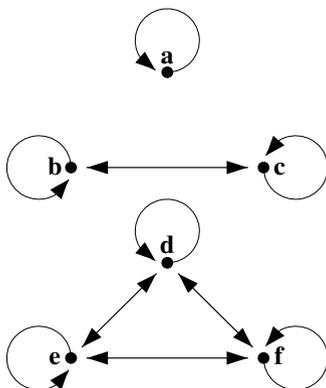(b) $X = \{4, 5, 6, 8, 11, 12, 35, 143\}$.

Fig. 10.3

Problem 10.16. Prove that if x and y are two vertices of a diagram that represents an equivalence relation on a set X and x is in the same equivalence class as y, then there are arrows pointing in both directions between x and y.

Problem 10.17. We say that a relation $\sim$ on a set X is antisymmetric if for all $x, y \in X$, whenever $x \sim y$ and $y \sim x$, then $x = y$. How can you see from the diagram of a relation that it is antisymmetric? Explain. Draw a nontrivial example (at least one arrow between two different vertices) of such a diagram using $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Problem# 10.18. Let X be a nonempty set and suppose that there is an equivalence relation defined on X. Prove that the union of all equivalence classes is X.

Problem# 10.19. Given an equivalence relation on a nonempty set X, prove that no equivalence class is empty.

Problem 10.20. Given three sets A, B, and C we can define a 3-adic relation among them in the following way. We first define the triple product of the sets A, B, and C by $A \times B \times C = \{(a, b, c) : a \in A, b \in B, \text{ and } c \in C\}$. This notation presumes that $(a_1, b_1, c_1) = (a_2, b_2, c_2)$ if and only if $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$. A 3-adic relation among sets A, B, and C is a subset of $A \times B \times C$. Give a meaningful example of a 3-adic relation on $A = B = C = \mathbb{Z}$. Give two triples that are related in your example and two triples that are not related.

Tips on Reading Mathematics

> Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?—Paul R. Halmos, [44]

- Be an active reader. Open to the page you need to read, get out some paper and a pencil.
- If notation is defined, make sure you know what it means. Your pencil and paper should come in handy here.
- Look up the definitions of all words that you do not understand.
- Read the statement of the theorem, corollary, lemma, or example. Can you work through the details of the proof by yourself? Try. Even if it feels like you are making no progress, you are gaining a better understanding of what you need to do.
- Once you truly understand the statement of what is to be proven, you may still have trouble reading the proof—even someone's well-written, clear, concise proof. Try to get the overall idea of what the author is doing, and then try (again) to prove it yourself.
- If a theorem is quoted in a proof and you don't know what it is, look it up. Check that the hypotheses apply, and that the conclusion is what the author claims it is.
- Don't expect to go quickly. You need to get the overall idea as well as the details. This takes time.
- If you are reading a fairly long proof, try doing it in bits.
- If you can't figure out what the author is doing, try to (if appropriate) choose a more specific case and work through the argument for that specific case.
- Draw a picture, if appropriate.
- If you really can't get it, do what comes naturally—put the book down and come back to it later. You might want to take this time to read similar proofs or some examples.
- After reading a theorem, see if you can restate it. Make sure you know what the theorem says, what it applies to, and what it does not apply to.
- After you read the proof, try to outline the technique and main idea the author used. Try to explain it to a willing listener. If you can't do this without looking back at the proof, you probably didn't fully understand the proof. Read it again.
- Can you prove anything else using a similar proof? Does the proof remind you of something else? What are the limits of this proof? This theorem?
- If your teacher is following a book, read over the proofs before you go to class. You'll be glad you did.

As we proceed, you will have plenty of opportunities to try these tips out and find some others of your own.