Chapter 19
Sequences

We have all seen lists of numbers. For example, we've all worked with a list of positive even integers presented in increasing order, $(2,4,6,8,\ldots,2n,\ldots)$, where $n = 1,2,3,4,\ldots$. The positive odd numbers $(1,3,5,7,\ldots,2n-1,\ldots)$ can also be presented in such a list, where $n = 1,2,3,4,\ldots$. What we are interested in here is a precise definition of "infinite list."

Here's an example from your childhood of a problem that yields such a list. Let n be a positive integer with $n \geq 2$. Suppose that we have n children arranged in a circle, and that rather than use their names, we number them $\{1,2,\ldots,n\}$. Say these children want to see who goes first in a game. They begin by eliminating the second child, and then proceed around the circle, eliminating every other child until there is only one child left. That happy child goes first. The question is, where should you stand in order to be the winning child? Let's start small: if there are two children, you should stand in the first spot. If there are three, you should stand in the third spot. If there are four, you should stand in the first spot. Where should you stand if there are n children? (This challenging problem is known as the Josephus problem. The answer appears at the end of the chapter. Of course, the children can count off by three or four, giving us a new problem to solve.) In this problem, for each group of n children, we have an answer. Thus we again have a list of numbers. We now turn to the definition of "list."

A **sequence** is a function f from the natural numbers $\mathbb{N}$ to a set X. In this chapter, we will concentrate on the case $X = \mathbb{R}$. It is standard to write $x_n = f(n)$, and to refer to x_n as a **term** in the sequence. The sequence will be denoted $(x_n)_{n=0}^{\infty}$, or just (x_n) when it is clear which n we are referring to or when it doesn't matter where the sequence starts. We can begin a sequence at an integer other than $n = 0$ when convenient, and we will often begin the sequence at $n = 1$ without much fanfare. We'll tell you where we are starting when it really matters.

Since sequences are functions, we can graph them as functions defined on the nonnegative (or positive) integers and then we can see what they are doing.

Example 19.1. The sequence (x_n) is defined by $x_n = 1 + 1/n$ for $n \in \mathbb{Z}^+$. We will write out the first few terms and graph the beginning of the sequence in Figure 19.1.

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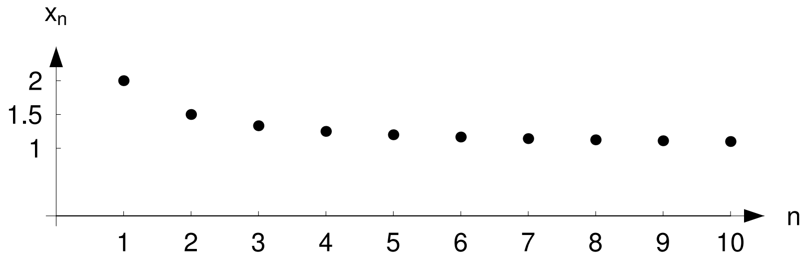

Fig. 19.1 Graph of (x_n), where $x_n = 1 + 1/n$ for $n \geq 1$

The first four terms are

$$x_1 = 2, \ x_2 = \frac{3}{2}, \ x_3 = \frac{4}{3}, \ x_4 = \frac{5}{4}.$$

Exercise 19.2. For each of the sequences given below, write out the first four terms and graph the beginning of the sequence.

(a) Let $x_n = 1 - (-1)^n$, for $n = 0, 1, 2, \ldots$.
(b) Let $x_n = n/(n+1)$, for $n = 0, 1, 2, \ldots$.
(c) Let $x_n = (n^2 + 1)/(1 - n)$, for $n = 2, 3, \ldots$. ○

When we look at sequences, we notice that different sequences behave in different ways. This is illustrated by our examples in Exercise 19.2. Some sequences approach some horizontal line as $n \to \infty$. Some seem to be shooting off to infinity, others jump around a lot. We need to understand how these sequences differ from each other, and the following definitions will help us do that.

You'll notice that a lot of terms that we introduced when we studied sets reappear here. What is really happening is that each sequence (x_n) (where order counts) gives rise to a nonempty set $S = \{x_n : n \in \mathbb{N}\}$ (where order doesn't count). For example, the sequence (z_n) where $z_n = (-1)^n$ gives rise to the set $S = \{-1, 1\}$, while the sequence $(y_n)_{n=1}^{\infty}$ where $y_n = n$ gives rise to the set $T = \{n : n \in \mathbb{Z}^+\} = \mathbb{Z}^+$. The set S that is associated with the sequence (x_n) is precisely the range of the sequence. Furthermore, once we understand this connection, we can ask the same questions about sequences that we have asked about sets: For example, we can ask when a sequence is bounded above, bounded below, bounded, has an infimum or supremum—and we can use everything we learned about sets to find an answer.

How important are sequences? Very important. In fact, we believe that these are so important to your future mathematical development that we will restate all the definitions in the specific setting of sequences. For example, a sequence of real numbers (x_n) is bounded if the set $S = \{x_n : n \in \mathbb{N}\}$ is bounded. Thus, according to our definition of bounded set, a sequence of real numbers (x_n) is **bounded above**, if there is a real number M such that $x_n \leq M$ for all n, and **bounded below**, if there exists a real number m such that $x_n \geq m$ for all n. A sequence is **bounded** if it is bounded above and below. That's how we defined it. But you may find that the following provides a more useful way to think about boundedness in $\mathbb{R}$.

Exercise 19.3. Let (x_n) be a sequence. Prove that (x_n) is bounded if and only if there exists a real number N such that $|x_n| \leq N$ for all n. ○

Just as before, a real number U satisfying $x_n \leq U$ for all n is called an **upper bound** of the sequence (x_n), and a real number L satisfying $L \leq x_n$ for all n is a **lower bound** of the sequence (x_n). In our illustration above, we considered the sequence (z_n), where $z_n = (-1)^n$. We see that this sequence is bounded above (1 is an upper bound) and bounded below (-1 is a lower bound), and therefore it is bounded. Alternatively, $|z_n| \leq 1$ for all n, and by the previous exercise we may conclude that the sequence (z_n) is bounded.

Exercise 19.4. (a) Give an example of a real number that is not an upper bound of the sequence given by $x_n = n/(n+1)$.
(b) Complete the following sentence: The real number U is not an upper bound of the sequence (x_n), if .... ○

If (x_n) is a sequence that is bounded below, the set $S = \{x_n : n \in \mathbb{N}\}$ is bounded below. Since S is a nonempty set of real numbers that is bounded below, the infimum version of the completeness axiom (see Exercise 12.9) implies that S has an infimum, and we call this the **infimum** of the sequence (x_n) (or **greatest lower bound** of (x_n)). We denote it by $\inf(x_n)$. Recall that you showed (Problem 12.17) that the infimum is unique. The definition of the infimum of a sequence, without reference to the set associated with the sequence, is given in the example below.

Example 19.5. Let (x_n) be a sequence that is bounded below. State the two properties that the infimum of (x_n) must satisfy.

The infimum of the sequence (x_n) is the real number m satisfying

(i) $m \leq x_n$ for all n, and
(ii) if p is a real number satisfying $p \leq x_n$ for all n, then $p \leq m$.

Each of the statements (i) and (ii) can be stated in the vernacular, and you should do so now. ○

Exercise 19.6. Guess the infimum for each of the cases below:

(a) $x_n = 1/n$, for $n = 1, 2, 3, \ldots$;
(b) $x_n = n^2$, for $n \in \mathbb{N}$;
(c) $x_n = n/(n+1)$, for $n \in \mathbb{N}$;
(d) $x_n = (-1)^n/(n^2+1)$, for $n \in \mathbb{N}$. ○

When you look for the infimum, remember that it may or may not appear in the sequence. Everything we do for the infimum can be done for the supremum. So the **supremum** of the sequence (x_n) (or **least upper bound** of (x_n)) is the supremum of the set $S = \{x_n : n \in \mathbb{N}\}$. We denote it by $\sup(x_n)$. The rest is left to you in the next exercise.

Exercise 19.7. Let (x_n) be a sequence that is bounded above. State the two properties that the supremum of (x_n) must satisfy, and say why it exists. ○

That takes care of how high and how low a sequence can go. Now we turn to how it gets where it is going.

A sequence (x_n) is **increasing** if $x_n \leq x_{n+1}$ for all n, and **decreasing** if $x_n \geq x_{n+1}$ for all n. We say the sequence (x_n) is **strictly increasing** if $x_n < x_{n+1}$ for all n. Likewise, a sequence (x_n) is **strictly decreasing** if $x_n > x_{n+1}$ for all n.

Exercise 19.8. The object of this exercise is to make sure you understand the definitions above. Either explain why you cannot give an example of the following, or give an example of

(a) a bounded sequence. Find an upper bound and a lower bound.
(b) a sequence that is bounded below, but not bounded above. Find a lower bound. Must the sequence be increasing?
(c) a sequence that is bounded above, but not bounded below. Find an upper bound.
(d) an increasing sequence that is neither bounded above nor below.
(e) a strictly increasing bounded sequence.
(f) a strictly decreasing sequence that is bounded above, but not below.
(g) a sequence that is neither strictly increasing nor strictly decreasing. ○

Exercise 19.9. What is your best guess for the supremum of the sequence

$$x_n = \underbrace{0.999...9}_{n\,9's}?$$

In Problem 20.18 of the next chapter you will give a rigorous proof of this fact. ○

Since sequences are functions, we can manipulate them algebraically. For example, to add two sequences together, we define the sum $(x_n) + (y_n)$ to be the sequence $(x_n + y_n)$. In the same way, we may subtract sequences.

Example 19.10. Suppose (x_n) and (y_n) are two bounded sequences. If $\sup(x_n) = l$ and $\sup(y_n) = m$, is $\sup(x_n + y_n) = l + m$?

We'll first show that $l + m$ is an upper bound of $(x_n + y_n)$. (Thus it makes sense to talk about the supremum of $(x_n + y_n)$.) Then we will try to show the second thing:

that $l+m$ is the least of all the upper bounds, in the sense defined above. Remember that since we don't know the answer here, our attempt might fail.

So let $l = \sup(x_n)$ and $m = \sup(y_n)$. Then $x_n \leq l$ for all n and $y_n \leq m$ for all n. Thus $x_n + y_n \leq l+m$ for all n. So far so good; we know that $l+m$ is an upper bound (and we know that the sequence $(x_n + y_n)$ is bounded above). But we still have to see whether or not $l+m$ is the least upper bound. So suppose that p is another upper bound. We are supposed to show that $p \geq l+m$. Well, $p \geq x_n + y_n$ for every n, but that doesn't seem to help since that doesn't (in general) imply anything about the relation between p and x_n or p and y_n. In fact, closer inspection reveals that if one of the sequences is negative, we can't say anything at all. So we abandon our attempt at a proof and search for a counterexample, using what we learned above.

Let (x_n) be a bounded nonconstant sequence, say $x_1 = 1$ and $x_n = 2$ for all other n. Thus $l = 2$. Now let $y_n = -x_n$. Then $m = -1$. So $x_n + y_n = 0$, and the supremum of $(x_n + y_n)$ is clearly 0, while $l+m = 1$. Hence the supremum of $(x_n + y_n)$ need not be $l+m$. ○

The lesson here is that in trying to prove something, we came up with an example that showed it wasn't true. This is a perfectly reasonable way to approach the problem as long as we are always on the lookout for what can go wrong with a proof.

Exercise 19.11. Let (x_n) be a sequence that is bounded below. Let $l = \inf(x_n)$. Show that $(-x_n)$ is bounded above and find the supremum of the sequence. ○

Recall that we introduced recursively defined functions in the last chapter: The recursively defined function had domain $\mathbb{N}$ and codomain X. Now X was allowed to be any nonempty set, but the domain had to be $\mathbb{N}$ (or a set of the form $\{x \in \mathbb{N} : x \geq m\}$ for some $m \in \mathbb{Z}$). So, these functions, our motivating example, $n!$, and every other function to which we applied the recursion theorem are all sequences. Many sequences are defined recursively, including several famous sequences, as we will see below. If a sequence is defined by giving a formula for x_n in terms of n, then we say that the sequence is defined explicitly. It is possible for a sequence to have both a recursive definition as well as an explicit one, as we will see below.

Exercise 19.12. Reconsider the function of Exercise 18.7, $g : \mathbb{N} \to \mathbb{R}$, defined recursively by $g(0) = 1$ and $g(n+1) = 5g(n)$ for every $n \in \mathbb{N}$. Then g is a familiar function. What is it? ○

One of the most famous examples of a sequence defined recursively is the Fibonacci sequence. Fibonacci, whose real name is Leonardo Pisano, was born in 1170 in Pisa. (One source you might consult for more information about Leonardo Pisano is L. Sigler's book [98].) The Fibonacci sequence is often presented with pictures of rabbits. So here is a version of Fibonacci's original rabbit problem: Suppose that rabbits live forever. Starting at the age of two months, each pair produces (exactly) one baby pair, and continues to do so every month thereafter. If we start with one brand new pair, how many pairs of rabbits will we have in the nth month? Now here's the sequence. See if you can figure out the reference to these rabbits.

Define $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. This sequence is called the **Fibonacci sequence** and the terms of the sequence are called the **Fibonacci numbers**. In this example, the recursion uses two preceding terms of the sequence. This complicates the application of the recursion theorem. In Problem 19.17 we will ask you to provide the details of this application of the recursion theorem.

Example 19.13. Find the first seven terms of the Fibonacci sequence.

The first seven Fibonacci numbers are: $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8$. ○

The Fibonacci sequence is extremely appealing to mathematicians and nonmathematicians. In fact, there are many web sites and books with information and problems about Fibonacci sequences, as well as the journal *The Fibonacci Quarterly*.

We present one of the many interesting patterns found in Fibonacci numbers below. Others can be found in the problems, as well as some of the references given in this chapter.

Exercise 19.14. Let (F_n) denote the Fibonacci sequence. Show that for every positive integer n the equation $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ holds.

Check the equation for the first few values of n to see if this is reasonable (but, of course, this is not a proof). The proof of Exercise 19.14 will use mathematical induction, and you can read our solution below when you are ready. ○

We now return to the solution of the Josephus problem mentioned at the beginning of the chapter. For each integer $n \geq 2$, we let $f(n)$ denote the number of the winning child. Then $f(2) = 1$, $f(3) = 3$, $f(2n) = 2f(n) - 1$, and $f(2n+1) = 2f(n) + 1$. Note that a variation on our "recursion theorem theme" made this solution possible! For more information on the Josephus problem, we recommend the article [97].

Definitions

Definition 19.1. A **sequence** is a function f from the natural numbers $\mathbb{N}$ to a set X. It is standard to write $x_n = f(n)$, and to refer to x_n as a **term** in the sequence. The sequence will be denoted (x_n).

Definition 19.2. A sequence of real numbers (x_n) is **bounded above**, if there is a real number M such that $x_n \leq M$ for all n, and **bounded below**, if there exists a real number m such that $x_n \geq m$ for all n. A sequence is **bounded** if it is bounded above and below.

Definition 19.3. A real number U satisfying $x_n \leq U$ for all n is called an **upper bound** of the sequence (x_n), and a real number L satisfying $L \leq x_n$ for all n is a **lower bound** of the sequence (x_n).

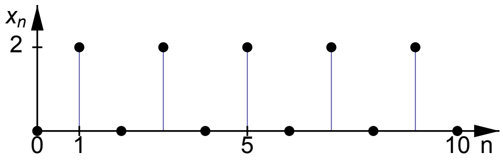

Fig. 19.2 Graph of (x_n), where $x_n = 1-(-1)^n$ for $n \geq 0$

Definition 19.4. The **infimum** (or **greatest lower bound**) of the sequence of real numbers (x_n) that is bounded below is the real number $\inf(x_n)$ satisfying

(i) $\inf(x_n) \leq x_m$ for all m, and
(ii) if p is a real number satisfying $p \leq x_m$ for all m, then $p \leq \inf(x_n)$.

Definition 19.5. The **supremum** (or **least upper bound**) of the sequence of real numbers (x_n) that is bounded above is the real number $\sup(x_n)$ satisfying

(i) $\sup(x_n) \geq x_m$ for all m, and
(ii) if p is a real number satisfying $p \geq x_m$ for all m, then $p \geq \sup(x_n)$.

Definition 19.6. A sequence (x_n) is **increasing** if $x_n \leq x_{n+1}$ for all n. It is **strictly increasing** if $x_n < x_{n+1}$ for all n.

Definition 19.7. A sequence (x_n) is **decreasing** if $x_n \geq x_{n+1}$ for all n. It is **strictly decreasing** if $x_n > x_{n+1}$ for all n.

Definition 19.8. Define $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. This sequence is called the **Fibonacci sequence** and the terms of the sequence are called the **Fibonacci numbers**.

Solutions to Exercises

Solution (19.2).

(a) $x_0 = 0$, $x_1 = 2$, $x_2 = 0$, and $x_3 = 2$. The first eleven terms are graphed in Figure 19.2.
(b) $x_0 = 0$, $x_1 = 1/2$, $x_2 = 2/3$, and $x_3 = 3/4$. The first eleven terms are graphed in Figure 19.3.

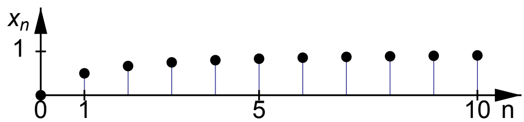

Fig. 19.3 Graph of (x_n), where $x_n = n/(n+1)$ for $n \geq 0$

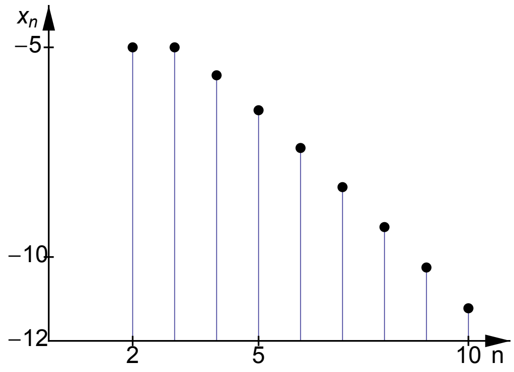

Fig. 19.4 Graph of (x_n), where $x_n = (n^2+1)/(1-n)$ for $n \geq 2$

(c) $x_2 = -5$, $x_3 = -5$, $x_4 = -17/3 \approx -5.67$, and $x_5 = -13/2 = -6.5$. The first nine terms are graphed in Figure 19.4.

Solution (19.3). If you have solved Problem 12.8, then you have solved this exercise as well. If not, here is a solution.

First we'll prove that if (x_n) is bounded, then there exists a real number N such that $|x_n| \leq N$ for all n. By the definition of bounded sequence, there exist real numbers m and M such that $m \leq x_n \leq M$, for all n. Hence $-|m| \leq m \leq x_n \leq M \leq |M|$ for all n. Letting $N = \max\{|m|, |M|\}$, we have $-N \leq x_n \leq N$. Thus $|x_n| \leq N$ for all n.

Finally, we prove that if there exists a real number N such that $|x_n| \leq N$ for all n, then (x_n) is bounded. Since $|x_n| \leq N$ for all n, we have $-N \leq x_n \leq N$ for all n. This shows that (x_n) is bounded below and bounded above. Hence the sequence (x_n) is bounded.

Solution (19.4). Many answers are possible for (a).

(a) Consider the number $m = -1$. Then m is not an upper bound of the sequence since $x_1 > m$.
(b) The real number U is not an upper bound of the sequence $(x_n)_{n\in\mathbb{N}}$ if there exists $n \in \mathbb{N}$ such that $x_n > U$.

Solution (19.6). The answers are: (a) 0, (b) 0, (c) 0, (d) $-1/2$.

Solution (19.7). A real number U is the supremum of a sequence (x_n) if (i) $x_n \leq U$ for all n, and (ii) if V is another upper bound of (x_n), then $U \leq V$.

The set $\{x_n\}$ is bounded above and thus, by the completeness axiom of $\mathbb{R}$, this set has a supremum. The supremum of this set is the supremum of the sequence (x_n).

Solution (19.8). You should be able to find examples for all parts of this problem, except part (d). An increasing sequence will always be bounded below, and its first term will serve as a lower bound. For parts (a) and (g), the sequence $((-1)^n)$ yields such an example. An upper bound for this sequence is 10 and a lower bound is

-10 (of course many other choices are possible). For (c) and (f), you can use the sequence $(-n)$, which is bounded above by 0 but is not bounded below. For (b), the sequence defined by $x_n = n + 2(-1)^n$ for $n \in \mathbb{N}$ is bounded below (by, for example, -100) and this sequence is not increasing (since $x_0 = 2$ and $x_1 = -1$). Finally, for (e) the sequence $(1 - 1/n)$ for $n \in \mathbb{Z}^+$ serves as an example.

Solution (19.9). Your guess was surely 1. Right?

Solution (19.11). Since $l = \inf(x_n)$, we know that $x_n \geq l$ for all n. Multiplying both sides by -1 we obtain $-x_n \leq -l$ for all n, and consequently $(-x_n)$ is bounded above and $-l$ is an upper bound. We claim that $-l$ is the supremum, too. Suppose that m is also an upper bound. Then $-x_n \leq m$ for all n. Multiplying by -1, we see that $x_n \geq -m$ for all n. But this implies that $-m$ is a lower bound for (x_n). Since $l = \inf(x_n)$, we know that $-m \leq l$. Thus $m \geq -l$, and $-l$ is the least of all the upper bounds. So $-l = \sup(-x_n)$.

Solution (19.12). We note that $g(0) = 1, g(1) = 5, g(2) = 5^2$, and $g(3) = 5^3$. We guess that $g(n) = 5^n$ for all $n \in \mathbb{N}$. We prove this guess using induction.

For the base step of $n = 0$ we have $g(0) = 1 = 5^0$.

Now for the induction step, we let $n \in \mathbb{N}$ and we suppose that $g(n) = 5^n$. Then $g(n+1) = 5g(n)$ by the recursive definition. Using the induction hypothesis we get $5(g(n)) = 5(5^n)$. We conclude that $g(n+1) = 5^{n+1}$, completing the induction step.

By mathematical induction the function is $g(n) = 5^n$, as we guessed above.

Solution (19.14). We will prove the validity of this equation using induction. We will consider two base steps here. For $n = 1$, we easily check that $F_2F_0 - F_1^2 = -1$. Similarly, for $n = 2$, we can check that $F_3F_1 - F_2^2 = 1$. For the induction step, let $n \in \mathbb{N}$ with $n \geq 2$ and suppose that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. In other words, we assume that $F_n^2 = F_{n+1}F_{n-1} - (-1)^n$. We will show that $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$. To see this, note that $F_{n+2} = F_{n+1} + F_n$ for all n. Thus (you should fill in reasons for each of the equalities):

$$\begin{aligned} F_{n+2}F_n - F_{n+1}^2 &= (F_{n+1} + F_n)F_n - F_{n+1}^2 \\ &= F_{n+1}F_n + F_n^2 - F_{n+1}^2. \end{aligned}$$

Use the induction hypothesis to replace the middle term, F_n^2, in the summand above to conclude that

$$\begin{aligned} F_{n+2}F_n - F_{n+1}^2 &= F_{n+1}F_n + F_{n+1}F_{n-1} - (-1)^n - F_{n+1}^2 \\ &= F_{n+1}(F_n + F_{n-1}) - F_{n+1}^2 + (-1)^{n+1} \\ &= (-1)^{n+1}, \end{aligned}$$

and the result now follows from the principle of mathematical induction.

The first 100 Fibonacci numbers can be found on the Web [60]. From there you can get to a very cute picture of little rabbits, as well as a set of puzzles based on these numbers. Fibonacci numbers also make an appearance in the popular children's book *The Number Devil* by Hans Magnus Enzensberger [25].

Problems

Problem 19.1. Graph the following sequences and briefly describe each of the graphs:

(a) $x_n = (-1)^n$, where $n \in \mathbb{N}$;
(b) $x_n = 1/2^n$, where $n \in \mathbb{N}$;
(c) $x_n = n/(n-1)$, where $n \in \mathbb{N}$ and $n \geq 2$;
(d) $x_n = (-1)^n/2^n$, where $n \in \mathbb{N}$;
(e) $x_n = (-1)^n(n^2/(n+1))$, where $n \in \mathbb{N}$.

Problem 19.2. Prove that the sequence (x_n), defined by $x_n = \frac{n}{n+1}$, is strictly increasing.

Problem 19.3. In what follows, no rigorous proofs are required. However, you should provide a brief explanation of your answers.

(a) Give an example of a sequence of rational numbers that is bounded above.
(b) Give an example of a sequence of rational numbers that has no upper bound, but does have a lower bound.
(c) Give an example of a strictly increasing sequence of numbers that has a supremum, but such that the supremum is not a term in the sequence. Can you find a strictly increasing sequence (x_n) such that the supremum is equal to x_n for some n? Why or why not?

Problem 19.4. The game of chess seems to have some of its roots in India. As the story goes, the emperor of India was smitten with the new game, and he asked the inventor of chess how he could reward him for this marvelous invention. The (apparently modest) reply was to ask for one grain of rice for the first square of the chess board, double that for the second, double that for the third, and so on. Thus, for each square after the first, the inventor receives double the number of grains he received on the previous square.

Let x_n = number of rice grains for the nth square of the chessboard. Then (x_n) is a sequence.

(a) Find the first four terms of (x_n).
(b) Find an explicit formula for x_n.

Consider the new sequence (a_n), where a_n = total number of rice grains for the first n squares. You will need to use the formula for a geometric sum: For a real number $a \neq 1$ and positive integer n we have $1 + a + \cdots + a^n = (1 - a^{n+1})/(1 - a)$. (If this formula is unfamiliar to you, prove it!)

(c) Find the first four terms of (a_n).
(d) Find an explicit formula for a_n.
(e) What is the mass of the rice the inventor asked for? (Use the fact that the mass of one grain of rice is approximately 25 mg.)
(f) The rice production of the whole world in 2004 was approximately 600 million metric tons of rice. How does this compare with the inventor's request?

Problem 19.5. Phenytoin is a medication designed to control seizures. It is administered twice daily at 8:00 a.m. and at 8:00 p.m. The dosage at each administration is 75 mg. The retention rate after 12 hours is 65% (that is, 12 hours after intake, 65% of the drug remains in the body). Let S_n denote the amount of phenytoin in the patient's body shortly after the nth administration of the drug.

(a) Calculate the first four terms of the sequence $(S_n)_{n=1}^{\infty}$.
(b) Give a recursive definition of this sequence.
(c) Prove that (S_n) is strictly increasing, bounded, and that $\sup(S_n)$ exists.

We let s_n denote the amount of phenytoin in the patient's body shortly *before* the nth administration of the drug.

(d) Express s_n in terms of S_n.
(e) Indicate why (s_n) is also increasing, bounded, and $\sup(s_n)$ exists.
(f) Find the minimum and maximum amount of phenytoin in the patient's blood following the 8 a.m. dose on day 3 and prior to the 8 a.m. dose on day 4.

Problem 19.6. Give an example of a sequence of rational numbers that has an irrational number as supremum. Prove your claim!

Problem 19.7. We define a sequence (a_n) by $a_1 = 1$ and $a_{n+1} = 3 - 1/a_n$ for all $n \geq 1$.

(a) Show that (a_n) is bounded.
(b) Prove that (a_n) is strictly increasing.

Problem 19.8. We define the sequence (x_n) by $x_n = \frac{n^3+n^2-1}{n^3+1}$ for $n \in \mathbb{N}$.

(a) Prove that (x_n) is bounded.
(b) Is (x_n) increasing? Prove it or give a counterexample.
(c) Find $\inf(x_n)$.

Problem 19.9. If $l = \sup(x_n)$, what is $\inf(-x_n)$? (You should know by now that the first thing to do is to try examples. Make up at least three different examples.) State your conjecture. Prove it.

Problem 19.10. If $l = \sup(x_n)$, what is $\sup(kx_n)$ where $k \in \mathbb{R}^+$? Prove your conjecture.

Problem 19.11. Let (x_n) be a bounded sequence such that $x_n \leq -2$ for all $n \in \mathbb{N}$.

(a) Prove that (x_n^2) is bounded.

(b) Let $l = \inf(x_n)$ and $m = \sup(x_n)$. Find $\inf(x_n^2)$ in terms of l or m, or both. Prove that your result is correct.

Problem 19.12. Suppose that (x_n) and (y_n) are bounded below.

(a) Show that $\inf(x_n + y_n) \geq \inf(x_n) + \inf(y_n)$.
(b) Is it always true that $\inf(x_n + y_n) = \inf(x_n) + \inf(y_n)$? Prove this or give a counterexample.

Problem 19.13. Suppose that (x_n) and (y_n) are bounded below. Is it always true that $\inf(x_n y_n) = \inf(x_n)\inf(y_n)$? Prove this or give a counterexample.

Problem 19.14. Let (x_n) and (y_n) be two sequences of real numbers. Assume that (y_n) is bounded above and that $x_n < y_n$ for all $n \in \mathbb{N}$.

(a) Prove that (x_n) is also bounded above.
(b) Prove that $\sup(x_n) \leq \sup(y_n)$.
(c) Do the assumptions imply that $\sup(x_n) < \sup(y_n)$? If yes, prove it; if no, find a counterexample.

Problem 19.15. Let (x_n) and (y_n) be two sequences of real numbers. Assume that (x_n) is bounded above and that $x_n < y_n < x_{n+1}$ for all $n \in \mathbb{N}$.

(a) What can you say about $\sup(x_n)$ and $\sup(y_n)$? Must they exist? If so, how do they compare?
(b) What can you say about $\inf(x_n)$ and $\inf(y_n)$? Must they exist? If so, how do they compare?

Problem 19.16. We define a sequence (x_n) recursively by letting $x_0 = 1000$ and $x_n = (.05)x_{n-1}$ for $n \geq 1$. Find another representation for this sequence. Have you seen this anywhere else before? If so, where?

Problem 19.17. With the help of the recursion theorem, prove that the Fibonacci sequence (see Definition 19.8) is well-defined; that is, specify a, X, and f appearing in the recursion theorem. (You can use Example 18.8 to suggest an approach to this problem.)

Problem 19.18. (a) Prove that the Fibonacci sequence is increasing.
(b) Prove that the Fibonacci sequence is unbounded.

Problem# 19.19. Let (F_n) be the Fibonacci sequence and $x_n = F_{n+1}/F_n$, for $n \geq 1$. Show that $x_n = 1 + 1/x_{n-1}$, for $n \geq 2$.

Problem 19.20. Prove the following explicit formula for the nth Fibonacci number:

$$F_n = \frac{a^n - b^n}{a - b}, \text{ where } a = \frac{1+\sqrt{5}}{2} \text{ and } b = \frac{1-\sqrt{5}}{2}.$$

Problem 19.21. Let $f(0) = 2$, $f(1) = 2$, and define $f(n+1) = f(n)f(n-1)$. Find an explicit formula for $f(n)$.

Problem 19.22. Let (F_n) denote the Fibonacci sequence. Define the Lucas sequence by $L_0 = 2$, $L_1 = 1$, and for $n \geq 1$ define $L_{n+1} = L_n + L_{n-1}$. (For some proofs you may want to use the second principle of induction stated in the problem section of Chapter 18 as Theorem 18.9.)

(a) Calculate $L_1, \ldots, L_{10}$.
(b) Calculate $L_n - F_{n-1}$, for $n \geq 1$. Find a remarkable pattern in this list of numbers. State it clearly and prove it by induction.
(c) Calculate $F_n + L_n$. Find a remarkable pattern in this list of numbers. State it clearly and prove it using part (b).