Chapter 4
Set Notation and Quantifiers

Consider the sentence "The equation $x^2 + 2x = 15$ has a unique solution." Thus far, we've approached such things intuitively. It's time now to tackle this head on. Is it true? False? It depends on which x we have in mind, of course. We turn to a rigorous way to make our sentence $x^2 + 2x = 15$ into a statement. But before we get to the heart of this chapter, it will be useful to have notation for the things with which we frequently work.

We will need to understand the possible set of values that the variable x can assume. Now we will follow the point of view that many mathematicians follow: while we think of a set as a collection of objects, we will define neither set nor object. What we will do instead is to say, carefully and precisely, how these two words can be used and we do so with axiomatic statements. (The system most people use now, the Zermelo–Fraenkel system together with the axiom of choice, is stated in the Appendix for reference. We will say much more about sets in Chapter 6 and in subsequent chapters.) At this point we will concentrate on understanding the notation and commonly used symbols.

We will write $x \in X$ to indicate that x is an element of X. (Some people read $x \in X$ as "x belongs to X," others read it "x is an element of X.") Usually we will be considering things of a particular type. The set of all possible objects that are considered in the context in which we work is called the **universe**, which is also sometimes called the *domain of discourse*. We will usually denote the universe by X. In some cases the universe may consist of all real numbers, or it may consist of all right triangles; it might even consist of all cows living in France. The set may consist of all positive real numbers, all isosceles right triangles, or all white cows living in France. And the elements might be the real number π, the isosceles right triangle with legs of length 1, or Farmer Boursin's white cow Elsie, which lives in Dijon, France, and produces a mighty fine cheese. Some people even allow the universe to be the "set of all sets," even though this universe is no set at all (see the Spotlight: Paradoxes on page 67).

When it is clear, implicitly, what the universe is, we may not mention it explicitly. But when there is any doubt at all, we will carefully state what the universe is. Once we do that, we can denote a set by writing $S = \{x \in X : x \text{ satisfies } P\}$. The brackets

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indicate that we are talking about a set of objects, called elements; $x \in X$ tells us where these elements live, and P is a property these elements have.

In this class, as well as others, some sets show up a lot and we have special notation for them. Notation should always be chosen carefully, as these have been. Most mathematicians agree on these, so don't make up your own notation and make sure you recognize what these are when they are used:

The natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

The integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

The rational numbers $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z} \text{ and } q \neq 0\}$.

The real numbers $\mathbb{R}$.

The complex numbers $\mathbb{C} = \{a + b\mathrm{i} : \mathrm{i}^2 = -1 \text{ and } a, b \in \mathbb{R}\}$.

If A is one of the sets $\mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$, then the set of the positive elements is denoted by $A^+ = \{x \in A : x > 0\}$ and the set of the negative elements is denoted by $A^- = \{x \in A : x < 0\}$. Thus we have defined $\mathbb{Z}^+, \mathbb{Z}^-, \mathbb{Q}^+, \mathbb{Q}^-, \mathbb{R}^+$, and $\mathbb{R}^-$.

The plane $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

For $n \in \mathbb{Z}^+$, Euclidean n-space $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, 2, \ldots, n\}$.

Some authors include zero in $\mathbb{N}$ and others don't. If you look in another text, make sure you know what convention they follow.

For real numbers a and b with $a \leq b$, the set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is called the closed interval from a to b. The sets $[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$ and $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$ are called unbounded closed intervals. For $a < b$, the set $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is called the open interval from a to b. We shall see that (a, b) can be interpreted several ways, and you should be able to decide which from the way it is used. You've done this in the past. For example, you have certainly had courses where (x, y) denotes a point and, in the same course, (x, y) might denote an open interval. Unbounded open intervals are defined, with appropriate changes, in the same way that we defined unbounded closed intervals above.

Exercise 4.1. Find a (different) useful way to describe the following sets (your useful way could be a sketch):

(a) $\{x \in \mathbb{Z} : x^2 = 1\}$;
(b) $\{x \in \mathbb{N} : x^2 = 1\}$;
(c) $\{(x, y) \in \mathbb{R}^2 : y = 0\}$;
(d) $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$;
(e) $\{x \in \mathbb{Z} : x \text{ is even}\}$;
(f) $\{(m, n) : m, n \in \mathbb{Z}\}$. ○

Now we can talk about slightly more complicated sentences. Think of the difference between the statements "In every box there is a prize" and "In some box there is a prize." Obviously, if you had to choose (and if it were the same prize), you would go with the first one. In mathematics, in order to determine the truth or falsity of a statement, we need to know whether we are talking about a particular x or all x. What we mean should be clear from the context. Letters like x that stand for elements of the universe are called variables. The phrases "for all," "for every," "for some," or "there exists," quantify variables. "For all," or $\forall$, is the universal quantifier and "there exists," or $\exists$, is the existential quantifier.

After agreeing that the universe consists of all real numbers, consider the following statement: "For all x it is the case that $x^2 - 1 \leq 0$." We know that we are asking that for every x, something must happen. It just so happens that this statement is false, but it is still a clear statement. For all x is usually written $\forall x$. So we could write

$$\forall x, x^2 - 1 \leq 0.$$

What follows the words "For all x" in our statement is another sentence that we could denote by p, but since p is a sentence involving x, we write $p(x)$. The statement above is of the form

$$\forall x, p(x).$$

One more remark about the example above. Suppose the universe is (still) the real numbers, but we want to make this a statement about positive integers only. In that case, we can express our statement symbolically as follows:

$$\forall x, (x \in \mathbb{Z}^+ \rightarrow (x^2 - 1 \leq 0)).$$

One very common error is to write $\forall x, (x \in \mathbb{Z}^+ \wedge (x^2 - 1 \leq 0))$ rather than what we have written above. But let's think about what this would mean: This would say that "all real numbers are positive integers and satisfy the inequality $x^2 - 1 \leq 0$." It is probably clear now that this is not what the original statement said.

For a different example, suppose that our universe is the set of integers and consider the sentence, "There is an integer x such that $x = 0$." This, too, is a statement, and happens to be true. This statement can be expressed symbolically by

$$\exists x, (x = 0)$$

and is read as "there exists x such that $x = 0$." This statement is of the form

$$\exists x, p(x).$$

One more remark about the last example. If we had chosen the set of the real numbers as the universe, we would express our statement symbolically as

$$\exists x, (x \in \mathbb{Z} \wedge x = 0).$$

Note that this time we are claiming that x exists, is an integer, and $x = 0$. To make all these things happen, we must use a conjunction rather than an implication. We have included tips on quantification that we hope will be a helpful guide (see Tips on Quantification on page 45), but remember to analyze your sentences carefully before translating them into symbols.

When you negate a statement, you must be 100% clear on what your universe is. You can easily see why, too: if you negate $x \in \mathbb{Z}$ and $\mathbb{Z}$ is your universe, then there are no x left, but if you negate $x \in \mathbb{Z}$ and $\mathbb{R}$ is your universe, there are still plenty of x left to worry about. So make sure that you give careful consideration to your universe before beginning a problem.

Let's return to the introductory example in this chapter and apply what we have learned.

Exercise 4.2. Our sentence was "The equation $x^2 + 2x = 15$ has a unique solution."

(a) Write this statement in formal language.
(b) Describe a universe in which this statement is true. Explain briefly why your answer is correct.
(c) Describe a universe in which this statement is false. Explain briefly why your answer is correct. ○

Before proceeding to negations, a little more practice with quantifiers might be helpful.

Exercise 4.3. Write the statements below in symbols, assuming that the universe is $\mathbb{R}$ throughout. Make sure that you quantify x; is it "all x" or "some x"?

(a) For all x, it is the case that x is an integer.
(b) There exists an integer x such that $x > 0$.
(c) There is a rational number x such that $x^2 + 1 = 0$.
(d) For every real number x, there exists a real number y such that $x < y$.
(e) There is a real number y such that $x < y$ for all x.
(f) If x is a rational number, then $x^2 - \pi \neq 0$.
(g) A real number x satisfies $x^2 > 0$, if $x \neq 0$.
(h) If $x > 0$, then $x > 4$ or $x < 6$. ○

We negated conjunctions, disjunctions, and implications. Now we will think about the negation of a quantified statement.

Suppose we have the statement "Every cow is black." How would we negate it? One pretty useless way is to say "Not every cow is black." It's better to say "Some cow is not black." So a useful negation of

$$\forall x, p(x)$$

is

$$\exists x, \neg p(x).$$

Similarly, if we say "There exists a black cow," a useful negation is "No cow is black." So a negation of

$$\exists x, p(x)$$

is

$$\forall x, \neg p(x).$$

You will find that sometimes you can negate a sentence directly and other times you need to convert to symbols. Here is another example.

Example 4.4. Negate the sentence "People who live in glass houses do not throw stones."

We will assume that the universe is the set of all people. What does this say? First, it says something about all people who live in glass houses. So we will use the quantifier "for all" and x will denote a person. The notation $g(x)$ will mean that x lives in a glass house. The notation $t(x)$ will mean that x throws stones. So our sentence becomes "For all x, if $g(x)$, then $\neg t(x)$." If you can negate it now, go ahead. If not, go through the steps below. You should provide reasons why each step below is correct:

1. $\neg(\forall x, (g(x) \to \neg t(x)))$;
2. $\exists x, \neg(g(x) \to \neg t(x))$;
3. $\exists x, \neg(\neg g(x) \vee \neg t(x))$;
4. $\exists x, (g(x) \wedge t(x))$.

The last sentence says that the negation of "People who live in glass houses do not throw stones" is "There exists a person who lives in a glass house and throws stones." There's another important thing to notice here. Though there is no obvious quantifier in the sentence "People who live in glass houses do not throw stones," we all interpret the quantifier as a universal quantifier. If you, or someone else, do not explicitly include a quantifier, (all!) people will assume you meant to insert a universal quantifier. ○

We emphasize that while it is good to practice these symbolic manipulations, it is also important to understand what you are doing. Sometimes you will find it easier to use the symbolic notation and sometimes you won't. Make sure you keep in mind what the sentence says, and whether or not your answer seems reasonable. Before you go off on your own, we'll do a fairly complicated example together.

Example 4.5. Suppose our universe is the set of real numbers and we wish to negate the statement "For every rational number x, there exists an integer n that is greater than x."

So let's try it. First we note that "For every rational number x" means that we are being told that "if x is a rational number" something will happen. What? There will exist an integer bigger than x. So this is an implication of the form "For all x, if x is a rational number, then there exists an n such that n is an integer and $n > x$." Sometimes it is easier to understand a statement if we replace the various subsentences with symbolic representations. We use

$p(x)$ for x is a rational number,

$q(n)$ for n is an integer, and
$r(n,x)$ for $n > x$.

Using this notation, we have

$$\forall x, (p(x) \to \exists n, (q(n) \wedge r(n,x))).$$

Let's try to negate this quantified statement form one step at a time, starting from the outside.

We know that when we negate "for all" it becomes "there exists." In other words, we can replace $\neg(\forall x, \cdots)$ with $\exists x, \neg(\cdots)$. So here's where we are now:

$$\neg(\forall x, (p(x) \to \exists n, (q(n) \wedge r(n,x))))$$

is equivalent to

$$\exists x, \neg(p(x) \to \exists n, (q(n) \wedge r(n,x))).$$

Now we negate the implication. From the last chapter we know that $\neg(P \to Q)$ is equivalent to $P \wedge \neg Q$. We're up to

$$\exists x, (p(x) \wedge \neg(\exists n, (q(n) \wedge r(n,x)))).$$

We still need to negate Q, which is the expression $\exists n, (q(n) \wedge r(n,x))$. At least this is simpler than what we started with! Now $\exists$ will change to $\forall$ and so we need only worry about $q(n) \wedge r(n,x)$. But that's a conjunction. So the final step is to negate that, and we know the negation of the conjunction will become $\neg q(n) \vee \neg r(n,x)$. So here's where we are now:

$$\exists x, (p(x) \wedge (\forall n, (\neg q(n) \vee \neg r(n,x)))).$$

We've done what we were asked to do, in a sense, but our answer is still in symbols. Let's translate back:

"There exists an x such that x is a rational number and for all n it is the case that n is not an integer or n is not greater than x."

Well, that's certainly a mouthful. Let's try again (explain how we get the following):

"There is a rational number x such that for all n, if n is an integer, then $n \leq x$."

And finally (explain!):

"There is a rational number x such that for all integers n, $n \leq x$." ○

Not all negations are this complicated, but even in simpler statements there are things of which you should be wary. Consider the two statements about real numbers: $\forall x, \exists y, x+y=0$ and $\exists y, \forall x, x+y=0$. Assuming the universe is the set of real numbers, what's the difference between these two statements? In the first, we say that for each x we can find a y with $x+y=0$. That's a statement you have known to be true for years, ever since you learned about $-x$. On the other hand, the second statement says that there exists a y such that for all x, we have $x+y=0$. That statement is false, because the same y would have to work for all x. What's the moral of this story? That the order of the quantifiers is very important.

Exercise 4.6. Negate the statements (a)–(h) of Exercise 4.3. ○

In order to provide you with lots of exercises, we will discuss the contrapositive, converse, and inverse of a statement with quantifiers, such as $\forall x, r(x)$ or of $\exists x, r(x)$. We use the following rules:

Given a statement of the form $\forall x, (p(x) \to q(x))$:

the converse is $\forall x, (q(x) \to p(x))$,
the contrapositive is $\forall x, (\neg q(x) \to \neg p(x))$,
and the inverse is $\forall x, (\neg p(x) \to \neg q(x))$.

Defined this way, the contrapositive will be true precisely when the original statement is true and, as before, this will not be true of the converse or inverse. For existential quantifiers, we follow the same procedure:

Given a statement of the form $\exists x, (p(x) \to q(x))$:

the converse is $\exists x, (q(x) \to p(x))$,
the contrapositive is $\exists x, (\neg q(x) \to \neg p(x))$,
and the inverse is $\exists x, (\neg p(x) \to \neg q(x))$.

Exercise 4.7. For each of the following state the contrapositive of the statement, the converse of the statement, the negation of the contrapositive of the statement, and the negation of the converse of the statement.

(a) $\forall x, ((p(x) \wedge q(x)) \to r(x))$.
(b) (Assume the universe for this is the real numbers.) If there is a real number strictly between 50 and 100, then that number is an integer with square root less than 8. ○

Solutions to Exercises

Solution (4.1). There are many possible answers. We list some below:

(a) $\{1, -1\}$;
(b) $\{1\}$;
(c) the x-axis in $\mathbb{R}^2$;
(d) the xy-plane in $\mathbb{R}^3$;
(e) $\{2n : n \in \mathbb{Z}\} = \{\ldots, -2, 0, 2, \ldots\}$;
(f) the set of all points in $\mathbb{R}^2$ such that both the x and y coordinates are integers.

Solution (4.2). Parts (b) and (c) have alternate solutions.

(a) $\exists x, ((x^2+2x=15) \wedge \forall y, (y^2+2y=15 \rightarrow y=x))$
(b) We choose $\mathbb{N}$ as the universe for both, x and y. The statement is true because 3 is the only natural number that solves the equation, which we leave to you to check.
(c) Now we use $\mathbb{R}$ as the universe for both variables. The statement is then false because 3 and -5 are two different solutions.

Some people use the symbol $\exists!$ to indicate the existence of a unique element in the universe. With that notation, part (a) reads: $\exists! x, x^2+2x=15$.

Solution (4.3). Note that the universe was assumed to be $\mathbb{R}$.

(a) $\forall x, x \in \mathbb{Z}$.
(b) $\exists x, ((x \in \mathbb{Z}) \wedge (x > 0))$.
(c) $\exists x, ((x \in \mathbb{Q}) \wedge (x^2+1=0))$.
(d) $\forall x, \exists y, (x < y)$.
(e) $\exists y, \forall x, (x < y)$.
(f) $\forall x, (x \in \mathbb{Q} \rightarrow x^2 - \pi \neq 0)$.
(g) $\forall x, (\neg(x=0) \rightarrow x^2 > 0)$.
(h) $\forall x, (x > 0 \rightarrow ((x > 4) \vee (x < 6)))$.

Solution (4.6). Note that the universe was assumed to be $\mathbb{R}$.

(a) There exists an x such that x is not an integer.
(b) For all x, the real number x is not an integer or x is nonpositive. This is equivalent to: For all x, if x is an integer, then x is nonpositive.
(c) For all x, if x is a rational number, then $x^2+1 \neq 0$.
(d) There exists an x such that for all y we have $x \geq y$.
(e) For all y, there exists an x such that $x \geq y$.
(f) There is a rational number x such that $x^2 - \pi = 0$.
(g) For some x, we have $x \neq 0$ and $x^2 \leq 0$.
(h) There exists a positive real number x such that $x \leq 4$ and $x \geq 6$.

Solution (4.7).

(a) For the contrapositive: $\forall x, (\neg r(x) \rightarrow (\neg p(x) \vee \neg q(x)))$.
For the converse: $\forall x, (r(x) \rightarrow (p(x) \wedge q(x)))$.
For the negation of the contrapositive: $\exists x, (\neg r(x) \wedge p(x) \wedge q(x))$.
For the negation of the converse: $\exists x, (r(x) \wedge (\neg p(x) \vee \neg q(x)))$.
(b) We begin by writing the statement in formal language. The universe for this problem is the set of real numbers: $\exists x, (50 < x < 100 \rightarrow (x \in \mathbb{Z} \wedge \sqrt{x} < 8))$.
Now for the contrapositive: $\exists x, ((x \notin \mathbb{Z} \vee \sqrt{x} \geq 8) \rightarrow (x \leq 50 \vee x \geq 100))$. In words: There is a real number such that if it is not an integer or its square root is at least 8, then the number is at most 50 or at least 100.
For the converse: $\exists x, ((x \in \mathbb{Z} \wedge \sqrt{x} < 8) \rightarrow 50 < x < 100)$. In words: There is a real number such that if it is an integer and its square root is less than 8, then it is strictly between 50 and 100.

For the negation of the contrapositive: $\forall x, ((x \notin \mathbb{Z} \vee \sqrt{x} \geq 8) \wedge 50 < x < 100)$. In words: Every real number is strictly between 50 and 100 and at least one of the following occur: x is not an integer or the square root of x is at least 8. For the negation of the converse: $\forall x, (x \in \mathbb{Z} \wedge \sqrt{x} < 8 \wedge (x \leq 50 \vee x \geq 100))$. In words: Every real number is an integer, its square root is less than 8, and it is not strictly between 50 and 100.

Problems

Tips on Quantification on page 45 summarizes many of the major points in this chapter. You may find it helpful to read these tips before working the problems below.

Problem 4.1. Write the following statements symbolically.

(a) For every x, there is a y such that $x = 2y$.
(b) For every y, there is an x such that $x = 2y$.
(c) For every x and for every y, it is the case that $x = 2y$.
(d) There exists an x such that for some y the equality $x = 2y$ holds.
(e) There exists an x and a y such that $x = 2y$.

Problem 4.2. Which of the statements in Problem 4.1 are true if the universe for both x and y is the set of the real numbers?

Problem 4.3. Which of the statements in Problem 4.1 are true if the universe for x is the set of the real numbers and the universe for y is the set of the integers?

Problem 4.4. Negate the statements in Problem 4.1.

Problem 4.5. Negate the following sentences. If you don't know how to negate it, change it to symbols and then negate. State the universe, if appropriate.

(a) For all $x \in \mathbb{R}$, we have $x^2 > 0$.
(b) Every odd integer is nonzero.
(c) If I am hungry, then I eat chocolate.
(d) For every girl there is a boy she doesn't like.
(e) There exists x such that $g(x) > 0$.
(f) For every x there is a y such that $xy = 1$.
(g) There is a y such that $xy = 0$ for every x.
(h) If $x \neq 0$, then there exists y such that $xy = 1$.
(i) If $x > 0$, then $xy^2 \geq 0$ for all y.
(j) For all $\varepsilon > 0$, there exists $\delta > 0$ such that if x is a real number with $|x-1| < \delta$, then $|x^2 - 1| < \varepsilon$.
(k) For all real numbers M, there exists a real number N such that $|f(n)| > M$ for all $n > N$.

Problem 4.6. What are the sets, A and B, described by the following statements?

(a) $\forall x, (x \in A \leftrightarrow \exists n, (n \in \mathbb{Z} \wedge x = 2n))$.
(b) $\forall x, (x \in B \leftrightarrow \exists n, (n \in \mathbb{Z} \wedge x = 2n + 1))$.

Problem 4.7. Consider the statement of Exercise 4.2 on page 36.

(a) Write the negation of that statement using symbols.
(b) Write the negation of that statement as an English sentence.

Problem 4.8. Consider the following statement.

For all positive integers x, there exists a real number y such that for all real numbers z, we have $y = z^x$ or $z = y^x$.

(a) Write this statement using symbols and appropriate quantification. Use $\mathbb{R}$ for the universe of all variables.
(b) Once you have written this statement in symbols, negate the (symbolic) statement that you obtained.

Problem 4.9. Consider the following statement:

$$\forall x, ((x \in \mathbb{Z} \wedge \neg(\exists y, (y \in \mathbb{Z} \wedge x = 7y))) \rightarrow (\exists z, (z \in \mathbb{Z} \wedge x = 2z))).$$

(a) Negate this statement.
(b) Write the original statement as an English sentence.
(c) Which statement is true, the original one or the negation? Explain your answer.

Problem 4.10. Write each of the statements below using symbolic notation. In this problem, use $\mathbb{R}$ as the universe for all variables involved.

(a) There is an integer that is bigger than its square.
(b) Every rational number is the product of two irrational numbers. (Note: A real number x is irrational if $x \notin \mathbb{Q}$.)
(c) There are integers m and n such that for each rational number x, we have $m < nx$ or $n < mx$.
(d) Every rational number is the solution of an equation $ax + b = 0$, where a and b are integers.

Problem 4.11. Why is this joke supposed to be funny? A physicist, a chemist, and a mathematician are traveling through Switzerland. From the train they spot a cow grazing in the field. The chemist gazes out the window and says, “Ah, all the cows in Switzerland are brown.” The physicist says, “No, no. You can’t conclude that. You can only say that some of the cows in Switzerland are brown.” The mathematician says, “No, no, no. All you can say is that there is a cow in Switzerland that is brown on one side.”

Problem 4.12. For each of the following, state

1. the negation of the statement;
2. the converse of the statement;
3. the negation of the converse;
4. the contrapositive of the statement; and
5. the negation of the contrapositive.

State the universe, if appropriate, and quantify anything that is quantifiable.

(a) Madeleine waters the rosebush only if it is Tuesday.
(b) If I ski, I will fall.
(c) Windows break if you throw balls through them.
(d) If I negate a sentence, then I always do it wrong.
(e) I will come only if you invite me.
(f) For all positive real numbers x, there exists an integer n such that $1/n < x$. (For the universe on x and n, use the real numbers.)
(g) If x is a nonzero real number, then $x^2 \neq 0$.
(h) If x is a nonzero real number, then there exists a real number y such that $x \cdot y = 1$.
(i) If x and y are even integers, then $x + y$ is an even integer.

Problem 4.13. Find a different useful description of each of the following:

(a) $\{x \in \mathbb{R} : x^2 = 2\}$;
(b) $\{(x,y) \in \mathbb{R}^2 : x = y\}$;
(c) $\{x \in \mathbb{N} : x \leq 0\}$;
(d) $\{x \in \mathbb{Z} : x^2 > 0\}$.

Problem 4.14. Write each of the following in set notation.

(a) The set of all odd integers.
(b) The set of all points in the xy-plane above the line $y = x$.
(c) The set of all points in the xy-plane that are inside the circle of radius one.
(d) The set of all irrational numbers.

Problem 4.15. Assume that the universe for the variables below is $\mathbb{R}$. Consider the statement form $\exists M, ((M \in \mathbb{Z}) \wedge \forall x, (x^2 \leq M))$.

(a) Negate this statement.
(b) Which is true, the statement or its negation?

Problem 4.16. Consider the statement "All odd positive integers are prime."

(a) Write this statement using logic symbols and standard set notation only. You may use $\mathscr{P}$ for the set of all prime numbers. Make sure you include the proper quantifiers.
(b) Negate the statement.
(c) Prove the statement of part (a) or (b), whichever is correct.

Problem 4.17. Let a, b, and c be fixed real numbers, and consider the statement "For all real numbers x, if x is greater than a or equal to b, then x does not equal c."

(a) Write this sentence in symbols.
(b) Negate the symbolic statement.
(c) Translate the statement back into a (coherent) English sentence.

Problem 4.18. For an integer x consider the statement: "If 8 does not divide $x^2 - 1$, then x is even."

(a) State an appropriate universe for x.
(b) Write the statement in symbols.
(c) Negate the statement.

Problem 4.19. Assume that the universe for all variables below is the set of the real numbers, $\mathbb{R}$. Consider the statement:

$$\forall x, (\exists y, (x^3 = y^2) \vee \forall z, (z^2 < 0 \rightarrow x^3 \neq z^2)).$$

(a) Negate this statement (keeping the symbolic notation).
(b) Which statement is true, the original or the negation? Give a very brief argument for your choice.

Problem 4.20. Decide whether statement (3) is true if statements (1) and (2) are both true. Give reasons for your answers.

(a) The three statements are:
  (1) Everyone who loves Bill loves Sam.
  (2) I don't love Sam.
  (3) I don't love Bill.

(b) The three statements are:
  (1) If Susie goes to the ball in the red dress, I will stay home.
  (2) Susie went to the ball in the green dress.
  (3) I did not stay home.

(c) The three statements are:
  (1) If l is a positive real number, then there exists a real number m such that $m > l$.
  (2) Every real number m is less than t.
  (3) The real number t is not positive.

(d) The three statements are:
  (1) Every little breeze seems to whisper Louise or my name is Igor.
  (2) My name is Stewart.
  (3) Every little breeze seems to whisper Louise.

(e) The three statements are:
  (1) There is a house on every street such that if that house is blue, the one next to it is black.

(2) There is no blue house on my street.
(3) There is no black house on my street.

(f) Let x and y be real numbers.

(1) If $x > 5$, then $y < 1/5$.
(2) We know $y = 1$.
(3) So $x \leq 5$.

(g) Let M and n be real numbers.

(1) If $n > M$, then $n^2 > M^2$.
(2) We know $n < M$.
(3) So $n^2 \leq M^2$.

(h) Let x, y, and z be real numbers.

(1) If $y > x$ and $y > 0$, then $y > z$.
(2) We know that $y \leq z$.
(3) Then $y \leq x$ or $y \leq 0$.

Tips on Quantification

- Check the universe for each of the variables. Write it down, if it is not self-evident.
- When a quantifier on a variable in a statement is a universal quantifier, writers sometimes omit it. For example, you may read a statement about real numbers such as "If x is negative, then x^2 is positive." Because the quantifier is not explicitly stated, we assume the author meant, "For every real number x, if x is negative, then x^2 is positive." If you want to say that *there exists* such an x, you must include the existential quantifier.
- Suppose a statement restricts the variable x to a proper subset A of the universe as in the statement form, "For all $x \in A$, property $p(x)$ holds." Since x is universally quantified, this is an implication of the form

$$\forall x, (x \in A \rightarrow p(x)).$$

- Suppose a statement restricts the variable x to a proper subset A of the universe as in the statement form, "For some $x \in A$, property $p(x)$ holds." Since x is existentially quantified, this is a conjunction of the form

$$\exists x, (x \in A \wedge p(x)).$$

- Simple statements are usually easy to negate. Just do it.

- Complicated statements will often resist a "just do." Write them out in symbols first. Make sure you know what the quantifier is on every variable. Check for the various ways one can say "if..., then...."
- Do not use logical connectives ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$) between quantifiers. (Do not write "$\forall x \vee \forall y \cdots$" or "$\forall x \wedge \forall y \cdots$.")
- Know the rules. You must know how to negate statements involving existential quantifiers, universal quantifiers, conjunctions, disjunctions, and implications. The most important negation is also the one students frequently forget: the negation of an implication.
- Practice: Every time you get a definition or theorem, try negating it. If you can't, this might indicate that you do not fully understand it.

If you think you need more practice, here it is. In what follows, unless otherwise stated, all variables are real numbers, and ε and δ represent positive real numbers. Negate all of these.

(a) Let a be a fixed element of $\mathbb{R}$. For every ε there exists δ such that for every $x \in \mathbb{R}$, if $|x-a| < \delta$, then $|x^2 - a^2| < \varepsilon$.
(b) For all x, we have $x < x + \varepsilon$ for every ε.
(c) For every integer n, there exists $x > n$ such that $x^2 > n^2$.
(d) For all x, y, and z, if $x < y$ and $z < 0$, then $zx > zy$.
(e) For every ε, there exists an integer N such that $1/n < \varepsilon$ for all $n \geq N$.
(f) For all x, we have $x < 0$ or $x > 0$.
(g) For all x, there exists an integer n such that $n > x$.
(h) For all x and y, if $x < y + \varepsilon$ for all ε, then $x \leq y$.
(i) For every ε, there exists δ such that $\delta < \varepsilon$.

We provide solutions (which you should not look at until you try the negation!) for the first four exercises below.

(a) Let a be a fixed element of $\mathbb{R}$. There exists ε such that for every δ there exists x for which $|x-a| < \delta$ and $|x^2 - a^2| \geq \varepsilon$.
(b) There exist x and ε such that $x \geq x + \varepsilon$.
(c) There exists an integer n such that for all x, if $x > n$, then $x^2 \leq n^2$.
(d) There exist x, y, and z such that $x < y$, $z < 0$, and $zx \leq zy$. ○