Chapter 6
Sets

In Chapter 4 we introduced the terms *set* and *element of a set*. In order to have some flexibility and to avoid the slightly awkward phrase *set of sets*, we will use the word *collection* as a synonym for set. In particular, we will usually speak of a collection of sets, which is just a set of sets. Just as you worked with points and lines in geometry without having a rigorous definition of those terms, we will ask you to use your intuition as you work with the terms *set* and *element*. It's important to note that you will need to exercise care when you use these words. Mathematics describes very carefully and exactly what you can do with *sets* and when you can use the words *element of a set*. In particular, the construction of "the set of all sets" is forbidden, as this would lead to contradictions. In order to get around such contradictions, mathematicians have developed axioms. These are listed in the Appendix as the Zermelo–Fraenkel system together with the axiom of choice (ZFC, for short). At this point, we will introduce our subject in a less formal way, leaving a more axiomatic treatment for a later course in set theory.

The objects that make up a set are called the elements or members of the set. A set has a defining property, and it is used to determine whether or not an element belongs to the set: To decide whether or not x is in the set S, you need to see whether x satisfies this defining property p. The **empty set** is the set with no elements, and is denoted by $\emptyset$.

Once we have the defining property, there are often several ways to describe a set. If there aren't too many elements in the set, then we can list all elements: $B = \{Benny, Betty, Billy, Bobby\}$. If the elements come from a well-known larger set X and satisfy a defining property p, we may write $\{x \in X : p(x)\}$. This is read "the set of all elements of X satisfying property p." We may think of X as the universe in this context.

Example 6.1. We will frequently use sets such as the set of all even integers, the set of all odd integers, or the set of integers that are divisible by three. We will denote these sets by $2\mathbb{Z}$, $2\mathbb{Z}+1$, or $3\mathbb{Z}$, respectively. Define these sets using the format suggested above; that is, by choosing appropriate X and p so that each set is described by $\{x \in X : p(x)\}$.

U. Daepp and P. Gorkin, *Reading, Writing, and Proving: A Closer Look at Mathematics*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4419-9479-0_6,

For the set of even integers, we write $2\mathbb{Z} = \{x \in \mathbb{Z} : x = 2n \text{ for some } n \in \mathbb{Z}\}$. The set of odd integers can be written as $2\mathbb{Z}+1 = \{x \in \mathbb{Z} : x = 2n+1 \text{ for some } n \in \mathbb{Z}\}$. Finally, the set of all integers that are divisible by three is $3\mathbb{Z} = \{x \in \mathbb{Z} : x = 3n \text{ for some } n \in \mathbb{Z}\}$. ○

Note that a set is described by its elements—not by the order we put the elements in the set, or whether we put an element in more than once. Thus the set $\{1,2,3\}$ is the same as the set $\{1,1,3,2\}$.

Exercise 6.2. For each of the following sets, say what the universe is and write out the defining property. For example, if we wish to describe the set of all women, the universe might be all people, and the defining property would be "x is a woman." Use complete sentences.

(a) The collection A of all members of the school band.
(b) The collection B of all irrational numbers.
(c) The collection of all prime numbers greater than or equal to 4 and less than 7. ○

Exercise 6.3. Care needs to be used when creating a defining property. What is wrong with each of the following?

(a) The collection C of all pretty people in Luxembourg.
(b) The collection D of all collections that do not contain themselves as an element. ○

The notation we have described so far in this chapter is not the only acceptable notation. For example, if we know what our universe is, there may be no reason to repeat it in the notation. Therefore, we may write $\{x \in X : p(x)\}$, or we may simply write $\{x : p(x)\}$. The next exercise introduces you to a slightly different way of describing a set.

Exercise 6.4. Let $S = \{x \in \mathbb{Z} : x = 2n+1 \text{ for some } n \in \mathbb{Z}\}$ and $T = \{s^2 : s \in S\}$. The notation for T is different from the notation we have discussed thus far in the chapter, yet you can still determine T. Write out a description of T using the same notation as the one used for S. Then write out a description of S using the same notation as the one used for T. ○

Exercise 6.5. Consider the set A of nonzero integers.

(a) Write this set using the notation $A = \{x \in S : p(x)\}$.

Use what you learned in previous chapters to answer the following questions.

Define a new "multiplication" on A by $x \star y = 2xy$ for $x, y \in A$. For parts (b) and (c) below, prove the statement or give a counterexample to it. (If you find you cannot answer the questions below, read the discussion following part (c).)

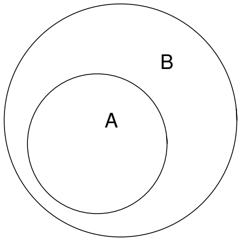

Fig. 6.1 $A \subseteq B$

(b) If $x, y \in A$, then $x \star y \in A$.
(c) There exists an element $y \in A$ such that $x \star y = x$ for every $x \in A$.

If you really can't get started, then you probably didn't understand the problem. One way to begin is to pick numbers for x and y and try them out until you get a feel for this new multiplication. Once you understand it, try rewriting the statements so that they make sense to you. For example, in (b), replace the conclusion $x \star y$ by its definition to obtain "If $x, y \in A$, then $2xy \in A$." All this should help. Remember, the most important thing is to get started. ○

A set A is a **subset** of a set B or, equivalently, A is **contained** in B, if every element of A is an element of B. We will write $A \subseteq B$ to indicate that A is a subset of B. This is depicted in Figure 6.1.

Notice that A is always a subset of itself: $A \subseteq A$. However, a subset can also be truly smaller, and we often find it necessary to use our notation to emphasize this. We say that A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$, and we will write $A \subset B$.

Showing that a set A is contained in another set B turns out to be one of the most important tasks in mathematics. One way to show that a set A is contained in a set B is to do exactly what the definition says; take an arbitrary element of the set A and then show that this element is in set B. We need not worry about whether or not there is an element in the set A. The definition of subset requires only the implication ($x \in A$ implies $x \in B$) to be true, not the antecedent ($x \in A$).

Example 6.6. In Exercise 6.4 we used two sets, S and T, where $S = \{2n+1 : n \in \mathbb{Z}\}$ and $T = \{s^2 : s \in S\}$. Show that $T \subseteq S$. Is T a proper subset of S?

Remember, to prove set inclusion we have to take an arbitrary element in the set T and then show that this element is in the set S. So, for our proof of containment, we will begin with $x \in T$, and attempt to end our proof with $x \in S$.

As we will see in future chapters, we can often devise a plan for a proof of this type by writing out what we know ($x \in T$) at the top of the page, and what we want to show ($x \in S$) at the bottom. You have probably attempted proving things this way before: you work from the top down, and from the bottom up. So our plan might look like

$$x \in T,$$

large space

$$x \in S.$$

But $x \in T$ means that $x = s^2$ for some $s \in S$, and $x \in S$ means that $x = 2n+1$ for some $n \in \mathbb{Z}$. So our plan (a few minutes later) might look like

$$x \in T,$$

$$x = s^2, \text{ for some } s \in S,$$

smaller space

$$x = 2n+1, \text{ for some } n \in \mathbb{Z},$$

$$x \in S.$$

We keep filling things in, making sure that each line follows logically from the previous one, until we see how to complete the proof. Here's what we end up with.

Proof. (Inclusion) We claim that $T \subseteq S$. Now, if $x \in T$, then $x = s^2$ for some $s \in S$. By the definition of S, there exists $n \in \mathbb{Z}$ such that $s = 2n+1$. Hence $x = s^2 = (2n+1)^2 = 4n^2+4n+1 = 2(2n^2+2n)+1$. Now let $m = 2n^2+2n$. Then $m \in \mathbb{Z}$ and $x = 2m+1$. Therefore $x \in S$. Thus $T \subseteq S$, as desired.

(Proper subset) In fact, T is a proper subset of S. To show this we need to exhibit an element that is in S, but not in T. Consider the number -1. Then $-1 = 2(-1)+1$ and $-1 \in \mathbb{Z}$. Thus, -1 satisfies the defining property for S, so $-1 \in S$. On the other hand, the elements of T are squares of real numbers. Consequently all of them are nonnegative. Hence $-1 \notin T$, and the inclusion is proper. □

If you remember the result of Problem 3.2, then you know that you already showed that s^2 is odd if and only if s is odd. If you refer the reader to this result (carefully referencing it, so the reader can find it easily), you can significantly shorten the proof of inclusion. Given two proofs written with equal clarity and insight, most people will prefer the shorter of the two. If the reader remembers the result, reading it again may detract from the proof. So, as long as you tell the reader what you are using and where to find it if he or she needs to, you can (and should) refer to previous results. ○

We now return to the subject of this chapter. Notice that we just told you how to show that a set A is contained in a set B. All you need to do is show that *for all x, if $x \in A$, then $x \in B$*. So we also just told you how to show that A is not contained in B—negate the definition of containment.

Exercise 6.7. Negate the statement: "For all x, if $x \in A$, then $x \in B$." ○

Two sets are equal if they have precisely the same elements. This can be defined a bit more formally as follows. A set A is **equal** to B, written $A = B$, if $A \subseteq B$ and $B \subseteq A$. To show that two sets are equal is therefore a two-step task: First you show that one set is contained in the other ($A \subseteq B$). Then you reverse the order of the sets and show inclusion again ($B \subseteq A$).

Note that when A is a subset of B we use the symbol $\subseteq$, but when x is an element of A we use the symbol $\in$. Choose your symbols carefully and don't mix them up! If x is not in A, then we write $x \notin A$. If A is not a subset of B, then we write $A \not\subseteq B$.

Exercise 6.8. Write a definition of set equality that reverts back to membership in a set, rather than set containment. ○

Example 6.9. Show that $\{(x,y) \in \mathbb{R}^2 : x^2 - x = y^2 = 0\} = \{(0,0),(1,0)\}$.

Devising a plan. According to the definition of equality above, we have to show two separate things. The first is to show that the set on the left is contained in the set on the right. For this part of the proof, we will begin with an arbitrary element $(w,z) \in \{(x,y) \in \mathbb{R}^2 : x^2 - x = y^2 = 0\}$ and we will try to show that $(w,z) = (0,0)$ or $(w,z) = (1,0)$. Then we must show that the set on the right is contained in the set on the left. So for this part, we will begin with $(a,b) = (0,0)$ or $(a,b) = (1,0)$ and try to show that it is in the set on the left.

Before beginning any problem, make sure you understand exactly what kind of objects are under consideration. In this example, our elements are in $\mathbb{R}^2$, and therefore they are "points" with an x- and a y-coordinate. Thus, when we discuss elements, we will not talk about an element like "z," but rather one of the form (x,y). You'll progress more quickly if you use the particular form of the elements in your problem.

Proof. If $(w,z) \in \{(x,y) \in \mathbb{R}^2 : x^2 - x = y^2 = 0\}$, then $w^2 - w = w(w-1) = 0$ and $z^2 = 0$. Hence $w = 0$ or $w = 1$ and in both cases, $z = 0$. Thus $(w,z) = (0,0)$ or $(w,z) = (1,0)$. This implies that $(w,z) \in \{(0,0),(1,0)\}$ and therefore we can conclude that $\{(x,y) \in \mathbb{R}^2 : x^2 - x = y^2 = 0\} \subseteq \{(0,0),(1,0)\}$.

If $(w,z) \in \{(0,0),(1,0)\}$, then $w = 0$ or $w = 1$ and thus $w(w-1) = w^2 - w = 0$. We also have $z = 0$ and thus $z^2 = 0$. Hence $(w,z) \in \{(x,y) \in \mathbb{R}^2 : x^2 - x = y^2 = 0\}$ and therefore $\{(0,0),(1,0)\} \subseteq \{(x,y) \in \mathbb{R}^2 : x^2 - x = y^2 = 0\}$.

By the definition of equality, $\{(x,y) \in \mathbb{R}^2 : x^2 - x = y^2 = 0\} = \{(0,0),(1,0)\}$. □

Exercise 6.10. Let $A = \{1,3,5\}, B = \{3,4,6\}, C = \{5\}$, and $D = \{1,3\}$. Which sets are subsets of the others? For which sets S do we have $1 \in S$? $1 \notin S$? Which sets are not subsets of each other? ○

Theorem 6.11. *Let A be a set. Then $\emptyset \subseteq A$.*

Proof. We must show that for every x, if $x \in \emptyset$, then $x \in A$. Since there are no elements in the empty set, the antecedent is always false. Therefore the implication is always true, completing the proof. □

Suppose A is a set and we wish to show that $A = \emptyset$. One inclusion, $A \subseteq \emptyset$, will suffice because the reverse inclusion is always true. In fact, such a proof will often be done by assuming that there exists $x \in A$ and then obtaining a contradiction. When showing that a set $A = \emptyset$, then, it will look like we are proving that two sets are equal by establishing only one containment. That's only because the other containment is obvious!

Exercise 6.12. Prove the following.

(a) For $A = \{x \in \mathbb{R} : x^2 + x + 1 = 0\}$, show that $A = \emptyset$, using the comments in the preceding paragraph.
(b) For $B = \{x \in \mathbb{N} : \exists y \in \mathbb{R}, x = y^2\}$, show that $B = \mathbb{N}$. ○

We will now present a list of very important definitions, using two sets, A and B, to create other sets. Some examples will be presented (briefly) here, and more can be found in the exercises. In what follows, we assume that all variables x belong to a universe, X.

The **union** of two sets A and B is defined by $A \cup B = \{x : x \in A \text{ or } x \in B\}$. For example, if A denotes the set of even integers, and B denotes the set of odd integers, then $A \cup B = \mathbb{Z}$.

The **intersection** of A and B is $A \cap B = \{x : x \in A \text{ and } x \in B\}$. If A and B are two sets such that $A \cap B = \emptyset$, then we say that A and B are **disjoint**. For example, if A is the set of even integers and B is the set of odd integers, then A and B are disjoint.

The **set difference** of B in A is $A \setminus B = \{x \in A : x \notin B\}$. A comment is in order here. We can never look for objects "not in B" unless we know where to start looking. So we use A to tell us where to look for elements not in B. If X is the universe, we will write B^c for $X \setminus B$. This is referred to as the **complement** of B. For example, let A be the set of integers. If $B = \mathbb{Z}^+$, then $A \setminus B$ is the set of elements of A (integers) that are not in B (that are not positive integers). Thus $A \setminus B = \{x \in \mathbb{Z} : x \leq 0\}$. On the other hand, if $A = \mathbb{N}$, then $A \setminus B = \{0\}$.

It is possible to visualize these sets using a representation called a Venn diagram. These diagrams are often helpful in sorting out the relationship between sets. The universe is usually indicated by a rectangle containing the sets. The idea is illustrated in Figures 6.2 and 6.3. You might enjoy the pretty Venn diagrams in [93] and the book *Cogwheels of the Mind: The Story of Venn Diagrams,* [24].

A word of warning: Be careful—pictures can be deceiving. Use the Venn diagram to get your intuition going, but check everything carefully using the techniques we have developed thus far.

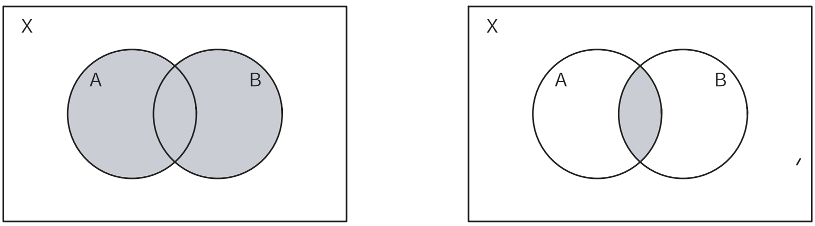

Fig. 6.2 $A \cup B$ and $A \cap B$

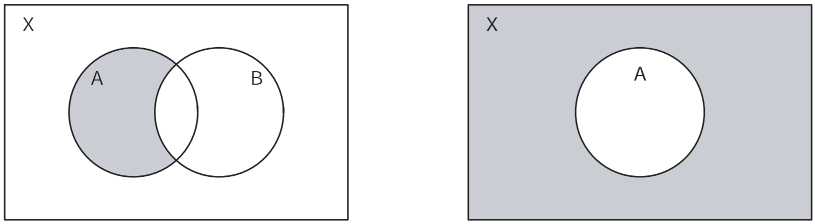

Fig. 6.3 $A \setminus B$ and A^c

Exercise 6.13. Use the sets in Exercise 6.10 to answer the following questions: What is $A \setminus B$? $A \setminus C$? Which sets are disjoint? If the universe is $\{1,2,3,4,5,6\}$, what is A^c? Find $A \cup B$ and $A \cap B$. ○

Exercise 6.14. Write a definition of union for three sets. Write a definition of intersection for three sets. Can you write a definition of set difference for three sets? Why or why not? ○

Definitions

Definition 6.1. The **empty set**, denoted $\emptyset$, is the set that contains no element.

Definition 6.2. The set A is a **subset** of set B, denoted $A \subseteq B$, if for all x it is the case that $x \in A$ implies that $x \in B$.

Definition 6.3. The set A is a **proper subset** of set B, denoted $A \subset B$, if $A \subseteq B$ and $A \neq B$.

Definition 6.4. Two sets A and B are **equal** if $A \subseteq B$ and $B \subseteq A$.

Definition 6.5. The **union** of the sets A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Definition 6.6. The **intersection** of the sets A and B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Definition 6.7. Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

Definition 6.8. The **set difference** of set B in set A is the set $A \setminus B = \{x \in A : x \notin B\}$.

Definition 6.9. If the set X is the universe and A is a subset of X, then the **complement** of A is the set $A^c = X \setminus A$.

Solutions to Exercises

Solution (6.2). Here is the answer to (b): The universe is the set of all real numbers. The defining property is "$x \in \mathbb{R} \setminus \mathbb{Q}$."

Solution (6.3).

(a) The adjective "pretty" is subjective and it is unclear whether a person from Luxembourg is a member of the set C or not.
(b) Consider the following question: Is the collection D an element of D or not? If it is an element of D, then it must satisfy the defining property, which says that D is not an element of D; in other words, in this case it would have to be both in the set and not in the set. On the other hand, if D is not an element of the collection D, then it does just what the defining property says. Thus it must be in the set D; in other words, in this case it would have to be both in the set and not in the set. Hence, this property is contradictory.

Solution (6.4). We can write $T = \{x \in \mathbb{Z} : x = (2n+1)^2 \text{ for some } n \in \mathbb{Z}\}$ and $S = \{2n+1 : n \in \mathbb{Z}\}$.

Solution (6.5). Let A be the set of nonzero integers.

(a) $A = \{x \in \mathbb{Z} : x \neq 0\}$.
(b) Let x and y be elements of A. Then $x \star y = 2xy$. Since x, y, and 2 are all integers, $x \star y \in \mathbb{Z}$. Furthermore, since x and y are elements of A, they are nonzero. Therefore $x \star y = 2xy \neq 0$. Consequently $x \star y \in A$, as desired.
(c) This is false. Suppose to the contrary that there were such an element y in A. Then $x \star y = x$ for every $x \in A$. Choosing $x = 1$, we see that $1 = 1 \star y = 2(1)(y) = 2y$. The only solution to this equation is $y = 1/2$, which is not an integer and therefore not an element of A. This contradiction shows that no such y can exist.

Solution (6.7). The negation is "There exists an x such that $x \in A$ and $x \notin B$."

Solution (6.8). Two sets A and B are equal if for all x we have $x \in A$ if and only if $x \in B$.

Solution (6.10). The following statements hold:

$C \subseteq A$ and $D \subseteq A$;
no other sets are subsets of each other;
$1 \in A$, $1 \in D$, $1 \notin B$, and $1 \notin C$.

Solution (6.12).

(a) Suppose there exists $x \in A$. Then $x \in \mathbb{R}$ and $x^2 + x + 1 = 0$. This implies that $x = (-1+\sqrt{3}\mathrm{i})/2$ or $x = (-1-\sqrt{3}\mathrm{i})/2$. In both cases, $x \notin \mathbb{R}$ leading to a contradiction. Hence $A = \emptyset$. Alternatively, we might recall the technique of completing the square to note that

$$x^2+x+1=x^2+x+\frac{1}{4}+\frac{3}{4}=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0\,,$$

establishing a contradiction.

(b) We need only show that $\mathbb{N}\subseteq B$ (why?). If $n\in\mathbb{N}$, then $n\in\mathbb{R}$ and $n\geq 0$. Thus $y=\sqrt{n}\in\mathbb{R}$ and $n=y^2$. Hence $y\in B$. This shows that $\mathbb{N}=B$.

Solution (6.13). The following statements hold:

$A\setminus B=\{1,5\}$, $A\setminus C=\{1,3\}$;
sets B and C are disjoint, and sets C and D are disjoint;
if the universe is as given, then $A^c=\{2,4,6\}$;
$A\cup B=\{1,3,4,5,6\}$, and $A\cap B=\{3\}$.

Solution (6.14). Let A,B, and C be sets and let the universe be denoted by X. Then $A\cup B\cup C=\{x\in X : x\in A$ or $x\in B$ or $x\in C\}$ and $A\cap B\cap C=\{x\in X : x\in A$ and $x\in B$ and $x\in C\}$. While the union and intersection of three sets makes sense, the set difference of three sets does not. In order to answer this question, we would need to reduce it to a set difference of two sets by including parentheses. For example, you can define the following set differences: $(A\setminus B)\setminus C$ and $A\setminus(B\setminus C)$ (try it!). Work out what these last two sets are when A, B, and C are as in Exercise 6.10.

Spotlight: Paradoxes

You may already have seen paradoxes in mathematics. For example, you may have seen Zeno's paradoxes, the most famous of which is the story of Achilles and the tortoise, in your calculus class. Another well-known paradox comes from the following: what is the sum of

$$1-1+1-1+1-\cdots?$$

You might argue that this sum should be $(1-1)+(1-1)+\cdots=0$. Or, you might just as well argue that this sum should be $1+(-1+1)+(-1+1)+\cdots=1$. You might even argue (as Luigi Guido Grandi did [27, p. 135]) that since the sums 0 and 1 are equally probable, the answer should be the average of 0 and 1; in other words, $1/2$. This paradox forces us to look closely at exactly what we mean by summing infinitely many numbers.

Betrand Russell pointed out a paradox in set theory. He also presented a popular form of this paradox, called the barber problem. The problem is the following. Suppose there is a town with one barber, and this barber says that he shaves those people, and only those, who do not shave themselves. The question is: Who shaves the barber? (You'll recognize the set theoretic form of this problem in Exercise 6.3.)

Paradoxes serve a very useful purpose. They point out where the foundations of mathematics are shaky (or even faulty!). To learn more about them, and how they have been handled, we recommend reading [26, Chapter 15], [57, Chapter 18], or [59, Chapter 51].

Problems

Problem 6.1. "Order" the following sets, using the appropriate symbol from among $=, \subseteq, \subset$:

$$\mathbb{N}, \mathbb{R}, \mathbb{Z}^+, \mathbb{Z}, \mathbb{Q}.$$

Problem# 6.2. Let A, B, and C be sets. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (We say that set containment is transitive.)

Problem 6.3. Recall that $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ the set of integers, and $\mathbb{R}$ the set of real numbers.

(a) Write the phrase "x belongs to $\mathbb{R}$" in symbols.
(b) Write the phrase "$\mathbb{Z}$ is a proper subset of $\mathbb{R}$" in symbols.
(c) Write the phrase "If x is an element of $\mathbb{Z}$, then x or $-x$ is an element of $\mathbb{N}$" in symbols.
(d) Use set notation to describe the set of squares of all multiples of 3.

Problem 6.4. In this problem our universe is $\mathbb{R}$, the set of real numbers.

(a) Give an example of subsets A and B of $\mathbb{R}$ that are disjoint.
(b) Give an example of subsets A and B of $\mathbb{R}$ that are not disjoint and find $A \setminus B$ and $B \setminus A$.
(c) Give an example of subsets A and B of $\mathbb{R}$ such that $A \subseteq B$.
(d) Give an example of subsets A, B, and C of $\mathbb{R}$ such that

$$A \cup (B \cap C) \neq (A \cup B) \cup (A \cup C).$$

Problem 6.5. The universe in this problem is $\mathbb{R}$. Let A be the closed interval $[0,2]$ and let B be the closed interval $[-1,1]$. Find $A \setminus B$, $B \setminus A$, A^c, B^c, $A^c \cap B^c$, $(A \cup B)$, and $(A \cup B)^c$.

Problem 6.6. Find an expression for each of the shaded sets in the Venn diagrams of Figure 6.4.

Problem 6.7. (a) Consider the set S of nonzero real numbers. Write S in set notation.
(b) Define a new "multiplication" on this set by $x \heartsuit y = x/y$. If $x, y \in S$, is $x \heartsuit y \in S$? Is there an element $y \in S$ such that $x \heartsuit y = x$ for all $x \in S$?
(c) Repeat parts (a) and (b), replacing the set S by the set T of negative real numbers.
(d) Repeat parts (a) and (b), replacing the set S by the set V of nonzero rational numbers.

Problem 6.8. Define two sets A and B as follows: $A = \{(2n+1)^3 : n \in \mathbb{Z}\}$ and $B = \{2n+1 : n \in \mathbb{Z}\}$.

(a) Prove that $A \subset B$.
(b) Suppose we redefine A and B, replacing $\mathbb{Z}$ by $\mathbb{R}$; in other words, let $A = \{(2n+1)^3 : n \in \mathbb{R}\}$ and $B = \{2n+1 : n \in \mathbb{R}\}$. What is the relation between these two sets? State and prove your answer.

Problem 6.9. Find an expression for each of the shaded sets in the Venn diagrams of Figure 6.5.

Problem 6.10. Is the following statement true or false: $\{\emptyset\} = \emptyset$? Why?

Problem 6.11. Show that $\{x \in \mathbb{R} : x^2 - 1 = 0\} = \{1, -1\}$.

Problem 6.12. Let $A = \{x \in \mathbb{Z} : 6 \text{ divides } x\}$, $B = \{x \in \mathbb{Z} : 21 \text{ divides } x\}$, and $C = \{x \in \mathbb{Z} : 42 \text{ divides } x\}$. Prove that $A \cap B = C$.

Problem 6.13. Let $A = \{(x,y) \in \mathbb{R}^2 : x - y = 0\}$, $B = \{(x,y) \in \mathbb{R}^2 : x + y = 0\}$, and $C = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 0\}$. Prove that $A \cup B = C$.

Problem 6.14. Let $A = \mathbb{Z}$, $B = \{x \in \mathbb{Z} : x = 2n+5 \text{ for some } n \in \mathbb{Z}\}$ and, $C = \{x \in \mathbb{Z} : x = -2m \text{ for some } m \in \mathbb{Z}\}$. Prove that $A \setminus B = C$.

Problem 6.15. Let S be the set of nonzero real numbers. Define a new "addition" on this set by $x \sharp y = x + y + 1$. If you add two numbers in S, do you end up with a number in S? (In other words, if $x, y \in S$, is $x \sharp y \in S$?)

Problem 6.16. Prove that $A = B$ in each of the following.

(a) Let A and B be the sets defined by $A = \{x \in \mathbb{R} : \sin(\pi x) = 0\}$ and $B = \mathbb{Z}$.
(b) Let $x \in \mathbb{R}$. Define $A = \{(ax+b)/(cx+d) : a,b,c,d \in \mathbb{Z} \text{ and } cx+d \neq 0\}$ and $B = \{(px+q)/(rx+s) : p,q,r,s \in \mathbb{Q} \text{ and } rx+s \neq 0\}$.

Problem 6.17. Let $A = \{x \in \mathbb{R} : ax^2 + bx + c = 0$ for some integers a, b, and c, with at least one of a, b, c nonzero$\}$ and let $B = \{x \in \mathbb{R} : px^2 + qx + r = 0$ for some rational numbers p, q, and r, with at least one of p, q, r nonzero$\}$.

(a) Prove that $2 \in A$.

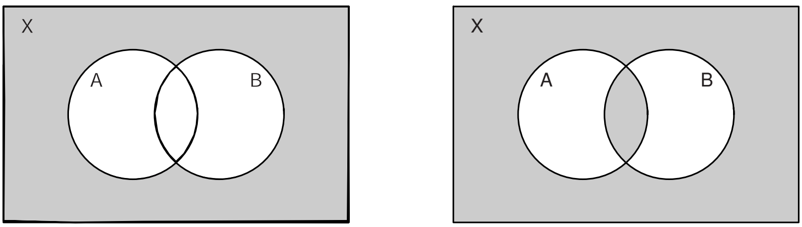

Fig. 6.4

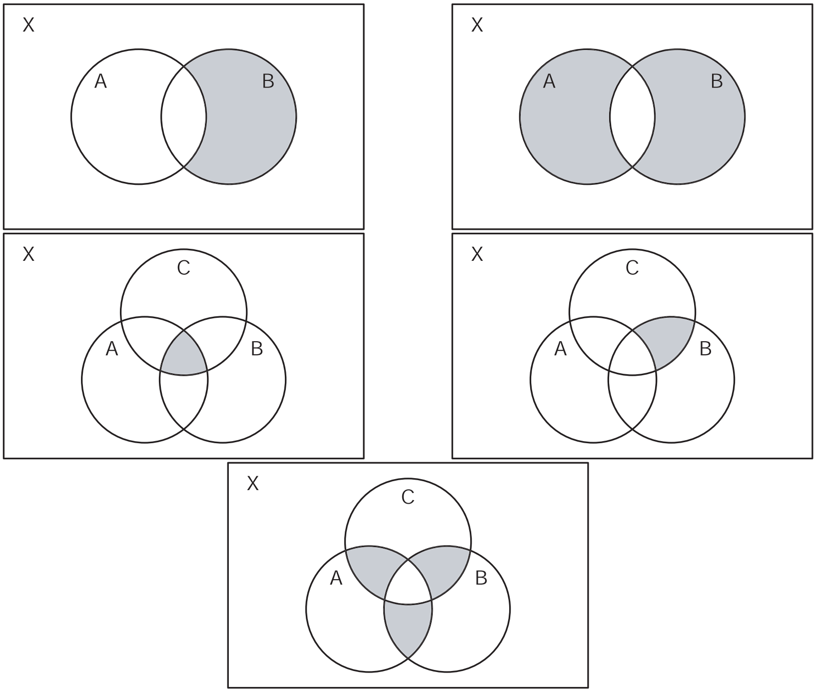

Fig. 6.5

(b) Prove that $\sqrt{2} \in A$.
(c) Give an example of a real number y such that $y \notin A$. (You do not need to prove that $y \notin A$.)
(d) Prove that $A = B$.
(e) Prove that $\mathbb{Q} \subseteq A$.

The following problems deal with sets of points in the plane. We remind you of the notation introduced in Chapter 4. The set of all points in the plane is denoted by $\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$. Subsets of $\mathbb{R}^2$ will be studied in more generality in Chapter 9.

Problem 6.18. Define a set A by $A = \{(x,y) \in \mathbb{R}^2 : y \neq 0\}$.

(a) Give a geometric description of A.
(b) Suppose we tell you that if you have two elements of this set A, you can "add" them according to the following rule:

$$(x,y) \diamond (z,w) = (xw + zy, wy).$$

The symbol $+$ here denotes usual addition. Show that the object that results when we add two elements of our set A is again an object in our set A.

(c) Continuing, find an element (a,b) in A such that $(a,b) \diamond (x,y) = (x,y)$ for every (x,y) in A.
(d) This "new" addition probably looks somewhat odd to you, but you have seen it before. What is it?

Problem 6.19. In each part of this problem, two sets, A and B, are defined. Prove that $A \subseteq B$ in each of the following:

(a) $A = \{x^2 : x \in \mathbb{Z}\}$ and $B = \mathbb{Z}$;
(b) $A = \mathbb{R}$ and $B = \{2x : x \in \mathbb{R}\}$;
(c) $A = \{(x,y) \in \mathbb{R}^2 : y = (5-3x)/2\}$ and $B = \{(x,y) \in \mathbb{R}^2 : 2y+3x = 5\}$.

Problem 6.20. Prove that $\{n^2+n+1 : n \in \mathbb{N}\} \subseteq \{2n+1 : n \in \mathbb{N}\}$.

Problem 6.21. Let

$$A = \{x \in \mathbb{N} : x < x^2 \text{ and for all } y \in \mathbb{Z}, \text{ if } y|x, \text{ then } y^2|x\}.$$

Prove that $A = \emptyset$.

Problem 6.22. Prove that one set is a proper subset of the other one in each of the following:

(a) $A = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$ and $B = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 0\}$;
(b) $A = \emptyset$ and $B = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 0\}$.

Problem 6.23. Consider the sets

$$A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\} \text{ and } B = \{(x,y) \in \mathbb{R}^2 : |x|+|y| \leq 1\}.$$

Is one of the two sets properly contained in the other one? Justify your answer.

Problem 6.24. Define the following three sets:

$A = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$;
$B = \{(x,y) \in \mathbb{R}^2 : y > |x|\}$;
$C = \{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$.

Carefully prove that $A \cap B = C$.