Chapter 9
The Power Set and the Cartesian Product

Now that we know about sets, we can construct some new ones from old ones in even more ways than we did before. In this section we look closely at two special sets: the first is called the power set, and the second is called the Cartesian product of two sets.

Let S be a set. Then the **power set** of S is the set of all subsets of S. We shall denote the power set by $\mathscr{P}(S)$. When working with the power set, the following observation will be extremely useful: $A \in \mathscr{P}(S)$ if and only if $A \subseteq S$.

Before we begin, note that the power set is again a set and its elements are also sets. The power set is never empty. Why?

Example 9.1. Consider the set $S = \{0, 1\}$. We look for all subsets of S. They are $\emptyset, \{0\}, \{1\}$, and $\{0, 1\}$. Then the power set of S is $\mathscr{P}(S) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. ○

As you probably noticed, the notation is tricky here. You have to distinguish very carefully between "element of" ($\in$) and "subset of" ($\subseteq$). Until you get really good at this distinction, we suggest that you ask yourself each time you use $\in$ or $\subseteq$ whether you chose the correct symbol. It's time for some practice with this notation.

Exercise 9.2. Let A be a set. Which of the following are true, which are false? Explain.

(a) $A \in \mathscr{P}(A)$;
(b) $\emptyset \subseteq \mathscr{P}(A)$;
(c) $\emptyset = \mathscr{P}(\emptyset)$;
(d) $\{\emptyset\} = \mathscr{P}(\emptyset)$;
(e) if $a \in A$, then $\{a\} \subseteq \mathscr{P}(A)$.

Now we are ready to explore the power set construction further.

Exercise 9.3. Let $A = \{1, 2, 3\}$, $B = \{2, 5\}$, $C = \{0, 1\}$.

(a) Find $\mathscr{P}(B)$ and $\mathscr{P}(C)$. Do these two sets have elements in common?
(b) Find $\mathscr{P}(A)$, $\mathscr{P}(B)$, $\mathscr{P}(A \cap B)$, and $\mathscr{P}(A \cup B)$.

(c) Compute $\mathscr{P}(A) \cup \mathscr{P}(B)$ and $\mathscr{P}(A) \cap \mathscr{P}(B)$.   ○

Remember to use element notation when you are thinking of the set as an element and subset notation when you are showing containment of sets.

In the next exercise, we will ask you to prove that two sets are equal. We've done this many times in the previous chapters, and so you know one way to begin: use an element-chasing argument. Ask yourself if your set plays the role of a set or the role of an element, and use the corresponding notation.

Exercise 9.4. Let A and B be sets. Prove that $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.   ○

When we talk about a set, it is understood that if we discuss the set $\{1,3\}$ we are discussing the set $\{3,1\}$ as well. A set is determined by its elements and there is no notion of order associated with what we have defined so far. When there is an order, such as when we plot points and need to know which is the x coordinate and which is the y coordinate, we use the notion of an ordered pair. The next set we will consider is called the Cartesian product of two sets X and Y, and it is constructed using ordered pairs.

Here is our informal definition: An **ordered pair** (x,y) is a pair of objects in which there is a first object x and a second object y. The very important property of ordered pairs is that $(x,y) = (z,w)$ if and only if $x = z$ and $y = w$.

We may now define the **Cartesian product** of X and Y, denoted $X \times Y$, to be the set of all ordered pairs in which the first element comes from X and the second from Y; that is,

$$X \times Y = \{(x,y) : x \in X, y \in Y\}.$$

For example, if $X = [0,1]$ and $Y = [0,2]$, then

$$X \times Y = \{(x,y) : 0 \le x \le 1, 0 \le y \le 2\}.$$

This is the rectangle in $\mathbb{R}^2$ with base along the interval $[0,1]$ and height along the interval $[0,2]$ sketched in

Perhaps you are wondering why we said "informal definition" when we presented our definition of ordered pair. Since this is probably the definition you were expecting, it most likely looks formal. It turns out that there is a rigorous definition of ordered pair; one that can be presented without referring to the "first" and "second" coordinates. The reason we do not present it here is that, in our opinion, a rigorous definition is mostly confusing rather than helpful at this point. If you have a strong desire to know more about this, you can work Problem 9.23 in this chapter.

It's time for a few more examples of Cartesian products.

Exercise 9.5.   (a) Write out all the elements in $\{0,1\} \times \{2,3\}$ and $\{2,3\} \times \{0,1\}$.
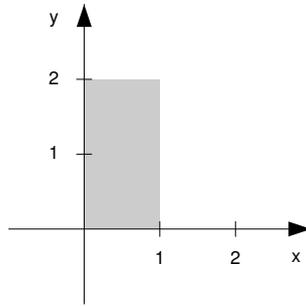 (b) Sketch the Cartesian products $[0,1] \times [2,3]$ and $[2,3] \times [0,1]$ as sets of points in the plane.

Fig. 9.1 $[0,1] \times [0,2]$

(c) Recall that we defined $\mathbb{R}^2 = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$. Write $\mathbb{R}^2$ using the Cartesian product notation.

(d) Having done that, can you describe $\mathbb{R}^3$ as a Cartesian product of two sets? (You might have more than one description that seems reasonable to you.)

(e) The set of even integers is denoted by $2\mathbb{Z}$ (see Example 6.1). Make sketches that describe the sets $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 2\mathbb{Z}$, and $2\mathbb{Z} \times \mathbb{Z}$.   ○

When we prove that two sets defined using Cartesian products are equal, we can still use the method of "element-chasing." Remember that we need to use the special form of the element, namely, that it looks like an ordered pair. If you don't use the form of an element, you may lose valuable information and, as a consequence, proving your result will be tougher than it has to be. As you read the proofs below, pay attention to where we use the fact that the element we consider is an ordered pair.

Theorem 9.6. *Let A be a set. Then $A \times \emptyset = \emptyset$.*

Proof. Suppose, to the contrary, that $A \times \emptyset \neq \emptyset$. Then there exists an element $(x,y) \in A \times \emptyset$. Therefore, by our definition of Cartesian product, $x \in A$ and $y \in \emptyset$. But this contradicts the fact that $\emptyset$ is the empty set. Thus $A \times \emptyset = \emptyset$.   □

Theorem 9.7. *Let A, B, C, and D be sets. Then*

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Proof. If $z \in (A \times B) \cup (C \times D)$, then $z = (x,y)$ where $(x,y) \in A \times B$ or $(x,y) \in C \times D$. Suppose first that $(x,y) \in A \times B$. Then $x \in A$ and $y \in B$. In this case $x \in A \cup C$ and $y \in B \cup D$, so by definition $(x,y) \in (A \cup C) \times (B \cup D)$. Now suppose that $(x,y) \in C \times D$. Then $x \in C$ and $y \in D$. Therefore $x \in A \cup C$ and $y \in B \cup D$. So $(x,y) \in (A \cup C) \times (B \cup D)$. Hence $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$, as desired.   □

Again, notice how quickly we changed from z to (x,y) in the proof. That's because we can't do anything if we don't realize that z is really an ordered pair.

Now consider the following nontheorem.

Nontheorem. *Let A, B, C, and D be sets. Then*

$$(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D).$$

Not a proof. If $(x, y) \in (A \cup C) \times (B \cup D)$, then $x \in A \cup C$ and $y \in B \cup D$. Thus $x \in A$ or $x \in C$, and $y \in B$ or $y \in D$. Hence $x \in A$ and $y \in B$ or $x \in C$ and $y \in D$. So $(x, y) \in A \times B$ or $(x, y) \in C \times D$. Thus $(x, y) \in (A \times B) \cup (C \times D)$.   ☐

Exercise 9.8. Find the error in the nonproof above and show that Nontheorem 9 really is not a theorem because the statement is false. (Find sets for which the statement does not hold.)   ○

   In these problems and all that follow, you will begin with an element in your set. It will be helpful to you to think about the form of your element. Is it a set? an ordered pair? If you rush through these proofs, as we did in Nontheorem 9, you will prove things that are false. This is generally frowned upon in mathematics. Go slowly, be careful, and check each step.
   We will now define relations. We will soon see that there is a connection between functions (something you probably feel familiar with) and relations (something you may not feel terribly familiar with). We begin with a definition.
   Suppose that X and Y are two sets. A **relation from X to Y** is a subset of $X \times Y$. A relation from X to X is called a **relation on X**.

Exercise 9.9. For the following, decide whether or not they are relations from a set X to a set Y. If they are, say what X is and what Y is. Then describe each set pictorially (as a set of points in the plane) or in words:

   (a) $\{(x, y) \in \mathbb{R}^2 : x \le y\}$;
   (b) $\{x/y : x, y \in \mathbb{Z} \text{ and } y \ne 0\}$;
   (c) $\{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Z} \text{ and } x + y = 0\}$.   ○

   We will learn more about relations in Chapter 10.

Definitions

Definition 9.1. The **power set** of a set S is the set of all subsets of S. It is denoted by $\mathscr{P}(S)$.

Definition 9.2 (see Problem 9.23 for a formal definition). The **ordered pair** of the objects x and y is the object (x, y) in which x is considered to be the first object and y the second object.

Definition 9.3. The **Cartesian product** of sets X and Y is the set of ordered pairs $X \times Y = \{(x, y) : x \in X, y \in Y\}$.

Definition 9.4. A **relation from a set X to a set Y** is a subset of $X \times Y$. If $Y = X$, we say the relation is a **relation on X**.

Solutions to Exercises

Solution (9.2).

(a) True. We have $A \subseteq A$ and thus $A \in \mathscr{P}(A)$.
(b) True (e.g., by Theorem 6.11).
(c) False. Since $\emptyset \subseteq \emptyset$ we get $\emptyset \in \mathscr{P}(\emptyset)$ and thus $\mathscr{P}(\emptyset) \neq \emptyset$.
(d) True. See the argument for (c) and note that if $B \subseteq \emptyset$, then $B = \emptyset$.
(e) False. Here is a counterexample: Let $A = \{0\}$, then $\mathscr{P}(A) = \{\emptyset, \{0\}\}$ and $0 \in A$ but $\{0\} \not\subseteq \mathscr{P}(A)$.

Solution (9.3).

(a) $\mathscr{P}(B) = \{\emptyset, \{2\}, \{5\}, \{2,5\}\}$, $\mathscr{P}(C) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$, and the empty set is an element of both sets.
(b) You can check that $\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ and $\mathscr{P}(A \cap B) = \{\emptyset, \{2\}\}$. We leave $\mathscr{P}(A \cup B)$ to you.
(c) $\mathscr{P}(A) \cap \mathscr{P}(B) = \{\emptyset, \{2\}\}$. We leave $\mathscr{P}(A) \cup \mathscr{P}(B)$ to you.

Solution (9.4). If $x \in \mathscr{P}(A \cap B)$, then $x \subseteq A \cap B$. Thus $x \subseteq A$ and $x \subseteq B$. This implies that $x \in \mathscr{P}(A)$ and $x \in \mathscr{P}(B)$, so $x \in \mathscr{P}(A) \cap \mathscr{P}(B)$. Since x was an arbitrarily chosen element, $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$. Each of these steps is reversible, so the containment $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$ follows as well.
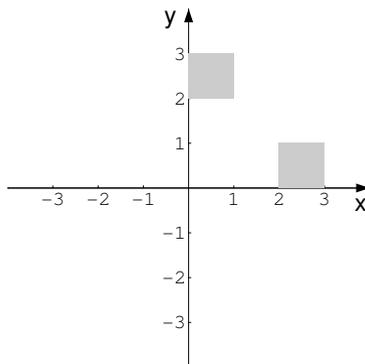

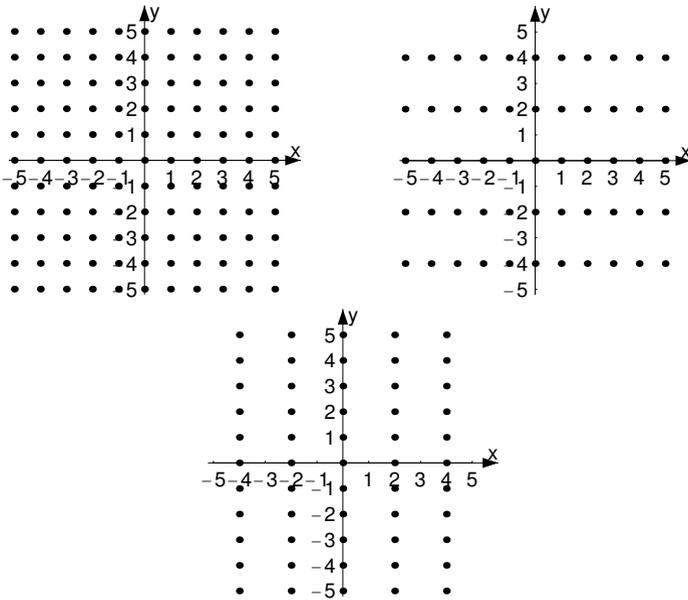**Fig. 9.2** $[0,1] \times [2,3]$ and $[2,3] \times [0,1]$

Fig. 9.3 $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 2\mathbb{Z}$, and $2\mathbb{Z} \times \mathbb{Z}$

Solution (9.5).

(a) The two sets are

$$\{0,1\} \times \{2,3\} = \{(0,2),(0,3),(1,2),(1,3)\}$$

and

$$\{2,3\} \times \{0,1\} = \{(2,0),(3,0),(2,1),(3,1)\}.$$

(b) The two sets are sketched in Figure 9.2.
(c) $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.
(d) One answer might be $\mathbb{R}^2 \times \mathbb{R}$.
(e) These are sketched in Figure 9.3.

Solution (9.8). Our "Not a proof" claims that "$x \in A$ or $x \in C$, and $y \in B$ or $y \in D$. Hence $x \in A$ and $y \in B$ or $x \in C$ and $y \in D$." This conclusion is not justified: it could also be that $x \in A$ and $y \in D$, or $x \in C$ and $y \in B$.

As an example, let $A = D = \emptyset$ and $B = C = \mathbb{R}$. Then $(A \cup C) \times (B \cup D) = \mathbb{R} \times \mathbb{R}$, while $(A \times B) \cup (C \times D) = \emptyset$.

Solution (9.9).

(a) This is a relation from $X = \mathbb{R}$ to $Y = \mathbb{R}$, consisting of the set of points in $\mathbb{R}^2$ lying above or on the line $y = x$.

(b) This is not a subset of $X \times Y$ for any choice of X and Y, hence this is not a relation.

(c) This is a relation from $X = \mathbb{Z}$ to $Y = \mathbb{Z}$, and consists of the points for which x is an integer and $y = -x$; that is, this is the set $\{(x, -x) : x \in \mathbb{Z}\}$.

Problems

Problem 9.1. Let $S = \{a, b, c\}$. Find $\mathscr{P}(S)$.

Problem 9.2. Replace ? by the proper symbol, choosing from among the following: $\in, \subseteq$, or $\subset$.

(a) $\mathbb{N}$? $\mathbb{Z}$;
(b) $\{1, 2, 3\}$? $\mathbb{Z}$;
(c) $\{5\}$? $\mathbb{Z}$;
(d) $\mathbb{Z}^+$? $\mathscr{P}(\mathbb{Z})$;
(e) $\{1, 2, 3\}$? $\mathscr{P}(\{1, 2, 3\})$;
(f) 5 ? $\mathbb{Z}$;
(g) $\emptyset$? $\mathbb{Z}$;
(h) $\emptyset$? $\mathscr{P}(\mathbb{Z})$.

Problem 9.3. Give an explicit description of $\mathscr{P}(\mathscr{P}(\{1\}))$ by listing all its elements.

Problem 9.4. Say whether the following are true or false and give a reason for your answer.

(a) $\{\emptyset\} \subseteq A$ for all sets A.
(b) $\emptyset \subset \mathscr{P}(A)$ for all sets A.
(c) If $A = \{x, y\}$, then $\mathscr{P}(A) = \{\{x\}, \{y\}, \{x, y\}, \{\emptyset\}\}$.
(d) If $A = \{x, y\}$, $\mathscr{P}(\{A\}) = \{\{x\}, \{y\}, \{x, y\}, \emptyset\}$.
(e) If $A_0 = \{\emptyset\}$, then $A_0 \in \mathscr{P}(A)$ for all sets A.

Problem 9.5.   (a) Show that $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$.
(b) Show that $\mathscr{P}(A) \cup \mathscr{P}(B) \neq \mathscr{P}(A \cup B)$ in general, by exhibiting two concrete sets, A and B, for which the aforementioned inequality holds.

Problem 9.6.   (a) Let X be a nonempty set in $\mathscr{P}(A \setminus B)$. Must $X \in \mathscr{P}(A) \setminus \mathscr{P}(B)$?
(b) Prove that it is never the case that $\mathscr{P}(A \setminus B) = \mathscr{P}(A) \setminus \mathscr{P}(B)$.

Problem 9.7. Let $2\mathbb{Z}$ denote the even integers and $2\mathbb{Z} + 1$ denote the odd integers. What is $\mathscr{P}(2\mathbb{Z}) \cap \mathscr{P}(2\mathbb{Z} + 1)$?

Problem 9.8. Show that $A \subseteq B$ if and only if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

Problem 9.9. For every set I and for every indexed collection of sets $\{A_\alpha : \alpha \in I\}$, prove that

$$\bigcup_{\alpha \in I} \mathscr{P}(A_\alpha) \subseteq \mathscr{P}\left(\bigcup_{\alpha \in I} A_\alpha\right).$$

Problem 9.10. Let $\{A_\alpha : \alpha \in I\}$ be a nonempty indexed collection of sets. Prove that $\mathscr{P}\left(\bigcap_{\alpha \in I} A_\alpha\right) = \bigcap_{\alpha \in I} \mathscr{P}(A_\alpha)$.

Problem 9.11. How many elements are there in the power set of $\{1,2,3,4\}$? How many elements are in the power set of $\{1,2,3,4,5\}$? State a general result. You'll be able to prove it later.

Problem 9.12. Describe the following relations pictorially (as a set of points in the plane) or in words:

(a) $\{(x,y) \in \mathbb{N} \times \mathbb{Z} : x \geq y\}$;
(b) $\{(x,y) \in \mathbb{R}^2 : x = y\}$;
(c) $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x+y \in 2\mathbb{Z}\}$;
(d) $\{0,1\} \times \mathbb{N}$;
(e) $\{(x,x^2) : x \in \mathbb{R}\}$;
(f) $\{(\sqrt{x},x) : x \in \mathbb{Z}^+\}$.

Problem 9.13. Describe the following Cartesian products:

(a) $\emptyset \times \mathbb{N}$;
(b) $\mathbb{Z} \times \emptyset$;
(c) $\mathbb{R} \times \mathbb{R}$;
(d) $\mathbb{R} \times \mathbb{Z}$.

Problem 9.14. Which of the following sets can be written as the Cartesian product of two subsets of $\mathbb{R}$? (Either give the two sets or explain why two such sets do not exist.)

(a) $\{(x,y) : 0 \leq y \leq 5\}$;
(b) $\{(x,y) : x > y\}$;
(c) $\{(x,y) : x^2 + y^2 = 1\}$.

Problem 9.15. Show that $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{Z} \times \mathbb{Z}$.

Problem 9.16.   (a) In the plane, sketch the set $[0,1] \times ([1,3] \cup [2,4])$.
(b) Sketch $([0,1] \cup [1,4]) \times ([0,1] \cup [2,4])$.

Problem 9.17. Let A, B, C, and D be nonempty sets. Then $A \times B = C \times D$ if and only if $A = C$ and $B = D$.

(a) Prove this statement.
(b) One of the two implications does not require the sets to be nonempty. Which one?
(c) If we do not require the sets to be nonempty, then the statement is false. Give examples of sets A, B, C, and D to show the necessity of the assumption that the sets be nonempty.

Problem 9.18. Suppose A, B, C, and D are four sets. If $A \times B \subseteq C \times D$, must $A \subseteq C$ and $B \subseteq D$? Why or why not?

Problem 9.19. Let A, B, and C be sets. If the statements below are true, prove them. If they are false, give a counterexample:

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Problem 9.20. Let $A = \{1, \{1\}, \{1, \{1\}\}\}$.

(a) Find $A \times A$.
(b) Find $A \cap \mathscr{P}(A)$.

Problem 9.21. Let $A = \{0, 1\}$. We define a relation R from A to the power set of A by $R = \{(x, y) \in A \times \mathscr{P}(A) : x \in y\}$. List all elements of R.

Problem 9.22. Let $\{A_\alpha : \alpha \in I\}$ and $\{B_\beta : \beta \in J\}$ be two indexed collections with nonempty index sets I and J. Prove that

$$\bigcap_{\alpha \in I} \left(\bigcup_{\beta \in J} (A_\alpha \times B_\beta) \right) = \left(\bigcap_{\alpha \in I} A_\alpha \right) \times \left(\bigcup_{\beta \in J} B_\beta \right).$$

Problem 9.23. This problem introduces rigorous definitions of an ordered pair and Cartesian product. Let A be a set and $a, b \in A$. We define the ordered pair of a and b with first coordinate a and second coordinate b as

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Using this definition prove the following.

(a) If $(a, b) = (x, y)$, then $a = x$ and $b = y$.
(b) If $a \in A$ and $b \in B$, then $(a, b) \in \mathscr{P}(\mathscr{P}(A \cup B))$.

Now we are able to define the Cartesian product of the two sets A and B as the set

$$A \times B = \{x \in \mathscr{P}(\mathscr{P}(A \cup B)) : x = (a, b) \text{ for some } a \in A \text{ and some } b \in B\}.$$

(c) Using the definitions introduced in this problem, prove that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

   This is a pretty complicated definition. It is also not our idea, but rather an idea that was born from axioms. P. Halmos's book, [41], is an excellent reference for this subject.

Tips on Writing Mathematics

> This letter is longer than usual simply because I could not spare the time to make it shorter.—Blaise Pascal, [80, p. 865][1]

After this point in the course the work will change. You'll find that you are writing more in words than in symbols. How you write is as important as what you write. Here are some things to think about as you write your proofs.

- In mathematics, it is always important that the reader know what the variables stand for. This was true in algebra in high school, geometry, and calculus, and it is true here too. If you use symbols—any symbols—make sure the meaning is clear to the reader *before* you use them.
- Think about your notation, and choose notation that is easy on the reader.
- A variable should only be assigned one meaning in your proof. For example, if you used C to denote the complex numbers, don't use C again to denote a different set.
- Try for a good blend of symbols and words. Don't juxtapose unrelated symbols if you don't have to. For example, consider the sentence "So $1 \leq p, q \geq 2$." You might find this confusing and (unnecessarily) difficult to read. If we say "So $1 \leq p$ and $q \geq 2$," the sentence is clear. It's often easier to read things if you put a word, even a little one, between symbols.
- Avoid starting a sentence with a symbol. This often confuses the reader unnecessarily. For example, consider the following sentence.

    Thus $x \in A$. A is a subset of B.

    First, the $A.A$ just doesn't look nice. Second, it's hard to read.
- Every sentence should start with a capital letter and end with a period, just like sentences are supposed to begin and end.
- All grammatical rules apply. Make sure your sentence has a noun and a verb, for example.
- Strive for clarity. Always keep the reader in mind. If something follows from a definition, say so. The reader will appreciate this and will know what you are thinking *and*, what's more, you will know why what you say is true. If something follows from Theorem 10.1, say so. It is extremely important for you to be aware of when you are using a result. For one thing, it means that you are more likely to notice if you are using a result that you do not have. (This would be wrong. Don't do it.) For another, it helps the reader who may not fully understand what you are doing.
- Certain phrases are particularly helpful in guiding a reader through your proof. For example, "Suppose to the contrary, ..." tells the reader that your proof will be done by contradiction. As a second example, if you are proving "A if and only if B," your reader will understand everything better if you say, "Suppose A. ... Then we have B." And then say, "Suppose B. ... Then we have A." You should

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[1] The translation is ours.

alert the reader to a proof that will be in cases, or a proof that will proceed using the contrapositive. You should not only tell the reader how you will begin the proof, you should also tell the reader when you believe you have completed the proof. Words like "thus, we have established the desired result" will let the reader know that you think you are done now and it's his or her turn to understand why. Other examples of phrases that you may use to guide your reader will come up as we learn new techniques.

- If you can find a shorter, clearer solution, do so.
- Perhaps the most difficult thing about writing a proof is to find a balance between the main ideas in the proof and the details. You'll often find that the more you explain, the more you hide the main ideas. On the other hand, if you don't explain enough, you might overlook an important detail or confuse your reader. It's not easy to strike the right balance. This is why we suggest waiting a bit, and then rereading your proof. If you can't figure out why you did something, it's unlikely that someone else will.
- If you have a partner in the class, it is an excellent idea to exchange papers and see if things are clear to each of you. (Check with your teacher to make sure this is allowed, of course.)

Exercise 9.10. Here's a student's proof of the following theorem: Let x and y be real numbers. Show that $xy \leq x^2/2 + y^2/2$.

Proof (Student version).

$$(x-y)^2 \geq 0$$
$$x^2 - 2xy + y^2 \geq 0$$
$$x^2 + y^2 \geq 2xy$$
$$x^2/2 + y^2/2 \geq xy \qquad \qquad \square$$

Criticize the student's solution and rewrite the proof, paying close attention to the tips presented here.   ○

Example 9.11 (Adapting an idea of [112]). If you are trying to publish a paper, you should think carefully about your exposition. Strive for clarity. Here's an example of how to improve mathematical writing:

Theorem (Steiner's Theorem I). Given any triangle, there is a unique ellipse inscribed in the triangle that passes through the midpoints of the sides of the triangle and is tangent to the sides of the triangle at these three midpoints.

Let's see. We've used the word "triangle" four times. Let's try to eliminate that repetition.

Theorem (Steiner's Theorem II). Given any triangle T, there is a unique ellipse inscribed in T that passes through the midpoints of the sides of T and is tangent to the sides of T at these three midpoints.

Now, if the ellipse is tangent to the sides of T, then it must pass through those points as well. So, it looks like that's a phrase we can omit. Let's try it.

Theorem (Steiner's Theorem III). *Given a triangle T, there is a unique ellipse inscribed in T that is tangent to each of the sides of T at the three midpoints.*

This looks simpler than our previous statement. Is there anything else we can do to shorten this? Well, yes. Yes there is. The "unique ellipse inscribed in T" is too wordy. If we say that the ellipse is tangent to each of the sides of T at the midpoints, then T must be the circumscribing triangle. Now, do we really have to say "three" midpoints? One argument in favor of "three" is that the reader will immediately see that we are talking about all possible midpoints. But we've got that covered: we said "tangent to each of the sides." So how about:

Theorem (Steiner's Theorem IV). *Given a triangle T, there is a unique inscribed ellipse that is tangent to each of the sides of T at the midpoints.*

Is this the shortest possible version? If a convex figure is tangent to each side of a triangle, then it is necessarily "inscribed." Thus we can drop this word, as it does not convey additional information.

Theorem (Steiner's Theorem V). *Given a triangle T, there is a unique ellipse that is tangent to each of the sides of T at the midpoints.*

Is the last statement really the "best" one? It certainly is the shortest. The reader may have to think for a moment before realizing that the ellipse is inscribed. If we state this explicitly rather than implicitly, it will most likely aid the reader. Most statements that we found of this theorem include the word "inscribed," perhaps for this reason. *Shortest* is not necessarily equivalent to *clearest!* In this case, we vote for version IV as the clearest.

Steiner's Theorem is, by the way, a beautiful theorem. You can find a lot about it on the Web by doing a search for "Steiner inellipse" or in the book [2, pp. 52–53].

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For other (not necessarily independent) views on writing see [42], [62], and [112].