Chapter 2
Basics of Particle Physics

This chapter introduces the basic techniques for the study of the intimate structure of matter, described in a historical context. After reading this chapter, you should understand the fundamental tools which led to the investigation and the description of the subatomic structure, and you should be able to compute the probability of occurrence of simple interaction and decay processes. A short reminder of the concepts of quantum mechanics and of special relativity needed to understand astroparticle physics is also provided.

2.1 The Atom

In the second half of the nineteenth century, the work by Mendeleev[1] on the periodic table of the elements provided the paradigm that paved the way for the experimental demonstration of the atomic structure. The periodic table is an arrangement of the chemical elements. Mendeleev realized that the physical and chemical properties of elements are related to their atomic mass in a quasiperiodic way. He ordered the 63 elements known at his time according to their atomic mass and arranged them in a table so that elements with similar properties would be in the same column. Figure 2.1 shows this arrangement. Hydrogen, the lightest element, is isolated in the first row of the table. The following light elements are then disposed in octets. Mendeleev found some gaps in his table and predicted that elements then unknown would be discovered which would fill these gaps. His predictions were successful.

[1]Dimitri Mendeleev (1834–1907) was a Russian chemist born in Tobolsk, Siberia. He studied science in St. Petersburg, where he graduated in 1856 and became full professor in 1863. Mendeleev is best known for his work on the periodic table, published in Principles of Chemistry in 1869, but also, according to a myth popular in Russia, for establishing that the minimum alcoholic fraction of vodka should be 40 %— this requirement was easy to verify, as this is the minimum content at which an alcoholic solution can be ignited at room temperature.

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Reihen	Gruppe I. — R^2O	Gruppe II. — RO	Gruppe III. — R^2O^3	Gruppe IV. RH^4 RO^2	Gruppe V. RH^3 R^2O^5	Gruppe VI. RH^2 RO^3	Gruppe VII. RH R^2O^7	Gruppe VIII. — RO^4
1	H=1							
2	Li=7	Be=9,4	B=11	C=12	N=14	O=16	F=19	
3	Na=23	Mg=24	Al=27,3	Si=28	P=31	S=32	Cl=35,5	
4	K=39	Ca=40	— =44	Ti=48	V=51	Cr=52	Mn=55	Fe=56, Co=59, Ni=59, Cu=63
5	(Cu=63)	Zn=65	— =68	— =72	As=75	Se=78	Br=80	
6	Rb=85	Sr=87	?Yt=88	Zr=90	Nb=94	Mo=96	— =100	Ru=104, Rh=104, Pd=106, Ag=108
7	(Ag=108)	Cd=112	In=113	Sn=118	Sb=122	Te=125	J=127	
8	Cs=133	Ba=137	?Di=138	?Ce=140	—	—	—	— — — —
9	(—)	—	—	—	—	—	—	
10	—	—	?Er=178	?La=180	Ta=182	W=184	—	Os=195, Ir=197, Pt=198, Au=199
11	(Au=199)	Hg=200	Tl=204	Pb=207	Bi=208	—	—	
12	—	—	—	Th=231	—	U=240	—	— — — —

Fig. 2.1 Mendeleev's periodic table as published in Annalen der Chemie 1872 [public domain]. The noble gases had not yet been discovered and are thus not displayed

Mendeleev's periodic table has been expanded and refined with the discovery of new elements and a better theoretical understanding of chemistry. The most important modification was the use of atomic number (the number of electrons, which indeed characterizes an element) instead of atomic mass to order the elements. Since atoms are neutral, the same number of positive charges (protons) should be present. Starting from the element with atomic number 3, Mendeleev conjectured that electrons are disposed in shells. The nth shell is complete with $2n^2$ electrons, and the external shell alone dictates the chemical properties of an element. As we know, the quantum mechanical view is more complete but not as simple.

The present form of the periodic table (Appendix A) is a grid of elements with 18 columns and 7 rows, with an additional double row of elements. The rows are called periods; the columns, which define the chemical properties, are called groups; examples of groups are "halogens" and "noble gases".

Thanks to Mendeleev's table, a solid conjecture was formulated that atoms are composite states including protons and loosely bound electrons. But how to understand experimentally the inner structure of the atom; i.e., How were protons and electrons arranged inside the atom? Were electrons "orbiting" around a positive nucleus, or were both protons and electrons embedded in a "plum pudding," with electrons (the "plums") more loosely bound? A technique invented around 1900 to answer this question has been influential throughout the history of particle physics.

2.2 The Rutherford Experiment

Collide a beam of particles with a target, observe what comes out, and try to infer the properties of the interacting objects and/or of the relevant interaction force. This is the paradigm of particle physics experiments. The first experiment was conducted

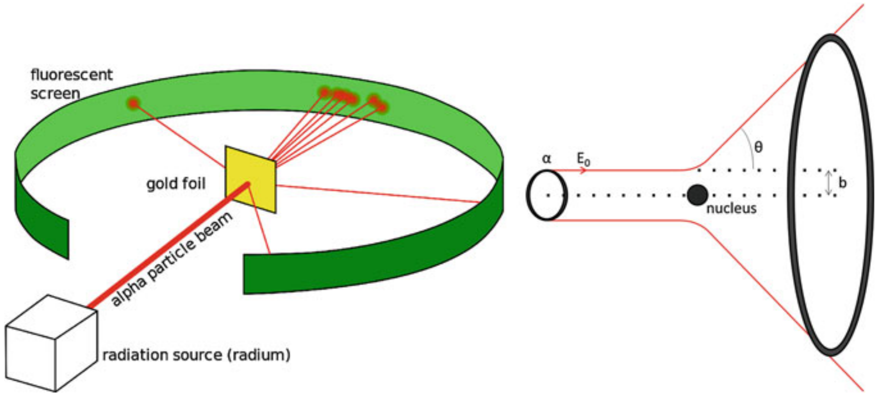

Fig. 2.2 Left: Sketch of the Rutherford experiment (by Kurzon [own work, CC BY-SA 3.0], via wikimedia commons). Right: trajectories of the α particles

by Marsden and Geiger starting in 1908 and is known as the Rutherford[2] experiment. The beam consisted of α particles (known today as helium nuclei); the target was a thin (some 400 nm) gold foil; the detector, a scintillating screen which could be read by a microscope. The result of the observation was that around 1 in 8000 α particles were deflected at very large angles (greater than 90°). A sketch of the experiment is shown in Fig. 2.2, left.

The interpretation of this result was given by Rutherford in 1911. It was based on a model in which the positive nucleus of the atom was a point fixed in space and the scattering of the α particles was due to the Coulomb force and obeyed classical mechanics (quantum mechanics was yet to be born). The α particles were thus supposed to follow Keplerian trajectories. As energy and angular momentum are conserved, for a given impact parameter b (the perpendicular distance between the beam particle and the nucleus, see Fig. 2.2, right) there will be a well-defined scattering angle θ, and:

$$b = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Q_1 Q_2}{2E_0}\cot\frac{\theta}{2} \tag{2.1}$$

[2]Ernest Rutherford (1871–1937) was a New Zealand-born physicist. In early works at McGill University in Canada, he proved that radioactivity involved the transmutation of one chemical element into another; he differentiated and named the α (helium nuclei) and β (electrons) radiations. In 1907, Rutherford moved to Manchester, UK, where he discovered (and named) the proton. In 1908, he was awarded the Nobel Prize in Chemistry "for his investigations into the disintegration of the elements, and the chemistry of radioactive substances." He became director of the Cavendish Laboratory at Cambridge University in 1919. Under his leadership, the neutron was discovered by James Chadwick in 1932. Also in 1932, his students John Cockcroft and Ernest Walton split for the first time the atom with a beam of particles. Rutherford was buried near Newton in Westminster Abbey, London. The chemical element rutherfordium—atomic number 104—was named after him in 1997.

where ϵ_0 is the vacuum dielectric constant, Q_1 and Q_2 are, respectively, the charges of the beam particle and of the target particle and E_0 is the kinetic energy of the beam particle.

If the number of beam particles per unit of transverse area n_{beam} does not depend on the transverse coordinates b and ϕ (the beam is uniform and wide with respect to the target size), the differential number of particles as a function of b is:

$$\frac{dN}{db} = 2\pi b \, n_{\text{beam}} \, . \tag{2.2}$$

Expressing the differential number of particles as a function of the scattering angle θ:

$$\frac{dN}{d\theta} = \frac{dN}{db}\frac{db}{d\theta} \tag{2.3}$$

we obtain using Eq. 2.1:

$$\frac{dN}{d\theta} = \pi \left(\frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{2E_0} \right)^2 \frac{\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} n_{\text{beam}} \tag{2.4}$$

or, in terms of the solid angle Ω, ($d\Omega = 2\pi \sin\theta d\theta$):

$$\frac{dN}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{4E_0} \right)^2 \frac{1}{\sin^4\frac{\theta}{2}} n_{\text{beam}} \, , \tag{2.5}$$

the well-known "Rutherford formula." This equation explained the observation of scattering at large angles and became the paradigm for particle diffusion of nuclei. According to gossip, like some experimentalists Rutherford disliked mathematics, and this formula was derived for him by the mathematician Ralph Fowler, who later married Rutherford's daughter, and finally became a professor of Theoretical Physics in Cambridge.

2.3 Inside the Nuclei: β Decay and the Neutrino

Beta (β) radioactivity, the spontaneous emission of electrons by some atoms, was discovered by Ernest Rutherford just a few years after the discovery by Henri Becquerel that uranium was able to impress photographic plates wrapped in black paper. It took then some years before James Chadwick in 1914 realized that the energy spectrum of the electrons originated in β decays was continuous and not discrete (Fig. 2.3). This was a unique feature in the new quantum world, in which decays were explained as transitions between well-defined energy levels. There was a missing energy problem, and many explanations were tried along the years, but none was

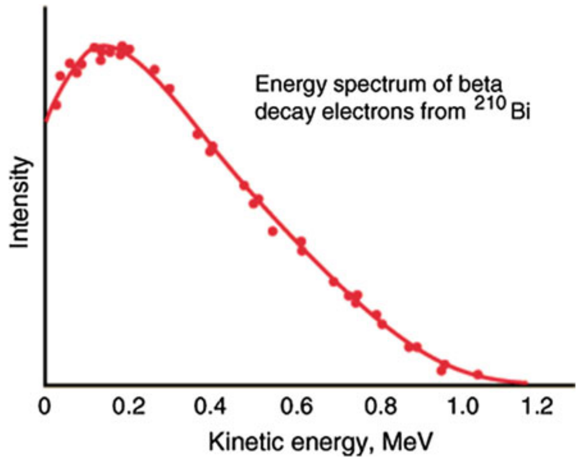

Fig. 2.3 Energy spectrum of electrons coming from the β decay of ^{210}Bi (called historically "Radium E") to ^{210}Po (called historically "Radium F"). From http://hyperphysics.phy-astr.gsu.edu/; the measurements are from G. J. Neary, Roy. Phys. Soc. (London) A175 (1940) 71

proved. In 1930, Niels Bohr went so far as to suggest that the energy conservation law could be violated.

In December 1930, in a famous letter, Wolfgang Pauli proposed as "desperate remedy" the existence of a new neutral particle with spin one-half and low mass named *neutron*: "The continuous β spectrum would then become understandable from the assumption that in the β decay a neutron is emitted along with the electron, in such way that the sum of the energies of the neutron and the electron is constant." This tiny new particle was later renamed *neutrino* by Enrico Fermi. The particle today known as neutron, constituent of the atomic nuclei, was discovered by James Chadwick in 1932, Nobel prize in Physics 1935. Then at the University of Cambridge, Chadwick found a radiation consisting of uncharged particles of approximately the mass of the proton. His group leader Rutherford had conjectured the existence of the neutron already in 1920, in order to explain the difference between the atomic number of an atom and its atomic mass, and he modeled it as an electron orbiting a proton.

Atomic nuclei were thus composed (in the modern language) by protons and neutrons, and the β radioactive decays were explained by the decay of one of the neutrons in the nucleus into one proton, one electron, and one neutrino (in fact, as it will be discussed later, an antineutrino):

$$n \to pe^-\bar{\nu}. \tag{2.6}$$

The β^+ decay, i.e., the decay of one proton in the nucleus into one neutron, one positron (the antiparticle of the electron), and one neutrino

$$p \to ne^+\nu \tag{2.7}$$

is also possible, although the neutron mass is larger than the proton mass—take into account that nuclei are bound in the nucleus and not free particles.

Neutrinos have almost no interaction with matter, and therefore, their experimental discovery was not an easy enterprise: intense sources and massive and performing detectors had to be built. Only in 1956, Reines and Cowan proved the existence of the neutrino, placing a water tank near a nuclear reactor. Some of the antineutrinos produced in the reactor interacted with a proton in the water, giving rise to a neutron and a positron, the so-called inverse beta process:

$$\bar{\nu} p \rightarrow n e^{+} . \tag{2.8}$$

The positron then annihilates with an ordinary electron, and the neutron is captured by cadmium chloride atoms dissolved in the water. Three photons were then detected (two from the annihilation and, 5 μs later, one from the de-excitation of the cadmium nucleus).

The mass of the neutrino is indeed very low (but not zero, as discovered by the end of the twentieth century with the observation of the oscillations between neutrinos of different families, a phenomenon that is possible only if neutrinos have nonzero mass) and determines the maximum energy that the electron may have in the beta decay (the energy spectrum end-point). The present measurements are compatible with neutrino masses below the eV.

A classical description of the neutron decay would be possible only if neutrons were a bound state of a proton, an electron and a neutrino—which experiments demonstrated not to be the case. In order to describe decays in a consistent way, we need to treat initial and final states as wavefunctions, and thus, to use the quantum mechanical formalism.

2.4 A Look into the Quantum World: Schrödinger's Equation

Schrödinger's[3] wave equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + U\Psi$$

[3]Erwin Schrödinger was an Austrian physicist who obtained fundamental results in the fields of quantum theory, statistical mechanics and thermodynamics, physics of dielectrics, color theory, electrodynamics, cosmology, and cosmic ray physics. He also paid great attention to the philosophical aspects of science, re-evaluating ancient and oriental philosophical concepts, and to biology and to the meaning of life. He formulated the famous paradox of the Schrödinger cat. He shared with P.A.M. Dirac the 1933 Nobel Prize in Physics "for the discovery of new productive forms of atomic theory."

can be seen as the translation into the wave language of the Hamiltonian equation of classical mechanics

$$H = \frac{p^2}{2m} + U,$$

where the Hamiltonian (essentially the total energy of the system, i.e., the kinetic energy plus the potential energy U) is represented by the operator

$$\hat{H} = i\hbar\frac{\partial}{\partial t}$$

and momentum by

$$\hat{\mathbf{p}} = -i\hbar\nabla .$$

The solutions of the equation are in general complex wavefunctions, which can be seen as probability density amplitudes (probability being the square of the modulus of the amplitude).

2.4.1 *Properties of Schrödinger's Equation and of its Solutions*

In "classical" quantum mechanics, physical states are represented by complex wave-functions $\Psi(\mathbf{r}, t)$ which are solutions of Schrödinger's equation. Here we recall briefly some of the main characteristics of these solutions; they can be extended in general to any "good" Hamiltonian equation. This is not meant to be a formal description, but will just focus on the concepts.

2.4.1.1 The Meaning of Wavefunctions

What is a wavefunction, and what can it tell us? In classical physics, an elementary particle, by its nature, is localized at a point, whereas its wavefunction is spread out in space. How can such an object be said to describe the state of a particle? The answer is given by the so-called Born's statistical interpretation. If Ψ is normalized such that

$$\int dV\, \Psi^*\Psi = 1 \tag{2.9}$$

(the integral is extended over all the volume), the probability to find the particle in an infinitesimal volume dV around a point $\mathbf{r}$ at a time t is

$$dP = \Psi^*(\mathbf{r}, t)\Psi(\mathbf{r}, t)\, dV = |\Psi(\mathbf{r}, t)|^2\, dV .$$

The left term in Eq. 2.9 is defined as the scalar product of the Ψ function by itself.

The statistical interpretation introduces an uncertainty into quantum mechanics: even if you know everything the theory can tell you about the particle (its wavefunction), you cannot predict with certainty the outcome of a simple experiment to measure its position: all the theory gives is statistical information about the possible results.

2.4.1.2 Measurement and Operators

The expectation value of the measurement of, say, position along the x coordinate is given by

$$\langle x \rangle = \int dV\, \Psi^* \, x \, \Psi \tag{2.10}$$

and one can easily demonstrate (see, e.g., [F2.1]) that the expectation value of the momentum along x is

$$\langle p_x \rangle = \int dV\, \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi \,. \tag{2.11}$$

In these two examples we saw that measurements are represented by operators acting on wavefunctions. The operator x represents position along x, and the operator $(-i\hbar\partial/\partial x)$ represents the x component of momentum, p_x. When ambiguity is possible, we put a "hat" on top of the operator to distinguish it from the corresponding physical quantity.

When we measure some quantity, we obtain a well-defined value: the wavefunction "collapses" to an eigenfunction, and the measured value is one of the eigenvalues of the measurement operator.

To calculate the expectation value of a measurement, we put the appropriate operator "in sandwich" between Ψ^* and Ψ, and integrate. If A is a quantity and $\hat{A}$ the corresponding operator,

$$\langle A \rangle = \int dV\, \Psi^* \left(\hat{A} \right) \Psi \,. \tag{2.12}$$

2.4.1.3 Dirac Notation

In the Dirac notation, the wavefunction is replaced by a state vector identified by the symbol $\mid \Phi \rangle$ and is called *ket*; the symbol $\langle \Psi \mid$ is called *bra*.

The *bracket* $\langle \Psi \mid \Phi \rangle$ is the scalar product of the two vectors:

$$\langle \Psi \mid \Phi \rangle = \int dV \Psi^* \Phi \,.$$

In this notation, an operator $\hat{A}$ acts on a ket $\mid \Phi >$, transforming it into a ket $\mid \hat{A}\, \Phi >$, and thus

$$< \Psi \mid \hat{A} \mid \Phi > = \int dV\, \Psi^*(\hat{A}\Phi)\,.$$

2.4.1.4 Good Operators Must be Hermitian

We define as Hermitian conjugate or adjoint of $\hat{A}$ an operator $\hat{A}^\dagger$ such that for any $|\Psi >$

$$\langle \hat{A}^\dagger \Psi | \Psi \rangle \;=\; \langle \Psi | \hat{A} \Psi \rangle\,.$$

Let $\hat{A}$ represent an observable. One has for the expectation value

$$\langle \Psi \mid \hat{A} \mid \Psi \rangle \;=\; \langle \Psi \mid \hat{A}\Psi \rangle \;=\; \langle \hat{A}\Psi \mid \Psi \rangle^* = \langle \Psi \mid \hat{A}^\dagger \Psi \rangle^* = \langle \Psi \mid \hat{A}^\dagger \mid \Psi \rangle^*$$

and thus, if we want all expectation values (and the results of any measurement) to be real, $\hat{A}$ must be a Hermitian operator (i.e., such that $\hat{A}^\dagger = \hat{A}$).

Now let us call Ψ_i the eigenvectors of $\hat{A}$ (which form a basis) and a_i the corresponding eigenvalues; for Ψ_m, Ψ_n such that $n \neq m$

$$\hat{A} \mid \Psi_m \rangle = a_m \mid \Psi_m \rangle\;; \quad \hat{A} \mid \Psi_n \rangle = a_n \mid \Psi_n \rangle$$

and thus

$$\begin{aligned} a_n \langle \Psi_n \mid \Psi_m \rangle &= \langle \Psi_n \mid \hat{A} \mid \Psi_m \rangle = a_m \langle \Psi_n \mid \Psi_m \rangle \\ &\Rightarrow 0 = (a_n - a_m)\langle \Psi_n \mid \Psi_m \rangle \\ &\Rightarrow \; \langle \Psi_n \mid \Psi_m \rangle = 0 \quad \forall\, m \neq n\,. \end{aligned}$$

If the Ψ_i are properly normalized

$$\langle \Psi_n \mid \Psi_m \rangle = \delta_{mn}\,.$$

Hermitian operators are thus good operators for representing the measurement of physical quantities: their eigenvalues are real (and thus can be the measurement of a quantity), and the solutions form an orthonormal basis.

2.4.1.5 Time-Independent Schrödinger's Equation

Schrödinger's equation is an equation for which the eigenvectors are eigenstates of defined energy. For a potential U not explicitly dependent on time, it can be split into two equations. One is a time-independent eigenvalue equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

and the other is an equation involving only time

$$\phi(t) = \exp(-iEt/\hbar)\,.$$

The complete solution is

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r})\phi(t)\,.$$

2.4.1.6 Time Evolution of Expectation Values

We define the *commutator* $[\hat{A}, \hat{B}]$ of two operators $\hat{A}$ and $\hat{B}$ as the operator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}\,,$$

and we say that the two operators commute when their commutator is zero. We can simultaneously measure observables whose operators commute, since such operators have a complete set of simultaneous eigenfunctions—thus one can have two definite measurements at the same time.

The time evolution of the expectation value of a measurement described by a Hermitian operator $\hat{A}$ is given by the equation

$$\frac{d}{dt}\langle\psi|\hat{A}|\psi\rangle = -\frac{i}{\hbar}\langle\psi|[\hat{H}, \hat{A}]|\psi\rangle\,. \tag{2.13}$$

2.4.1.7 Probability Density and Probability Current; Continuity Equation

The probability current $\mathbf{j}$ associated to a wavefunction can be defined as

$$\mathbf{j} = \frac{\hbar}{2mi}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)\,. \tag{2.14}$$

A continuity equation holds related to the probability density P to find a particle at a given time in a given position:

$$\frac{\partial P}{\partial t} + \nabla\cdot\mathbf{j} = 0\,.$$

2.4.1.8 Spectral Decomposition of an Operator

Since the eigenfunctions $\{|\Psi_i\rangle\}$ of a Hermitian operator $\hat{A}$ form a basis, we can write

$$\sum_j |\Psi_j\rangle \langle\Psi_j| = I$$

where I is the unity operator.

This means that any wavefunction can be represented in this orthonormal basis by a unique combination:

$$|\Psi\rangle = \sum_m |\Psi_m\rangle\langle\Psi_m|\Psi\rangle = \sum_m c_m |\Psi_m\rangle$$

where $c_m = \langle\Psi_m|\Psi\rangle$ are complex numbers.

The normalization of $|\Psi\rangle$ to 1 implies a relation on the c_m:

$$1 = \langle\Psi|\Psi\rangle = \sum_m \langle\Psi|\Psi_m\rangle\langle\Psi_m|\Psi\rangle = \sum_m |c_m|^2 ,$$

and the probability to obtain from a measurement the eigenvalue a_m is

$$P_m = |\langle\Psi_m|\Psi\rangle|^2 = |c_m|^2 .$$

In addition, we can determine coefficients $a_{mn} = \langle\Psi_m|\hat{A}|\Psi_n\rangle$ such that

$$\hat{A}|\Psi\rangle = \sum_{mn} |\Psi_m\rangle\langle\Psi_m|\hat{A}|\Psi_n\rangle\langle\Psi_n|\Psi\rangle = \sum_{m,n} a_{mn}c_n |\Psi_m\rangle .$$

$[a_{mn}]$ is a square matrix representing $\hat{A}$ in the vector space defined by eigenvectors; the c_n are an n-tuple of components representing $|\Psi\rangle$.

2.4.1.9 Uncertainty Relations

Pairs of noncommuting operators cannot give rise to simultaneous measurements arbitrarily precise for the associated quantities (this is usually called Heisenberg's uncertainty principle, but in fact it is a theorem).

Let us define as spread of an operator the operator:

$$\Delta\hat{A} = \hat{A} - \langle A\rangle .$$

Let $\hat{A}$ and $\hat{B}$ be two Hermitian operators; we define $\hat{C}$ such that

$$[\hat{A}, \hat{B}] = i\hat{C}$$

($\hat{C}$ is Hermitian; you can demonstrate it). One has

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{\langle C^2\rangle}{4}\,. \tag{2.15}$$

In particular, when a simultaneous measurement of position and momentum along an axis, say x, is performed, one has

$$\Delta x \Delta p_x \geq \frac{\hbar}{2} \sim \hbar\,.$$

Somehow linked to this is the fact that energy is not defined with absolute precision, but, if measured in a time Δt, has an uncertainty ΔE such that

$$\Delta E \Delta t \sim \hbar$$

(energy conservation can be violated for short times). The value of Planck's constant $\hbar \simeq 6.58 \times 10^{-22}$ MeV s is small with respect to the value corresponding to the energies needed to create particles living for a reasonable (detectable) time.

2.4.2 Uncertainty and the Scale of Measurements

If we want to investigate a structure below a length scale Δx, we are limited by the uncertainty theorem. Since a wavelength

$$\lambda \simeq \frac{\hbar}{p} \tag{2.16}$$

can be associated with a particle of momentum p, this means that particles of energy (energy is close to momentum times c for high-energy particles):

$$E > \frac{\hbar c}{\Delta x} \tag{2.17}$$

must be used. For example, X-rays with an energy of $\sim$1 keV can investigate the structure of a target at a scale

$$\Delta x > \frac{\hbar c}{E} \simeq 2 \times 10^{-11}\ \mathrm{m}, \tag{2.18}$$

an order of magnitude smaller than the atomic radius. A particle with an energy of 7 TeV, the running energy of the Large Hadron Collider (LHC) accelerator at CERN can investigate the structure of a target at a scale

$$\Delta x > \frac{\hbar c}{E} \simeq 3 \times 10^{-20}\,\text{m}. \tag{2.19}$$

Since one can extract only a finite energy from finite regions of the Universe (and maybe the Universe itself has a finite energy), there is an intrinsic limit to the investigation of the structure of matter, below which the quest makes no more sense. However, as we shall see, there are practical limits much more stringent than that. Does the concept of elementary particle have a meaning below these limits? The question is more philosophical than physical, since one cannot access infinite energies.

The maximum energies attainable by human-made accelerators are believed to be of the order of the PeV. However, nature gives us for free beams of particles with much larger energies, hitting the Earth from extraterrestrial sources: cosmic rays.

2.5 The Description of Scattering: Cross Section and Interaction Length

Particle physicists observe and count particles, as pioneered by the Rutherford experiment. They count, for instance, the number of particles of a certain type with certain characteristics (energy, spin, scattering angle) that result from the interaction of a given particle beam at a given energy with a given target. It is then useful to express the results as quantities independent from the number of particles in the beam, or in the target. These quantities are called cross sections.

2.5.1 Total Cross Section

The total cross section σ measured in a collision of a beam with a single object (Fig. 2.4) is defined as

$$\sigma_{\text{tot}} = \frac{N_{\text{int}}}{n_{\text{beam}}} \tag{2.20}$$

where N_{int} is the total number of measured interactions and n_{beam} is, as previously defined, the number of beam particles per unit of transverse area.

A cross section has thus dimensions of an area. It represents the effective area with which the interacting particles "see" each other. The usual unit for cross section is the barn, b ($1\,\text{b} = 10^{-24}\,\text{cm}^2$) and its submultiples (millibarn—mb, microbarn—μb, nanobarn—nb, picobarn—pb, femtobarn—fb, etc.). To give an order of magnitude, the total cross section for the interaction of two protons at a center-of-mass energy of around 100 GeV is 40 mb (approximately the area of a circle with radius 1 fm).

We can write the total cross section with a single target as

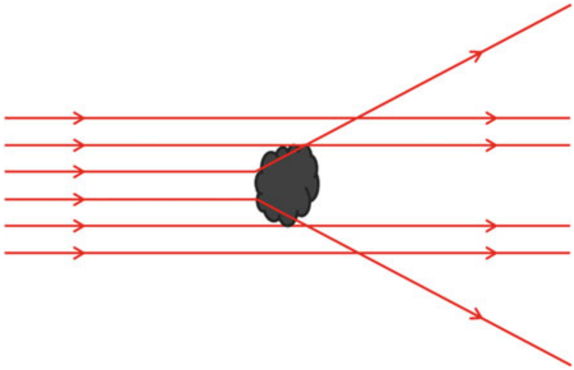

Fig. 2.4 Interaction of a particle beam with a single object target. Lines represent different particles in the beam

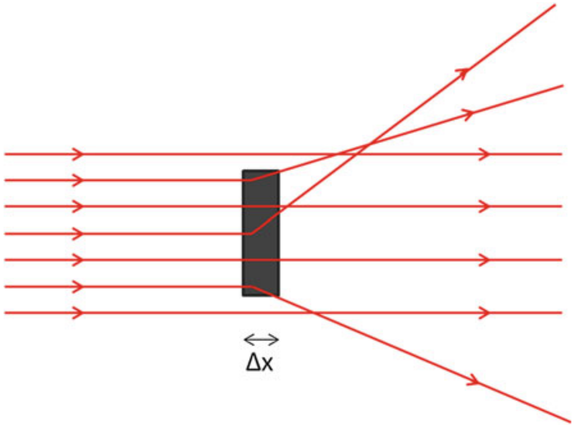

Fig. 2.5 Interaction of a particle beam with a target composed of many sub-targets

$$\sigma_{\text{tot}} = \frac{W_{\text{int}}}{J}\,, \tag{2.21}$$

in terms of the interaction rate W_{int} (number of interactions per unit of time) and of the flux of incident particles J (number of beam particles that cross the unit of transverse area per unit of time). J is given as

$$J = \rho_{\text{beam}}\, v, \tag{2.22}$$

where ρ_{beam} is the density of particles in the beam and v is the beam particle velocity in the rest frame of the target.

In real life, most targets are composed of N_t small sub-targets (Fig. 2.5) within the beam incidence area. Considering as sub-targets the nuclei of the atoms of the target with depth Δx, and ignoring any shadowing between them, N_t is given by:

$$N_t = \mathcal{N}\frac{\rho \Delta x}{w_a}\,, \tag{2.23}$$

where $\mathcal{N}$ is Avogadro's number, ρ is the specific mass of the target, w_a is its atomic weight. Note that N_t is a dimensionless number: it is just the number of sub-targets that are hit by a beam that has one unit of transverse area. In the case of several sub-targets, the total cross section can thus be written as:

$$\sigma_{\text{tot}} = \frac{W_{\text{int}}}{J N_t} = \frac{W_{\text{int}}}{\mathcal{L}} , \tag{2.24}$$

where $\mathcal{L}$ is the luminosity.

The total number of interactions occurring in an experiment is then simply the product of the total cross section by the integral of the luminosity over the run time T of the experiment:

$$N_{\text{tot}} = \sigma_{\text{tot}} \int_T \mathcal{L} dt . \tag{2.25}$$

The units of integrated luminosity are therefore inverse barn, b^{-1}.

In this simplified model we are neglecting the interactions between the scattered particles, the interactions between beam particles, the binding energies of the target particles, the absorption, and the multiscattering of the beam within the target.

2.5.2 Differential Cross Sections

In practice, detectors often cover only a given angular region and we do not measure the total cross section in the full solid angle. It is therefore useful to introduce the differential cross section

$$\frac{d\sigma(\theta, \phi)}{d\Omega} = \frac{1}{\mathcal{L}} \frac{dW_{\text{int}}(\theta, \phi)}{d\Omega} \tag{2.26}$$

and

$$\sigma_{\text{tot}} = \int \int \frac{d\sigma(\theta, \phi)}{d\Omega} d\phi \, d\cos\theta . \tag{2.27}$$

The Rutherford formula (2.5) expressed as a differential cross section is then

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{4E_0} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} . \tag{2.28}$$

2.5.3 Cross Sections at Colliders

In colliders, beam–target collisions are replaced by beam–beam collisions (Fig. 2.6). Particles in the beams come in bunches. The luminosity is thus defined as

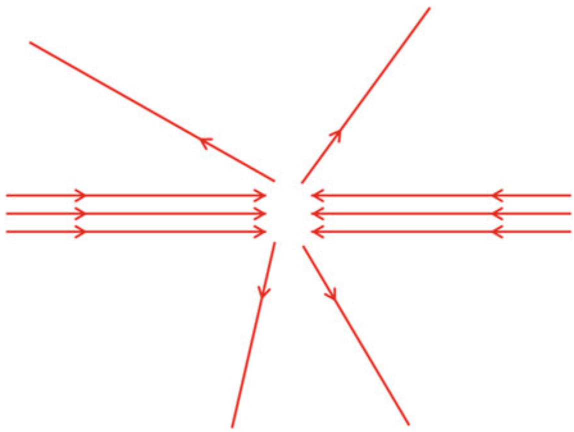

Fig. 2.6 Beam–beam interaction

$$\mathcal{L} = \frac{N_1 N_2}{A_T} N_b\, f \tag{2.29}$$

where N_1 and N_2 are the number of particles in the crossing bunches, N_b is the number of bunches per beam, A_T is the intersection transverse area, and f is the beam revolution frequency. In case of two Gaussian beams, 1 and 2, one can approximate

$$A_T \simeq 2\pi \sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2} \sqrt{\sigma_{y_1}^2 + \sigma_{y_2}^2} \tag{2.30}$$

where x and y are othornormal coordinates transverse to the beam. In case of equal and symmetric beams

$$A_T \simeq 4\pi {\sigma_b}^2 \,. \tag{2.31}$$

2.5.4 *Partial Cross Sections*

When two particles collide, it is often the case that there are many possible outcomes. Quantum mechanics allows us to compute the occurrence probability for each specific final state. Total cross section is thus a sum over all possible specific final states

$$\sigma_{\rm tot} = \sum_i \sigma_i \tag{2.32}$$

where σ_i is defined as the partial cross section for channel i.

A relevant partial cross section is the elastic cross section, σ_{el}. In an elastic process, the particles in the final state and in the initial state are the same—there is simply an exchange of energy–momentum. Whenever there is no available energy to create

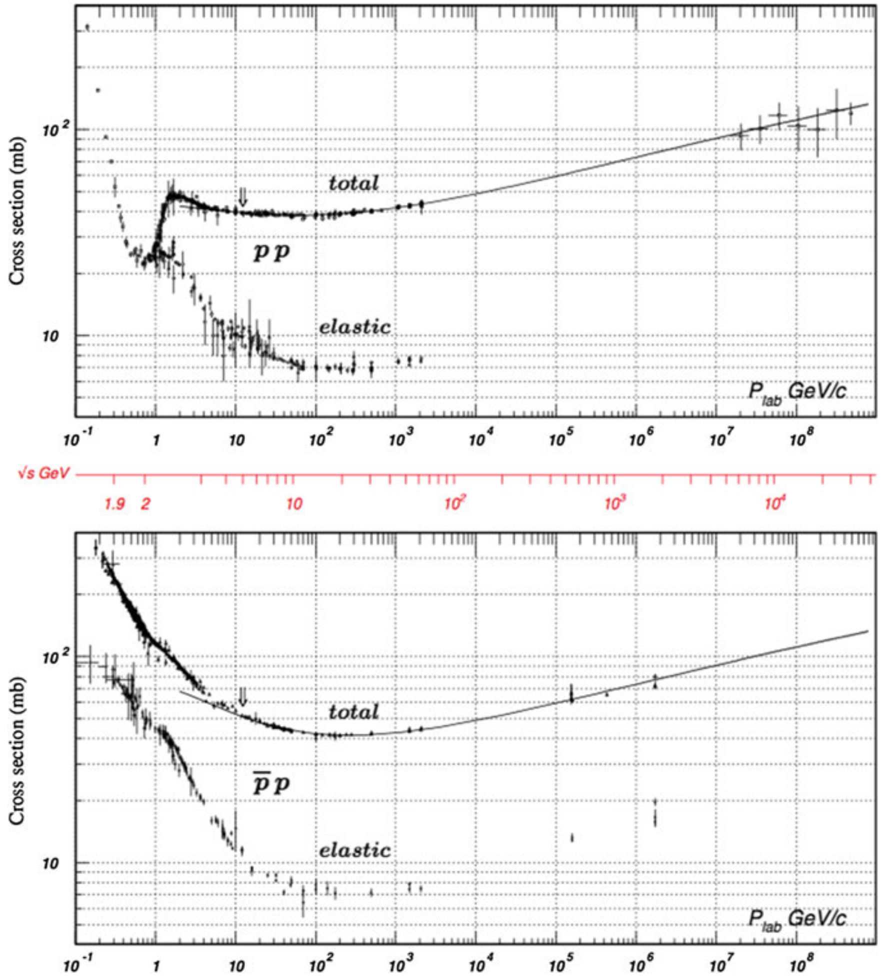

Fig. 2.7 Total and elastic cross sections for pp and $\bar{p}p$ collisions as a function of beam momentum in the laboratory reference frame and total center-of-mass energy. From the Review of Particle Physics, K.A. Olive et al. (Particle Data Group), Chin. Phys. C 38 (2014) 090001

new particles, $\sigma_{\rm tot} = \sigma_{el}$. This is shown in Fig. 2.7 for the case of proton–proton and antiproton–proton interactions.

2.5.5 Interaction Length

When a beam of particles crosses matter, its intensity is reduced. Using the definition of total cross section (Eqs. 2.21 and 2.24), the reduction when crossing a slice of thickness Δx is:

$$\frac{\Delta N}{N} = \frac{W_{\text{int}}}{J} = \left(\mathcal{N}\frac{\rho}{w_A}\,\Delta x\right)\sigma_{\text{tot}} \tag{2.33}$$

where w_A is the atomic weight of the target. Defining the interaction length L_{int} as

$$L_{\text{int}} = \frac{w_A}{\sigma_{\text{tot}}\mathcal{N}\rho} \tag{2.34}$$

then

$$\frac{dN}{dx} = -\frac{1}{L_{\text{int}}}N \tag{2.35}$$

and

$$N = N_0\, e^{-x/L_{\text{int}}}. \tag{2.36}$$

L_{int} has units of length (usually cm). However, this quantity is often redefined as

$$L'_{\text{int}} = L_{\text{int}}\,\rho = \frac{w_A}{\sigma_{\text{tot}}\,\mathcal{N}} \tag{2.37}$$

and its units will then be g cm^{-2}. This way of expressing L_{int} is widely used in cosmic ray physics. In fact, the density of the atmosphere has a strong variation with height. For this reason, to study the interaction of cosmic particles in their path in the atmosphere, the relevant quantity is not the path length but rather the amount of matter that has been traversed, $\int \rho dx$.

In a rough approximation, the atmosphere is isothermal; under this hypothesis, its depth x in g cm^{-2} varies exponentially with height h (km), according to the formula

$$x = Xe^{-h/H} \tag{2.38}$$

where $H \simeq 6.5$ km, and $X \simeq 1030$ g/cm^2 is the total vertical atmospheric depth.

2.6 Description of Decay: Width and Lifetime

Stable particles like (as far as we know) the proton and the electron are the exception, not the rule. The lifetime of most particles is finite, and its value spans many orders of magnitude from, for instance, 10^{-25} s for the electroweak massive bosons (Z and W) to around 900 s for the neutron, depending on the strength of the relevant interaction and on the size of the decay phase space.

In order to describe decays we must use quantum mechanical language, given that they are a genuine quantum process whose statistical nature cannot be properly explained by classical physics. We shall use, thus, the language of wavefunctions.

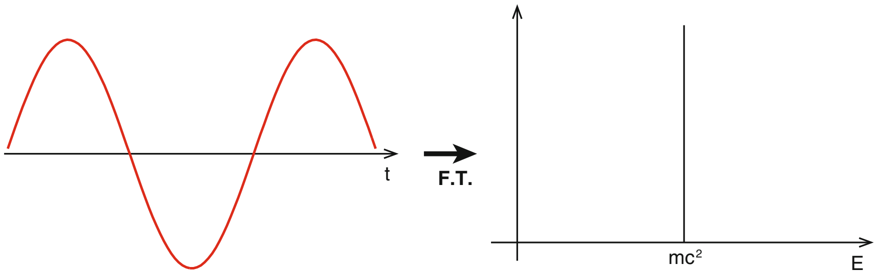

Fig. 2.8 Wavefunction of a stable particle and its energy spectrum

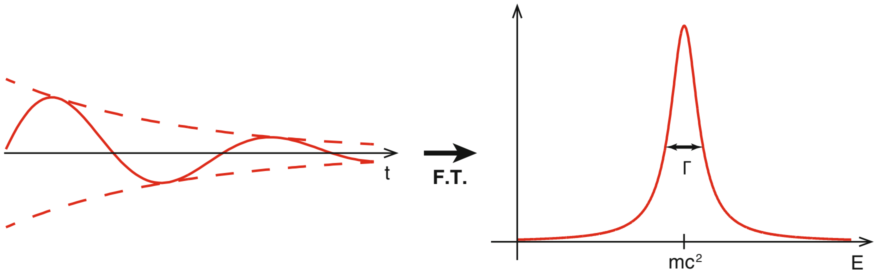

Fig. 2.9 Wavefunction of an unstable particle and its energy spectrum

$|\Psi(x, y, z, t)|^2\, dV$ is the probability density for finding a particle in a volume dV around point (x, y, z) at time t.

Stable particles are described by pure harmonic wavefunctions, and their Fourier transforms are functions centered in well-defined proper energies—in the rest frame, $E = mc^2$ (Fig. 2.8):

$$\Psi(t) \propto \Psi(0)\, e^{-\imath \frac{E}{\hbar} t} \tag{2.39}$$

$$\Psi(E) \propto \delta(E - mc^2)\,. \tag{2.40}$$

Unstable particles are described by damped harmonic wavefunctions and therefore their proper energies are not well-defined (Fig. 2.9):

$$\Psi(t) \propto \Psi(0) e^{-\imath \frac{E}{\hbar} t} e^{-\frac{\Gamma}{2\hbar} t} \implies |\Psi(t)|^2 \propto |\Psi(0)|^2 e^{-t/\tau} \tag{2.41}$$

$$\Psi(E) \propto \frac{1}{(E - mc^2) + i\Gamma/2} \implies |\Psi(E)|^2 \propto \frac{1}{(E - mc^2)^2 + \Gamma^2/4} \tag{2.42}$$

which is a Cauchy function (physicists call it a Breit–Wigner function) for which the width Γ is directly related to the particle lifetime τ:

$$\tau = \frac{\hbar}{\Gamma}. \tag{2.43}$$

If a particle can decay through different channels, its total width will be the sum of the partial widths Γ_i of each channel:

$$\Gamma_t = \sum \Gamma_i \,. \tag{2.44}$$

An unstable particle may thus have several specific decay rates, but it has just one lifetime:

$$\tau = \frac{\hbar}{\sum \Gamma_i} \,. \tag{2.45}$$

Therefore, all the Breit–Wigner functions related to the decays of the same particle have the same width Γ_t but different normalization factors, which are proportional to the fraction of the decays in each specific channel, also called the branching ratio, BR_i, defined as

$$BR_i = \frac{\Gamma_i}{\Gamma_t} \,. \tag{2.46}$$

2.7 Fermi Golden Rule and Rutherford Scattering

Particles interact like corpuscles but propagate like waves. This was the turmoil created in physics in the early twentieth century by Einstein's photoelectric effect theory. In the microscopic world, deterministic trajectories were no longer possible. Newton's laws had to be replaced by wave equations. Rutherford formulae, classically deduced, agree anyway with calculations based on quantum mechanics.

In quantum mechanics, the scattering of a particle due to an interaction that acts only during a finite time interval can be described as the transition between an initial and a final stationary states characterized by well-defined momenta. The probability λ of such a transition is given, if the perturbation is small, by Fermi's[4] "golden rule" (see [F2.1] among the recommended readings at the end of the chapter):

[4]Enrico Fermi (Rome 1901–Chicago 1954) studied in Pisa and became full professor of Analytical Mechanics in Florence in 1925, and then of Theoretical Physics in Rome from 1926. Soon he surrounded himself by a group of brilliant young collaborators, the so-called via Panisperna boys (E. Amaldi, E. Majorana, B. Pontecorvo, F. Rasetti, E. Segré, O. D'Agostino). For Fermi, theory and experiment were inseparable. In 1934, he discovered that slow neutrons catalyzed a certain type of nuclear reactions, which made it possible to derive energy from nuclear fission. In 1938, Fermi went to Stockholm to receive the Nobel Prize, awarded for his fundamental work on neutrons, and from there he emigrated to the USA, where he became an American citizen in open dispute with the Italian racial laws. He actively participated in the Manhattan Project for the use of nuclear power for the atomic bomb, but spoke out against the use of this weapon on civilian targets. Immediately after the end of World War II, he devoted himself to theoretical physics of elementary particles and to the origin of cosmic rays. Few scientists of the twentieth century impacted as profoundly as Fermi in different areas of physics: Fermi stands for elegance and power of thought in the group of immortal geniuses like Einstein, Landau, Heisenberg, and later Feynman.

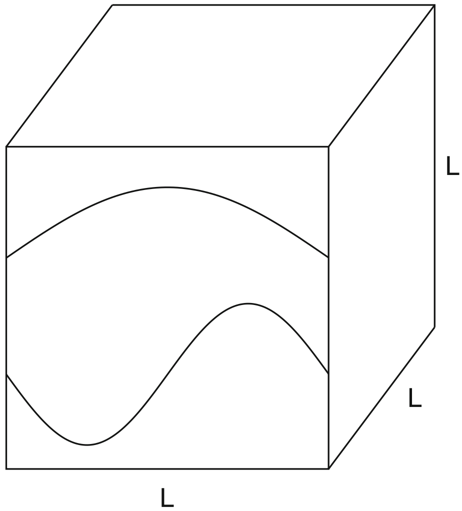

Fig. 2.10 Normalization box

$$\lambda = \frac{2\pi}{\hbar} |H'_{if}|^2 \rho(E_i) \tag{2.47}$$

where H'_{if} is the transition amplitude[5] between states i and f ($H'_{if} = \langle f|H'_{int}|i\rangle$, where H'_{int} is the interaction Hamiltonian) and $\rho(E_i)$ is the density of final states for a given energy $E_i = E_f$. The cross section is, as it was seen above, the interaction rate per unit of flux J. Thus,

$$\sigma_{\text{tot}} = \frac{\lambda}{J} \,. \tag{2.48}$$

To compute the cross section one then needs to determine the transition amplitude, the flux, and the density of final states.

2.7.1 Transition Amplitude

Rutherford scattering can be, to a first approximation, treated as the nonrelativistic elastic scattering of a single particle by a fixed static Coulomb potential. The initial and final time-independent state amplitudes may be written as plane waves normalized in a box of volume L^3 (Fig. 2.10) and with linear momenta $\mathbf{p_i} = \hbar\, \mathbf{k_i}$ and $\mathbf{p_f} = \hbar\, \mathbf{k_f}$, respectively ($k = |\mathbf{k_i}| = |\mathbf{k_f}|$):

---

[5]Depending on the textbook, you might encounter the notation H_{if} or H_{fi}.

$$u_i = L^{-\frac{3}{2}} \exp(i\, \mathbf{k_i} \cdot \mathbf{r}) \tag{2.49}$$

and

$$u_f = L^{-\frac{3}{2}} \exp(i\, \mathbf{k_f} \cdot \mathbf{r})\,. \tag{2.50}$$

Assuming a scattering center at the origin of coordinates, the Coulomb potential is written as

$$V\,(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r} \tag{2.51}$$

where ε_0 is the vacuum dielectric constant and Q_1 and Q_2 are the charges of the beam and of the target particles. The transition amplitude can thus be written as

$$H'_{if} = L^{-3} \int \exp\,(-\,i\, \mathbf{k_f} \cdot \mathbf{r})\; V\,(r) \exp\,(-\,i\, \mathbf{k_i} \cdot \mathbf{r})\; d^3\, x\,. \tag{2.52}$$

Introducing the momentum transfer:

$$\mathbf{q} = \hbar\;\; (\mathbf{k_f} - \mathbf{k_i}) \tag{2.53}$$

the transition amplitude given by:

$$H'_{if} = L^{-3} \int V\,(r) \exp\left(-\,\frac{i}{\hbar}\,\mathbf{q}\;\cdot \mathbf{r}\right) d^3\, x \tag{2.54}$$

is just the Fourier transform of $V\,(r)$ and then

$$H'_{if} = -\frac{4\,\pi\hbar^2}{L^3}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{|\mathbf{q}|^2}\right). \tag{2.55}$$

Expressing $|\mathbf{q}|^2$ as a function of the scattering angle θ as

$$|\mathbf{q}|^2 = 4\,\hbar^2\, k^2 \sin^2\frac{\theta}{2}\,, \tag{2.56}$$

the transition amplitude may finally be written as

$$H'_{if} = -\frac{4\pi\hbar^2}{L^3}\left(\frac{1}{4\pi\varepsilon_0}\;\frac{Q_1\; Q_2}{4\hbar^2\, k^2 \sin^2\frac{\theta}{2}}\right). \tag{2.57}$$

2.7.2 Flux

The flux, as seen in Eq. 2.22, is $J = \rho_{\text{beam}} v$, which in the present case may be written as

$$J = \frac{v}{L^3} = \frac{\hbar\, k}{m\, L^3}\,. \tag{2.58}$$

2.7.3 Density of States

The density of final states $\rho(E_i)$ is determined by the dimension of the normalization box. At the boundaries of the box, the wavefunction should be zero and so only harmonic waves are possible in the case of free particles. Therefore, the projections of the wave number vector κ along each axis should also obey

$$k_x = \frac{2\pi\, n_x}{L}\ ;\ k_y = \frac{2\pi\, n_y}{L}\ ;\ k_z = \frac{2\pi\, n_z}{L} \tag{2.59}$$

where n_x, n_y and n_z are the integer harmonic numbers.

Considering now a given wave number vector in its vector space, the volume associated to each possible state defined by a particular set of harmonic numbers is just

$$\frac{dk_x}{dn_x}\,\frac{dk_y}{dn_y}\,\frac{dk_z}{dn_z} = \left(\frac{2\pi}{L}\right)^3, \tag{2.60}$$

while the elementary volume d^3k in spherical coordinates is

$$d^3k = k^2\, dk\, d\Omega. \tag{2.61}$$

Then, the number of states dn in the volume d^3k is

$$dn = \left(\frac{L}{2\pi}\right)^3 k^2\, dk\, d\Omega\,. \tag{2.62}$$

Remembering that in nonrelativistic quantum mechanics

$$E = \frac{(\hbar k)^2}{2m}\,, \tag{2.63}$$

the density of states $\rho\,(E_i)$ is therefore given as

$$\rho\,(E_i) = \frac{dn}{dE} = \left(\frac{L}{2\pi\hbar}\right)^3 \frac{(\hbar k)^2}{v} d\Omega\,, \tag{2.64}$$

where v is the velocity of the particle.

2.7.4 Rutherford Cross Section

Replacing all the terms in (2.47) and (2.48):

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{4E_0}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}} \tag{2.65}$$

and this is exactly the Rutherford formula.

In fact, the minimum distance at which a nonrelativistic beam particle with energy E_0 can approach the target nucleus is:

$$d_{\text{min}} = \frac{Q_1Q_2}{4\pi\varepsilon_0 E_0} \tag{2.66}$$

while the de Broglie wavelength associated to that particle is

$$\lambda = \frac{h}{\sqrt{2mE_0}}\,. \tag{2.67}$$

In the particular case of the Rutherford experiment (α particles with a kinetic energy of 7.7 MeV against a golden foil) $\lambda \ll d_{\text{min}}$ and the classical approximation is, by chance, valid.

2.8 Particle Scattering in Static Fields

The Rutherford formula was deduced assuming a static Coulomb field created by a fixed point charge. These assumptions can be either too crude or just not valid in many cases. Hereafter, some generalizations of the Rutherford formula are discussed.

2.8.1 Extended Charge Distributions (Nonrelativistic)

Let us assume that the source of the static Coulomb field has some spatial extension $\rho(r')$ (Fig. 2.11) with

$$\int_0^\infty \rho\left(r'\right)\, dr' = 1\,. \tag{2.68}$$

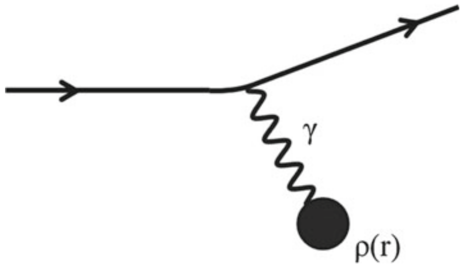

Fig. 2.11 Scattering by an extended source

Then,

$$H'_{if} = L^{-3} \int\int \left(\frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2 \, \rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}\,|} \right) \exp\left(- \frac{i}{\hbar} \mathbf{q} \cdot \mathbf{r} \right) d^3x \, d^3x' =$$

$$= L^{-3} \int\int \left(\frac{1}{4\pi\varepsilon_0} \frac{Q_1 Q_2 \, \rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}\,|} \right) \exp\left(- \frac{i}{\hbar} \mathbf{q} \cdot (\mathbf{r} - \mathbf{r}') \right) \exp\left(- \frac{i}{\hbar} \mathbf{q} \cdot \mathbf{r}' \right) d^3x \, d^3x' \tag{2.69}$$

and defining the electric form factor $F(q)$ as

$$F(q) = \int \rho(\mathbf{r}') \exp\left(- \frac{i}{\hbar} \mathbf{q} \cdot \mathbf{r}' \right) d^3x' \tag{2.70}$$

the modified scattering cross section is

$$\frac{d\sigma}{d\Omega} = |F(\mathbf{q})|^2 \left(\frac{d\sigma}{d\Omega} \right)_0 \tag{2.71}$$

where $\left(\frac{d\sigma}{d\Omega}\right)_0$ is the Rutherford cross section.

In the case of the proton, the differential *ep* cross section at low transverse momentum is described by such a formula, and the form factor is given by the *dipole formula*

$$F(q) \propto \left(1 + \frac{|\mathbf{q}|^2}{\hbar^2 b^2} \right)^{-2} . \tag{2.72}$$

The charge distribution is the Fourier transform $\rho(r) \propto e^{-r/a}$, where $a = 1/b \simeq 0.2$ fm corresponds to a root mean square charge radius of 0.8–0.9 fm. The size of the proton is then determined to be at the scale of 1 fm.

2.8.2 Finite Range Interactions

The Coulomb field, as the Newton gravitational field, has an infinite range. Let us now consider a field with an exponential attenuation (Yukawa potential)

$$V(r) = \frac{g}{4\pi r} \exp\left(-\frac{r}{a}\right) \tag{2.73}$$

where g is the interaction strength, and a is the interaction range scale. Then,

$$H'_{if} = L^{-3} \int \left(\frac{g}{4\pi r} \exp(-\frac{r}{a})\right) \exp\left(-\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{r}\right) d^3x, \tag{2.74}$$

giving

$$H'_{if} = -\frac{\hbar^2}{L^3}\left(\frac{g}{\mathbf{q}^2 + \frac{\hbar^2}{a^2}}\right). \tag{2.75}$$

Using now the Fermi golden rule,

$$\frac{d\sigma}{d\Omega} = g^2\left(\frac{\mathbf{q}^2}{\mathbf{q}^2 + M^2c^2}\right)^2 \left(\frac{d\sigma}{d\Omega}\right)_0, \tag{2.76}$$

where $\left(\frac{d\sigma}{d\Omega}\right)_0$ is the Rutherford cross section. $M = \hbar/(a\,c)$ was interpreted by Hideki Yukawa,[6] as it will be discussed in Sect. 3.2.4, as the mass of a particle exchanged between nucleons and responsible for the strong interaction which ensures the stability of nuclei. The scale $a = 1$ fm corresponds to the size of nucleons, and the mass of the exchanged particle comes out to be $M \simeq 200$ MeV$/c^2$ (see Sect. 2.10 for the conversion).

2.8.3 Electron Scattering

Electrons have nonzero spin $\left(S = \frac{1}{2}\hbar\right)$, and thus a nonzero magnetic moment

$$\mu = \frac{Q_e}{m_e}\mathbf{S} \tag{2.77}$$

where Q_e and m_e are, respectively, the charge and the mass of the electron.

The electron scattering cross section is given by the Mott cross section (its derivation is beyond the scope of the present chapter as it implies relativistic quantum mechanics):

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(1 - \beta^2 \sin^2\frac{\theta}{2}\right). \tag{2.78}$$

When the velocity $\beta \to 0$, the Rutherford scattering formula is recovered as

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[6]Hideki Yukawa (Tokyo, 1907–Kyoto, 1981), professor at Kyoto University, gave fundamental contributions to quantum mechanics. For his research he won the prize Nobel Prize for Physics in 1949.

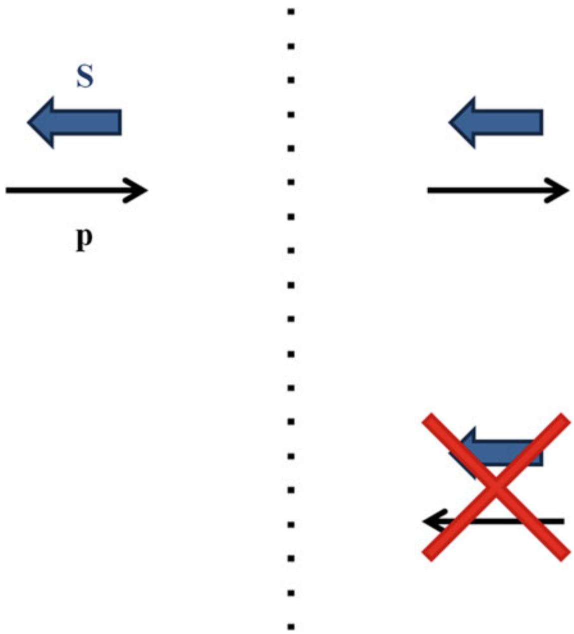

Fig. 2.12 Schematic representation of helicity conservation in the limit $\beta = 1$. The initial left helicity state (on the left) is conserved and thus the final right helicity state (on the right), corresponding to backscattering, is not allowed

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 . \tag{2.79}$$

When $\beta \to 1$,

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \cos^2\frac{\theta}{2} , \tag{2.80}$$

which translates the fact that, for massless particles, the projection of the spin **S** over the direction of the linear momentum **p** is conserved, as it will be discussed in Sect. 6.3.4 (Fig. 2.12). The helicity quantum number h is defined as

$$\mathrm{h} = \mathbf{S} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} . \tag{2.81}$$

A massless electron, thus, could not be backscattered.

2.9 Special Relativity

Physical laws are, since Galilei and Newton, believed to be the same in all inertial reference frames (i.e., in all frames moving with constant speed with respect to a frame in which they hold—classical mechanics postulates with the law of inertia the existence of at least one such frame). This is called the principle of special relativity, and it has been formulated in a quantitative way by Galilei. According to the laws of transformations of coordinates between inertial frames in classical physics (called Galilean transformations), accelerations are invariant with respect to a change of reference frame—while speeds are trivially noninvariant. Since the equations of classical physics (Newton's equations) are based on accelerations only, this automatically guarantees the principle of relativity.

Something revolutionary happened when Maxwell's equations[7] were formulated. Maxwell's equations

$$\nabla \cdot \boldsymbol{\mathcal{E}} = \frac{\rho}{\epsilon_0} \tag{2.82}$$

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.83}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.84}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} + \mu_0 \mathbf{j} \tag{2.85}$$

together with the equation describing the motion of a particle of electric charge q in an electromagnetic field

$$\mathbf{F} = q(\boldsymbol{\mathcal{E}} + \mathbf{v} \times \mathbf{B}), \tag{2.86}$$

the Lorentz[8] force, provide a complete description of electromagnetic field and of its dynamical effects. Such laws contain explicitly a speed, the speed of light c, 299792458 m/s ($\sim$300 000 km/s, with a relative accuracy better than 10^{-3}). This speed is also present when the equations are written in vacuum, i.e., where neither charges ρ nor currents $\mathbf{j}$ are present: if they hold, thus, the classical formulation of relativity, based on the Galilei transformations, is not invariant in all inertial frames. One can easily create some paradoxes based on this (see the Exercises at the end of the chapter): electromagnetism is not consistent with classical mechanics.

To solve the problem, maintaining the speed of light c as an invariant in nature, and guaranteeing the covariant formulation of the laws of mechanics, a deep change in our perception of space and time was needed: it was demonstrated that time and length intervals are not absolute. Two simultaneous events in one reference frame

[7]James Clerk Maxwell (1831–1879) was a Scottish physicist. His most prominent achievement was formulating classical electromagnetic theory. Maxwell's equations, published in 1865, demonstrate that electricity, magnetism, and light are all manifestations of the same phenomenon: the electromagnetic field. Maxwell also contributed to the Maxwell–Boltzmann distribution, which gives the statistical distribution of velocities in a classical perfect gas in equilibrium. Einstein had a photograph of Maxwell, one of Faraday and one of Newton in his office.

[8]Hendrik Antoon Lorentz (1853–1928) was a Dutch physicist who made important contributions in electromagnetism. He also wrote explicitly the equations subsequently used by Albert Einstein to describe the transformation of space and time coordinates in different inertial reference frames. He was awarded the 1902 Nobel Prize in Physics.

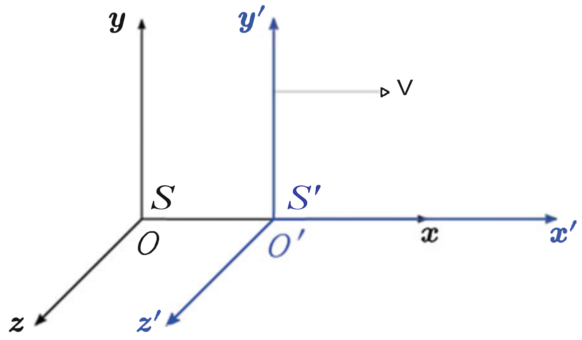

Fig. 2.13 Inertial reference frames

are not simultaneous in any other reference frame that moves with nonzero velocity with respect to the first one; the Galilean transformations had to be replaced by new ones, the Lorentz transformations. Another striking consequence of this revolution was that mass is just a particular form of energy; kinematics at velocities near c is quite different from the usual one, and particle physics is the laboratory to test it.

2.9.1 Lorentz Transformations

Let S and S' be two inertial reference frames. S' moves with respect to S at a constant velocity $\mathbf{V}$ along the common S and S' x-axis (Fig. 2.13). The coordinates in one reference frame transform into new coordinates in the other reference frame (Lorentz transformations) as:

$$\begin{aligned} ct &= \gamma(ct' + \beta x') \\ x &= \gamma(x' + \beta ct') \\ y &= y' \\ z &= z' \end{aligned}$$

where $\beta = V/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

It can be verified that applying the above transformations, the speed of light is an invariant between S and S'.

A conceptually nontrivial consequence of these transformations is that for an observer in S the time interval ΔT is larger than the time measured by a clock in S' for two events happening at the same place, the so-called *proper time*, $\Delta T'$ (time dilation):

$$\Delta T = \gamma\, \Delta T'\,, \tag{2.87}$$

while the length of a ruler that is at rest in S' is shorter when measured in S (length contraction):

$$\Delta L = \Delta L'/\gamma \,. \tag{2.88}$$

Lorentz transformations of coordinates guarantee automatically the invariance of the squared interval

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{2.89}$$

We now extend the properties of the quadruple, or 4-ple, (cdt, dx, dy, dz) to other 4-ples behaving in a similar way, introducing representations such that the equations become covariant with respect to transformations—i.e., the laws of physics hold in different reference frames, similarly to what happens in classical physics.

Let us introduce a simple convention: in the 4-ple (cdt, dx, dy, dz), the elements will be numbered from 0 to 3. Greek indices like μ will run from 0 to 3 ($\mu = 0, 1, 2, 3$), and Roman symbols will run from 1 to 3 ($i = 1, 2, 3$) as in the usual three-dimensional case.

We define as four-vector a quadruple

$$A^\mu = \left(A^0, A^1, A^2, A^3\right) = \left(A^0, \mathbf{A}\right) \tag{2.90}$$

which transforms like (cdt, dx, dy, dz) for changes of reference systems. The A^μ (with high indices) is called *contravariant* representation of the four-vector.

Correspondingly, we define the 4-ple

$$A_\mu = (A_0, A_1, A_2, A_3) = \left(A^0, -A^1, -A^2, -A^3\right) = \left(A^0, -\mathbf{A}\right) \tag{2.91}$$

which is called *covariant* representation.

The coordinates of an event (ct, x, y, z) can be considered as the components of a four-dimensional radius vector in a four-dimensional space. So we shall denote its components by x^μ, where the index μ takes the values 0, 1, 2, 3 and

$$x^0 = ct \quad x^1 = x \quad x^2 = y \quad x^3 = z \,. \tag{2.92}$$

By our definition, the quantity $\sum_\mu A_\mu A^\mu \equiv A_\mu A^\mu$ is invariant. Omitting the sum sign when an index is repeated once in contravariant position and once in covariant position is called Einstein summation convention. Sometimes, when there is no ambiguity, this quantity is also indicated as A^2.

By analogy to the square of a four-vector, one forms the *scalar product* of two different four-vectors:

$$A^\mu B_\mu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \,.$$

It is clear that it can be written either as $A^\mu B_\mu$ or $A_\mu B^\mu$—the result is the same.

The product $A^\mu B_\mu$ is a *four-scalar*: it is invariant under rotations of the four-dimensional coordinate system.

The component A^0 is called the *time component* and $(A^1,\ A^2,\ A^3)$ the *space components* of the four-vector. Under purely spatial rotations the three space components of the four-vector A^i form a three-dimensional vector $\mathbf{A}$.

The square of a four-vector can be positive, negative, or zero; accordingly, the four-vector is called *timelike-*, *spacelike-* and *null-vector*, respectively.

We can write $A^\mu = g^{\mu\nu} A_\nu$, where

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.93}$$

is called *metric tensor* (sometimes Minkowski tensor or Minkowski metric tensor), a symmetric matrix which transforms the contravariant A^μ in the covariant A_μ and vice versa.

Indeed, we can also write $A_\mu = g_{\mu\nu} A^\mu$, where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.94}$$

is the covariant representation of the same metric tensor.

$g^{\mu\nu}$ is the completely contravariant metric tensor, $g_{\mu\nu}$ is the completely covariant metric tensor. The scalar product of two vectors can therefore be written in the form

$$A^\mu A_\mu = g_{\mu\nu} A^\mu A^\nu = g^{\mu\nu} A_\mu A_\nu \, . \tag{2.95}$$

Besides, we have that $g_{\mu\nu} g^{\mu\rho} = \delta^\rho_\mu = 1$. In this way we have enlarged the space adding a fourth dimension A^0: the time dimension.

The generic transformation between reference frames can be written expressing Lorentz transformations by means of a four-matrix Λ:

$$A'_\mu = \Lambda^\nu_\mu A_\nu \tag{2.96}$$

where in the case of two frames moving along x

$$\Lambda^\nu_\mu = \begin{pmatrix} \gamma & -\gamma\,\beta & 0 & 0 \\ -\gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.97}$$

2.9.1.1 Tensors

A four-dimensional tensor of the second rank is a set of 16 quantities $A^{\mu\nu}$, which transforms like products of components of two four-vectors (i.e., they enter in covariant equations). We could similarly define four-tensors of higher rank. For example, we could have the expression $A^\mu B_{\mu\sigma} \equiv C_\sigma$ where A^μ transforms as a vector, $B_{\mu\sigma}$ transforms as a product of vectors, and C_σ is a four-vector.

A second-rank tensor can be written in three ways: covariant $A_{\mu\nu}$, contravariant $A^{\mu\nu}$, and mixed A^μ_ν. The connection between different types of components is determined from this general rule: raising or lowering a space index (1, 2, 3) changes the sign of the component, while raising or lowering the time index (0) does not. The quantity $A^\mu_\mu = tr\left(A^\nu_\mu\right)$ is the trace of the tensor.

Our aim is now to rewrite the physical laws using these four-vectorial entities. To do that we introduce the *completely antisymmetric tensor* of rank 4, $\varepsilon^{\mu\nu\rho\sigma}$. Like for tensors $g_{\mu\nu}$, $g^{\mu\nu}$, its components are the same in all coordinate systems.

By definition,

$$\varepsilon^{0123} = 1. \tag{2.98}$$

The components change sign under interchange of any pair of indices, and thus, the nonzero components are those for which all four indices are different. Every permutation with an odd rank changes the sign. The number of component with nonzero value is $4! = 24$.

We have

$$\varepsilon_{\mu\nu\rho\sigma} = g_{\alpha\mu} g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma} \varepsilon^{\alpha\beta\gamma\delta} = -\varepsilon^{\mu\nu\rho\sigma}.$$

Thus, $\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = -24$ (number of nonzero elements changed of sign).

In fact with respect to rotations of the coordinate system, the quantities $\varepsilon^{\alpha\beta\gamma\delta}$ behave like the components of a tensor, but if we change the sign of one or three of the coordinates the components $\varepsilon^{\alpha\beta\gamma\delta}$, being defined as the same in all coordinate systems, do not change, whereas some of the components of a tensor should change sign.

2.9.1.2 An Example: The Metric Tensor

The invariant interval can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.99}$$

Under a Lorentz transformation,

$$ds^2 = g_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma dx^\rho dx^\sigma. \tag{2.100}$$

Since the interval is invariant,

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma dx^\rho dx^\sigma = g_{\rho\sigma} dx^\rho dx^\sigma \Longrightarrow \left(g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma - g_{\rho\sigma}\right) dx^\rho dx^\sigma = 0. \tag{2.101}$$

As the last equation must be true for any infinitesimal interval, the quantity in parentheses must be zero, so

$$g_{\rho\sigma} = g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}. \tag{2.102}$$

2.9.1.3 Covariant Derivatives

As a consequence of the total differential theorem, $(\partial s/\partial x^{\mu})dx^{\mu}$ is equal to the scalar ds. Thus $(\partial s/\partial x^{\mu})$ is a four-vector and since x^{μ} is contravariant, it is covariant, because ds is a scalar. We call the operator $\partial_{\mu} = \dfrac{\partial}{\partial x^{\mu}}$ four-gradient.

We can write $\dfrac{\partial\phi}{\partial x^{\mu}} = \left(\dfrac{1}{c}\dfrac{\partial\phi}{\partial t}, \nabla\phi\right)$. In general, the operators of differentiation with respect to the coordinates $x^{\mu} \equiv (ct, x, y, z)$, should be regarded as the covariant components of the operator four-gradient. For example,

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \;\; \partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right). \tag{2.103}$$

We can build from covariant quantities the operator

$$\partial_{\mu}\partial^{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box\,; \tag{2.104}$$

this is called the D'Alembert operator.

2.9.2 Space–Time Interval

Two events in spacetime can have a spacetime difference such that

$$\begin{aligned}
&\Delta s^2 > 0 \,(\text{time} - \text{like interval})\\
&\Delta s^2 = 0\\
&\Delta s^2 < 0 \,(\text{space} - \text{like interval}).
\end{aligned}$$

We remind the reader that, due to the invariance of c, Δs^2 is an invariant.

The space-time is divided thus into two regions by the hypercone of the $\Delta s^2 = 0$ events (the so-called light cone, Fig. 2.14). If the interval between two causally connected events is "time-like" (no time travels, sorry) then the maximum speed at which information can be transmitted is c.

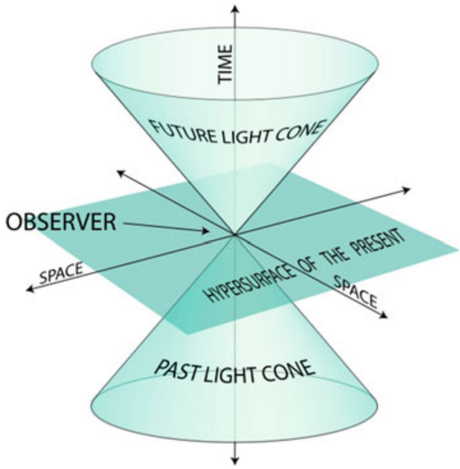

Fig. 2.14 Light cone. By stib at en.wikipedia, via wikimedia commons

2.9.3 Velocity Four-Vector

The classical laws of transformation of velocities between inertial frames S and S' (the "intuitive" Galilei law of the addition of the velocities) cannot hold at high velocities since there is strong experimental evidence that the speed of light in vacuum is the same for all observers, regardless of their relative motion or of the motion of the light source (the second postulate of Einstein's special relativity).

Transformation laws consistent with the theory of relativity can be deduced in a very simple way introducing the velocity quadruple u:

$$u = \lim_{\Delta t_0 \to 0} \frac{\Delta R}{\Delta t_0} \tag{2.105}$$

where ΔR, the displacement four-vector, is defined as the difference between two four-vectors representing the coordinates of successive events on the spacetime trajectory of a body i in the frame S,

$$\Delta R = (c(t + \Delta t),\ x + \Delta x, y + \Delta y, z + \Delta z) - (ct,\ x, y, z)) = (c\Delta t,\ \Delta x, \Delta y, \Delta z)\,, \tag{2.106}$$

and Δt_0 is the difference of the proper times of the two events. Note that, since ΔR is a four-vector due to the linearity of the Lorentz transformations, and Δt_0 a Lorentz invariant, u is indeed a four-vector.

Using now the time dilation relation

$$\Delta t_0 = \frac{\Delta t}{\gamma_i} = \Delta t\,\sqrt{\left(1 - {\beta_i}^2\right)},$$

where β_i and γ_i are the normalized velocity and the Lorentz factor of the body i in the frame S, the velocity four-vector u can then be written as:

$$u = \left(\gamma_i c,\, \gamma_i u_x, \gamma_i u_y, \gamma_i u_z\right) = (\gamma_i c,\, \gamma_i \mathbf{u}) \tag{2.107}$$

where $\mathbf{u}$ is the three-dimensional velocity of the body in the reference frame S and u_x, u_y, u_z are its components.

In a similar way the velocity four-vector between the same two events can be written in the S' frame as:

$$u' = \left(\gamma'_i c,\, \gamma'_i u'_x, \gamma'_i u'_y, \gamma'_i u'_z\right) = (\gamma'_i c,\, \gamma'_i \mathbf{u'})$$

and since both u and u' are four-vectors, they transform one into the other through the Lorentz transformation:

$$\begin{pmatrix} \gamma'_i c \\ \gamma'_i u'_x \\ \gamma'_i u'_y \\ \gamma'_i u'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_i c \\ \gamma_i u_x \\ \gamma_i u_y \\ \gamma_i u_z \end{pmatrix}, \tag{2.108}$$

where γ and $\beta = V/c$ are, as usual, the Lorentz boost and the relative velocity between the two frames.

Solving the matrix equation

$$\gamma'_i = \gamma\, \gamma_i \left(1 - V\, u_x/c^2\right) \tag{2.109}$$

one has that

$$u'_x = \frac{u_x - V}{1 - V\, u_x/c^2} \tag{2.110}$$

$$u'_y = \frac{u_y}{\gamma\, \left(1 - V\, u_x/c^2\right)} \tag{2.111}$$

$$u'_z = \frac{u_z}{\gamma\, \left(1 - V\, u_x/c^2\right)}\,. \tag{2.112}$$

2.9.4 Energy and Momentum

Energy and momentum conservation have a deep meaning for physicists. They are closely connected to the invariance of the laws of physics with respect to time and space translations.

"Classical" momentum, is, however, not conserved in special relativity. One can demonstrate that the conservation of energy and momentum can be recovered with an improved definition of energy and momentum:

$$E = \gamma mc^2 \; ; \; \mathbf{p} = \gamma m\mathbf{v} \,. \tag{2.113}$$

The product of the mass, which is a scalar and thus an invariant, by the velocity four-vector, is itself a four-vector. We call it momentum four-vector:

$$p = mu = (m\gamma c \,, \gamma m\mathbf{v}) \;\; . \tag{2.114}$$

The space component is the relativistic definition of the three-vector linear momentum, and recovers the Newtonian definition whenever $v \ll c$.

The Newtonian definition of three-vector force

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{2.115}$$

can still be retained but now $\mathbf{p}$ is the relativistic three-vector momentum,

$$\mathbf{F} = \frac{d(\gamma m\mathbf{v})}{dt} \;\; .$$

In the same way, the kinetic energy K of a body is still the result of the work W applied to that body. Considering, for simplicity, a body initially at rest that moves under the influence of a force F aligned along the x-axis:

$$K = W = \int F dx = \int \frac{d\,(\gamma m v)}{dt} dx = \int m v d\,(\gamma v)\,, \tag{2.116}$$

but

$$\frac{d\gamma}{dv} = -\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} \left(-\frac{2v}{c^2}\right)$$

and

$$d(\gamma v) = \frac{dv}{\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}} \,.$$

The kinetic energy acquired by the body is then

$$K = m \int_0^v \frac{v dv}{\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}} = \gamma mc^2 - mc^2. \tag{2.117}$$

Expanding the new definition of the kinetic energy in powers of v:

$$K = \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \cdots \tag{2.118}$$

the classical kinetic energy is recovered as a low-speed limit. The total energy of a body can then be defined as:

$$E = \gamma mc^2 \tag{2.119}$$

while the energy of the body at rest is given by:

$$E_0 = mc^2 \, . \tag{2.120}$$

Mass is thus proportional to the "internal" energy or, in the words of Einstein, "mass and energy are both but different manifestations of the same thing." The dream of the alchemists is possible, but the recipe is very different.

On the other hand, the "Lorentz boost" γ of a particle and its velocity β normalized to the speed of light can now be obtained as:

$$\gamma = E/(mc^2) \; ; \; |\beta| = |pc|/E \, . \tag{2.121}$$

Energy and momentum form thus a four-vector p^μ whose components are:

$$p^0 = E/c \, ; \; p^1 = p_x \, ; \; p^2 = p_y \, ; \; p^3 = p_z \, . \tag{2.122}$$

Since Lorentz transformations are valid for any four-vector, the transformations of energy and of momentum from one reference frames to another are just:

$$E/c = \gamma(E'/c + \beta \, p'_x) \tag{2.123}$$

$$p_x = \gamma(p'_x + \beta \, E'/c) \tag{2.124}$$

$$p_y = p'_y \tag{2.125}$$

$$p_z = p'_z \tag{2.126}$$

The scalar product of p^μ by itself is by definition invariant, and the result is:

$$p^2 = p_\mu p^\mu = (E/c)^2 - |\mathbf{p}|^2 = m^2c^2 \tag{2.127}$$

and thus,

$$E^2 = m^2c^4 + |\mathbf{p}|^2c^2 \, . \tag{2.128}$$

While in classical mechanics you cannot have a particle of zero mass, this is possible in relativistic mechanics. The particle will have four-momentum

$$p^\mu = (E/c, \mathbf{p}) \tag{2.129}$$

with

$$E^2 - p^2c^2 = 0\,, \tag{2.130}$$

and thus will move at the speed of light. The converse is also true: if a particle moves at the speed of light, its rest mass is zero—but the particle still carries a momentum E/c. The photon is such a particle.

2.9.5 *Examples of Relativistic Dynamics*

2.9.5.1 Decay

Let a particle of mass M spontaneously decay into two particles with masses m_1 and m_2, respectively. In the frame of reference in which the initial particle is at rest, energy conservation gives

$$Mc^2 = E_1 + E_2\,. \tag{2.131}$$

where E_1 and E_2 are the energies of the final-state particles. Since, for a particle of mass m, $E \geq m$, this requires that $M \geq (m_1+m_2)$: a particle can decay spontaneously only into particles for which the sum of masses is smaller or equal to the mass of the initial particle. If $M < (m_1 + m_2)$, the initial particle is stable (with respect to that particular decay), and if we want to generate that process, we have to supply from outside an amount of energy at least equal to its "binding energy" $(m_1 + m_2 - M)c^2$.

Momentum must be conserved as well in the decay: in the rest frame of the decaying particle, $p_1 + p_2 = 0$. Consequently, $p_1^2 = p_2^2$ or

$$E_1^2 - m_1^2c^2 = E_2^2 - m_2^2c^2\,. \tag{2.132}$$

Solving the two equations above, one gets

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}c^2 \quad ; \quad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}c^2\,. \tag{2.133}$$

2.9.5.2 Elastic Scattering

Let us consider, from the point of view of relativistic mechanics, the elastic collision of particles. We denote the momenta and energies of the two colliding particles (with masses m_1 and m_2) as p_1^i and p_2^i, respectively; we use primes for the corresponding quantities after collision. The laws of conservation of momentum and energy in the collision can be written together as the equation for conservation of the four-momentum:

$$p_1^\mu + p_2^\mu = p_1^{'\mu} + p_2^{'\mu}\,. \tag{2.134}$$

We rewrite it as $p_1^\mu + p_2^\mu - p_1^{'\mu} = p_2^{'\mu}$ and square:

$$p_1^\mu + p_2^\mu - p_1^{'\mu} = p_2^{'\mu} \Longrightarrow m_1^2c^4 + p_1^\mu p_{2\mu} - p_{1\mu}p_1^{'\mu} - p_{2\mu}p_1^{'\mu} = 0\,. \tag{2.135}$$

Similarly,

$$p_1^\mu + p_2^\mu - p_2^{'\mu} = p_1^{'\mu} \Longrightarrow m_2^2c^4 + p_1^\mu p_{2\mu} - p_{2\mu}p_2^{'\mu} - p_{1\mu}p_2^{'\mu} = 0\,. \tag{2.136}$$

Let us consider the collision in a reference frame in which one of the particles (m_2) was at rest before the collision. Then, $\mathbf{p_2} = 0$, and $p_1^\mu p_{2\mu} = E_1 m_2 c^2$, $p_{2\mu}p_1^{'\mu} = m_2 E_1' c^2$, $p_{1\mu}p_1^{'\mu} = E_1 E_1' - p_1 p_1' c^2 \cos\theta_1$ where $\cos\theta_1$ is the angle of scattering of the incident particle m_1. Substituting these expressions into Eq. 2.135, we get

$$\cos\theta_1 = \frac{E_1'(E_1 + m_2c^2) - E_1 m_2 c^2 - m_1^2c^4}{p_1 p_1' c^2}\,. \tag{2.137}$$

We note that if $m_1 > m_2$, i.e., if the incident particle is heavier than the target particle, the scattering angle θ_1 cannot exceed a certain maximum value. It is easy to find that this value is given by:

$$\sin\theta_{1\,\max} = m_2/m_1 \tag{2.138}$$

which coincides with the familiar classical result.

2.9.6 Mandelstam Variables

The kinematics of two-to-two particle scattering (two incoming and two outgoing particles, see Fig. 2.15) can be expressed in terms of Lorentz invariant scalars, the Mandelstam variables s, t, u, obtained as the square of the sum (or subtraction) of the four-vectors of two of the particles involved These variables were introduced by the South-African physicist Stanley Mandelstam.

If p_1 and p_2 are the four-vectors of the incoming particles and p_3 and p_4 are the four-vectors of the outgoing particles, the Mandelstam variables are defined as

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ t &= (p_1 - p_3)^2 \\ u &= (p_1 - p_4)^2\,. \end{aligned}$$

The variable s is the square of the center-of-mass energy. In the center-of-mass reference frame S^*:

$$s = \left((E_1^*, \mathbf{p^*}) + (E_2^*, -\mathbf{p^*})\right)^2 = E_{CM}^2 = (E_1^* + E_2^*)^2. \tag{2.139}$$

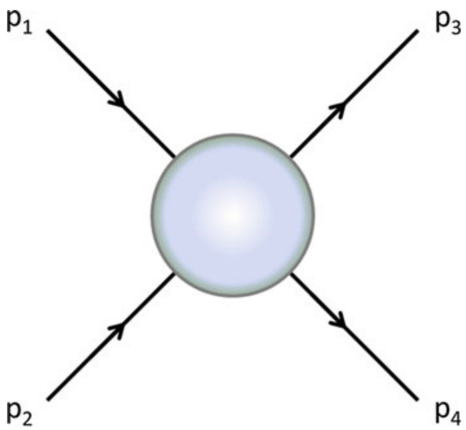

Fig. 2.15 Two-to-two particle scattering

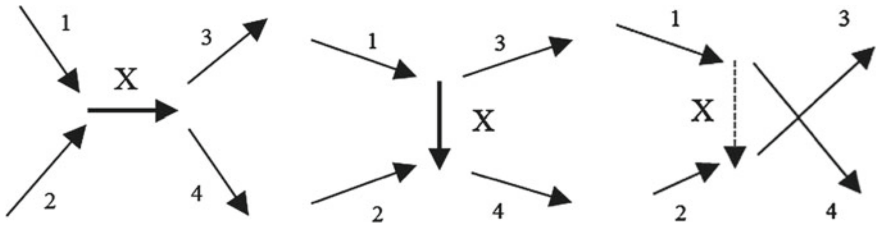

Fig. 2.16 Two-to-two particles interaction channels: left s-channel; center t-channel; right u-channel

In the laboratory reference frame S:

$$s = E_{CM}^2 = \left((E_{\rm beam}, \mathbf{p}_{\rm beam}) + (M_{\rm target}c^2, 0)\right)^2 =$$
$$= M_{\rm beam}^2 c^4 + M_{\rm target}^2 c^4 + 2E_{\rm beam} M_{\rm target} c^2 \,.$$

The center-of-mass energy is then proportional to the beam energy in a collider and (asymptotically for very high energies) to the square root of the beam energy in a fixed target experiment.

If the interaction is mediated by an intermediate particle X resulting from the "fusion" of particles 1 and 2 (s-channel, see Fig. 2.16 left),

$$1 + 2 \to X \to 3 + 4. \tag{2.140}$$

s is the square of the X particle energy–momentum four-vector, and one must have

$$s \geq M_X c^2 \tag{2.141}$$

so that particle X can live in our real world.

If the interaction is mediated by a particle X emitted by particle 1 and absorbed by particle 2 (t-channel, see Fig. 2.16 center):

$$t = ((E_1, \mathbf{p_1}) - (E_3, \mathbf{p_3}))^2 = (E_1 - E_3)^2 - (\mathbf{p_1} - \mathbf{p_3})^2 . \quad (2.142)$$

If $mc^2 \ll E$

$$t = q^2 \simeq -4E_1 E_3 \sin^2 \frac{\theta}{2} \quad (2.143)$$

where q, the energy-momentum four-vector of particle X, is the generalization in four dimensions of the momentum transfer previously introduced. Due to its space-like character, q^2 is negative and

$$q^2 \simeq -\mathbf{q}^2 . \quad (2.144)$$

To avoid the negative sign, a new variable Q^2 is defined as $Q^2 = -q^2$.

Finally, the u-channel is equivalent to the t-channel with the roles of particle 3 and 4 interchanged. It is relevant mainly in backward scattering processes.

In two-to-two scattering processes, there are eight outgoing variables (two four-vectors), four conservation equations (energy and momentum), and a relation between the energy of each outgoing particle and its momentum (see previous section). Then, there are only two independent outgoing variables and s, t, u must be related. In fact,

$$s + t + u = m_1c^2 + m_2c^2 + m_3c^2 + m_4c^2 . \quad (2.145)$$

2.9.7 Lorentz Invariant Fermi Rule

In nonrelativistic quantum mechanics, the probability density $|\psi(\mathbf{r})|^2$ is usually normalized to 1, in some arbitrary box of volume V. However, V is not a Lorentz invariant and therefore the transition amplitude H'_{if}, the density of final states $\rho(E_i)$, and the flux J as defined previously are not Lorentz invariant. The adopted convention is to normalize the density of probability to $2E$ (EV is a Lorentz invariant and the factor 2 is historical). The transition rate (2.47) is then redefined as

$$\lambda = \frac{2\pi}{\hbar} \frac{|\mathcal{M}|^2}{\prod_{i=1}^{n_i} 2E_i} \rho_{n_f}(E) \quad (2.146)$$

where the square of the scattering amplitude is

$$|\mathcal{M}|^2 = |H'_{if}|^2 \prod_{i=1}^{n_i} 2E_i \prod_{f=1}^{n_f} 2E_f \tag{2.147}$$

and the relativistic phase space is

$$\rho_{n_f}(E) = \frac{1}{(2\pi\hbar)^{3n_f}} \int \prod_{f=1}^{n_f} 2E_f \delta\left(\sum_{f=1}^{n_f} \mathbf{p_f} - \mathbf{p_0}\right) \delta\left(\sum_{f=1}^{n_f} E_f - E_0\right) . \tag{2.148}$$

n_i and n_f are the number of particles in the initial and final states, respectively; p_0 and E_0 are the total initial linear momentum and energy, and the δ functions ensure the conservation of linear momentum and energy.

In the case of a two-body final state, the phase space in the center-of-mass frame is simply

$$\rho_2(E^*) = \frac{\pi}{(2\pi\hbar)^6} \frac{|\mathbf{p^*}|}{E^*} \tag{2.149}$$

where $\mathbf{p^*}$ and E^* are the linear momentum and the energy of each final state particle in the center-of-mass reference frame, respectively. The flux is now defined as

$$J = 2E_a 2E_b v_{ab} = 4F \tag{2.150}$$

where v_{ab} is the relative velocity of the two interacting particles and F is called the Möller's invariant flux factor. In terms of the four-vectors p_a and p_b of the incoming particles:

$$F = \sqrt{(p_a.p_b)^2 - {m_a}^2 {m_b}^2 c^4} \tag{2.151}$$

or, in terms of invariant variables,

$$F = \sqrt{\left(s - \left(m_a c^2 + m_b c^2\right)^2\right)\left(s - \left(m_a c^2 - m_b c^2\right)^2\right)} . \tag{2.152}$$

Putting together all the factors, the cross section for the two-particle interaction is given as

$$\sigma_{a+b\to 1+2+\cdots+n_f} = \frac{1}{4F} \frac{S\hbar^2}{(2\pi)^{3n_f-4}} \int |\mathcal{M}|^2 \prod_{f=1}^{n_f} \frac{d^3 p_f}{2E_f} \delta\left(\sum_{f=1}^{n_f} \mathbf{p_f} - \mathbf{p_0}\right) \delta\left(\sum_{f=1}^{n_f} E_f - E_0\right) \tag{2.153}$$

where S is a statistical factor that corrects for double counting whenever there are identical particles and also accounts for spin statistics.

In the special case of a two-to-two body scattering in the center-of-mass frame, a simple expression for the differential cross section $\frac{d\sigma}{d\Omega}$ can thus be obtained (if $|\mathcal{M}|^2$ is a function of the final momentum, the angular integration cannot be carried out):

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S\,|\mathcal{M}|^2}{s} \frac{|\mathbf{p_f}|}{|\mathbf{p_i}|}\,. \tag{2.154}$$

The partial width can be computed using again the relativistic version of the Fermi golden rule applied to the particular case of a particle at rest decaying into a final state with n_f particles

$$\Gamma_i = \frac{1}{2\hbar m_i}\frac{S}{(2\pi)^{(3n_f-4)}}\int |\mathcal{M}|^2 \prod_{f=1}^{n_f}\frac{d^3p_f}{2E_f}\delta\left(\sum_{f=1}^{n_f}\mathbf{p_f}-\mathbf{p_0}\right)\delta\left(\sum_{f=1}^{n_f}E_f - E_0\right). \tag{2.155}$$

In the particular case of only two particles in the final state, it simplifies to

$$\Gamma_i = \frac{S|\mathbf{p}^*|}{8\pi\hbar m_i^2 c}|\mathcal{M}|^2 \tag{2.156}$$

where $\mathbf{p}^*$ is the linear momentum of each final state in the center-of-mass reference frame.

2.9.8 The Electromagnetic Tensor and the Covariant Formulation of Electromagnetism

In order to make the electromagnetism equations covariant, we introduced a new formalism, the quadrivector formalism. But the electromagnetic field is expressed in terms of 3-ples $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$, which do not allow us to express the equations in a four-vectorially covariant way.

We shall now write the equations of electromagnetism, Maxwell's equations (2.82 – 2.86), in a completely covariant form.

First, let us express the electric and magnetic fields through the vector and scalar potentials.

We begin examining Maxwell equation $\nabla\cdot\mathbf{B}=0$—the simplest of the equations. We know that it implies that $\mathbf{B}$ can be expressed as the curl of a vector field. So, we write it in the form:

$$\mathbf{B} = \nabla\times\mathbf{A}\,. \tag{2.157}$$

Next, we take Faraday's law, $\nabla\times\boldsymbol{\mathcal{E}} = -\partial\mathbf{B}/\partial t$. If we express $\mathbf{B}$ as a function of the vector potential, and differentiate with respect to it, we can write Faraday's law in the form $\nabla\times\boldsymbol{\mathcal{E}} + \partial(\nabla\times\mathbf{A})/\partial t = 0$. Since we can differentiate either with respect to time or to space first, we can also write this equation as

$$\nabla \times \left(\boldsymbol{\mathcal{E}} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0\,. \tag{2.158}$$

We see that $\boldsymbol{\mathcal{E}} + \partial \mathbf{A}/\partial t$ is a vector whose curl is equal to zero. Therefore that vector can be expressed as the gradient of a scalar field. In electrostatics, we take $\boldsymbol{\mathcal{E}}$ to be the gradient of $-\phi$. We do the same thing for $\boldsymbol{\mathcal{E}} + \partial \mathbf{A}/\partial t$ and set

$$\boldsymbol{\mathcal{E}} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi\,. \tag{2.159}$$

We use the same symbol ϕ, so that in the electrostatic case the relation $\boldsymbol{\mathcal{E}} = -\nabla \phi$ still holds. Faraday's law can thus be put in the form

$$\boldsymbol{\mathcal{E}} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}\,. \tag{2.160}$$

We have solved two of Maxwell's equations already, and we have found that to describe the electromagnetic fields $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$, we need four potential functions: a scalar potential ϕ, and a vector potential $\mathbf{A}$, which is, of course, three functions.

Now that $\mathbf{A}$ determines part of $\boldsymbol{\mathcal{E}}$, as well as $\mathbf{B}$, what happens when we change $\mathbf{A}$ to $\mathbf{A}' = \mathbf{A} + \nabla \psi$? Although $\mathbf{B}$ does not change since $\nabla \times \nabla \psi = 0$, in general $\boldsymbol{\mathcal{E}}$ would change. We can, however, still allow $\mathbf{A}$ to be changed without affecting the electric and magnetic fields—that is, without changing the physics—if we always change $\mathbf{A}$ and ϕ together by the rules

$$\mathbf{A}' = \mathbf{A} + \nabla \psi\ ;\ \ \phi' = \phi - \frac{\partial \psi}{\partial t}\,. \tag{2.161}$$

Let's now turn to the two remaining Maxwell equations, which will give us relations between the potentials and the sources. Once we determine $\mathbf{A}$ and ϕ from the currents and charges, we can always get $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$ from Eqs. (2.157) and (2.160), so we will have another form of Maxwell's equations.

We begin by substituting Eq. 2.160 into $\nabla \cdot \boldsymbol{\mathcal{E}} = \rho/\epsilon_0$; we get

$$\nabla \cdot \left(-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right) = \frac{\rho}{\epsilon_0} \Longrightarrow -\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = \frac{\rho}{\epsilon_0}\,. \tag{2.162}$$

This equation relates ρ and $\mathbf{A}$ to the sources.

Our final equation will be the most complicated one. Thanks to Eqs. (2.157) and (2.160), the fourth Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \tag{2.163}$$

can be written as

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}\right) \tag{2.164}$$

and since $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ we can write

$$\begin{aligned} &-\nabla^2\mathbf{A} + \left(\nabla(\nabla\cdot\mathbf{A}) + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi\right) + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{j} \\ \Longrightarrow &-\nabla^2\mathbf{A} + \left(\nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)\right) + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{j}\,. \end{aligned} \tag{2.165}$$

Fortunately, we can now make use of our freedom to choose arbitrarily the divergence of **A**, which is guaranteed by Eq. 2.161. What we are going to do is to use our choice to fix things so that the equations for **A** and for ϕ are separate but have the same form. We can do this by taking (this is called the Lorenz[9] gauge):

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}\,. \tag{2.166}$$

When we do that, the two terms in brackets in **A** and ϕ in Eq. 2.165 cancel, and that equation becomes much simpler:

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{j} \Longrightarrow \Box\,\mathbf{A} = \mu_0\mathbf{j} \tag{2.167}$$

and also the equation for ϕ takes a similar form:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = \frac{\rho}{\epsilon_0} \Longrightarrow \Box\,\phi = \frac{\rho}{\epsilon_0}\,. \tag{2.168}$$

These equations are particularly fascinating. We can easily obtain from Maxwell's equations the continuity equation for charge

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0\,.$$

If a net electric current is flowing out of a region, then the charge in that region must be decreasing by the same amount. Charge is conserved. This provides a proof that $j^\mu = (\rho/c, \mathbf{j})$ is a four-vector, since we can write

$$\partial_\mu j^\mu = 0\,. \tag{2.169}$$

[9] Ludvig Lorenz (1829–1891), not to be confused with Hendrik Antoon Lorentz, was a Danish mathematician and physicist, professor at the Military Academy in Copenhagen.

If we define the 4-ple $A^\mu = (\phi/c, \mathbf{A})$, Eqs. 2.167 and 2.168 can be written together as

$$\Box A^\mu = \mu_0 j^\mu \,. \tag{2.170}$$

Thus the 4-ple A^μ is also a four-vector; we call it the four-potential of the electromagnetic field. Considering this fact, it appears clearly that the Lorenz gauge (2.166) is covariant and can be written as

$$\partial_\mu A^\mu = 0 \,. \tag{2.171}$$

In regions where there are no longer any charges and currents, the solution of Eq. 2.171 is a four-potential, which is changing in time but always moving out at speed c. This four-field travels onward through free space.

Since A^μ is a four-vector, the antisymmetric matrix

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{2.172}$$

it is thus a four-tensor. Obviously, the diagonal elements of this tensor are null. The 0th row and column are, respectively,

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \frac{1}{c}\frac{\partial A^i}{\partial t} + \frac{\partial \phi}{\partial x^i} = -\mathcal{E}^i/c \tag{2.173}$$

$$F^{i0} = -F^{0i} = \mathcal{E}^i/c \,. \tag{2.174}$$

The 1…3 elements of the matrix are

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -(\nabla \times \mathbf{A})_z = -B_z \tag{2.175}$$

$$F^{13} = \partial^1 A^3 - \partial^3 A^1 = (\nabla \times \mathbf{A})_y = B_y \tag{2.176}$$

$$F^{23} = \partial^2 A^3 - \partial^3 A^2 = -(\nabla \times \mathbf{A})_x = -B_x \tag{2.177}$$

and correspondingly for the symmetric components.

Finally, the *electromagnetic tensor is*

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\mathcal{E}_x/c & -\mathcal{E}_y/c & -\mathcal{E}_z/c \\ \mathcal{E}_x/c & 0 & -B_z & B_y \\ \mathcal{E}_y/c & B_z & 0 & -B_x \\ \mathcal{E}_z/c & -B_y & B_x & 0 \end{pmatrix} . \tag{2.178}$$

The components of the electromagnetic field are thus elements of a tensor, the electromagnetic tensor.

The nonhomogeneous Maxwell equations have been written as Eq. 2.170:

$$\Box A^{\mu} = (\partial_{\nu}\partial^{\nu})A^{\mu} = j^{\mu}\,. \tag{2.179}$$

We can write

$$\partial_{\nu}F^{\nu\mu} = \partial_{\nu}(\partial^{\nu}A^{\mu} - \partial^{\mu}A^{\nu}) = (\partial_{\nu}\partial^{\nu})A^{\mu} - \partial^{\mu}(\partial_{\nu}A^{\nu}) \tag{2.180}$$

and since $\partial_{\nu}A^{\nu} = 0$,

$$\partial_{\nu}F^{\nu\mu} = (\partial_{\nu}\partial^{\nu})A^{\mu} = \Box A^{\mu} = j^{\mu}\,. \tag{2.181}$$

The covariant equation

$$\partial_{\nu}F^{\nu\mu} = j^{\mu} \tag{2.182}$$

is equivalent to the nonhomogeneous Maxwell equations.

In the same way, one has for the homogeneous equations:

$$\nabla \times \boldsymbol{\mathcal{E}} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad ; \quad \nabla \cdot \mathbf{B} = 0$$

the following result (four equations):

$$\partial^2 F^{03} + \partial^3 F^{20} + \partial^0 F^{32} = 0 \qquad \cdots \qquad \partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0$$

and thus

$$\left(\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \mathbf{B}}{\partial t} \;\&\; \nabla \cdot \mathbf{B} = 0\right) \Longleftrightarrow \epsilon_{\alpha\beta\gamma\delta}\partial^{\beta}F^{\gamma\delta} = 0 \quad (\alpha = 0, 1, 2, 3)\,.$$

Due to the tensor nature of $F_{\mu\nu}$, the two following quantities are invariant for transformations between inertial frames:

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = B^2 - \mathcal{E}^2/c^2$$

$$\frac{c}{8}\epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta} = \mathbf{B} \cdot \boldsymbol{\mathcal{E}}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric unit tensor of rank four.

2.10 Natural Units

The international system of units (SI) can be constructed on the basis of four fundamental units: a unit of length (the meter, m), a unit of time (the second, s), a unit of mass (the kilogram, kg), and a unit of charge (the coulomb, C).[10]

These units are inappropriate in the world of fundamental physics: The radius of a nucleus is of the order of 10^{-15} m (also called one femtometer or one fermi, fm); the mass of an electron is of the order of 10^{-30} kg; the charge of an electron is (in absolute value) of the order of 10^{-19} C. Using such units, we would carry along a lot of exponents. Thus, in particle physics, we prefer to use units like the electron charge for the electrostatic charge, and the electron-volt eV and its multiples (keV, MeV, GeV, TeV) for the energy:

Length	1 fm	10^{-15} m
Mass	1 MeV/c^2	$\sim 1.78 \times 10^{-30}$ kg
Charge	$\|e\|$	$\sim 1.602 \times 10^{-19}$ C.

Note the unit of mass, in which the relation $E = mc^2$ is used implicitly: what one is doing here is to use $1\,\text{eV} \simeq 1.602 \times 10^{-19}$ J as the new fundamental unit of energy. In these new units, the mass of a proton is about 0.938 GeV/c^2, and the mass of the electron is about 0.511 MeV/c^2. The fundamental energy level of a hydrogen atom is about -13.6 eV.

In addition, nature provides us with two constants which are particularly appropriate in the world of fundamental physics: the speed of light $c \simeq 3.00 \times 10^8$ m/s $= 3.00 \times 10^{23}$ fm/s, and Planck's constant (over 2π) $\hbar \simeq 1.05 \times 10^{-34}$ J s $\simeq 6.58 \times 10^{-16}$ eV s.

It seems then natural to express speeds in terms of c, and angular momenta in terms of $\hbar$. We then switch to the so-called natural units (NUs). The minimal set of natural units (not including electromagnetism) can then be chosen as

Speed	1 c	3.00×10^8 m/s
Angular momentum	1 $\hbar$	1.05×10^{-34} J s
Energy	1 eV	1.602×10^{-19} J

After the convention $\hbar = c = 1$, one single unit can be used to describe the mechanical Universe: we choose energy, and we can thus express all mechanical quantities in terms of eV and of its multiples. It is immediate to express momenta and masses directly in NU. To express 1 m and 1 s, we can write[11]

---

[10] For reasons related only to metrology (reproducibility and accuracy of the definition) in the standard SI the unit of electric current, the ampere A, is used instead of the coulomb; the two definitions are however conceptually equivalent.

[11] $\hbar c \simeq 1.97 \times 10^{-13}$ MeV m $= 3.15 \times 10^{-26}$ J m.

$$\begin{aligned} 1\text{ m} &= \frac{1\text{m}}{\hbar c} \simeq 5.10 \times 10^{12}\text{ MeV}^{-1} \\ 1\text{ s} &= \frac{1\text{s}}{\hbar} \simeq 1.52 \times 10^{21}\text{ MeV}^{-1} \\ 1\text{ kg} &= 1\text{J}/c^2 \simeq 5.62 \times 10^{29}\text{ MeV}. \end{aligned}$$

Both length and time are thus, in natural units, expressed as inverse of energy. The first relation can also be written as 1 fm $\simeq$ 5.10 GeV^{-1}. Note that when you have a quantity expressed in MeV^{-1}, in order to express it in GeV^{-1}, you must multiply (and not divide) by a factor of 1000.

Let us now find a general rule to transform quantities expressed in natural units into SI, and vice versa. To express a quantity in NU back in SI, we first restore the $\hbar$ and c factors by dimensional arguments and then use the conversion factors $\hbar$ and c (or $\hbar c$) to evaluate the result. The dimensions of c are [m/s]; the dimensions of $\hbar$ are [kg m^2 s^{-1}].

The converse (from SI to NU) is also easy. A quantity with meter-kilogram-second [m k s] dimensions $M^p L^q T^r$ (where M represents mass, L length, and T time) has the NU dimensions $[E^{p-q-r}]$, where E represents energy. Since $\hbar$ and c do not appear in NU, this is the only relevant dimension, and dimensional checks and estimates are very simple. The quantity Q in SI can be expressed in NU as

$$Q_{\rm NU} = Q_{\rm SI}\left(5.62 \times 10^{29}\frac{\rm MeV}{\rm kg}\right)^p \left(5.10 \times 10^{12}\frac{\rm MeV^{-1}}{\rm m}\right)^q \times \left(1.52 \times 10^{21}\frac{\rm MeV^{-1}}{\rm s}\right)^r \text{MeV}^{p-q-r}$$

The NU and SI dimensions are listed for some important quantities in Table 2.1.

Note that, choosing natural units, all factors of $\hbar$ and c may be omitted from equations, which leads to considerable simplifications (we will profit from this in the next chapters). For example, the relativistic energy relation

Table 2.1 Dimensions of different physical quantities in SI and NU

	SI			NU
Quantity	p	q	r	n
Mass	1	0	0	1
Length	0	1	0	−1
Time	0	0	1	−1
Action ($\hbar$)	1	2	−1	0
Velocity (c)	0	1	−1	0
Momentum	1	1	−1	1
Energy	1	2	−2	1

$$E^2 = p^2c^2 + m^2c^4 \tag{2.183}$$

becomes

$$E^2 = p^2 + m^2. \tag{2.184}$$

Finally, let us discuss how to treat electromagnetism. To do so, we must introduce a new unit, of charge for example. We can redefine the unit charge by observing that

$$\frac{e^2}{4\pi\epsilon_0} \tag{2.185}$$

has the dimension of [J m], and thus is a pure, dimensionless, number in NU. Dividing by $\hbar c$ one has

$$\frac{e^2}{4\pi\epsilon_0\hbar c} \simeq \frac{1}{137}\,. \tag{2.186}$$

Imposing for the electric permeability of vacuum $\epsilon_0 = 1$ (thus automatically $\mu_0 = 1$ for the magnetic permeability of vacuum, since from Maxwell's equations $\epsilon_0\mu_0 = 1/c^2$), we obtain the new definition of charge, and with this definition:

$$\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}\,. \tag{2.187}$$

This is called the Lorentz–Heaviside convention. Elementary charge in NU becomes then a pure number

$$e \simeq 0.303\,. \tag{2.188}$$

Let us make now some applications.

The Thomson Cross Section. Let us express a cross section in NU. The cross section for Compton scattering of a photon by a free electron is, for $E \ll m_e c^2$ (Thomson regime),

$$\sigma_T \simeq \frac{8\pi\alpha^2}{3m_e^2}\,. \tag{2.189}$$

The dimension of a cross section is, in SI, [m^2]. Thus, we can write

$$\sigma_T \simeq \frac{8\pi\alpha^2}{3m_e^2}\hbar^a c^b \tag{2.190}$$

and determine a and b such that the result has the dimension of a length squared. We find $a = 2$ and $b = -2$; thus,

$$\sigma_T \simeq \frac{8\pi\alpha^2}{3(m_e c^2)^2}(\hbar c)^2 \tag{2.191}$$

and thus $\sigma_T \simeq 6.65 \times 10^{-29}\,\mathrm{m}^2 = 665\,\mathrm{mb}$.

The Planck Mass, Length, and Time. According to quantum theory, a length called the Compton wavelength, λ_C, can be associated to any mass m. λ_C is defined as the wavelength of a photon with an energy equal to the rest mass of the particle:

$$\lambda_C = \frac{h}{mc} = 2\pi\frac{\hbar}{mc}\,. \tag{2.192}$$

The Compton wavelength sets the distance scale at which quantum field theory becomes crucial for understanding the behavior of a particle: wave and particle description become complementary at this scale.

On the other hand, we can compute for any mass m the associated Schwarzschild radius, R_S, such that compressing it to a size smaller than this radius we form a black hole. The Schwarzschild radius is the scale at which general relativity becomes crucial for understanding the behavior of the object:

$$R_S = \frac{2Gm}{c^2}\,, \tag{2.193}$$

where G is the gravitational constant.[12]

We call Planck mass the mass at which the Schwarzschild radius of a particle becomes equal to its Compton length, and Planck length their common value when this happens. The probe that could locate a particle within this distance would collapse to a black hole, something that would make measurements very strange. In NU, one can write

$$\frac{2\pi}{m_P} = 2Gm_P \rightarrow m_P = \sqrt{\frac{\pi}{G}} \tag{2.194}$$

which can be converted into

$$m_P = \sqrt{\frac{\pi\hbar c}{G}} \simeq 3.86 \times 10^{-8}\mathrm{kg} \simeq 2.16 \times 10^{19}\,\mathrm{GeV/c^2}. \tag{2.195}$$

Since we are talking about orders of magnitude, the factor $\sqrt{\pi}$ is often neglected and we take as a definition:

$$m_P = \sqrt{\frac{\hbar c}{G}} \simeq 2.18 \times 10^{-8}\mathrm{kg} \simeq 1.22 \times 10^{19}\,\mathrm{GeV/c^2}. \tag{2.196}$$

[12] A classical derivation of this formula proceeds by computing the radius for which the escape velocity from a spherical distribution of mass with zero angular momentum is equal to c.

Besides the Planck length ℓ_P, we can also define a Planck time $t_P = \ell_P/c$ (their value is equal in NU):

$$\ell_P = t_P = \frac{1}{m_P} = \sqrt{G} \tag{2.197}$$

(this corresponds to a length of about 1.6×10^{-20} fm, and to a time of about 5.4×10^{-44} s).

Both general relativity and quantum field theory are needed to understand the physics at mass scales about the Planck mass or distances about the Planck length, or times comparable to the Planck time. Traditional quantum physics and gravitation certainly fall short at this scale; since this failure should be independent of the reference frame, many scientists think that the Planck scale should be an invariant irrespective of the reference frame in which it is calculated (this fact would of course require important modifications to the theory of relativity).

Note that the shortest length you may probe with the energy of a particle accelerated by the LHC is about 10^{15} times larger than the Planck length scale. Cosmic rays, which can reach center-of-mass energies beyond 100 TeV, are at the frontier of the exploration of fundamental scales.

Further Reading

[F2.1] J.S. Townsend, "A modern approach to quantum mechanics", McGraw-Hill 2012. An excellent quantum mechanics course at an advanced undergraduate level.

[F2.2] W. Rindler, "Introduction to Special Relativity", Second Edition, Oxford University Press 1991. A classic textbook on special relativity for undergraduates.

Exercises

1. *Rutherford formula.* Consider the Rutherford formula.
   (a) Determine the distance of closest approach of an α particle with an energy of 7.7 MeV to a gold target.
   (b) Determine the de Broglie wavelength of that α particle.
   (c) Explain why the classical Rutherford formula survived the revolution of quantum mechanics.

   You can find the numerical values of particle data and fundamental constants in the Appendices, or in your Particle Data Book(let).
2. *Cross section at fixed target.* Consider a fixed target experiment with a monochromatic proton beam with an energy of 20 GeV and a 2-m-long liquid hydrogen (H_2) target ($\rho = 60\,\text{kg/m}^3$). In the detector placed just behind the target beam fluxes of 7×10^6 protons/s and 10^7 protons/s are measured, respectively, with the target full and empty. Determine the proton–proton total cross section at this energy and its statistical error:

(a) without taking into account the attenuation of the beam inside the target;
(b) taking into account the attenuation of the beam inside the target.

3. *LHC collisions.* The LHC running parameters in 2012 were, for a c.m. energy $\sqrt{s} \simeq 8$ TeV: number of bunches = 1400; time interval between bunches $\simeq$50 ns; number of protons per bunch $\simeq 1.1 \times 10^{11}$; beam width at the crossing point $\simeq$16 μm.

   (a) Determine the maximum instantaneous luminosity of the LHC in 2012.
   (b) Determine the number of interactions per collision ($\sigma_{pp} \sim 100$ mb).
   (c) As you probably heard, LHC found a particle called Higgs boson, which Leon Lederman called the "God particle" (a name the news like very much). If Higgs bosons are produced with a cross section $\sigma_H \sim 21$ pb, determine the number of Higgs bosons decaying into 2 photons ($BR(H \to \gamma\gamma) \simeq 2.28 \times 10^{-3}$) which might have been produced in 2012 in the LHC, knowing that the integrated luminosity of the LHC (luminosity integrated over time) during 2012 was around 20 fb^{-1}. Compare it to the real number of detected Higgs bosons in this particular decay mode reported by the LHC collaborations (about 400). Discuss the difference.

4. *Experimental determination of cross sections.* A thin (1.4 mg/cm^2) target made of ^{22}Na is bombarded with a 5 nA beam of α particles. A detector with area 16 cm^2 is placed at 1 m from the target perpendicular to the line between the detector and the target. The detector records 45 protons/s, independently of its angular position (θ, ϕ). Find the cross section in mb for the $^{22}\text{Na} + \alpha \to p + X$–also written as ^{22}Na(α, p)– reaction.
5. *Uncertainty relations.* Starting from Eq. 2.15, demonstrate the uncertainty principle for position and momentum.
6. *Classical electromagnetism is not a consistent theory.* Consider two electrons at rest, and let r be the distance between them. The (repulsive) force between the two electrons is the electrostatic force

$$F = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} ,$$

   where e is the charge of the electron, and is directed along the line joining the two charges. But an observer is moving with a velocity v perpendicular to the line joining the two charges will measure also a magnetic force (still directed as F)

$$F' = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} - \frac{\mu_0}{2\pi r}v^2 e^2 \neq F .$$

   The expression of the force is thus different in the two frames of reference. But masses, charges, and accelerations are classically invariant. Comment.
7. *Classical momentum is not conserved in special relativity.* Consider the completely inelastic collision of two particles, each of mass m, in their c.m. system

(the two particles become one particle at rest after the collision). Now observe the same collision in the reference frame of one particle. What happens if you assume that the classical definition of momentum holds in relativity as well?

8. *Energy is equivalent to mass.* How much more does a hot potato weigh than a cold one (in kg)?
9. *Mandelstam variables.* Demonstrate that, in the $1 + 2 \to 3 + 4$ scattering, $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.
10. *GZK threshold.* The cosmic microwave background fills the Universe with photons with a peak energy of 0.37 meV and a density of $\rho \sim 400/\text{cm}^3$. Determine:

    (a) The minimal energy (known as the GZK threshold) that a proton should have in order that the reaction $p\gamma \to \Delta$ may occur.
    (b) The interaction length of such protons in the Universe considering a mean cross section above the threshold of 0.6 mb.

11. *$\bar{p}$ production at the Bevatron.* Antiprotons were first produced in laboratory in 1955, in proton–proton fixed target collisions at an accelerator called Bevatron (it was named for its ability to impart energies of billions of eV, i.e., Billions of eV Synchrotron), located at Lawrence Berkeley National Laboratory, USA. The discovery resulted in the 1959 Nobel Prize in physics for Emilio Segrè and Owen Chamberlain.

    (a) Describe the minimal reaction able to produce antiprotons in such collisions.
    (b) When a proton is confined in a nucleus, it cannot have arbitrarily low momenta, as one can understand from the Heisenberg principle; the actual value of its momentum is called the "Fermi momentum." Determine the minimal energy that the proton beam must have in order that antiprotons were produced considering that the target protons have a Fermi momentum of around 150 MeV/c.

12. *Photon conversion.* Consider the conversion of one photon in one electron–positron pair. Determine the minimal energy that the photon must have for this conversion to be possible if the photon is in the presence of:

    (a) one proton;
    (b) one electron;
    (c) when no charged particle is around.

13. *π^-decay.* Consider the decay of a flying π^- into $\mu^-\bar{\nu_\mu}$ and suppose that the μ^- was emitted along the line of flight of the π^-. Determine:

    (a) The energy and momentum of the μ^- and of the $\bar{\nu_\mu}$ in the π^- frame.
    (b) The energy and momentum of the μ^- and of the $\bar{\nu_\mu}$ in the laboratory frame, if the momentum $P_\pi^- = 100$ GeV/c.
    (c) Same as the previous question but considering now that was the $\bar{\nu_\mu}$ that was emitted along the flight line of the π^-.

14. *π^0 decay.* Consider the decay of a π^0 into $\gamma\gamma$ (with pion momentum of 100 GeV/c). Determine:

(a) The minimal and the maximal angles between the two photons in the laboratory frame.
(b) The probability of having one of the photons with an energy smaller than an arbitrary value E_0 in the laboratory frame.
(c) Same as (a) but considering now that the decay of the π^0 is into e^+e^-.
(d) The maximum momentum that the π^0 may have in order that the maximal angle in its decay into $\gamma\gamma$ and in e^+e^- would be the same.

15. *Three-body decay.* Consider the decay $K^+ \to \pi^+\pi^+\pi^-$. Determine:

(a) the minimum and maximum values of the π^- energy and momentum in the K^+ rest system;
(b) the maximum value of the momentum in the laboratory system, assuming a K^+ with a momentum $p_K = 100$ GeV/c.

Do the same for the electron in the decay $n \to pe\bar{\nu}_e$.

16. *A classical model for the electron.* Suppose we interpret the electron as a classical solid sphere of radius r and mass m, spinning with angular momentum $\hbar/2$. What is the speed, v, of a point on its "equator"? Experimentally, it is known that r is less than 10^{-18} m. What is the corresponding equatorial speed? What do you conclude from this?
17. *Invariant flux.* In a collision between two particles a and b the incident flux is given by $F = 4|\mathbf{v_a} - \mathbf{v_b}|E_a E_b$ where $\mathbf{v_a}$, $\mathbf{v_b}$, E_a and E_b are, respectively, the vectorial speeds and the energies of particles a and b.

(a) Verify that the above formula is equivalent to: $F = 4\sqrt{(P_a P_b)^2 - (m_a m_b)^2}$ where P_a and P_b are, respectively, the four-vectors of particles a and b, and m_a and m_b their masses.
(b) Relate the expressions of the flux in the center of mass and in the laboratory reference frames.

18. *Lifetime and width of a particle.* The lifetime of the π^0 meson is $\simeq 0.085$ fs. What is the width of the π^0 (absolute, and relative to its mass)?
19. *Width and lifetime of a particle.* The width of the $\rho(770)$ meson is $\simeq 149$ MeV. What is the lifetime of the $\rho(770)$?
20. *Classical Schwarzschild radius for a Black Hole.* Compute the radius of a spherical planet of mass M for which the escape velocity is equal to c.
21. *Units.* Determine in natural units:

(a) Your own dimensions (height, weight, mass, age).
(b) The mean lifetime of the muon ($\tau_\mu = 2.2\,\mu s$).

22. *Units.* In natural units the expression of the muon lifetime is

$$\tau_\mu = \frac{192\pi^3}{G_F^2 m_\mu^5}$$

where G_F is the so-called Fermi constant describing phenomenologically the strength of weak interactions.

(a) Is the Fermi constant dimensionless? If not compute its dimension in NU and in SI.
(b) Obtain the conversion factor for transforming G_F from SI to NU.