Chapter 8
Intuitionism and Intuitionistic Logic

H.C.M. (Harrie) de Swart

Abstract Brouwer's intuitionism is based on quite different philosophical ideas about the nature of mathematical objects than classical mathematics. This intuitionistic point of view results in a different use of language and in a corresponding different intuitionistic logic which is far more subtle than the classical use of language and corresponding classical logic. Nevertheless an intuitionistic deduction system and a notion of intuitionistic deducibility was developed by A. Heyting and it is amazing to see that a small change in the logical axioms, replacing the logical axiom $\neg\neg A \rightarrow A$ by $\neg A \rightarrow (A \rightarrow B)$, may have such far reaching consequences. Since finding (intuitionistic) formal deductions may be difficult, an intuitionistic tableaux based formal deduction system is presented in which the construction of intuitionistic deductions is rather straightforward. The semantics of intuitionistic logic and the notion of (intuitionistic) valid consequence are given in terms of (intuitionistic) Kripke models and it is shown that the three notions of intuitionistic valid consequence, intuitionistic deducibility and intuitionistic tableau-deducibility are equivalent. Intuitionistic sets are either finite constructions or otherwise, they are (subsets of) construction projects. Spreads are a particular kind of construction project, inducing specific principles which typically do not hold for other sets.

8.1 Intuitionism vs Platonism; basic ideas

A *classical mathematician* studies the properties of mathematical objects like an astronomer, who studies the properties of celestial bodies. Mathematical objects are like celestial bodies in the sense that they exist independently of us; they are created by God. And mathematicians are like astronomers who try to discover properties of these objects.

An *intuitionist* creates the mathematical objects himself. According to *Brouwer's intuitionism*, mathematical objects, like 5, 7, 12 and $+$, are mental constructions. A proposition about mathematical objects (like $5+7=12$) is true if one has a proof-construction that establishes it. Such a proof is again a mental construction.

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> Mathematics is created by a free action, independent of experience. [L.E.J. Brouwer [3], p. 97]

In order to better understand the intuitionistic point of view, let us consider the classical or platonistic standpoint more closely. Using Dummett's terminology, one might say that from a classical or platonistic point of view mathematical objects exist in some external realm of mathematical reality; this realm of mathematical reality, existing objectively and independently of our knowledge, renders our mathematical statements (like $5+7=12$) true or false.

> From a classical or platonistic standpoint, the understanding of a mathematical statement consists in a grasp of what it is for that statement to be true, where truth may attach to it even when we have no means of recognizing the fact. [M. Dummett [4], p. 6-7]

The following quotation, translated from German, from P. Bernays [1] may further illuminate the classical or platonistic standpoint.

> One considers the objects of a theory as the elements of a totality and concludes from that: for every property, which can be expressed by means of the notions of the theory, it is an established fact whether there is an element in the totality which has this property or not. From this way of seeing things follow also the following alternatives: either all elements of a set have a given property or there is at least one which does not have this property.
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> One finds in the axiomatics of geometry – in the form Hilbert has given to it – an example for this way of building a theory. When we compare Hilbert's axiomatics with those of Euclid, where we waive that in the case of the Greek mathematicians still some axioms are missing, we notice that Euclid talks about figures, which should be constructed, while for Hilbert the systems of points, of lines and of planes already exist right from the beginning. Euclid postulates: one can connect two points by a straight line; Hilbert, on the contrary, formulates the axiom: Given two arbitrary points, then there exists a straight line, on which both points are located. Existence refers here to the system of straight lines. This example already shows that the tendency, we are talking about, goes in the direction to consider all objects as detached from any connection with the thinking subject.
>
> Since this tendency has become valid most of all in the philosophy of Plato, let me be allowed, to call it Platonism. [P. Bernays [1], pp. 62-63]

The contrast between 'Platonism' and 'Intuitionism' already figures in essence in the comment of Proclos (450 A.D.) on the first book of Euclid's *Elements*, in which a distinction is made between Speusippos and his adherents (350 B.C.), who maintain that all construction problems are theorems, and Menaichmos and the people around him (350 B.C.), who maintain that all theorems are construction problems. (See Paul Ver Eecke [19], pp. 69-70.)

The different views underlying classical and intuitionistic mathematics also result in a different view of the infinite. In classical mathematics, the infinite (for instance, the set $\mathbb{N}$ of the natural numbers $0,1,2,3,\ldots$) is treated as *actual* or *completed*. Quoting S.C. Kleene,

> An infinite set is regarded as existing as a completed totality, prior to or independently of any human process of generation or construction, and as though it could be spread out completely for our inspection. [S.C. Kleene [8], p. 48]

Since for an intuitionist mathematical objects are mental constructions, in intuitionism the infinite is treated only as *potential* or *becoming* or *constructive*. Intuitionistically, the set $\mathbb{N}$ of natural numbers is identified with the *construction project* for its elements: start with 0, and add 1 to each natural number which has already been constructed before. And it was one of the main achievements of L.E.J. Brouwer (1881-1966) to solve the problem how we can talk constructively about the non-enumerable totality $\mathbb{R}$ of the real numbers (see Section 8.7).

For this purpose Brouwer introduced his notion of *spread*, which is again a construction project for producing the elements of the spread, the elements being (potentially) infinite sequences of natural numbers rather than natural numbers themselves. And since real numbers can be represented by infinite sequences of natural numbers, more precisely, by infinite sequences of intervals with rational endpoints, Brouwer's notion of spread enables us to talk constructively about the set $\mathbb{R}$ of the real numbers. (See Section 8.7 for a more elaborate discussion of sets in intuitionism and spreads in particular.)

Since, according to Brouwer, both the mathematical objects themselves and the proofs establishing properties of them are mental constructions, doing mathematics is in principle language-less. Nevertheless, language may be introduced for reasons of communication.

> People try by means of sounds and symbols to originate in others copies of mathematical constructions and reasonings which they have made themselves; by the same means they try to aid their own memory. In this way *mathematical language* comes into being, and as its special case *the language of logical reasoning*. [L.E.J. Brouwer, [3], p. 73]

8.1.1 Language

The following is a quotation from L.E.J. Brouwer [3].

> The immediate companion of the intellect is language. From life in the Intellect follows the impossibility to communicate directly, instinctively, by gesture or looks, or, even more spiritually, through all separation of distance. People then try and train themselves and their offspring in some form of communication by means of crude sounds, laboriously and helplessly, for never has anyone been able to communicate his soul by means of language. . . .
>
> Only in those very narrowly delimited domains of the imagination such as the exclusively intellectual sciences – which are completely separated from the world of perception and therefore touch the least upon the essentially human – only there may mutual understanding be sustained for some time and succeed reasonably well. Little confusion is possible about the meaning of such words as 'equal' or 'triangle', but even then two different people will never think of them in exactly the same way. Even in the most restricted sciences, logic and mathematics (a sharp distinction between these two is hardly possible), no two different people will have the same conception of the fundamental notions of which these two sciences are constructed; and yet, they have a common will, and in both there is a small, unimportant part of the brain that forces the attention in a similar way. . . .
>
> Language becomes ridiculous when one tries to express subtle nuances of will which are not a living reality to the speakers concerned, when for example so-called philosophers or metaphysicians discuss among themselves morality, God, consciousness, immortality or

> the free will. These people do not even love each other, let alone share the same subtle movements of the soul. Sometimes they do not even know each other personally. They either talk at cross purposes or they each build their own little logical system that lacks any connection with reality. For logic is life in the human brain; it may accompany life outside it: it can never guide it by virtue of its own power. ...
>
> Language by itself has no meaning; any philosophy, which in this way tried to find a firm foundation has come to grief. Lulled into sleep by the mistaken belief in its certainty one later hits upon deficiencies and contradictions. A language which does not derive its certainty from the human will, which claims to live on in the 'pure concept' is an absurdity. To be able to go on talking without being caught in contradiction or without making a silent assumption is an art to be valued only in an acrobat. [L.E.J. Brouwer [3], pp. 6-7]

8.1.2 First Steps in Intuitionistic Reasoning

Since, intuitionistically, the truth of a mathematical proposition is established by a proof – which is a particular kind of mental construction –, the meaning of the logical connectives has to be explained in terms of proof-constructions:

A proof of $A \wedge B$ is anything that is a proof of A and of B.

A proof of $A \vee B$ is, in fact, a proof either of A or of B, or yields an effective means, at least in principle, for obtaining a proof of one or other disjunct.

A proof of $A \to B$ is a construction of which we can recognize that, applied to any proof of A, it yields a proof of B. Such a proof is therefore an operation carrying proofs (of A) into proofs (of B).

Intuitionists consider $\neg A$ as an abbreviation for $A \to \bot$, postulating that nothing is a proof of $\bot$ (falsity).

Also the existential quantifier has a constructive meaning in intuitionism: $\exists x[P(x)]$ (there exists an x with the property P) means that I have an algorithm to construct an x in the given domain and next prove that this x has the property P. Consequently, $\exists x[P(x)]$ has a much stronger meaning than $\neg\forall x[\neg P(x)]$ (not every x has the property $\neg P$, i.e., the assumption $\forall x[\neg P(x)]$ yields a contradiction). The constructive reading of $A \vee B$ may be rendered by $\exists x[(x = 0 \wedge A) \vee (x = 1 \wedge B)]$.

The constructive meaning of the intuitionistic connectives causes that many familiar laws from classical logic are no longer valid. Below we shall make clear that the different philosophical points of view, the intuitionistic and the platonistic one, concerning the nature of mathematical objects, result in a quite different use of language.

1. It is reckless to affirm the validity of $A \vee \neg A$. Classically, the validity of $A \vee \neg A$ means that the state of affairs expressed by proposition A is either true or false, without necessarily having a method to decide which of these two. But intuitionistically the validity of $A \vee \neg A$ means that we have a method adequate in principle to solve *any* mathematical problem. Consider *Goldbach's conjecture* G, which states that each even number is the sum of two odd primes: $2 = 1 + 1$, $4 = 3 + 1$, $6 = 5 + 1$, $8 = 7 + 1$, $10 = 7 + 3$, $12 = 7 + 5$, $14 = 7 + 7$, $16 = 13 + 3$, $18 = 13 + 5$, ... One can check only finitely many individual instances,

while Goldbach's conjecture is a statement about infinitely many (even) natural numbers. So far neither Goldbach's conjecture, G, nor its negation, $\neg G$, has been proved. We are therefore not in a position to affirm $G \vee \neg G$. Someone who does, claims that he or she can provide a proof either of G or of $\neg G$; such a person is called *reckless*. Of course, an intuitionist can prove that $(2+3=5) \vee \neg(2+3=5)$. But the validity of $A \vee \neg A$ means that he can give a proof of A or can give a proof of $\neg A$ for *any* mathematical proposition A. And this is a reckless statement. Note that the proposition $\neg(G \vee \neg G)$ implies a contradiction. From $G \rightarrow G \vee \neg G$ it follows that $\neg(G \vee \neg G) \rightarrow \neg G$. And from $\neg G \rightarrow G \vee \neg G$ it follows that $\neg(G \vee \neg G) \rightarrow \neg\neg G$. So, $\neg(G \vee \neg G)$ implies both $\neg G$ and $\neg\neg G$, i.e., a contradiction.

2. $\neg\neg(A \vee \neg A)$ is intuitionistically valid, and hence it is false to assert $\neg(A \vee \neg A)$.
   Proof: $\neg(A \vee B)$ is intuitionstically equivalent to $\neg A \wedge \neg B$. So, since $\neg(\neg A \wedge \neg\neg A)$ holds intuitionistically, it follows that $\neg\neg(A \vee \neg A)$ is intuitionistically valid.
3. It is reckless to affirm the validity of $\neg\neg E \rightarrow E$. For taking $E = A \vee \neg A$, we have seen in 2) that $\neg\neg(A \vee \neg A)$ is intuitionistically valid, while $A \vee \neg A$ is not, as we have seen in 1).
4. $E \rightarrow \neg\neg E$ is intuitionistically valid.
   Proof: We have a proof of $\neg\neg E$ when we can show that we shall never have a proof of $\neg E$, that is, when we show that we shall never have a proof that E will never be proved. Clearly, in general this does not amount to a proof of E itself, as we have seen in 3) by taking $E = A \vee \neg A$. On the other hand, a proof of E does count as a proof that E will never be disproved, for otherwise the possibility of deriving a contradiction would remain open; hence $E \rightarrow \neg\neg E$ is intuitionistically valid.
5. $\neg D \rightarrow \neg\neg\neg D$ is intuitionistically valid.
   Proof: From 4), taking $E = \neg D$.
6. $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ is intuitionistically valid. But the converse, $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$ is not intuitionistically valid.
   Proof: Given $A \rightarrow B$, if we have a proof that B can never be proved, then clearly A can never be proved either, since we could transform any proof of A into a proof of B. We may thus always infer $\neg B \rightarrow \neg A$ from $A \rightarrow B$.
   From 6) follows immediately that $(\neg B \rightarrow \neg A) \rightarrow (\neg\neg A \rightarrow \neg\neg B)$ and hence also that $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow \neg\neg B)$ is intuitionistally valid, because $A \rightarrow \neg\neg A$ is intuitionistically valid. But because $\neg\neg B \rightarrow B$ is not intuitionistically valid, we may not conclude the validity of $(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$.
7. $\neg\neg\neg D \rightarrow \neg D$ is intuitionistically valid.
   Proof: By 4) $D \rightarrow \neg\neg D$ is intuitionistically valid. Hence by 6) $\neg\neg\neg D \rightarrow \neg D$ is intuitionistically valid. From 5) and 7) it follows that $\neg D$ and $\neg\neg\neg D$ are intuitionistically equivalent; in other words, three negation signs can be reduced to one, four negation signs can be reduced to two, five to one, and so on. However, recall that in general $\neg\neg E$ is not equivalent to E, as we have seen in 3).
8. It is reckless to assert the validity of $(A \rightarrow B) \vee (B \rightarrow A)$. Explanation: Let G be Goldbach's conjecture and F be another unsolved mathematical problem; then we are neither in the position to assert $F \rightarrow G$ nor in the position to assert $G \rightarrow F$. Notice that $(A \rightarrow B) \vee (B \rightarrow A)$ is classically valid; see Exercise 8.7.

In the preceding subsection we have seen how *a different philosophical point of view concerning the nature of mathematical objects, results automatically in a different use of language and in a different logic*.

The classical meaning of, for instance, $A \vee \neg A$, may be rendered intuitionistically by $\neg\neg(A \vee \neg A)$. In Exercise 8.12 we shall give a translation from classical formulas to intuitionistic formulas, preserving the actual meaning of the formulas.

While the classical connectives can all be defined in terms of $\neg$ and any one other (cf. Section 2.5), all three connectives listed above $\rightarrow$, $\wedge$ and $\vee$ are intuitionistically independent. Notice that $A \rightleftarrows B$ can be defined as $(A \rightarrow B) \wedge (B \rightarrow A)$.

In Section 8.2 a proof-theoretic formulation of intuitionistic propositional logic is given; we present a Hilbert-type proof system for intuitionistic propositional logic, consisting of (logical) axioms and one rule Modus Ponens (MP), and a tableaux system which given formulas $A_1, \ldots, A_n$ and B will give either an intuitionistic deduction of B from $A_1, \ldots, A_n$ or a counterexample showing that such a deduction cannot exist. In Section 8.4 we give a model-theoretic description of intuitionistic propositional logic in terms of Kripke models.

In classical propositional logic each formula built by means of connectives from only one atomic formula P is equivalent to either $P \wedge \neg P$, P, $\neg P$ or $P \rightarrow P$ (see Exercise 2.14). Nishimura [12] showed that in intuitionistic propositional logic, however, an infinite number of non-equivalent formulas can be built from only one atomic formula P (see Exercise 8.13). So by the intuitionistic refinement of the propositional language, formulas which are indistinguishable classically (i.e., equivalent in classical propositional logic) become different intuitionistically (i.e., non-equivalent in intuitionistic propositional logic). For instance, the formulas $\neg\neg A \rightarrow B$, $A \rightarrow B$ and $A \rightarrow \neg\neg B$ are classically equivalent, but not intuitionistically. Summarizing: *the language of the intuitionist is richer and more subtle than the language of the classical mathematician*.

Exercise 8.1. For the following pairs of formulas, which can be inferred from which intuitionistically?

(a) $A \rightarrow B$	and	$\neg B \rightarrow \neg A$
(b) $\neg(A \wedge B)$	and	$\neg A \vee \neg B$
(c) $A \rightarrow B$	and	$\neg(A \wedge \neg B)$
(d) $A \rightarrow B$	and	$\neg A \vee B$
(e) $A \rightarrow B \vee C$	and	$(A \rightarrow B) \vee (A \rightarrow C)$
(f) $A \vee \neg A$	and	$\neg\neg A \rightarrow A$

Note that the two formulas in each pair are classically equivalent (and hence classically indistinguishable), but not so intuitionistically. So, in this sense, the intuitionistic language is richer than the classical one.

Exercise 8.2. Verify that the following inferences are intuitionistically correct.

1. If $A \rightarrow C$, then $\neg\neg A \rightarrow \neg\neg C$.
2. If $A \rightarrow (B \rightarrow C)$ and B, then $\neg\neg A \rightarrow \neg\neg C$.
3. If $A \rightarrow (B \rightarrow C)$ and $\neg\neg A$, then $\neg\neg B \rightarrow \neg\neg C$.
4. If $\neg\neg(A \rightarrow B)$, then $\neg\neg A \rightarrow \neg\neg B$.
   (Hint: use 3 with $A \rightarrow B$, A, B instead of A, B, C, respectively.)

5. $\neg\neg A \to \neg\neg B$ iff $\neg\neg(A \to B)$.
6. If $\neg\neg A \wedge \neg\neg B$, then $\neg\neg(A \wedge B)$.
   (Hint: use 3 with $A,\ B,\ A \wedge B$ instead of $A,\ B,\ C$, respectively.)
7. $\neg\neg A \wedge \neg\neg B$ iff $\neg\neg(A \wedge B)$.
8. If $\neg\neg A \vee \neg\neg B$, then $\neg\neg(A \vee B)$.
9. Show that conversely *not* for all formulas $A,\ B$, if $\neg\neg(A \vee B)$, then $\neg\neg A \vee \neg\neg B$ intuitionistically.

Exercise 8.3. Note that if A is decidable, i.e., $A \vee \neg A$ is intuitionistically true, then also $\neg\neg A \to A$ is intuitionistically true.

8.2 Intuitionistic Propositional Logic: Syntax

The alphabet for intuitionistic propositional logic looks the same as the one for classical propositional logic, but the atomic formulas and the connectives now have a constructive interpretation, different from the classical interpretation.

Definition 8.1 (Alphabet). The alphabet for intuitionistic propositional logic consists of the following symbols:
1. $P_1, P_2, P_3, \ldots$, called atomic formulas or propositional variables, to be interpreted as (atomic) propositions.
2. $\to$, $\wedge$, $\vee$ and $\neg$, called connectives, to be interpreted constructively in terms of proofs.
3. (and), called brackets.

Definition 8.2 (Constructive interpretation of the connectives).
$A \to B$: I have a construction which transforms any proof of A into a proof of B.
$A \wedge B$: I can construct a proof of A and I can construct a proof of B.
$A \vee B$: I have an algorithm that yields a proof of A or a proof of B.
$\neg A$: $A \to \bot$, where $\bot$ is a special atomic formula, denoting falsity.
$\bot$: falsity; a proof of this formula implies a proof of any formula.

Definition 8.3 (Formulas). 1. If P is any of the atomic formulas $P_1, P_2, P_3, \ldots$, then P is an (atomic) formula.
2. If A and B are formulas, then $(A \to B)$, $(A \wedge B)$, $(A \vee B)$ and $(\neg A)$ are (composite) formulas.

We apply the usual convention for leaving out brackets, see Section 2.1.

A proof theoretic formulation of *intuitionistic propositional logic* was given in 1928 by Arend Heyting (1898-1980) and is obtained by replacing axiom schema 8, $\neg\neg A \to A$ of classical logic (see Section 2.6) by axiom schema 8^i: $\neg A \to (A \to B)$. So, the axiom schemata for intuitionistic propositional logic are the following ones:

1. $A \to (B \to A)$
2. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$

3. $A \to (B \to A \wedge B)$
4a $A \wedge B \to A$
4b $A \wedge B \to B$
5a $A \to A \vee B$
5b $B \to A \vee B$
6. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$
7. $(A \to B) \to ((A \to \neg B) \to \neg A)$
8^i. $\neg A \to (A \to B)$

In addition, like in classical propositional logic, Modus Ponens $\frac{A \quad A \to B}{B}$, is the only rule of inference for intuitionistic propositional logic.

Definition 8.4 (Intuitionistic Deducibility and Provability).
a) A *deduction* of B from $A_1, \ldots, A_n$ in intuitionistic propositional logic := a finite list of formulas with B as last one, such that each formula in the list is either:
1. one of the premisses $A_1, \ldots, A_n$, or
2. an instance of one of the axiom schemata for intuitionistic propositional logic, or
3. obtained from two earlier formulas in the list by an application of Modus Ponens.
b) In case there are no premisses $A_1, \ldots, A_n$, we speak of a (formal) *proof* of B.
c) B is *deducible* from $A_1, \ldots, A_n$ in intuitionistic propositional logic := there exists a deduction of B from $A_1, \ldots, A_n$ in intuitionistic propositional logic.
Notation: $A_1, \ldots, A_n \vdash_i B$. By $A_1, \ldots, A_n \not\vdash_i B$ we mean: not $A_1, \ldots, A_n \vdash_i B$.
d) B is (formally) *provable* in intuitionistic propositional logic := there is a (formal) proof of B in intuitionistic propositional logic. **Notation**: $\vdash_i B$.

Since the intuitionistic axiom schema 8^i, $\neg A \to (A \to B)$ is provable in classical propositional logic, it follows that all propositional formulas provable intuitionistically are also provable classically; see Exercise 8.4. In order to formally prove $A \vee \neg A$ classically, we proved that $\neg\neg(A \vee \neg A)$ and next used $\neg\neg B \to B$ with $B = A \vee \neg A$. However, such a proof is no longer available in intuitionistic logic. In Section 8.4 we shall show that no intuitionistic formal proof of $A \vee \neg A$ can exist.

Since searching for deductions and proofs in terms of logical axioms and Modus Ponens may be difficult, we shall introduce a tableaux system for intuitionistic propositional logic in Section 8.3. In this system searching for an intuitionistic deduction of a putative conclusion from given premisses is straightforward and a systematic search either yields such a deduction or provides us with a counterexample showing that such a deduction cannot exist.

Gentzen's natural deduction rules for *intuitionistic* propositional logic are obtained from Gentzen's natural deduction rules for classical propositional logic (see Section 2.7.2) by leaving out the double negation elimination rule $\frac{\neg\neg A}{A}$.

Exercise 8.4. (i) Show that all formulas provable in intuitionistic propositional logic are also provable in classical propositional logic.
(ii) Show that the *deduction theorem* also holds for intuitionistic propositional logic: if $A_1, \ldots, A_n, A \vdash_i B$, then $A_1, \ldots, A_n \vdash_i A \to B$.

Exercise 8.5. Show that the deductions in Gentzen's system of natural deduction, found in Exercise 2.60 a i), b i) and c i) are intuitionistically correct, but not so are the deductions found in Exercise 2.60 a ii), b ii) and c ii).

8.3 Tableaux for Intuitionistic Propositional Logic

Definition 8.5 (Signed Formula). A *signed formula* is any expression of the form $T(A)$ or $F(A)$, where A is a formula.

Intuitionistically, $T(A)$ is read as: I have a proof of A; and $F(A)$ as: I do not have a proof of A (which is weaker than 'I have a proof of $\neg A$'!). If no confusion is possible. the brackets may be left out: so, we frequently write TA instead of $T(A)$ and FA instead of $F(A)$.

Definition 8.6 (Sequent). A *sequent* S is a finite set of signed formulas.

A *tableaux system* for *intuitionistic propositional logic* is obtained from the tableaux system in Section 2.8 for classical propositional logic by replacing the tableaux rules $F \rightarrow$ and $F\neg$ by

$$F \rightarrow_i \ \frac{S,\ F\ B \rightarrow C}{S_T,\ TB,\ FC} \qquad \text{and} \qquad F\neg_i \ \frac{S,\ F\ \neg B}{S_T,\ TB}.$$

respectively, where $S_T = \{TA \mid TA \in S\}$, i.e., S_T is the set of all T-signed formulas in S. We have drawn a line in the rules $F \rightarrow_i$ and $F\neg_i$ in order to stress that in the transition from S to S_T all F-signed formulas in S, if any, are lost.

For the sake of completeness we list here the *tableaux rules for intuitionistic propositional logic* (see also Fitting [5, 6]):

$T\wedge$	$S,\ T\ B \wedge C$ $S,\ TB,\ TC$	$F\wedge$	$S,\ F\ B \wedge C$ $S,\ FB \mid S,\ FC$
$T\vee$	$S,\ T\ B \vee C$ $S,\ TB \mid S,\ TC$	$F\vee$	$S,\ F\ B \vee C$ $S,\ FB,\ FC$
$T\rightarrow$	$S,\ T\ B \rightarrow C$ $S,\ FB \mid S,\ TC$	$F\rightarrow_i$	$\dfrac{S,\ F\ B \rightarrow C}{S_T,\ TB,\ FC}$
$T\neg$	$S,\ T\ \neg B$ $S,\ FB$	$F\neg_i$	$\dfrac{S,\ F\ \neg B}{S_T,\ TB}$

Notation: S,TA stands for $S \cup \{TA\}$, i.e., the set containing all signed formulas in S and in addition TA; and S,FA similarly stands for $S \cup \{FA\}$. Instead of $\{TB_1,\ldots,TB_m,\ FC_1,\ldots,FC_n\}$ we often simply write $TB_1,\ldots,TB_m,\ FC_1,\ldots,FC_n$. For example, by $\{TD,\ FE\},\ TA$ we mean $\{TD,\ FE,\ TA\}$, but we will usually write $TD,\ FE,\ TA$.

Reading the tableaux rules from the top down, rule $F \to_i \ \dfrac{S,\ F\,B \to C}{S_T,\ TB,\ FC}$ is interpreted as follows: if I am in a proof-situation (this notion is analogous to the notion of possible world) in which I do not have a proof of $B \to C$, then there is a proof-situation accessible from the given one, in which I do have a proof of B without having a proof of C. The change from S to S_T (where S_T is the set of all T-expressions in S) is explained by noting that formulas of which I did not have a proof in the former situation may have been proved by me in the latter situation, while sentences once proved remain proved forever (an ideal mathematician does not forget); so F-signed formulas in S may not be copied down below the line.

In a similar way there is a shift of one proof-situation to another in the interpretation of rule $F\neg_i \ \dfrac{S,\ F\,\neg B}{S_T,\ TB}$, while in interpreting the other intuitionistic tableaux rules there is no such shift. For instance, rule

$$
T \to \qquad \begin{array}{c} S,\ T\,B \to C \\ S,\ FB \ \mid \ S,\ TC \end{array}
$$

is read as follows: if I am in a (proof-)situation in which I have a proof of $B \to C$, then in that same (proof-)situation I do not have a proof of B or I do have a proof of C. So the intuitionistic tableaux rules in which there is a shift from S to S_T (the rules which have a bar in it) are precisely the rules the interpretation of which requires a shift from a former to a later proof-situation.

Notice that the rules for intuitionistic propositional logic still have the property that in any application of the rules all T-signed formulas in the upper half of the rule may be repeated in its lower half; because of the rules $F \to_i$ and $F\neg_i$ the same does not hold any more for the F-signed formulas.

Example 8.1. Below is an intuitionistic tableau-deduction of $\neg Q \to \neg P$ from $P \to Q$. The tableau starts with the premisses T-signed and the putative conclusion F-signed. Informally, we check the possibility to have a proof of the premisses without having a proof of the putative conclusion. Next we apply the tableaux rules and if all possibilities turn out to be closed, i.e., after all to be impossible, then we say that we have a tableau-deduction of the putative conclusion from the given premisses.

$$
\begin{array}{rcl}
 & \underline{T\,P \to Q,\ F\,\neg Q \to \neg P} & \\
 & T\,P \to Q,\ T\neg Q,\ F\neg P & \\
 & T\,P \to Q,\ FQ,\ F\neg P & \\
 & \underline{FP,\ FQ,\ F\neg P} \mid TQ,\ FQ,\ F\neg P & \\
T\,P \to Q,\ T\neg Q,\ TP \mid \text{closure} & & \\
T\,P \to Q,\ FQ,\ TP & & \\
FP,\ FQ,\ TP \mid TQ,\ FQ,\ TP & & \\
\text{closure} \mid \text{closure} & &
\end{array}
$$

So, we start with the supposition that we have a proof of $P \to Q$ without having a proof of $\neg Q \to \neg P$. That might be possible in three different ways, but all of these three turn out to be impossible, in other words give closure. Therefore we shall say that $P \to Q \vdash_i' \neg Q \to \neg P$. The schema above is called a (closed) tableau $\mathscr{T}$ with initial branch $\mathscr{B}_0 = \{T\ P \to Q,\ F\ \neg Q \to \neg P\}$.

Let tableau $\mathscr{T}_0 = \{\mathscr{B}_0\}$ and $\mathscr{B}_1 = \mathscr{B}_0^* \cup \{T\neg Q,\ F\neg P\}$, where the * in $\mathscr{B}^*$ indicates that only T-signed formulas in $\mathscr{B}$ may count towards closure. Then we call tableau $\mathscr{T}_1 = \{\mathscr{B}_1\}$ a *one-step expansion* of $\mathscr{T}_0$, corresponding with the application of rule $F \to_i$ to $F\ \neg Q \to \neg P$ in $\mathscr{B}_0$. Next let $\mathscr{B}_2 = \mathscr{B}_1 \cup \{FQ\}$, then we call $\mathscr{T}_2 = \{\mathscr{B}_2\}$ a *one-step expansion* of $\mathscr{T}_1$, corresponding with the application of rule $T\neg$ to $T\neg Q$ in $\mathscr{B}_1$. Next let branch $\mathscr{B}_{20} = \mathscr{B}_2 \cup \{FP\}$ and branch $\mathscr{B}_{21} = \mathscr{B}_2 \cup \{TQ\}$, then we call tableau $\mathscr{T}_3 = \{\mathscr{B}_{20}, \mathscr{B}_{21}\}$ a *one-step expansion* of $\mathscr{T}_2$, corresponding with the application of rule $T \to$ to $T\ P \to Q$ in $\mathscr{B}_2$. Let branch $\mathscr{B}_{200} = \mathscr{B}_{20}^* \cup \{TP\}$, then tableau $\mathscr{T}_4 = \{\mathscr{B}_{200}, \mathscr{B}_{21}\}$ is called a *one-step expansion* of $\mathscr{T}_3$, corresponding with the application of rule $F\neg_i$ to $F\neg P$ in $\mathscr{B}_{20}$. Let $\mathscr{B}_{2000} = \mathscr{B}_{200} \cup \{FQ\}$, then $\mathscr{T}_5 = \{\mathscr{B}_{2000}, \mathscr{B}_{21}\}$ is a one step expansion of $\mathscr{T}_4$.

Tableau $\mathscr{T}$ above consists of three branches and is closed since all of its branches are closed, i.e., contain for some formula A both TA and FA. Informally this means that the assumption that it is possible to have a proof of $P \to Q$ without having a proof of $\neg Q \to \neg P$ turns out to be untenable.

Definition 8.7 ((Tableau) Branch). (a) A *tableau branch* is a set of signed formulas. A branch is *closed* if it contains signed formulas TA and FA for some formula A. A branch that is not closed is called *open*.
(b) Let $\mathscr{B}$ be a branch and TA, resp. FA, a signed formula occurring in $\mathscr{B}$. TA, resp. FA, is *fulfilled* in $\mathscr{B}$ if (i) A is atomic, or (ii) $\mathscr{B}$ contains the bottom formulas in the application of the corresponding rule to A, and in case of the rules $T\vee$, $F\wedge$ and $T \to$, $\mathscr{B}$ contains one of the bottom formulas in the application of these rules.
(c) A branch $\mathscr{B}$ is *completed* if $\mathscr{B}$ is closed or every signed formula in $\mathscr{B}$ is fulfilled in $\mathscr{B}$.

Definition 8.8 (Tableau). (a) A set $\mathscr{T}$ of branches is a *tableau* with initial branch $\mathscr{B}_0$ if there is a sequence $\mathscr{T}_0, \mathscr{T}_1, \ldots, \mathscr{T}_n$ such that $\mathscr{T}_0 = \{\mathscr{B}_0\}$, each $\mathscr{T}_{i+1}$ is a one-step expansion of $\mathscr{T}_i$ ($0 \le i < n$) and $\mathscr{T} = \mathscr{T}_n$.
(b) We say that a finite $\mathscr{B}$ has tableau $\mathscr{T}$ if $\mathscr{T}$ is a tableau with initial branch $\mathscr{B}$.
(c) A tableau $\mathscr{T}$ is *open* if some branch $\mathscr{B}$ in it is open, otherwise $\mathscr{T}$ is *closed*.
(d) A tableau is *completed* if each of its branches is completed, i.e., no application of a tableau rule can change the tableau.

Definition 8.9 (Tableau-deduction/proof). (a) A (logical) *tableau-deduction* of B *from* $A_1, \ldots, A_n$ (in intuitionistic propositional logic) is a tableau $\mathscr{T}$ with $\mathscr{B}_0 = \{TA_1, \ldots, TA_n, FB\}$ as initial branch, such that all branches of $\mathscr{T}$ are closed.

In case $n = 0$, i.e., there are no premisses $A_1, \ldots, A_n$, this definition reduces to:
(b) A (logical) *tableau-proof* of B (in intuitionistic propositional logic) is a tableau $\mathscr{T}$ with $\mathscr{B}_0 = \{FB\}$ as initial sequent, such that all branches of $\mathscr{T}$ are closed.

Definition 8.10 (Tableau-deducible). (a) B is *tableau-deducible from* $A_1, \ldots, A_n$ (in intuitionistic propositional logic) := there exists a tableau-deduction of B from $A_1, \ldots, A_n$. **Notation**: $A_1, \ldots, A_n \vdash_i' B$. $A_1, \ldots, A_n \nvdash_i' B$ means: not $A_1, \ldots, A_n \vdash_i' B$.

(b) B is *tableau-provable* (in intuitionistic propositional logic) := there exists a tableau-proof of B. **Notation**: $\vdash_i' B$.
(c) For Γ a (possibly infinite) set of formulas, B is *tableau-deducible from* Γ := there exists a finite list $A_1, \ldots, A_n$ of formulas in Γ such that $A_1, \ldots, A_n \vdash_i' B$.
Notation: $\Gamma \vdash_i' B$.

Example 8.2. We check whether we can show that $\neg Q \to \neg P \ \vdash_i' \ P \to Q$ or equivalently $\vdash_i' (\neg Q \to \neg P) \to (P \to Q)$. So, we start a tableau with $T\ \neg Q \to \neg P,\ F\ P \to Q$:

$$\begin{array}{c}
\underline{T\ \neg Q \to \neg P,\ F\ P \to Q} \\
\underline{T\ \neg Q \to \neg P,\ TP,\ FQ} \\
F\ \neg Q,\ TP,\ FQ \mid T\ \neg P,\ TP,\ FQ \\
\underline{F\ \neg Q,\ TP,\ FQ} \mid FP,\ TP,\ FQ \\
T\ Q,\ TP \quad \mid \text{closure}
\end{array}$$

The right branch does close, but the left branch does not; so we did not construct an intuitionistic tableau-deduction of $P \to Q$ from $\neg Q \to \neg P$. From this open branch we shall construct in Section 8.5 an intuitionistic Kripke counterexample in which $\neg Q \to \neg P$ is true, but in which $P \to Q$ is false. Since the intuitionistic proof system is sound, i.e., all formulas that are intuitionistically tableau-provable are also true in all intuitionistic Kripke models (cf. Theorem 8.2), it follows that there does not exist an intuitionistic tableau-deduction of $P \to Q$ from $\neg Q \to \neg P$.

In order to show that the intuitionistic notions of tableau-deducibility (Definition 8.10) and (Hilbert-type) deducibility (Definition 8.4) are equivalent, we first prove Theorem 8.1: if $A_1, \ldots, A_n \vdash_i' B$, then $A_1, \ldots, A_n \vdash_i B$. In Section 8.4 it is shown that the Hilbert type proof system for intuitionistic propositional logic is sound, i.e., if $A_1, \ldots, A_n \vdash_i B$, then also $A_1, \ldots, A_n \models_i B$ (B is an intuitionistically valid (or logical) consequence of $A_1, \ldots, A_n$). And in Section 8.5 we show completeness: if $A_1, \ldots, A_n \models_i B$, then $A_1, \ldots, A_n \vdash_i' B$.

Theorem 8.1. *(i) If B is tableau-deducible from $A_1, \ldots, A_n$ (in intuitionistic propositional logic), i.e., $A_1, \ldots, A_n \vdash_i' B$, then B is deducible from $A_1, \ldots, A_n$ (in intuitionistic propositional logic), i.e., $A_1, \ldots, A_n \vdash_i B$. In particular, for $n = 0$:*
(ii) If $\vdash_i' B$, then $\vdash_i B$.

Proof. Suppose $A_1, \ldots, A_n \vdash_i' B$, i.e., B is intuitionistically tableau-deducible from $A_1, \ldots, A_n$. It suffices to show:

for every sequent $S = \{TD_1, \ldots, TD_k,\ FE_1, \ldots, FE_m\}$ in an intuitionistic tableau-deduction of B from $A_1, \ldots, A_n$ it holds that $D_1, \ldots, D_k \vdash_i E_1 \vee \ldots \vee E_m$. (*)

Consequently, because $\{TA_1, \ldots, TA_n,\ FB\}$ is the first (upper) sequent in any given intuitionistic tableau-deduction of B from $A_1, \ldots, A_n$, we have that $A_1, \ldots, A_n \vdash_i B$.
The proof of (*) is tedious, but has a simple plan: the statement is true for the closed sequents at the bottom of an intuitionistic tableau-deduction, and the statement remains true if we go up in the intuitionistic tableau-deduction via the T and F rules.
Basic step: Any closed sequent in an intuitionistic tableau-deduction of B from $A_1, \ldots, A_n$ is of the form $\{TD_1, \ldots, TD_k,\ TP,\ FP,\ FE_1, \ldots, FE_m\}$. So, we have to

show that $D_1,\ldots,D_k,\ P\ \vdash_i P\vee E_1\vee\ldots\vee E_m$. And this is straightforward: $D_1,\ldots,D_k,\ P\vdash_i P$ and $P\vdash_i P\vee E_1\vee\ldots\vee E_m$.
Induction step: We have to show that for all rules the following is the case: if (*) holds for all lower sequent(s) in the rule (induction hypothesis), then (*) holds for the upper sequent in the rule. For convenience, we will suppose that $S=\{TD,\ FE\}$ in all rules.

Rule $T\rightarrow$:
$$\frac{TD,\ FE,\ T\ B\rightarrow C}{TD,\ FE,\ FB\ \mid\ TD,\ FE,\ TC}$$
Suppose $D\vdash_i E\vee B$ and $D,\ C\vdash_i E$ (induction hypothesis). To show: $D,\ B\rightarrow C\vdash_i E$.
Because $E\vee B,\ B\rightarrow C\vdash_i E\vee C$ (see Exercise 2.50), by the first induction hypothesis, $D,\ B\rightarrow C\vdash_i E\vee C$. (1)
From the second induction hypothesis, by the intuitionistic deduction theorem (see Exercise 8.4), $D\vdash_i C\rightarrow E$. (2)
Because $E\vee C,\ C\rightarrow E\vdash_i E\vee E$ (see Exercise 2.50), it follows from (1) and (2): $D,\ B\rightarrow C\vdash_i E\vee E$. But (by $\vee$-elimination) $E\vee E\vdash_i E$. Hence $D,\ B\rightarrow C\vdash_i E$.

Rule $F\rightarrow_i$:
$$\frac{TD,\ FE,\ F\ B\rightarrow C}{TD,\ TB,\ FC}$$
Suppose $D,\ B\vdash_i C$ (induction hypothesis). Then by the (intuitionistic) deduction theorem (see Exercise 8.4), $D\vdash_i B\rightarrow C$ and hence $D\vdash_i E\vee(B\rightarrow C)$, what was to be shown.

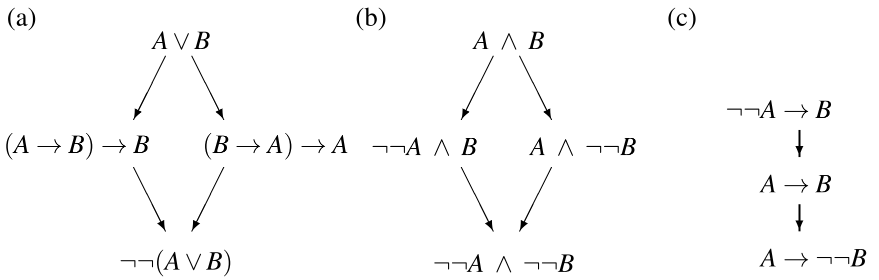
The other tableaux rules are treated similarly, see Theorem 2.27. Notice that the proof for rule $F\rightarrow$ with $S=\{TD,FE\}$ instead of $S_T=\{TD\}$ in rule $F\rightarrow_i$, would not intuitionistically hold anymore: from $D,B\vdash_i E\vee C$ it does not follow that $D\vdash_i E\vee(B\rightarrow C)$, since in order to show the latter one needs the assumption $B\vee\neg B$. □

Exercise 8.6. a) Show that all axioms for intuitionistic propositional logic (see Section 8.2) are tableau-provable (in intuitionistic propositional logic). b) verify that:
1) $\vdash_i' A\rightarrow\neg\neg A$, but trying to show $\vdash_i' \neg\neg A\rightarrow A$ fails;
2) $\vdash_i' \neg\neg(A\vee\neg A)$, but trying to show $\vdash_i' A\vee\neg A$ fails;
3) $\neg A\vee B\ \vdash_i'\ A\rightarrow B$, but trying to show $A\rightarrow B\ \vdash_i'\ \neg A\vee B$ fails.
4) $\neg A\vee\neg B\ \vdash_i' \neg(A\wedge B)$, but trying to show $\neg(A\wedge B)\ \vdash_i'\ \neg A\vee\neg B$ fails.
5) $\vdash_i' \neg\neg\neg A\rightarrow\neg A$.
c) Check that it is not possible to construct an intuitionistic tableau-deduction of B from $A\rightarrow B$ and $\neg A\rightarrow B$; the intuitive reason for this is, of course, that $A\vee\neg A$ does not hold intuitionistically.

Exercise 8.7. Prove: $\vdash'\ (P\rightarrow Q)\vee(Q\rightarrow P)$ classically, but not intuitionistically.

Exercise 8.8. Prove the *Disjunction Property* for intuitionistic propositional logic: if $\vdash_i' B\vee C$, then $\vdash_i' B$ or $\vdash_i' C$. Notice that the corresponding statement for classical propositional logic does not hold.

Exercise 8.9. Show that the implications in the following diagrams all hold intuitionistically, but not in the converse direction. Note that the formulas in each diagram are classically equivalent.



Exercise 8.10. Show that the following pairs of formulas are intuitionistically equivalent. 1) $\neg\neg A \wedge \neg\neg B$ and $\neg\neg(A \wedge B)$; 2) $\neg\neg A \to \neg\neg B$, $\neg\neg(A \to B)$ and $A \to \neg\neg B$.

A is *stable* := $\vdash_i' \neg\neg A \to A$. So 1) and 2) above say that if both A and B are stable, then $A \wedge B$ and $A \to B$ are stable too.
3) Show that $\neg A$ is stable for each formula A.
4) For $A \underline{\vee} B := \neg(\neg A \wedge \neg B)$ show that $A \underline{\vee} B$ is stable, if both A and B are stable.
5) Show that $\neg\neg A \vee \neg\neg B$ and $\neg\neg(A \vee B)$ are *not* intuitionistically equivalent and hence we *cannot* conclude: if A and B are stable, then $A \vee B$ is stable too.

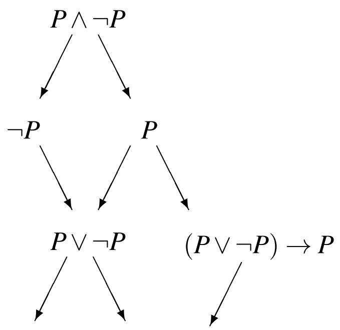
Exercise 8.11. A is *decidable* := $\vdash_i' A \vee \neg A$.
1) Prove that $\vdash_i' (A \vee \neg A) \to (\neg\neg A \to A)$. Hence, if A is decidable, then A is stable (see Exercise 8.10).
2) Prove that not $\vdash_i' (\neg\neg A \to A) \to (A \vee \neg A)$. Find a formula B which is stable such that we cannot say that it is decidable. (Hint: see Exercise 8.10, 3.)

Exercise 8.12. Let E^* come from E by replacing (or 'translating') each part of E of the form shown below in the first line by the respective expression shown in the second.

	P	$A \to B$	$A \wedge B$	$A \vee B$	$\neg A$
second.	$\neg\neg P$	$A \to B$	$A \wedge B$	$\neg(\neg A \wedge \neg B)$	$\neg A$

1. Note that E^* is stable, i.e., $\vdash_i' \neg\neg E^* \to E^*$, for each formula E (see Exercise 8.10).
2. Using $\vdash_i \neg\neg E^* \to E^*$, prove that *classical propositional logic can be defined within the intuitionistic one*: if $A_1, \ldots, A_n \vdash B$ (classically), then $A_1^*, \ldots, A_n^* \vdash_i B^*$.

Exercise 8.13. Show that no two of the formulas $P \to P$, $P \wedge \neg P$, P, $\neg P$, $P \vee \neg P$, $(P \vee \neg P) \to P$ are intuitionistically equivalent. Confer this with the classical case where each formula built from only one atomic formula P is equivalent to either $P \to P$, $P \wedge \neg P$, P or $\neg P$ (see Exercise 2.14). In fact, the formulas mentioned above are just the initial formulas of an infinite list of formulas built from only one atomic formula P such that 1) no two of the formulas in the list are intuitionistically equivalent to each other, and 2) any formula built from only one atomic formula P is equivalent (intuitionistically) to one of the formulas in the list. See I. Nishimura [12].



Exercise 8.14. * Let A be a formula and let Γ be a set of formulas of intuitionistic propositional logic. $\Gamma \mid A$ is defined by induction on A as follows.
$\Gamma \mid P$:= $\Gamma \vdash_i P$.
$\Gamma \mid B \vee C$:= $\Gamma \mid\!\vdash B$ or $\Gamma \mid\!\vdash C$ where $\Gamma \mid\!\vdash A := \Gamma \mid A$ and $\Gamma \vdash_i A$.
$\Gamma \mid B \wedge C$:= $\Gamma \mid B$ and $\Gamma \mid C$.
$\Gamma \mid B \to C$:= if $\Gamma \mid\!\vdash B$, then $\Gamma \mid C$.
$\Gamma \mid \neg B$:= if $\Gamma \mid\!\vdash B$, then Γ is inconsistent.
Prove the following *theorem* (S.C. Kleene, 1962): if $\Gamma \mid C$ for all formulas C in Γ and $\Gamma \vdash_i A$, then $\Gamma \mid A$. Conclude the following *corollary*: If $H \mid H$ and $H \vdash_i B \vee C$, then $H \vdash_i B$ or $H \vdash_i C$.

Exercise 8.15. * Prove the following intuitionistic analogue of Theorem 2.18, introduced by S.A. Kripke (oral communication, August, 1977). Every consistent formula A is intuitionistically provably equivalent to a disjunction $A_1 \vee \ldots \vee A_n$, where each A_j, $1 \leq j \leq n$, is a consistent conjunction of atomic formulas, of negations $\neg B$, where not $A_j \vdash_i B$, and of implications $B \to C$, where not $A_j \vdash_i B$; hence for each such A_j, $A_j \mid A_j$ (see Exercise 8.14). Hint: if $A_j = (B \to C) \wedge A'_j$ and $A_j \vdash_i B$, then $\vdash_i A_j \rightleftarrows C \wedge A'_j$.

8.4 Intuitionistic Propositional Logic: Semantics

In this section we define a notion of *intuitionistically (Kripke-)valid consequence*, $A_1, \ldots, A_n \models_i B$, for intuitionistic propositional logic and we shall show that this (semantic) notion of valid consequence for intuitionistic propositional logic is equivalent to the (syntactic) notions of deducibility, $A_1, \ldots, A_n \vdash_i B$ and $A_1, \ldots, A_n \vdash'_i B$, for intuitionistic propositional logic.

In the intuitionistic tableaux rules (see Section 8.3) we interprete TA as: I have a proof of A, and FA as: I do not (yet) have a proof of A. Then, reading these rules from the top down, rule $F \to_i \dfrac{S,\ FB \to C}{S_T,\ TB,\ FC}$ may be interpreted as follows: if I am in a proof-situation (this notion is analogous to the notion of possible world in Chapter 6) in which I do not (yet) have a proof of $B \to C$, then there is a proof-situation accessible from the given one, in which I do have a proof of B without having a proof of C. The change from S to S_T, where S_T is the set of all T-signed formulas in S, is explained by noting that formulas of which I did not have a proof in the former situation may have been proved by me in the latter situation, while sentences once proved remain proved forever (an ideal mathematician does not forget); so F-signed formulas in S may not be copied down below the line.

In a similar way there is a shift of one proof-situation to another in the interpretation of rule $F\neg_i\ \frac{S,\ F\neg B}{S_T,\ TB}$, while in interpreting the other intuitionistic tableaux rules there is no such shift. For instance, rule $T\rightarrow_i\ \frac{S,\ T\ B\rightarrow C}{S,\ FB\ |\ S,\ TC}$ is read as follows: if I am in a (proof-)situation in which I have a proof of $B \rightarrow C$, then in that same (proof-)situation I may not have a proof of B or I may have a proof of C. So, the intuitionistic tableaux rules in which there is a shift from S to S_T (the rules which have a bar in it) are precisely the rules the interpretation of which requires a shift from a former to a later proof-situation.

These considerations form the basis for S.A. Kripke's semantics for intuitionistic logic; see Kripke [11]. In fact, this semantics has grown out of the possible world semantics for modal logics (see Chapter 6) developed by Kripke [10] two years earlier in 1963. And although E.W. Beth [2] in 1947 had already developed a semantics for intuitionistic logic which is very close to the later Kripke-semantics, the latter one has become more popular because it is easier to work with.

Definition 8.11 (Kripke model). $M = \langle S,\ R,\ \models_i \rangle$ is a *Kripke model* for intuitionistic propositional logic :=
1. S is a non-empty set; the elements of S are called possible proof-situations.
2. R is a binary relation on S, which is *reflexive*, i.e., for all s in S, sRs, and *transitive*, i.e., for all $s,\ s',\ s''$ in S, if sRs' and $s'Rs''$, then sRs''. sRs' is read as: the proof situation s' is accessible from and later in time than the proof-situation s; R is called the *accessibility relation*.
3. $\models_i$ is a relation between the elements of S and the atomic formulas (of intuitionistic propositional logic) such that for all $s,\ s'$ in S, if $s \models_i P$ and sRs', then $s' \models_i P$. This condition is evident if one reads $s \models_i P$ as: P has been proved in the proof-situation s; once proved, it remains proved.

Definition 8.12 ($M, s \models_i A$)**.** Given a Kripke model $M = \langle\ S,\ R,\ \models_i\ \rangle$, we define $M, s \models_i A$, to be read as: A has been proved (or holds) in the situation s of the model M, for arbitrary s in S and for arbitrary formulas A as follows:

$M, s \models_i P$:= $s \models_i P$ (P atomic)
$M, s \models_i B \wedge C$:= $M, s \models_i B$ and $M, s \models_i C$
$M, s \models_i B \vee C$:= $M, s \models_i B$ or $M, s \models_i C$
$M, s \models_i B \rightarrow C$:= for all t in S, if sRt, then not $M, t \models_i B$ or $M, t \models_i C$,
or, equivalently, if sRt and $M, t \models_i B$, then $M, t \models_i C$,
$M, s \models_i \neg B$:= for all t in S, if sRt, then not $M, t \models_i B$.

Note that the definition of $M, s \models_i \neg B$ results from the one of $M, s \models_i B \rightarrow C$ by identifying $\neg B$ with $B \rightarrow \bot$ and postulating that for all s in S, not $M, s \models_i \bot$ ($\bot$ is the so-called false formula).

Lemma 8.1. *Let $M = \langle\, S,\ R,\ \models_i\, \rangle$ be a Kripke model, s and t elements of S and A a formula. If $M,s \models_i A$ and sRt, then $M,t \models_i A$.*

Proof. For atomic formulas P this follows immediately from condition 3 in Definition 8.11. Now suppose that the lemma has been proved for the formulas B and C (induction hypothesis), i.e., a) if $M,s \models_i B$ and sRt, then $M,t \models_i B$, and b) if $M,s \models_i C$ and sRt, then $M,t \models_i C$.
Then for $A = B \wedge C$ and $A = B \vee C$ the induction proposition follows immediately from the definition of $M,s \models_i A$ and from a) and b).
Now suppose $A = B \to C$, $M,s \models_i B \to C$ and sRt. We have to show that $M,t \models_i B \to C$, i.e., for all t' in S, if tRt' and $M,t' \models_i B$, then $M,t' \models_i C$. So suppose tRt' and $M,t' \models_i B$. Then since sRt and tRt', by the transitivity of R, sRt' and hence, since $M,s \models_i B \to C$ and $M,t' \models_i B$, it follows that $M,t' \models_i C$.
The case $A = \neg B$ is treated similarly. □

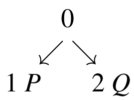
Definition 8.13 **($A_1,\ldots,A_n \models_i B$).**
1. Let $M = \langle\, S,\ R,\ \models_i\, \rangle$ be a Kripke model and A a formula. M is an *intuitionistic model* of A (or *A is true (intuitionistically) in M*) := for all s in S, $M,s \models_i A$.
Notation: $M \models_i A$. Otherwise M is called an *intuitionistic countermodel for A* (or an *intuitionistic counterexample to A*). **Notation**: $M \not\models_i A$.
2. A is *intuitionistically (Kripke-)valid* := for all Kripke models M, $M \models_i A$.
Notation: $\models_i A$.
3. B is an *intuitionistically (Kripke-)valid consequence* of $A_1,\ldots,A_n$:= for all Kripke models $M = \langle\, S,\ R,\ \models_i\, \rangle$ and for all s in S, if for all $j = 1,\ldots,n$, $M,s \models_i A_j$, then $M,s \models_i B$. **Notation**: $A_1,\ldots,A_n \models_i B$.
Note that $A_1,\ldots,A_n \models_i B$ iff $\models_i A_1 \wedge \ldots \wedge A_n \to B$.

Example 8.3. Let $M = \langle \{0,1\},\ R,\ \models_i\, \rangle$ be the intuitionistic (Kripke) model with $\{0,1\}$ being the set consisting of the natural numbers 0 and 1 only, R being defined by $0R0$, $1R1$ and $0R1$ (and not $1R0$), and $\models_i$ being defined by $1 \models_i P$ (and not $0 \models_i P$). This model can be completely characterized by the following picture:

$$\begin{array}{c} 0 \\ \downarrow \\ 1\ P \end{array}$$

Note that not $M,0 \models_i P$ and not $M,0 \models_i \neg P$ and hence that not $M,0 \models_i P \vee \neg P$. So M is an intuitionistic counterexample to $P \vee \neg P$ and therefore $\not\models_i P \vee \neg P$. Also notice that for this Kripke model M, for any formula A, $M \models_i A$ iff $M,0 \models_i A$.
Notice that $M,0 \not\models_i \neg P$ and $M,1 \not\models_i \neg P$ and therefore $M,0 \models_i \neg\neg P$, while $M,0 \not\models_i P$. So, M is an intuitionistic countermodel to $\neg\neg P \to P$. Hence $\neg\neg P \to P$ is not intuitionistically valid, i.e., $\not\models_i \neg\neg P \to P$.

Example 8.4. Let $M = \langle \{0,1,2\}, R,\ \models_i\, \rangle$ be the intuitionistic (Kripke) model with $\{0,1,2\}$ being the set consisting of the natural number 0, 1 and 2, R being defined by $0R0$, $1R1$, $2R2$; $0R1$, $0R2$ (and not $1R2$, not $2R1$, not $1R0$, not $2R0$), and $\models_i$ being defined by $1 \models_i P$, $2 \models_i Q$ (and not $1 \models_i Q$, not $2 \models_i P$, not $0 \models_i P$, not $0 \models_i Q$). This model can be completely characterized by the following picture:



Note that $M, 0 \models_i \neg(P \wedge Q)$, but not $M, 0 \models_i \neg P$ and not $M, 0 \models_i \neg Q$. Hence, $M, 0 \not\models_i \neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$. So M is a counterexample to $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$ and therefore $\not\models_i \neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$. Also notice that for this Kripke model M, for any formula A, $M \models_i A$ iff $M, 0 \models_i A$.
Notice that $M, 0 \not\models_i P \rightarrow Q$ and $M, 0 \not\models_i Q \rightarrow P$. Therefore, M is an intuitionistic (Kripke) counterexample to $(P \rightarrow Q) \vee (Q \rightarrow P)$. Hence, $\not\models_i (P \rightarrow Q) \vee (Q \rightarrow P)$.

The reader can easily check that, for instance, $P \rightarrow \neg\neg P$ and $\neg P \vee \neg Q \rightarrow \neg(P \wedge Q)$ are Kripke valid, i.e., true in all intuitionistic Kripke models, in particular in the Kripke models of Example 8.3 and 8.4. Let us prove this for the formula $P \rightarrow \neg\neg P$. We have to show that $M, s \models_i P \rightarrow \neg\neg P$ for all Kripke models $M = \langle\, S,\, R,\, \models_i \,\rangle$ and for all s in S. So suppose sRt and $M, t \models_i P$. Then we must prove that $M, t \models_i \neg\neg P$, i.e., for all t' in S, if tRt', then $M, t' \not\models_i \neg P$. So suppose tRt'. Then, since $M, t \models_i P$, it follows from Lemma 8.1 that $M, t' \models_i P$ and consequently that $M, t' \not\models_i \neg P$. □

In Theorem 8.1 we have shown: if $A_1, \ldots, A_n \vdash_i' B$, then $A_1, \ldots, A_n \vdash_i B$, i.e., any formula which is intuitionistically tableau-deducible from given premisses $A_1, \ldots, A_n$ is also intuitionistically deducible from these premisses (in terms of the intuitionistic logical axioms and Modus Ponens).
Now we shall show the *soundness theorem for intuitionistic propositional logic*: if $A_1, \ldots, A_n \vdash_i B$, then $A_1, \ldots, A_n \models_i B$, i.e., each formula which is intuitionistically deducible from given premisses $A_1, \ldots, A_n$ is also an intuitionistically (Kripke) valid consequence of these premisses.
In Section 8.5 we shall close the circle and prove the *completeness of intuitionistic propositional logic*, that is, the intuitionistic logical axioms together with Modus Ponens are complete with respect to the intuitionistic (Kripke) semantics, i.e., if $A_1, \ldots, A_n \models_i B$, then $A_1, \ldots, A_n \vdash_i' B$.

Theorem 8.2 (Soundness). *If $A_1, \ldots, A_n \vdash_i B$, then $A_1, \ldots, A_n \models_i B$.*

Proof. It is easy to see that every intuitionistic logical axiom is intuitionistically (Kripke) valid. Let us check the intuitionistic logical axioms $A \rightarrow (B \rightarrow A)$ and $\neg A \rightarrow (A \rightarrow B)$. So, let $M = \langle\, S,\, R,\, \models_i \,\rangle$ be an intuitionistic (Kripke) model.
1. To show: for all s in S, $M, s \models_i A \rightarrow (B \rightarrow A)$. So, suppose sRt and $M, t \models_i A$ (1). Then we have to show that $M, t \models_i B \rightarrow A$. So suppose tRt' (2) and $M, t' \models_i B$. To show: $M, t' \models_i A$. This follows from (1), (2) and Lemma 8.1.
2. To show: for all s in S, $M, s \models_i \neg A \rightarrow (A \rightarrow B)$. So, suppose sRt and $M, t \models_i \neg A$, i.e., for all t' in S, if tRt', then $M, t' \not\models_i A$. Therefore, for all t' in S with tRt', if $M, t' \models_i A$, then $M, t' \models_i B$, i.e., $M, t \models_i A \rightarrow B$.
3. Next we have to show that Modus Ponens is sound with respect to the intuitionistic Kripke semantics, i.e., if $M, s \models_i A$ (1) and $M, s \models_i A \rightarrow B$ (2), then $M, s \models_i B$. So suppose (1) and (2). We have to show that $M, s \models_i B$. This follows from (1), sRs and (2). □

Exercise 8.16. Prove:
a) If M_1 is a Kripke counterexample to B and M_2 is a Kripke counterexample to C, then from M_1 and M_2 one can construct a Kripke counterexample to $B \vee C$.
b) Conclude from a): if $\models_i B \vee C$ ($B \vee C$ is intuitionistically Kripke valid), then $\models_i B$ or (in its classical meaning) $\models_i C$ (B is Kripke valid or C is Kripke valid).

8.5 Completeness of Intuitionistic Propositional Logic

We shall prove completeness of intuitionistic propositional logic, i.e., that any intuitionistically (Kripke-)valid consequence of given premisses may be logically deduced by the intuitionistic tableaux rules from those premisses: if $A_1, \ldots, A_n \models_i B$, then $A_1, \ldots, A_n \vdash_i' B$ (Theorem 8.5).

In order to prove completeness of intuitionistic propositional logic, we define a *procedure to construct a counterexample* to a given conjecture that $A_1, \ldots, A_n \vdash_i' B$ with the following property: if the procedure fails, i.e., does not yield a Kripke counterexample, we have in fact constructed an intuitionistic tableau-deduction of B from $A_1, \ldots, A_n$. The procedure makes use of the tableaux rules and produces 'trees' which we shall call *(intuitionistic) search trees*.

Definition 8.14 (Procedure to construct a counterexample). In order to construct a (Kripke-)counterexample to the conjecture that $A_1, \ldots, A_n \vdash_i' B$, we must construct an intuitionistic Kripke model M such that for some proof situation s in M, $M, s \models_i A_1 \wedge \ldots \wedge A_n$, but $M, s \not\models_i B$.

Step 1: Start with $\{TA_1, \ldots, TA_n, FB\}$ and apply all intuitionistic tableaux rules for the propositional connectives, except the rules $F \rightarrow_i$ and $F\neg_i$, as frequently as possible. However, in case one of the split-rules $T \rightarrow$, $T\vee$ and $F\wedge$ is applied, we make two search trees: one with the left split and one with the right split. Notice that for an intuitionistic tableau-deduction both search trees have to close.

For instance, consider the conjecture $\neg(P \wedge Q) \vdash_i \neg P \vee \neg Q$:

search tree (1)	search tree (2)
$T\ \neg(P \wedge Q),\ F\ \neg P \vee \neg Q$	$T\neg(P \wedge Q),\ F\ \neg P \vee \neg Q$
$F\ P \wedge Q,\ F\ \neg P \vee \neg Q$	$F\ P \wedge Q,\ F\ \neg P \vee \neg Q$
$F\ P \wedge Q,\ F\ \neg P,\ F\neg Q$	$F\ P \wedge Q,\ F\ \neg P,\ F\ \neg Q$
$FP,\ F\ \neg P,\ F\neg Q$	$FQ,\ F\ \neg P,\ F\ \neg Q$

In the transition from the third to the fourth line we apply the rule $F\wedge$ to $F\ P \wedge Q$, which causes a split. At that stage we make two search trees, one with the left split signed formula FP and one with the right split signed formula FQ. One continues to apply all possible rules, except the $F\neg_i$ and $F \rightarrow_i$ rules, as frequently as possible.

At this stage we have partially constructed one, two (or more) search trees, each consisting of one node labeled with signed formulas. A labeled node s in which all tableaux rules except the $F\neg_i$ and $F \rightarrow_i$ rules have been applied as frequently as possible will be called *logically complete*. Intuitively, this means that one has fully described which formulas have been proved and which formulas have not (yet) been proved in the present proof situation s. Next we continue to expand each search tree by one or more applications of the $F\neg_i$ and $F \rightarrow_i$ rules.

Step 2 Each labeled node s in a search tree τ which is logically complete may contain one or more signed formulas of the form $F\ \neg B$ or $F\ B \rightarrow C$. For each of the signed formulas of the form $F\ \neg B$ and $F\ B \rightarrow C$ in a labeled node s we construct a new node s', declare s' accessible from s in the given search tree τ, i.e., $sR_\tau s'$, and label this node s' with the formulas $S_T,\ FB$ or $S_T,\ TB,\ FC$ respectively, which result from applying the rule $F\neg_i$ to $S,\ F\ \neg B$ or the rule $F \rightarrow_i$ to $S,\ F\ B \rightarrow C$, respectively. It is important to copy all T-signed formulas occurring in s to the new node s' (formulas once proved remain proved). Notice that F-signed formulas that occur in labeled node s may not occur anymore in node s' and that for closure it suffices that one of the successor nodes contains TA and FA for some formula A.

Next we apply step 1 again, but now starting with $S_T,\ FB$ or $S_T,\ TB,\ FC$, depending on whether rule $F\ \neg_i$ or $F \rightarrow_i$ has been applied, resulting in one or more logically complete nodes (proof situations) s'.

Step 1 and 2 are repeated as frequently as possible.

For search tree (1) above one may apply the $F\ \neg_i$ rule to $F\ \neg P$, losing all other F signed formulas, and one may apply the $F\ \neg_i$ rule to $F\ \neg Q$, again losing all other F signed formulas. Similarly for search tree (2) above.

search tree (1)

$T\ \neg(P \wedge Q),\ F\ \neg P \vee \neg Q$
$F\ P \wedge Q,\ F\ \neg P \vee \neg Q$
$F\ P \wedge Q,\ F\ \neg P,\ F\neg Q$
$FP,\ F\ \neg P,\ F\neg Q$

/ \

$T\ \neg(P \wedge Q),\ TP$ | $T\ \neg(P \wedge Q),\ TQ$
$F\ P \wedge Q,\ TP$ | $F\ P \wedge Q,\ TQ$

search tree (2)

$T\neg(P \wedge Q),\ F\ \neg P \vee \neg Q$
$F\ P \wedge Q,\ F\ \neg P \vee \neg Q$
$F\ P \wedge Q,\ F\ \neg P,\ F\ \neg Q$
$FQ,\ F\ \neg P,\ F\ \neg Q$

/ \

$T\ \neg(P \wedge Q),\ TP$ | $T\ \neg(P \wedge Q),\ TQ$
$F\ P \wedge Q,\ TP$ | $F\ P \wedge Q,\ TQ$

Application of rule $F\wedge$ to search tree (1) yields four different search trees; three of them are closed, i.e. contain a branch that is closed (a branch is *closed* if it contains TA and FA for some formula A), but one of them, search tree τ below, is not closed:

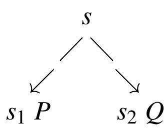
search tree τ

$T\ \neg(P \wedge Q),\ F\ \neg P \vee \neg Q$
$F\ P \wedge Q,\ F\ \neg P \vee \neg Q$
$F\ P \wedge Q,\ F\ \neg P,\ F\neg Q$
$FP,\ F\ \neg P,\ F\neg Q$

/ \

$T\ \neg(P \wedge Q),\ TP$ | $T\ \neg(P \wedge Q),\ TQ$
$F\ P \wedge Q,\ TP$ | $F\ P \wedge Q,\ TQ$
$FQ,\ TP$ | $FP,\ TQ$

Kripke model M_τ



The open (i.e., not closed) search tree τ yields a (Kripke-)counterexample M_τ to the conjecture that $\neg(P \wedge Q) \vdash_i' \neg P \vee \neg Q$, as depicted in the right column above.

The Kripke model $M_\tau = \langle\{s, s_1, s_2\}, R_\tau, \models_i\rangle$ is defined as follows: $sR_\tau s_1$, $sR_\tau s_2$, $s_1 \models_i P$, corresponding with the fact that TP occurs in s_1, and $s_2 \models_i Q$ corresponding with the fact that TQ occurs in s_2. Clearly, $M_\tau, s \models_i \neg(P \wedge Q)$, corresponding with the fact that $T\ \neg(P \wedge Q)$ occurs in s, but $M_\tau, s \not\models_i \neg P$ and $M_\tau, s \not\models_i \neg Q$, corresponding with the fact that $F\ \neg P$, respectively $F\ \neg Q$, occurs in s.

Definition 8.15 (Search tree).
A search tree τ for the conjecture $A_1, \ldots, A_n \vdash_i' B$ is a set of nodes, labeled with signed formulas, with a relation R_τ between the nodes, such that:
0. The upper node contains $TA_1, \ldots, TA_n, FB$.
1. $sR_\tau s' := s' = s$ or s' is an immediate successor of s, i.e., s' results from applying the rule $F\ \neg_i$ or $F \rightarrow_i$ to a formula in s of the form $F\ \neg C$, respectively $F\ C \rightarrow D$.
2. For each node s in the search tree τ:
a) if TP occurs in s and $sR_\tau s'$, then TP occurs in s'.
b) if $T\ C \rightarrow D$ occurs in s, then for all s' in τ, if $sR_\tau s'$, then FC occurs in s' **or** TD occurs in s';
c) if $F\ C \rightarrow D$ occurs in s, then there is a node s' in τ with $sR_\tau s'$, TC occurs in s' **and** FD occurs in s';
d) if $T\ C \wedge D$ occurs in s, then TC occurs in s **and** TD occurs in s;
e) if $F\ C \wedge D$ occurs in s, then FC occurs in s **or** FD occurs in s;
f) if $T\ C \vee D$ occurs in s, then TC occurs in s **or** TD occurs in s.
g) if $F\ C \vee D$ occurs in s, then FC occurs in s **and** FD occurs in s;
h) if $T\ \neg C$ occurs in s, then for every node s' in τ, if $sR_\tau s'$, then FC occurs in s';
i) if $F\ \neg C$ occurs in s, then there is a node s' in τ with $sR_\tau s'$ and TC occurs in s'.

Definition 8.16 (Closed branch; closed search tree).
a) A branch in a search tree τ is *closed* if it contains at least one node labeled with TA and FA for some formula A. Otherwise, the branch is called *open*.
b) A search tree τ is *closed* if it contains at least one closed branch. Otherwise, the search tree is called *open*.

Theorem 8.3. *Let τ be an open search tree for the conjecture $A_1, , \ldots, A_n \vdash_i' B$ with upper node s_0. Let S_τ the set of nodes in τ and let R_τ be defined as in Definition 8.15. Define $s \models_i P :=$ TP occurs in s. Then $M_\tau = \langle\ S_\tau, R_\tau, \models_i\ \rangle$ is an (intuitionistic) Kripke countermodel to the conjecture that $A_1, \ldots, A_n \vdash_i' B$. More precisely, $M_\tau, s_0 \models A_1 \wedge \ldots \wedge A_n$, but $M_\tau, s_0 \not\models B$.*

Proof. Let τ be an open search tree with s_0 as upper node, containing $TA_1, \ldots, TA_n$, FB. Let $M_\tau = \langle\ S_\tau, R_\tau, \models_i\ \rangle$ be the corresponding Kripke model, as defined in the theorem. We shall prove by induction:
1) If TA occurs in s, then $M_\tau, s \models_i A$; and 2) If FA occurs in s, then $M_\tau, s \not\models_i A$.
Since $TA_1, \ldots, TA_n, FB$ occur in the top node s_0, it follows that $M_\tau, s_0 \models_i A_1 \wedge \ldots \wedge A_n$, but $M_\tau, s_0 \not\models_i B$. Therefore, $A_1, \ldots, A_n \not\models_i B$.
Induction basis Let $A = P$ be atomic. If TP occurs in s, then by definition $s \models_i P$, i.e., $M_\tau, s \models_i P$. If FP occurs in s, then - since τ is open - TP does not occur in s and hence by definition $s \not\models_i P$, i.e., $M_\tau, s \not\models_i P$.
Induction step Suppose 1) and 2) hold for C and D (induction hypothesis). We shall prove that 1) and 2) hold for $C \rightarrow D$, $C \wedge D$, $C \vee D$ and $\neg C$.
Let $A = C \rightarrow D$ and suppose $T\ C \rightarrow D$ occurs in s. Then according to Definition

8.15 b, for all s' in τ, if $sR_\tau s'$, then FC is in s' or TD is in s'. So, by the induction hypothesis, for all s' in τ, if $sR_\tau s'$, then $M_\tau, s' \not\models_i C$ or $M_\tau, s' \models_i D$. Consequently, $M_\tau, s \models_i C \to D$.
Let $A = C \to D$ and suppose $F\ C \to D$ occurs in s. Then according to Definition 8.15 c, there is a node s' in τ with $sR_\tau s'$ such that TC is in s' and FD is in s'. So, by the induction hypothesis, $M_\tau, s' \models_i C$ and $M_\tau, s' \not\models_i D$. Consequently, $M_\tau, s \not\models C \to D$.
The cases that $A = C \wedge D$, $A = C \vee D$ and $A = \neg C$ are treated similarly. □

Example 8.5. We wonder whether $\vdash_i' P \vee \neg P$. So, in the left column below we start a search tree τ beginning with $F\ P \vee \neg P$:

$$\begin{array}{c} F\ P \vee \neg P \\ FP,\ F\neg P \\ | \\ TP \end{array} \qquad\qquad \begin{array}{c} s \\ \downarrow \\ s'\ P \end{array}$$

In the application of rule $F\neg_i$ to $F\neg P$ the F-signed formulas are not be copied to the next proof-situation. We do not find a tableau-proof of $P \vee \neg P$. Instead we have actually constructed a search tree τ for $P \vee \neg P$ which is open, i.e., no node in it contains both TB and FB for some formula B; and from this open search tree one can read off an intuitionistic Kripke counterexample to $P \vee \neg P$, as is shown in the right-column above. Confer Example 8.3.

Example 8.6. We wonder whether $\vdash_i' (P \to Q) \vee (Q \to P)$. So, in the left column below we start a search tree τ beginning with $F\ (P \to Q) \vee (Q \to P)$:

$$\begin{array}{c} F\ (P \to Q) \vee (Q \to P) \\ F\ P \to Q,\ F\ Q \to P \\ \diagup \quad \diagdown \\ TP,\ FQ \qquad TQ,\ FP \end{array} \qquad\qquad \begin{array}{c} s \\ \swarrow \quad \searrow \\ s_1\ P \qquad s_2\ Q \end{array}$$

At the second line of the search tree in the left column above, we can apply rule $F \to_i$ to $F\ P \to Q$ losing the expression $F\ Q \to P$ or apply rule $F \to_i$ to $F\ Q \to P$ losing the expression $F\ P \to Q$. In neither case we find a tableau-proof of $(P \to Q) \vee (Q \to P)$. Instead we have actually constructed a search tree for $(P \to Q) \vee (Q \to P)$ which is open, i.e., no node in it contains both TB and FB for some formula B; and from this open search tree one can read off a Kripke counterexample to $(P \to Q) \vee (Q \to P)$. Confer Example 8.4.

Example 8.7. We wonder whether $P \vee \neg Q \vdash_i' (R \to P) \vee (Q \to R)$. Application of the procedure to construct a counterexample to this conjecture yields two different search trees which both turn out to be closed.

$$\begin{array}{l} T\ P \vee \neg Q,\ F\ (R \to P) \vee (Q \to R) \\ T\ P \vee \neg Q,\ F\ R \to P,\ F\ Q \to R \\ TP,\ T\ P \vee \neg Q,\ F\ R \to P,\ F\ Q \to R \\ \qquad\qquad \diagup \quad \diagdown \\ TP,\ TR,\ FP \qquad TP,\ TQ,\ FR \\ \text{closed} \end{array}$$

$$\begin{array}{l} T\ P \vee \neg Q,\ F\ (R \to P) \vee (Q \to R) \\ T\ P \vee \neg Q,\ F\ R \to P,\ F\ Q \to R \\ T\neg Q,\ T\ P \vee \neg Q,\ F\ R \to P,\ F\ Q \to R \\ \quad FQ,\ T\ P \vee \neg Q,\ F\ R \to P,\ F\ Q \to R \\ \qquad\qquad \diagup \quad \diagdown \\ T\neg Q,\ TR,\ FP \qquad T\neg Q,\ TQ,\ FR \\ \quad FQ,\ TR,\ FP \qquad FQ,\ TQ,\ FR \\ \qquad\qquad\qquad\qquad\quad \text{closed} \end{array}$$

Note that the two branches which give closure together yield an intuitionistic tableau-deduction of $(R \to P) \vee (Q \to R)$ from $P \vee \neg Q$:

$$T\ P \vee \neg Q,\ F\ (R \to P) \vee (Q \to R)$$
$$T\ P \vee \neg Q,\ F\ R \to P,\ F\ Q \to R$$

$T\ P \vee \neg Q,\ TP,\ \underline{F\ R \to P},\ F\ Q \to R$	$T\ P \vee \neg Q,\ T\ \neg Q,\ F\ R \to P,\ \underline{F\ Q \to R}$
$T\ P \vee \neg Q,\ TP,\ TR,\ FP$	$T\ P \vee \neg Q,\ T\ \neg Q,\ TQ,\ FR$
closure	$T\ P \vee \neg Q,\ FQ,\ TQ,\ FR$
	closure

The correctness of this remark is not accidental and follows immediately from the definition of a tableau-deduction and the structure of the procedure in Definition 8.14 to construct a counterexample.

Theorem 8.4. *If all search trees for the conjecture $A_1, \ldots, A_n \vdash_i' B$ are closed, i.e., contain closure in one of their branches, then $A_1, \ldots, A_n \vdash_i' B$.*

Proof. Suppose all search trees for the conjecture $A_1, \ldots, A_n \vdash_i' B$ are closed. Then it follows from the procedure to construct a counterexample to this conjecture that the closed branches together form an intuitionistic tableau-deduction of B from $A_1, \ldots, A_n$. □

Theorem 8.5 (Completeness). *If $A_1, \ldots, A_n \models_i B$, then $A_1, \ldots, A_n \vdash_i' B$.*

Proof. Suppose $A_1, \ldots, A_n \models_i B$. Apply the procedure to construct a counterexample to the conjecture $A_1, \ldots, A_n \vdash_i' B$. If one of the resulting search trees is open, say τ, then by Theorem 8.3, $M_\tau, s_0 \models_i A_1 \wedge \ldots \wedge A_n$, while $M_\tau, s_0 \not\models_i B$. This contradicts the assumption $A_1, \ldots, A_n \models_i B$. Hence, there can be no open search tree for the conjecture $A_1, \ldots, A_n \vdash_i' B$. That is, all search trees for this conjecture are closed. So, by Theorem 8.4, $A_1, \ldots, A_n \vdash_i' B$. □

The procedure to construct a counterexample to the conjecture $A_1, \ldots, A_n \vdash_i' B$ will stop after finitely many steps and then either yield an intuitionistic Kripke-counterexample or an intuitionistic tableau-deduction of B from $A_1, \ldots, A_n$. Therefore, intuitionistic propositional logic is *decidable*.

Theorem 8.6 (Decidability). *Intuitionistic propositional logic is decidable, i.e., there is a procedure to decide in finitely many steps whether $A_1, \ldots, A_n \vdash_i' B$.*

Proof. Given a conjecture $A_1, \ldots, A_n \vdash_i' B$, the procedure (in Definition 8.14) to construct a counterexample yields only finitely many different search trees, each of which will be completed in a finite number of steps. If they all close, by Theorem 8.4 they actually give us a tableau-deduction of B from $A_1, \ldots, A_n$, showing that $A_1, \ldots, A_n \vdash_i' B$ and hence $A_1, \ldots, A_n \models_i B$; if one of them is open, by Theorem 8.3 it actually yields a Kripke counterexample to the given conjecture, showing that $A_1, \ldots, A_n \not\models_i B$ and hence $A_1, \ldots, A_n \not\vdash_i' B$. □

Note that while the decision procedure for $A_1, \ldots, A_n \models B$ in classical propositional logic was immediately evident from the definition of $A_1, \ldots, A_n \models B$ (the truth table of B has 1 in all lines in which all of $A_1, \ldots, A_n$ are 1), this is not so for the notion of Kripke valid consequence ($A_1, \ldots, A_n \models_i B$) in intuitionistic propositional logic.

Exercise 8.17. Construct either an intuitionistic tableau-proof or an intuitionistic Kripke counterexample for the following formulas (confer Exercise 8.1):
(a) $(P \to Q) \to (\neg Q \to \neg P)$ and $(\neg Q \to \neg P) \to (P \to Q)$.
(b) $\neg(P \wedge Q) \to (\neg P \vee \neg Q)$ and $(\neg P \vee \neg Q) \to \neg(P \wedge Q)$.
(c) $(P \to Q) \to \neg(P \wedge \neg Q)$ and $\neg(P \wedge \neg Q) \to (P \to Q)$.
(d) $(P \to Q) \to (\neg P \vee Q)$ and $(\neg P \vee Q) \to (P \to Q)$.
(e) $(P \to Q \vee R) \to (P \to Q) \vee (P \to R)$ and $(P \to Q) \vee (P \to R) \to (P \to Q \vee R)$.
(f) $(P \vee \neg P) \to (\neg\neg P \to P)$ and $(\neg\neg P \to P) \to (P \vee \neg P)$.

Exercise 8.18. * (S.A. Kripke, oral communication, 1977)
Let $M = \langle S,\ R,\ \models_i \rangle$ be defined as follows: S is the set of all formulas A of intuitionistic propositional logic such that $A \mid A$ (see Exercise 8.14). For $H,\ H'$ in S, let $HRH' := H' \vdash_i H$. For H in S let $H \models_i P := H \vdash_i P$ (P atomic).
Verify that $M = \langle S,\ R,\ \models_i \rangle$ is a Kripke model for intuitionistic propositional logic such that for all formulas A, $M, H \models_i A$ iff $H \vdash_i A$. Hint: use Exercise 8.14 and 8.15. As a corollary we have the *completeness theorem* for intuitionistic propositional logic: if $\models_i A$, then $\vdash_i A$.

8.6 Quantifiers in Intuitionism; Intuitionistic Predicate Logic

We have already seen in Section 8.1 that in classical mathematics the infinite – for instance, the set $\mathbb{N}$ of the natural numbers – is treated as actual or completed. On the other hand, since for an intuitionist mathematical objects are mental constructions, the set $\mathbb{N}$ of the natural numbers intuitionistically cannot be regarded as a completed totality, but only as potential or becoming or constructive. As explained in Section 8.1, the set $\mathbb{N}$ of the natural numbers is intuitionistically a construction project: start with 0 and for every natural number n already constructed earlier construct $n+1$. As a consequence the quantifiers have intuitionistically a meaning different from the classical one.

The classical meaning of *there exists an n such that* $P(n)$ ($\exists n[n \in \mathbb{N} \wedge P(n)]$), that somewhere in the completed infinite totality of the natural numbers there occurs an n such that $P(n)$, is not available to the intuitionist, since he does not conceive the set $\mathbb{N}$ of the natural numbers as a completed totality. The intuitionistic meaning of the proposition *there exists a natural number n such that* $P(n)$ is that one can *construct* a natural number n which one can prove has the property P. So, an intuitionistic proof of the proposition in question must be *constructive*, i.e., it must indicate a concrete natural number with the property P, or at least indicate a method by which one can construct such a natural number.

The intuitionistic meaning of *all natural numbers n have the property P* ($\forall n[n \in \mathbb{N} \to P(n)]$), or briefly *for all n,* $P(n)$, is the following: I have a method (construction), which applied to any natural number n provides a proof of $P(n)$. Note that the classical concept of a completed infinity of the natural numbers does not occur in this intuitionistic interpretation of a universal quantification over the natural numbers.

Propositions of the form *for all natural numbers n, $P(n)$* may be proved intuitionistically by using the principle of mathematical induction: if (1) $P(0)$ and (2) for all $n \in \mathbb{N}$, if $P(n)$, then $P(n+1)$, then for all n, $P(n)$. In order to arrive at an intuitionistic proof of the proposition *for all natural numbers n, $P(n)$*, the proofs of both the induction basis (1) and the induction step (2) should, of course, be intuitionistic too.

That intuitionistic methods are to be distinguished from non-intuitionistic ones is explained by S.C. Kleene [8] as follows:

> In classical mathematics there occur *non-constructive* or *indirect* existence proofs, which the intuitionists do not accept. For example, to prove *there exists an n such that $P(n)$*, the classical mathematician may deduce a contradiction from the assumption *for all n, not $P(n)$*. Under both the classical and the intuitionistic logic, by reductio ab absurdum this gives *not for all n, not $P(n)$*. The classical logic allows this result to be transformed into *there exists an n such that $P(n)$*, but not (in general) the intuitionistic. Such a classical existence proof leaves us no nearer than before the proof was given to having an example of a number n such that $P(n)$ (though sometimes we may afterwards be able to discover one by another method). [S.C. Kleene [8], p. 49]

Intuitionistic methods are to be distinguished from non-intuitionistic ones not only in the case of proofs, but also in the case of definitions. For instance, suppose one can show that the number 3 has a given property P if Goldbach's conjecture (G) is true and that if Goldbach's conjecture (G) is false, then the number 5 has the property P. From a classical point of view one may say that one has shown the existence of a natural number n with the property P. But because Goldbach's conjecture is an open, i.e., not solved, problem, from an intuitionistic point of view no construction of such a natural number n has been given. Neither 3 nor 5 is an example as long as Goldbach's conjecture has not been solved. From an intuitionistic point of view one has only proved the implication *if G or not G, then there exists an n such that $P(n)$*, where G is Goldbach's conjecture. From a classical point of view, the premiss $G \vee \neg G$ of this implication is available and consequently from a classical point of view one may infer its conclusion that there is a natural number n with the property P. However, in the present state of knowledge, an intuitionist does not accept the principle *G or not G* and hence *the number n which is equal to 3 if G, and equal to 5 if not G* is intuitionistically not a valid definition of a natural number n: one has no method to construct this natural number.

We have just seen that the quantifiers in intuitionism have a meaning quite different from their classical one. Let V be an (intuitionistic) set. $\forall x \in V[A(x)]$ (for every x in V, $A(x)$) means intuitionistically: I have a construction which assigns to each object a in V a proof of $A(a)$. And $\exists x \in V[A(x)]$ (for some x in V, $A(x)$) means intuitionistically: I can construct an object a in V and give a proof(-construction) of $A(a)$.

The reader should verify for himself that for the intuitionistic quantifiers we still have $\neg\exists x \in V[A(x)] \rightleftarrows \forall x \in V[\neg A(x)]$, and also $\exists x \in V[\neg A(x)] \to \neg\forall x \in V[A(x)]$, but not anymore the converse, $\neg\forall x \in V[A(x)] \to \exists x \in V[\neg A(x)]$: from the assumption that $\neg\forall x \in V[A(x)]$, i.e., that $\forall x \in V[A(x)]$ yields a contradiction, one can in general not construct a particular element x in V such that $\neg A(x)$. An intuitionistic (weak) counterexample to $\neg\forall x \in V[A(x)] \to \exists x \in V[\neg A(x)]$ is obtained as follows:

Let $C(k)$:= in the decimal expansion of π a sequence of the form 0 1 2 3 4 5 6 7 8 9 occurs before the k^{th} decimal. And let $\rho = 0.\ a_1\ a_2\ a_3 \ldots$, where $a_k = 3$ if $\neg C(k)$, and $a_k = 0$ if $C(k)$. Then if $\rho \neq \frac{1}{3}$, i.e., $\rho \neq 0.333\ldots$, then $\neg\forall k \in \mathbb{N}[\neg C(k)]$, and if $\rho \neq 0.33\ldots\ 0\ 0\ 0\ldots$, then $\neg\exists k \in \mathbb{N}[C(k)]$.

Let $\mathbb{Q}$ be the set of all rational numbers. Then
(1) $\neg\forall x \in \mathbb{Q}[x \neq \rho]$, i.e., it is not the case that ρ is irrational; for if ρ is irrational, then both $\rho \neq \frac{1}{3}$ and $\rho \neq 0.33\ldots\ 000\ldots$ and therefore both $\neg\forall k \in \mathbb{N}[\neg C(k)]$ and $\neg\exists k \in \mathbb{N}[C(k)]$ or equivalently $\forall k \in \mathbb{N}[\neg C(k)]$; contradiction.
(2) But it is reckless to assume that $\exists x \in \mathbb{Q}[\neg(x \neq \rho)]$: for indicating a rational number equal to ρ implies $\forall k \in \mathbb{N}[\neg C(k)] \vee \exists k \in \mathbb{N}[C(k)]$, or equivalently, $\neg\exists k \in \mathbb{N}[C(k)] \vee \exists k \in \mathbb{N}[C(k)]$, which clearly is a reckless statement since it states that I know whether in the decimal expansion of π a sequence of the form 0 1 2 3 4 5 6 7 8 9 occurs or not.
From (1) and (2) it follows that $\neg\forall x \in \mathbb{Q}[x \neq \rho] \to \exists x \in \mathbb{Q}[\neg(x \neq \rho)]$ is reckless. Note that $\neg\forall x \in V[A(x)] \to \exists x \in V[\neg A(x)]$ is a generalization of $\neg(P \wedge Q) \to \neg P \vee \neg Q$, of which we have already seen in Section 8.5 that it was intuitionistically reckless.

We are not able to give an intuitionistic proof of $\forall x \in V[\neg\neg A(x)] \to \neg\neg\forall x \in V[A(x)]$. In Section 8.7 we shall show that this formula does not hold intuitionistically in the case that $V = \{0,1\}^{\mathbb{N}}$: Let $A(\alpha)$ express that α is the infinite sequence consisting of only 0's, which we denote by $\alpha = \underline{0}$, i.e., $\forall n \in \mathbb{N}[\alpha(n) = 0]$. Then $\forall\alpha \in \{0,1\}^{\mathbb{N}}[\neg\neg(\alpha = \underline{0} \vee \alpha \neq \underline{0})]$, but $\neg\forall\alpha \in \{0,1\}^{\mathbb{N}}[\alpha = \underline{0} \vee \alpha \neq \underline{0}]$.
However, in the case that $V = \mathbb{N}$, whether $\forall x \in \mathbb{N}[\neg\neg A(x)] \to \neg\neg\forall x \in \mathbb{N}[A(x)]$ holds intuitionistically or not is still an open problem.

8.6.1 Deducibility for Intuitionistic Predicate Logic

The language of intuitionistic predicate logic is the same as the one for classical predicate logic (see Chapter 4), the difference being that the connectives and quantifiers now have another, constructive and intuitionistic meaning. For the sake of completeness we give the alphabet of predicate logic and mention the rules according to which the well formed expressions or formulas of predicate logic are formed.

Definition 8.17 (Alphabet for Predicate Logic).
The alphabet for (intuitionistic) predicate logic consists of the following symbols:
individual constants: $c_1, c_2, c_3, \ldots$
predicate symbols: $P_1, P_2, P_3, \ldots$, where P_i is supposed to be n_i-ary, i.e., taking n_i arguments.
free individual variables: $a_1, a_2, a_3, \ldots$; bound individual variables: $x_1, x_2, x_3, \ldots$
connectives: $\rightleftarrows$, $\to$, $\wedge$, $\vee$, $\neg$; quantifiers: $\exists$, $\forall$
brackets: (,), [,]

We shall use a, b to range over free individual variables, x, y, z to range over bound individual variables, and P, Q, R to range over predicate symbols.

Definition 8.18 (Formulas of Predicate Logic).
If P is a n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, i.e., individual constants or free individual variables, then $P(t_1, \ldots, t_n)$ is an (atomic) formula.
If B and C are formulas, then also $(B \rightleftarrows C)$, $(B \rightarrow C)$, $(B \wedge C)$, $(B \vee C)$ and $(\neg B)$ are formulas.
If $A(a)$ is a formula containing the free variable a, then $\forall x[A(x)]$ and $\exists x[A(x)]$ are formulas, where $A(x)$ results from $A(a)$ by replacing all occurrences of a by x.

A *Hilbert-type* proof-theoretic formulation of intuitionistic predicate logic is obtained by replacing axiom 8, $\neg\neg A \rightarrow A$ for classical predicate logic (see Section 4.4) by axiom 8^i, $\neg A \rightarrow (A \rightarrow B)$ (see Section 8.2). The other axioms and rules for the connectives and quantifiers remain unchanged and the reader should verify intuitively that they all hold true for the intuitionistic interpretation of $\rightleftarrows$, $\rightarrow$, $\wedge$, $\vee$, $\neg$, $\forall$ and $\exists$.

Definition 8.19 (Axioms and Rules for Intuitionistic Predicate Logic).
The (logical) axioms and rules for intuitionistic predicate logic consist of:
1. the axiom schemata for intuitionistic propositional logic, given in Section 8.2.
2. the axiom schemata for the quantifiers: $\forall x[A(x)] \rightarrow A(t)$ and $A(t) \rightarrow \exists x[A(x)]$, where t is a term, i.e., an individual constant or free individual variable.
3. the logical rules for $\rightarrow$, $\forall$ and $\exists$: $\dfrac{A \quad A \rightarrow B}{B}(MP)$, $\dfrac{C \rightarrow A(a)}{C \rightarrow \forall x[A(x)]}$, $\dfrac{A(a) \rightarrow C}{\exists x[A(x)] \rightarrow C}$, provided C does not contain a.

Definition 8.20 (Deduction; Deducible).
1. An intuitionistic (Hilbert-type) *deduction* of B *from* $A_1, \ldots, A_n$ (in predicate logic) is a finite list $B_1, \ldots, B_k$ of formulas, such that
(a) $B = B_k$ is the last formula in the list, and
(b) each formula in the list is either one of $A_1, \ldots, A_n$, or an axiom of intuitionistic predicate logic (i.e., an instance of one of the axiom schemata), or is obtained by application of one of the rules to formulas preceding it in the list, such that
(c) *all free variables of* $A_1, \ldots, A_n$ *are held constant*, i.e., the $\forall$-rule and the $\exists$-rule are not applied with respect to a free variable a occurring in $A_1, \ldots, A_n$, except preceding the first occurrence of $A_1, \ldots, A_n$ in the deduction.
2. B is intuitionistically *deducible from* $A_1, \ldots, A_n$:= there exists an intuitionistic (Hilbert-type) deduction of B from $A_1, \ldots, A_n$. **Notation**: $A_1, \ldots, A_n \vdash_i B$. The symbol $\vdash_i$ may be read 'yields intuitionistically'. $A_1, \ldots, A_n \nvdash_i B$ abbreviates: not $A_1, \ldots, A_n \vdash_i B$.
3. For Γ a (possibly infinite) set of formulas, B is intuitionistically *deducible from* Γ := there is a finite list $A_1, \ldots, A_n$ of formulas in Γ such that $A_1, \ldots, A_n \vdash_i B$. **Notation**: $\Gamma \vdash_i B$.

Example 8.8. $\exists x[\neg A(x)] \vdash_i \neg\forall x[A(x)]$. Proof: $\forall x[A(x)] \rightarrow A(a)$ is an (intuitionistic) axiom. Hence, $\vdash_i \neg A(a) \rightarrow \neg\forall x[A(x)]$, and hence, by application of the rule for $\exists$, $\vdash_i \exists x[\neg A(x)] \rightarrow \neg\forall x[A(x)]$. Consequently, $\exists x[\neg A(x)] \vdash_i \neg\forall x[A(x)]$.
However, conversely, the classical deduction of $\exists x[\neg A(x)]$ from $\neg\forall x[A(x)]$ is not intuitionistically valid: $A(a) \rightarrow \exists x[A(x]$ is an axiom. Consequently, $\vdash \neg\exists x[A(x)] \rightarrow$

$\neg A(a)$ and hence $\vdash \neg\exists x[A(x)] \to \forall x[\neg A(x)]$ and $\vdash \neg\forall x[\neg A(x)] \to \neg\neg\exists x[A(x)]$. Now classically, but not intuitionistically, one may remove the double negation signs, obtaining $\vdash \neg\forall x[\neg A(x)] \to \exists x[A(x)]$ classically, but not intuitionistically.

Without proof we mention here the following generalization of Exercise 8.12, that *classical predicate logic can be defined within intuitionistic predicate logic*. For a proof of this theorem we refer the reader to S.C. Kleene [8], Section 82.

Theorem 8.7. *If $A_1,\dots,A_n \vdash B$ (classically), then $A_1^*,\dots,A_n^* \vdash_i B^*$ (intuitionistically), where A^* is defined as follows: For A atomic, $A^* := \neg\neg A$.*

$(B\wedge C)^* := B^* \wedge C^*$	$(\forall x[B(x)])^* := \forall x[B(x)^*]$,
$(B\vee C)^* := \neg(\neg B^* \wedge \neg C^*)$	$(\exists x[B(x)])^* := \neg\forall x[\neg B(x)^*]$,
$(A\to B)^* := A^* \to B^*$	$(\neg A)^* := \neg A^*$.

8.6.2 Tableaux for Intuitionistic Predicate Logic

A *tableaux* system for intuitionistic predicate logic is obtained by replacing the rules $F\to$, $F\neg$ and $F\,\forall$ for classical predicate logic (see Subsection 4.4.2) by the rules $F\to_i$, $F\neg_i$ and $F\,\forall_i$:

$$F\to_i \; \frac{S,\, F\; B\to C}{S_T,\; TB, FC} \qquad F\neg_i \; \frac{S,\, F\;\neg B}{S_T,\; TB} \qquad F\,\forall_i \; \frac{S,\, F\;\forall x[A(x)]}{S_T,\; FA(a)}$$

where a is new, i.e., does not occur in $S,\ F\,\forall x[A(x)]$, and where $S_T = \{TB \mid TB\in S\}$ is the set of all T-expressions in S. So, the tableaux rules for intuitionistic predicate logic are obtained by adding to the tableaux rules for the intuitionistic connectives, presented in Section 8.3, the intuitionistic tableaux rules for the quantifiers:

$$T\,\exists \quad \frac{S,\; T\,\exists x[A(x)]}{S,\; TA(a)} \qquad\qquad F\,\exists \quad \frac{S,\; F\,\exists x[A(x)]}{S,\; F\,\exists x[A(x)],\; FA(t)}$$

a new: a does not occur in $S,\ T\exists x[A(x)]$ — t being any term

$$T\forall \quad \frac{S,\; T\,\forall x[A(x)]}{S,\; T\,\forall x[A(x)],\; TA(t)} \qquad\qquad F\forall_i \quad \frac{S,\; F\,\forall x[A(x)]}{S_T,\; FA(a)}$$

t being any term — a new: a does not occur in $S,\ F\,\forall x[A(x)]$

The definitions of *an intuitionistic tableau-deduction of B from* $A_1,\dots,A_n$ and of *B is intuitionistically tableau-deducible from* $A_1,\dots,A_n$, denoted by $A_1,\dots,A_n \vdash_i' B$, are similar to the ones given in Section 8.3.

Example 8.9. $\exists x[\neg A(x)] \vdash_i' \neg\forall x[A(x)]$. Proof: the tableau in the left column below is an intuitionistic tableau-deduction of $\neg\forall x[A(x)]$ from $\exists x[\neg A(x)]$:

$T\;\exists x[\neg A(x)],\, F\;\neg\forall x[A(x)]$	$T\;\neg\forall x[A(x)],\, F\;\exists x[\neg A(x)]$
$T\;\exists x[\neg A(x)],\, T\;\forall x[A(x)]$	$F\;\forall x[A(x)],\, F\;\exists x[\neg A(x)]$
$T\;\neg A(a_1),\, T\;\forall x[A(x)]$	$F\;\forall x[A(x)],\, F\;\neg A(a_1)]$
$F\;A(a_1),\, T\;\forall x[A(x)]$	$F\;\forall x[A(x)],\; T\;A(a_1)]$
$F\;A(a_1),\, T\;A(a_1)$	$F\;A(a_2),\; T\;A(a_1)$
closure	no closure

But, conversely, we do not find a tableau-deduction of $\exists x[\neg A(x)]$ from $\neg\forall x[A(x)]$: the tableau in the right column above fails to be an intuitionistic tableau deduction of $\exists x[\neg A(x)]$ from $\neg\forall x[A(x)]$.

Without proof we mention that Theorem 8.1 for intuitionistic propositional logic can be generalized to intuitionistic predicate logic.

Theorem 8.8. *(i) If B is tableau-deducible from* $A_1,\ldots,A_n$ *(in intuitionistic predicate logic), i.e.,* $A_1,\ldots,A_n \vdash_i' B$*, then B is deducible from* $A_1,\ldots,A_n$ *(in intuitionistic predicate logic), i.e.,* $A_1,\ldots,A_n \vdash_i B$*. In particular, for* $n = 0$*:*
(ii) If $\vdash_i' B$*, then* $\vdash_i B$*.*

8.6.3 Kripke Semantics for Intuitionistic Predicate Logic

In the definition of a Kripke model for intuitionistic predicate logic below, we shall, for the sake of simplicity in notation, assume that our language contains no individual constants. The definitions below generalize Definition 8.11 for intuitionistic propositional logic.

Definition 8.21 (Kripke Model for Intuitionistic Predicate Logic).
A *Kripke model M* for intuitionistic predicate logic is a quadruple $\langle\, S,\ R,\ U,\ \models_i\,\rangle$ such that
1. S is a non-empty set (of possible proof-situations);
2. R is a binary relation on S (regarded as the accessibility relation), which is reflexive and transitive (see Definition 8.11);
3. U assigns to each s in S a non-empty set $U(s)$, such that if sRs', then $U(s)$ is a subset of $U(s')$. $U(s)$ can be conceived as the Universe of the proof-situation s, i.e., the set of objects constructed in the situation s;
4. $\models_i$ is a relation between elements of S and expressions of the form $P(a_1,\ldots,a_k)$ $[n_1,\ldots,n_k]$, such that
i) if $s \models P(a_1,\ldots,a_k)[n_1,\ldots,n_k]$, then $n_1,\ldots,n_k$ are elements of $U(s)$, and
ii) if $s \models P(a_1,\ldots,a_k)[n_1,\ldots,n_k]$ and sRs', then $s' \models P(a_1,\ldots,a_k)[n_1,\ldots,n_k]$.
$s \models P(a_1,\ldots,a_k)[n_1,\ldots,n_k]$ is to be read as: in the proof situation s it has been shown that $(n_1,\ldots,n_k)$ has the property P.

Definition 8.22. $(M,s \models_i A(a_1,\ldots,a_k)[n_1,\ldots,n_k])$
Given a Kripke model $M = \langle\, S,\ R,\ U,\ \models_i\,\rangle$, s in S, a formula $A(a_1,\ldots,a_k)$ and elements $n_1,\ldots,n_k$ in $U(s)$, $M,s \models_i A(a_1,\ldots,a_k)[n_1,\ldots,n_k]$ is defined as follows:
$M,s \models_i P(a_1,\ldots,a_k)[n_1,\ldots,n_k] := s \models_i P(a_1,\ldots,a_k)[n_1,\ldots,n_k]$;
$M,s \models_i B \wedge C[n_1,\ldots,n_k] := M,s \models_i B[n_1,\ldots,n_k]$ and $M,s \models_i C[n_1,\ldots,n_k]$;
The definition for $B \vee C$, $B \rightarrow C$ and $\neg B$ is analogous to the one (Definition 8.12) for intuitionistic propositional logic.
$M,s \models_i \forall x[B(x,a_1,\ldots,a_k)][n_1,\ldots,n_k]$:= for all s' in S such that sRs' and for all n in $U(s')$, $M,s' \models_i B(a,a_1,\ldots,a_k)[n,n_1,\ldots,n_k]$ (where a is new);
$M,s \models_i \exists x[B(x,a_1,\ldots,a_k)][n_1,\ldots,n_k] := M,s \models B(a,a_1,\ldots,a_k)[n,n_1,\ldots,n_k]$ for at least one n in $U(s)$ (a being a new free variable).

Example 8.10. Let $M = \langle \{s, s'\},\ R,\ U,\ \models_i \rangle$ be the Kripke model for intuitionistic predicate logic defined by sRs', $U(s) = \{1\}$, $U(s') = \{1,2\}$, $s \models_i P(a)[1]$, $s' \models_i P(a)[1]$, and $s' \not\models_i P(a)[2]$.

$$\begin{array}{ll} \{1\} & s\ P(a)[1] \\ & \quad\downarrow \\ \{1,2\} & s'\ P(a)[1] \end{array}$$

Then $M, s \models_i \neg\forall x[P(x)]$, because $M, s \not\models_i \forall x[P(x)]$ (since $M, s' \not\models_i P(a)[2]$) and $M, s' \not\models_i \forall x[P(x)]$. But $M, s \not\models_i \exists x[\neg P(x)]$, because $M, s \not\models_i \neg P(a)[1]$.
Hence, $M, s \not\models_i \neg\forall x[P(x)] \to \exists x[\neg P(x)]$.

Example 8.11. Let $M = \langle\ S,\ R,\ U,\ \models_i\ \rangle$ be the Kripke model for intuitionistic predicate logic defined by:
$S = \{s_1, s_2\}$, $s_1 R s_1, s_1 R s_2$ and $s_2 R s_2$,
$U(s_1) = \{1\}$, $U(s_2) = \{1,2\}$,
$s_1 \models_i P(a)[1]$, $s_2 \models_i P(a)[1]$, $s_2 \models_i Q$, $s_2 \not\models_i P(a)[2]$,

$$\begin{array}{lcl} & s_1 & \\ \{1\} & \big| & P(a)[1] \\ \{1,2\} & \big| & P(a)[1], Q \\ & s_2 & \end{array}$$

Then $M, s_1 \models_i \forall x[P(x) \vee Q]$, but $M, s_1 \not\models_i \forall x[P(x)] \vee Q$ (because $M, s_1 \not\models_i \forall x[P(x)]$ and $M, s_1 \not\models Q$). Hence, $M, s_1 \not\models_i \forall x[P(x) \vee Q] \to \forall x[P(x)] \vee Q$.

Example 8.12. Let $M = \langle\ S,\ R,\ U,\ \models_i\ \rangle$ be the Kripke model for intuitionistic predicate logic defined by:
$S = \{s_1, s_2, \ldots\}$,
for all i, j, if $1 \leq i \leq j$, then $s_i R s_j$,
for all $i = 1, 2, \ldots, U(s_i) = \{1, \ldots, i\}$,
for all $i \geq 2, s_i \models P(a)[1], \ldots, s_i \models P(a)[i-1]$,
$s_i \not\models P(a)[i]$,

$$\begin{array}{r|l} \{1\}\ s_1 & \\ \{1,2\}\ s_2 & P(1) \\ \{1,2,3\}\ s_3 & P(1), P(2) \\ & \\ & \\ & \end{array}$$

Then for each s in S, $M, s \not\models_i \forall x[P(x) \vee \neg P(x)]$. Hence, $M, s_1 \models_i \neg\forall x[P(x) \vee \neg P(x)]$.

The reader can easily verify that the analogue of Lemma 8.1 holds again for intuitionistic predicate logic.

Lemma 8.2. *Let* $M = \langle\ S,\ R,\ U,\ \models_i \rangle$ *be a Kripke model, s and t elements of S and* $A = A(a_1, \ldots, a_k)$ *a formula.*
If $M, s \models_i A[n_1, \ldots, n_k]$ *and sRt, then* $M, t \models_i A[n_1, \ldots, n_k]$.

Definition 8.23 (Intuitionistically Kripke-valid Consequence).
Let $M = \langle\ S,\ R,\ U,\ \models_i\ \rangle$ be a Kripke model for intuitionistic predicate logic and $A = A(a_1, \ldots, a_k)$ a formula.
(a) M is a *model* for A, or A *holds in* M := for all s in S and for all $n_1, \ldots, n_k$ in $U(s)$, $M, s \models_i A(a_1, \ldots, a_k)[n_1, \ldots, n_k]$. **Notation**: $M \models_i A$.
(b) A is intuitionistically *Kripke-valid* := for all Kripke models M, $M \models_i A$.
Notation: $\models_i A$.
(c) B is an intuitionistically *Kripke-valid consequence* of $A_1, \ldots, A_n$:=
$\models_i A_1 \wedge \ldots A_n \to B$. **Notation**: $A_1, \ldots A_n \models_i B$.

Example 8.13. For the Kripke model M of Example 8.10, $M \not\models_i \neg\forall x[P(x)] \rightarrow \exists x[\neg P(x)]$. For the Kripke model M of Example 8.11, $M \not\models_i \forall x[P(x) \vee Q] \rightarrow \forall x[P(x)] \vee Q$. And for the Kripke model M of Example 8.12, $M \not\models \neg\neg\forall x[P(x) \vee \neg P(x)]$. Since $\models_i \forall x[\neg\neg(P(x) \vee \neg P(x))]$, it follows that $\forall x[\neg\neg A(x)] \not\models_i \neg\neg\forall x[A(x)]$.

8.6.4 Soundness and Completeness

The intuitionistic Hilbert-type proof system in Subsection 8.6.1 for intuitionistic predicate logic is sound with respect to the intuitionistic Kripke semantics given in Subsection 8.6.3.

Theorem 8.9 (Soundness). *If $A_1, \ldots, A_n \vdash_i B$, then $A_1, \ldots, A_n \models_i B$.*

Proof. The proof is a generalization of the proof of the soundness theorem (Theorem 8.2) for intuitionistic propositional logic.

The procedure to construct a counterexample to a given conjecture $A_1, \ldots, A_n \vdash_i' B$, given in Definition 8.14 for intuitionistic propositional logic, may be adapted to intuitionistic predicate logic, taking also the quantifiers into account. We shall illustrate this procedure in the Examples 8.14 and 8.15. Again, if this procedure yields an open search tree, then we have actually constructed an intuitionistic Kripke counterexample to the given conjecture. And if all search trees are closed, then the closed branches form together a tableau-deduction of B from $A_1, \ldots, A_n$. Hence, again we may conclude that the tableaux rules for intuitionistic predicate logic are complete with respect to the Kripke semantics for intuitionistic predicate logic:

Theorem 8.10 (Completeness). *If $A_1, \ldots, A_n \models_i B$, then $A_1, \ldots, A_n \vdash_i' B$.*

Finally, we may generalize the proof of Theorem 8.1 to intuitionistic predicate logic:

Theorem 8.11. *If $A_1, \ldots, A_n \vdash_i' B$, then $A_1, \ldots, A_n \vdash_i B$.*

Hence, the three notions of $A_1, \ldots, A_n \vdash_i B$, $A_1, \ldots, A_n \models_i B$ and $A_1, \ldots, A_n \vdash_i' B$ turn out to be equivalent.

Example 8.14. We wonder whether $\neg\forall x[P(x)] \vdash_i' \exists x[\neg P(x)]$. Our procedure to construct a counterexample to this conjecture yields the search tree in the left column below:

$T\ \neg\forall x[P(x)],\ F\ \exists x[\neg P(x)]$	
$T\ \neg\forall x[P(x)],\ F\ \neg P(a_1)$	
$T\ \neg\forall x[P(x)],\ T\ P(a_1)$	$\{1\}\quad s\ \ P(a)[1]$
$F\ \forall x[P(x)],\ T\ P(a_1)$	$\vert$
$\vert$	$\downarrow$
$T\ \neg\forall x[P(x)],\ F\ P(a_2),\ T\ P(a_1)$	$\{1,2\}\ s'\ P(a)[1]$

Although we may proceed with developing this search tree, it is clear that we will never find closure. In fact, we have constructed the Kripke counterexample M described in Example 8.10, as depicted in the right column above. For instance, by definition $s \models_i P(a)[1]$, corresponding with the fact that $T\ P(a_1)$ occurs in s.

Example 8.15. We wonder whether $\forall x[P(x) \vee Q] \vdash_i' \forall x[P(x)] \vee Q$. Our procedure to construct a counterexample to this conjecture yields among others the search tree in the left column below:

$$
\begin{array}{c}
T\ \forall x[P(x) \vee Q],\ F\ \forall x[P(x)] \vee Q \\
T\ P(a_1) \vee Q,\ T\ \forall x[P(x) \vee Q],\ F\ \forall x[P(x)],\ FQ \\
TP(a_1),\ T\ \forall x[P(x) \vee Q],\ F\ \forall x[P(x)],\ FQ \\
| \\
TP(a_1),\ T\ \forall x[P(x) \vee Q],\ FP(a_2) \\
TP(a_1),\ T\ P(a_2) \vee Q,\ T\ \forall x[P(x) \vee Q],\ FP(a_2) \\
TP(a_1),\ TQ,\ T\ \forall x[P(x) \vee Q],\ FP(a_2)
\end{array}
\qquad
\begin{array}{r|l}
\{1\}\ s_1 & P(a)[1] \\
& \\
\{1,2\}\ s_2 & P(a)[1],\ Q
\end{array}
$$

From this open search tree one may easily read off the the Kripke counterexample described in Example 8.11, as depicted in the right column above. For instance, by definition $s_1 \models_i P(a)[1]$ corresponding to the fact that $T\ P(a_1)$ occurs in s_1 and $s_2 \models_i Q$ corresponding to the fact that TQ occurs in s_2.

The proofs of the *soundness* and *completeness* of intuitionistic predicate logic with respect to (intuitionistic) Kripke semantics are generalizations of the corresponding proofs for intuitionistic propositional logic (see the proofs of Theorem 8.2 and 8.5), and are analogous to the corresponding proofs for classical predicate logic (see Chapter 4).

There is however one complication. Also for intuitionistic predicate logic we have: if A is Kripke valid, then there is no open search tree starting with $F\ A$. Thus, if A is Kripke valid, we can conclude that each search tree starting with $F\ A$ is not open, i.e., each search tree starting with $F\ A$ is *not not* closed. Classically, we can conclude from this that each search tree starting with FA is closed (and hence that A is formally provable), but not so intuitionistically. So the completeness theorem for intuitionistic predicate logic with respect to (intuitionistic) Kripke semantics can only be proved if we use a classical metalanguage and not if we want to use intuitionistic metamathematics.

W. Veldman [16] has discovered a generalization of the notion of an intuitionistic Kripke model and hence a somewhat different notion of Kripke validity, such that completeness with respect to this generalized Kripke semantics can be proved intuitionistically. The essence is to allow that $\perp$ (falsum) is true in one or more proof situations and to demand that if $\perp$ is true in situation s ($s \models_i \perp$), then every formula A is true in s ($s \models_i A$). Next this idea was used by de Swart [14] to give another intuitionistic completeness proof with respect to a different semantics.

Note that the transition from 'not not closed' to 'closed' is intuitionistically correct in the case of intuitionistic propositional logic, because in that case we can for each search tree decide whether it is closed or not. And from $\forall t[C(t) \vee \neg C(t)]$ and $\forall t[\neg\neg C(t)]$, where $C(t)$ stands for 'the search tree t is closed', it follows intuitionistically that $\forall t[C(t)]$.

Like classical predicate logic, also intuitionistic predicate logic is *undecidable*. The search trees starting with $F\ A$ may become infinitely long, each time introducing new variables and we may not know whether we can stop at some stage.

Exercise 8.19. Verify (intuitively) that intuitionistically the following formulas hold true: a) $\neg\exists x \in V[\neg A(x)] \rightleftarrows \forall x \in V[\neg\neg A(x)]$; b) $\neg\neg\forall x \in V[A(x)] \to \forall x \in V[\neg\neg A(x)]$.
c) Verify (intuitively) that the following formula does not hold intuitionistically:
$\forall x \in V[\neg\neg A(x)] \to \forall x \in V[A(x)]$.
d1) Show that a) is intuitionistically tableau-provable and d2) that a) is also formally provable (in the intuitionistic Hilbert-type proof system).

Exercise 8.20. Prove that for A a formula in prenex normal form (see Subsection 4.3.5) $\vdash_i' A$ (in intuitionistic predicate logic) is decidable. Since intuitionistic predicate logic is undecidable, it follows that not every formula has a prenex normal form to which it is equivalent in intuitionistic predicate logic.

Exercise 8.21. Prove that if $\vdash_i' \exists x[A(x)]$ (intuitionistically) and in $A(a)$ there occur no other free variables than a, then $\vdash_i' \forall x[A(x)]$.

Exercise 8.22. Prove that $\forall x[P(x) \vee Q] \to \forall x[P(x)] \vee Q$ holds in all Kripke models (for intuitionistic predicate logic) $M = \langle\ S,\ R,\ U,\ \models_i\ \rangle$ with *constant domain*, i.e., $U(s) = U(s')$ for all $s,\ s'$ in S. (Compare Example 8.11.)

Exercise 8.23. For each formula A of intuitionistic predicate logic we define a formula A' of modal predicate logic by induction as follows: $P' := \Box P$, $(B \to C)' := \Box(B' \to C')$, $(\neg B)' := \Box\neg(B')$, $(B \wedge C)' := B' \wedge C'$, $(B \vee C)' := B' \vee C'$, $(\forall x[B(x)])' := \Box\forall x[B(x)']$, $(\exists x[B(x)])' := \exists x[B(x)']$. Prove that $\models_i A$ iff $\models A'$ in $S4$, i.e., A is intuitionistically Kripke valid iff A' is valid in the modal logic $S4$ (see Chapter 6).

8.7 Sets in Intuitionism: Construction Projects and Spreads

Intuitionistically, a set – like any other mathematical object – should be a mental construction. Natural numbers can be conceived as objects which are finitely constructible. Intuitionistically, the set of all natural numbers is identified with the following *construction project*: a) 0 is a natural number, and b) if n is a natural number, then n' is a natural number too. (The term ‘construction project’ was coined by Johan J. de Iongh.)

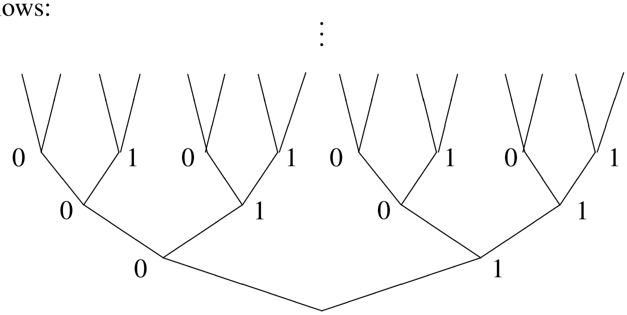
The set $\mathbb{N}$ of the natural numbers is intuitionistically not regarded as a completed totality, but only as potential or becoming or constructive. The construction project can be stated in only two clauses, but it generates the potentially infinite set $\mathbb{N}$ of the natural numbers. At each stage only finitely many elements of $\mathbb{N}$ will actually have been constructed; but also at each stage the construction project tells us how to continue the construction of new natural numbers.

In classical mathematics one accepts the Powerset axiom: if V is a set, then $P(V)$ is a set too. It follows that $P(\mathbb{N})$, $PP(\mathbb{N})$, $PPP(\mathbb{N})$, ... are sets of ever increasing cardinality. However, these sets are not surveyable in the most literal meaning of the word; more precisely, no construction project is known of which we could reasonably say that it generates the elements of such a set in the course of time. For that reason, Brouwer rejected the powerset axiom and refused to accept the existence of these sets.

The notion of construction project is a primitive, i.e., undefined, one. But we certainly want to say that the clauses a) and b) above together define a construction project for $\mathbb{N}$. In what follows we introduce the notion of spread, which we want to consider as a particular kind of construction project. We do not exclude the possibility that one may discover other kinds of construction projects in the future, although we are not aware of them now.

The intuitionist constructs the integers from the natural numbers ($\mathbb{Z}$ is enumerable) and the rationals from the integers ($\mathbb{Q}$ is enumerable), just like this is done in classical set theory. A more difficult question is whether there is an intuitionistic set which can reasonably be called $\mathbb{R}$; in other words, if we can generate the elements of such an $\mathbb{R}$ by an appropriate construction project. Only if this is the case, does quantification over the reals ('for all $x \in \mathbb{R} \ldots$' and 'for some $x \in \mathbb{R} \ldots$') make sense. One could say that Brouwer invented the spread concept in order to answer this question.

Now the construction project for $\mathbb{R}$ is rather complicated. For that reason we first indicate a construction project for $\{0,1\}^{\mathbb{N}}$, i.e., the set of all (potentially) infinite sequences of zeros and ones. Schematically, the construction project for $\{0,1\}^{\mathbb{N}}$ looks as follows:

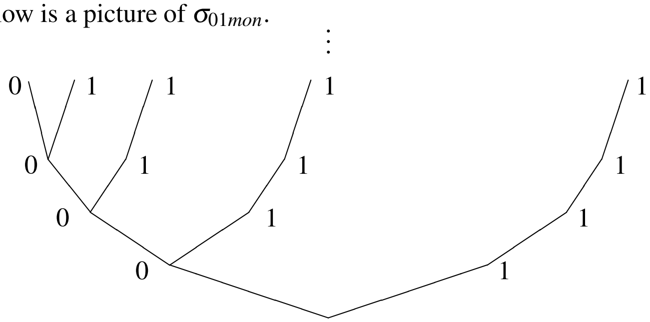


As an introduction one might think of a construction project as a mental project generating all possibilities to swim from Amsterdam to 'the end of the world', where at each stage one has the choice of going to the left or going to the right.

By choosing an element from $\{0,1\}$ at successive moments or stages, potentially infinite sequences of zero's and one's come into being. These sequences are generated in the course of time by a simple precept, called the choice-law: at each stage choose either a zero or a one. We identify the (intuitionistic) set $\{0,1\}^{\mathbb{N}}$ with this precept; and we call the potentially infinite sequences of zero's and one's the elements of this set, since they are generated in accordance with the corresponding choice-law. One does have an overall picture of how the elements of this set come into being.

The elements of a *spread* are generated by choosing natural numbers consecutively, with due observation of a *choice-law* (corresponding to the given spread), which prescribes which natural numbers may be chosen given the choices already made before.

In the case of $\{0,1\}^{\mathbb{N}}$, the choice-law dictates that to each finite sequence of natural numbers already chosen before, we may choose only an element of $\{0,1\}$. This spread is also called σ_{01} or the *binary spread*. A different choice-law is the one that dictates that given a finite sequence of natural numbers chosen before, one may choose an element of $\{0,1\}$ if the given finite sequence does not contain 1 and that otherwise one has to choose an element of $\{1\}$. The spread belonging to this choice-law generates the monotone non-decreasing elements of σ_{01} and is called σ_{01mon}. Below is a picture of σ_{01mon}.



The elements of the spread, called *choice-sequences*, are the potentially infinite sequences of natural numbers which are admitted by the choice-law of the spread. So the elements of $\{0,1\}^{\mathbb{N}}$ are the potentially infinite sequences of zero's and one's.

σ_ω is the *universal spread*, i.e., $\mathbb{N}^{\mathbb{N}}$. This choice-law dictates that to each finite sequence of natural numbers already chosen before, one may choose any natural number n in $\mathbb{N}$.

Some authors define a spread simply as a tree in which each node has at least one successor. This is not in the spirit of intuitionism, but rather in the spirit of classical mathematics.

We may consider every element of $\{0,1\}^{\mathbb{N}}$ as the characteristic function of a subset of $\mathbb{N}$ (see Theorem 3.12). Note that from an intuitionistic point of view, the characteristic function K_U (Definition 3.27) of a subset U of $\mathbb{N}$ is only well defined if U is decidable, i.e., if for each $n \in \mathbb{N}$ one can decide whether $n \in U$ or not. So, the intuitionist also has at his disposal the set $P_{dec}(\mathbb{N})$, i.e., the set of all decidable subsets of $\mathbb{N}$.

The set containing just 0 if Goldbach's conjecture holds and containing just 1 if Goldbach's conjecture does not hold, is a subset of $\mathbb{N}$, but not a decidable one. In fact, the intuitionist does not know of any spread to which we could reasonably give the name $P(\mathbb{N})$. Up till now no one has succeeded to present a construction project which might be said to generate in the course of time *all* subsets of $\mathbb{N}$.

As he needed a construction project for $\mathbb{R}$, and not just for $\mathbb{N}^{\mathbb{N}}$, Brouwer generalized the notion of spread just introduced. A *dressed spread* consists of

1. a *choice-law*, prescribing – given a finite sequence of natural numbers already chosen before – which natural numbers may be chosen next, and

2. a *correlation-law*, which after each choice correlates effectively an object from a fixed countable set to the finite sequence of natural numbers chosen up till now.

The elements of a dressed spread, again called *choice-sequences*, are the potentially infinite sequences of objects which have been assigned by the correlation-law to the sequences of natural numbers chosen according to the choice-law.

In Brouwer's applications those correlated objects can be natural numbers, but also rational numbers and intervals with rational endpoints. Defining real numbers as infinite sequences of intervals with rational endpoints (for instance, $\sqrt{2} = [1,2],[1.4,1.5],[1.41,1.42],[1.414,1.415],\ldots$), Brouwer indicated a specific choice-law and a specific correlation-law such that the corresponding spread has precisely the reals as elements.

A (dressed) spread is not thought of intuitionistically as the 'totality' of its elements, but rather as the pair consisting of the choice-law and the correlation-law, which together govern the generation process under which its elements grow. So a (dressed) spread is a construction project, which generates the elements of the spread in the course of time. These elements are potentially infinite sequences (for instance, of intervals with rational endpoints, in the case of $\mathbb{R}$), of which at each stage only a finite initial segment has been completed, but of which also is prescribed at each stage how the finite sequence already constructed can be continued. Among the elements of a spread there are choice sequences which are extensionally the same as individual choice sequences defined by a particular law or otherwise.

For sets which can be obtained by means of a construction project, in particular for spreads, some surprising axioms are defended. One of them is *Brouwer's Continuity Principle*, which we explain below.

Brouwer's Continuity Principle for natural numbers: Let σ be a spread.
If $\forall\alpha \in \sigma\ \exists k \in \mathbb{N}\ [A(\alpha,k)]$, then

$$\forall\alpha \in \sigma\ \exists m \in \mathbb{N}\ \exists k \in \mathbb{N}\ \forall\beta \in \sigma\ [\bar{\beta}m = \bar{\alpha}m \rightarrow A(\beta,k)].$$

i.e., for each α in σ there is an initial segment of length m and a natural number k such that to all β in σ having the same initial segment of length m as α the same natural number k is correlated; $\bar{\beta}m = \bar{\alpha}m := \forall n < m[\beta(n) = \alpha(n)]$.

Justification: Although Brouwer used this principle without further justification, we now try to give a justification. Suppose $\forall\alpha \in \sigma\ \exists k \in \mathbb{N}\ [A(\alpha,k)]$. Because intuitionistically the elements of a spread are considered as continuously growing with new choices and not as being completed, and because natural numbers themselves are finite constructions, the correlation that associates with each α in σ a natural number k can intuitionistically only consist in such a way that the correlated natural numbers will be determined effectively at a certain finite stage in the growth of the choice sequences. That is, intuitionistically the correlated natural numbers will have to be determined by finite initial segments of the choice sequences. The justification of Brouwer's principle ultimately rests on the insight that each element α in σ can be thought of as being given step by step, also in the case that some particular α is determined by a finite law. And the truth of $A(\alpha,k)$ does not depend on the manner in which α has been generated. □

We now derive a consequence from Brouwer's principle.

Theorem 8.12. *Let* $\sigma = \sigma_{01}$ *and* $\underline{0}$ *be the element in* σ_{01} *such that* $\forall n \in \mathbb{N}\ [\underline{0}(n) = 0]$. *Then* $\neg \forall \alpha \in \sigma\ [\alpha = \underline{0} \vee \neg(\alpha = \underline{0})]$.

Proof. Suppose $\forall \alpha \in \sigma\ [\alpha = \underline{0} \vee \neg(\alpha = \underline{0})]$, i.e.,

$$\forall \alpha \in \sigma\ \exists k \in \mathbb{N}\ [(k = 0 \wedge \alpha = \underline{0}) \vee (k = 1 \wedge \neg(\alpha = \underline{0}))].$$

By Brouwer's principle, $\forall \alpha \in \sigma\ \exists m \in \mathbb{N}\ \exists k \in \mathbb{N}\ \forall \beta \in \sigma\ [\bar{\beta} m = \bar{\alpha} m \rightarrow$

$$(k = 0 \wedge \beta = \underline{0}) \vee (k = 1 \wedge \neg(\beta = \underline{0}))].$$

Now consider $\alpha = \underline{0}$. Then there is $m \in \mathbb{N}$ such that $\forall \beta \in \sigma\ [\bar{\beta} m = \bar{\alpha} m \rightarrow \beta = \underline{0}]$. However, let β be such that $\bar{\beta} m = \bar{\alpha} m = \bar{\underline{0}} m$ and $\beta(m) = 1$. Then $\beta \neq \underline{0}$. Contradiction. Therefore $\neg \forall \alpha \in \sigma\ [\alpha = \underline{0} \vee \neg(\alpha = \underline{0})]$. □

Notice that although for each $\alpha \in \sigma_{01}$ the statement $\alpha = \underline{0} \vee \alpha \neq \underline{0}$ itself does not yield a contradiction – in fact, $\neg\neg(\alpha = \underline{0} \vee \alpha \neq \underline{0})$ – , the simultaneous quantification over all $\alpha \in \sigma_{01}$ of the expression $\alpha = \underline{0} \vee \alpha \neq \underline{0}$ does lead to a contradiction. Summarizing: $\neg\neg(\alpha = \underline{0} \vee \alpha \neq \underline{0})$, but $\neg \forall \alpha \in \sigma_{01}\ [\alpha = \underline{0} \vee \alpha \neq \underline{0}]$.

From Brouwer's principle it also follows that there do not exist bijective mappings from either σ_{01} or σ_{01mon} to $\mathbb{N}$ (Brouwer 1918; see Exercise 8.24). Note that from a classical point of view σ_{01mon} is an enumerable set.

Given a construction project which results in an intuitionistic set V and given a well defined extensional property $A(x)$ concerning the elements x of V, an intuitionist also accepts $W := \{x \in V \mid A(x)\}$ as a set, for quantification over W may be explained in the usual way as a restricted quantification over V : $\forall x \in W\ [E(x)] := \forall x \in V\ [A(x) \rightarrow E(x)]$ and $\exists x \in W\ [E(x)] := \exists x \in V\ [A(x) \wedge E(x)]$.

For a more extensive treatment of spreads and the axioms holding for them see Gielen, Veldman and de Swart [7].

Choice Sequences *Choice-sequences* are the potentially infinite sequences of natural numbers which are generated by the choice-law of the spread. And in the case of a dressed spread *choice sequences* are the potentially infinite sequences of objects which have been assigned by the correlation-law to the sequences of natural numbers chosen according to the choice-law.

Some particular choice sequences may be called *lawlike*, for instance, the sequence $\underline{0}$ which is generated by the choice-law that dictates that given a finite sequence of choices already made before one has to choose a 0 and nothing else. Other particular choice sequences may be called *lawless*, for instance, the sequence which is generated by the choice-law that dictates that given any finite sequence of choices already made before one has to choose any natural number and of which we have determined in advance that the choice-law will never impose any restriction on further choices.

However, the expression 'α is lawlike' is not a well defined propositional function for $\alpha \in \sigma_\omega$ for the following reasons:

1. The notion of finite law has not been defined precisely.
2. Let $A(\alpha) := \alpha$ is lawlike. Then $A(\underline{0})$ is true. But $A(0, 0, 0, \ldots)$ is not true for the

sequence 0, 0, 0, ... which 'accidentally' only contains zero's. But a well defined propositional function should be a function in the sense that if α and β are extensionally equal, then $A(\alpha)$ and $A(\beta)$ should be the same well defined assertion.

'$\underline{0}$ is lawlike' is a well defined proposition (what-ever 'lawlike' may mean precisely). But as long as α has not been specified by some specific finite law, the expression 'α is lawlike' has no clear meaning, because the notion of 'finite law' has not been defined. Consequently, we are not able to speak about the set of all lawlike sequences and hence we cannot quantify over them. Similar observations hold for the expressions 'α is lawless' and 'the set of all lawless sequences'. We have no construction project that generates in the course of time all lawlike (respectively, all lawless) sequences! See de Swart [15].

$A(\beta)$, e.g., $\forall n[\beta(n) = 0]$, is a propositional function, rather than a well defined proposition. If we have a construction determining all values of β, then this construction together with our understanding of $A(\beta)$ would result in a well defined proposition. But as long as such a finite law for β is not given, we have neither a proof of $A(\beta)$, nor an insight in the impossibility of experiencing the truth of $A(\beta)$.

Summarizing: 1. Intuitionistic objects, in particular sets, are finite constructions or *construction projects*. 2. Construction projects for non-denumerable sets technically boil down to spreads. A *dressed spread* consists of i) a choice law and ii) a correlation law. 3. *Choice sequences* are just the elements of a spread. 4. Brouwer defines the set $\mathbb{R}$ as a dressed spread whose choice sequences are infinite sequences of (decreasing) intervals with rational endpoints. 5. Brouwer's principle is proved by reflection on what it means to have a proof of $\forall\alpha \in \sigma\exists k \in \mathbb{N}\ [\ A(\alpha,k)\ ]$, rather than the result of the peculiar epistemological status of choice sequences. 6. Quantification only makes sense if it is a quantification over an intuitionistic set, i.e., a set for which one has a construction(-project).

Starting from these philosophically sound principles it is quite possible to develop intuitionistic mathematics, enough for the purposes of science (physics, economics, etc.). See, for instance, Veldman [17, 18] and de Swart [13].

Exercise 8.24. Using Brouwer's principle, prove that there is no bijection from σ_{01} to $\mathbb{N}$, neither from σ_{01mon} to $\mathbb{N}$. However, from a classical point of view σ_{01mon} is enumerable; explain the difference.

8.8 The Brouwer Kripke axiom

Brouwer-Kripke axiom (Brouwer, 1948) Let P be a *determinate* proposition. Then there is an α in σ_{01} such that $P \leftrightarrows \exists n[\alpha(n) = 1]$.

Justification Given a determinate proposition P it can be pondered again and again in my mathematical life. We construct α as follows: $\alpha(n) = 1$ if at stage n I did succeed in proving P; otherwise, $\alpha(n) = 0$. □

Johan de Iongh stressed that P should be *determinate*, i.e., P should not depend on infinite objects which are still under construction. As long as information about P has not yet been completed, I cannot really start to think about its truth. In particular, P should not be of the form $\forall n[\beta(n) = 0]$, in which case one would obtain a contradiction from the Brouwer-Kripke axiom and what Kleene [9] called Brouwer's principle for functions. Wim Veldman [17] calls this principle AC_{11}, where AC stands for Axiom of Choice.

Theorem: The Brouwer Kripke axiom and AC_{11} (Brouwer's principle for functions) are contradictory when applied to the expression $\forall n[\beta(n) = 0]$ with β in σ_{01}.

For a precise formulation of AC_{11} and for a proof of this theorem we refer the reader to [17] or [7]. Several authors have blamed this contradiction on AC_{11} (Brouwer's principle for functions); however, the restriction proposed by Johan de Iongh that P should be determinate seems rather natural and self-evident, while AC_{11} has a good justification.

Application of the Brouwer-Kripke axiom: Let $\alpha(n) = 1$ if at stage n I have a proof of $G \vee \neg G$, where G is Goldbach's conjecture. Then $G \vee \neg G \leftrightarrow \exists n[\alpha(n) = 1]$. Because of $\neg\neg(G \vee \neg G)$ we know $\neg\neg\exists n[\alpha(n) = 1]$. But $G \vee \neg G$, i.e., $\exists n[\alpha(n) = 1]$ cannot be asserted.

8.9 Solutions

Solution 8.1. (a) If $A \to B$, then $\neg B \to \neg A$. Conversely, if $\neg B \to \neg A$, then $A \to \neg\neg B$. But we are not entitled to infer $A \to B$ from $\neg B \to \neg A$. For $\neg B \to \neg(\neg\neg B)$ is intuitionistically valid and $\neg\neg B \to B$ is not.
(b) $(\neg A \vee \neg B) \to \neg(A \wedge B)$ is intuitionistically valid: $(\neg A \vee \neg B)$ and $(A \wedge B)$ together yield a contradiction. The converse formula $\neg(A \wedge B) \to (\neg A \vee \neg B)$ is not valid intuitionistically: interpreting A as Goldbach's conjecture and B as $\neg A$ we have $\neg(A \wedge \neg A)$, but not $\neg A \vee \neg\neg A$.
(c) $(A \to B) \to \neg(A \wedge \neg B)$ is intuitionistically valid: $(A \to B)$ and $(A \wedge \neg B)$ together yield a contradiction. But $\neg(A \wedge \neg B) \to (A \to B)$ is not valid intuitionistically: interpreting A as $\neg\neg B$ we have $\neg(\neg\neg B \wedge \neg B)$, but not $\neg\neg B \to B$, as we have seen before.
(d) $(\neg A \vee B) \to (A \to B)$ is intuitionistically valid: $\neg A \to (A \to B)$ and $B \to (A \to B)$, hence $(\neg A \vee B) \to (A \to B)$. But $(A \to B) \to (\neg A \vee B)$ is not valid intuitionistically: interpreting B as $\neg\neg A$ we have $A \to \neg\neg A$, but not $\neg A \vee \neg\neg A$.
(e) $((A \to B) \vee (A \to C)) \to (A \to B \vee C)$ is intuitionistically valid: $(A \to B) \to (A \to B \vee C)$ and $(A \to C) \to (A \to B \vee C)$, hence $((A \to B) \vee (A \to C)) \to (A \to B \vee C)$. But the converse formula is not valid intuitionistically: interpreting A as $B \vee C$ we have $B \vee C \to B \vee C$, but not $(B \vee C \to B) \vee (B \vee C \to C)$.
(f) $(A \vee \neg A) \to (\neg\neg A \to A)$ is intuitionistically valid: for if $A \vee \neg A$, then $\neg\neg A$ rules out the second possibility $\neg A$; so, only the first possibility A is left. $(\neg\neg A \to A) \to (A \vee \neg A)$ is not intuitionistically valid: interpreting A as $\neg\neg P$ we have $\neg\neg\neg\neg P \to \neg\neg P$, but not $\neg\neg P \vee \neg\neg\neg P$ which is equivalent to $\neg\neg P \vee \neg P$.

Solution 8.2. 1. If $A \to C$, then $\neg C \to \neg A$ and hence $\neg(\neg A) \to \neg(\neg C)$.
2. If $A \to (B \to C)$ and B, then $A \to C$ and so by 1) $\neg\neg A \to \neg\neg C$.
3. If $A \to (B \to C)$ and $\neg\neg A$, then by 2) $B \to \neg\neg C$ and hence by 1) $\neg\neg B \to \neg\neg\neg\neg C$, i.e., $\neg\neg B \to \neg\neg C$.
4. From 3) with $A \to B,\ A, B$ instead of $A,\ B,\ C$ respectively, if $(A \to B) \to (A \to B)$ and $\neg\neg(A \to B)$, then $\neg\neg A \to \neg\neg B$.
5. Suppose $\neg\neg A \to \neg\neg B$ and $\neg(A \to B)$. Then $\neg\neg A$ and $\neg B$, for $A \to B$ follows both from $\neg A$ and from B. So, $\neg\neg A$ and $\neg\neg A \to \neg\neg B$. Therefore, $\neg\neg B$. Contradiction with $\neg B$. So, if $\neg\neg A \to \neg\neg B$, then $\neg\neg(A \to B)$.
6. From 3) with $A,\ B,\ A \wedge B$ instead of $A,\ B,\ C$ respectively, if $A \to (B \to A \wedge B)$ and $\neg\neg A$, then $\neg\neg B \to \neg\neg(A \wedge B)$. Hence, if $\neg\neg A \wedge \neg\neg B$, then $\neg\neg(A \wedge B)$.
7. $A \wedge B \to A$. Hence by 1), $\neg\neg(A \wedge B) \to \neg\neg A$. Similarly, $\neg\neg(A \wedge B) \to \neg\neg B$.
8. $A \to A \vee B$. Hence by 1), $\neg\neg A \to \neg\neg(A \vee B)$. Similarly, $\neg\neg B \to \neg\neg(A \vee B)$. Therefore, if $\neg\neg A \vee \neg\neg B$, then $\neg\neg(A \vee B)$.
9. Take $B = \neg A$. Then $\neg\neg(A \vee \neg A)$, but $\neg\neg A \vee \neg A$ is not intuitionistically valid.

Solution 8.3. If A is decidable, i.e., $A \vee \neg A$, then $\neg\neg A$ eliminates the second option $\neg A$ and consequently only the first option A is left.

Solution 8.4. Axiom 8^i, $\neg A \to (A \to B)$, of intuitionistic propositional logic is (formally) provable in classical propositional logic. This follows from $\neg A,\ A \vdash B$ (weak negation elimination) by two applications of the deduction theorem. Hence, we have (i): all formulas provable in the intuitionistic system are provable in the classical system. Since the proof of the deduction theorem only uses the axiom schemas 1 and 2 and applications of Modus Ponens and since all these tools are available in intuitionistic propositional logic, we have (ii): the deduction theorem also holds for intuitionistic propositional logic.

Solution 8.5. Note that the deductions found in Exercise 2.60 a ii), b ii) and c ii) do use the rule of double negation elimination $(d\neg E)$, while those in a i), b i) and c i) do not.

Solution 8.6. a) We restrict ourselves to a tableau-proof of $A \to (B \to A)$ and of $\neg A \to (A \to B)$:

$F\ A \to (B \to A)$	$F\ \neg A \to (A \to B)$
$TA,\ F\ B \to A$	$T\ \neg A,\ F\ A \to B$
$TA,\ TB,\ FA$	$T\ \neg A,\ FA,\ F\ A \to B$
closure	$T\ \neg A,\ TA,\ FB$
	$T\ \neg A,\ FA,\ TA,\ FB$
	closure

b) 1) In the left column below is an intuitionistic tableau-proof of $A \to \neg\neg A$, while in the right column there is a failed attempt to give an intuitionistic tableau-proof of $\neg\neg A \to A$:

$F\ A \to \neg\neg A$	$F\ \neg\neg A \to A$
$TA,\ F\ \neg\neg A$	$T\ \neg\neg A,\ FA$
$TA,\ T\ \neg A$	$T\ \neg\neg A,\ F\ \neg A,\ FA$
$TA,\ FA$	$T\ \neg\neg A,\ TA$
closure	no closure

b) 2) Below is in the left column an intuitionistic tableau-proof of $\neg\neg(A \vee \neg A)$, while in the right column there is a failed attempt to give an intuitionistic tableau-proof of $A \vee \neg A$:

$\underline{F\ \neg\neg(A \vee \neg A)}$	$F\ A \vee \neg A$
$T\ \neg(A \vee \neg A)$	$\underline{FA,\ F\neg A}$
$T\ \neg(A \vee \neg A),\ F\ A \vee \neg A$	TA
$\underline{T\ \neg(A \vee \neg A),\ FA,\ F\neg A}$	no closure
$T\ \neg(A \vee \neg A),\ TA$	
$T\ \neg(A \vee \neg A),\ F\ A \vee \neg A,\ TA$	
$T\ \neg(A \vee \neg A),\ FA,\ F\neg A,\ TA$	
closure	

c) It is not possible to construct an intuitionistic tableau-deduction of B from $A \to B$ and $\neg A \to B$:

$T\ A \to B,\ T\ \neg A \to B,\ FB$

$FA,\ T\ A \to B,\ T\ \neg A \to B,\ FB \quad | \quad TB,\ T\ \neg A \to B,\ FB$

$\underline{FA,\ T\ A \to B,\ F\neg A,\ FB} \mid FA,\ T\ A \to B,\ TB,\ FB \mid TB,\ T\ \neg A \to B,\ FB$

$T\ A \to B,\ TA$

$FA,\ TA \mid TB,\ TA$

no closure

Solution 8.7. Below in the left column there is a classical tableau-proof of $(P \to Q) \vee (Q \to P)$, and in the right column there are two failed attempts to construct an intuitionistic tableau proof of $(P \to Q) \vee (Q \to P)$.

$F\ (P \to Q) \vee (Q \to P)$	$F\ (P \to Q) \vee (Q \to P)$
$F\ P \to Q,\ F\ Q \to P$	$\underline{F\ P \to Q,\ F\ Q \to P}$
$TP,\ FQ,\ F\ Q \to P$	$\swarrow \quad \searrow$
$TP,\ FQ,\ TQ,\ FP$	$TP,\ FQ \qquad TQ,\ FP$
	no closure no closure

Solution 8.8. In all F-rules, except in the rules $F \to$ and $F\neg$, going from the top downwards, only F-formulas are introduced, while in the intuitionistic rules $F \to_i$ and $F\neg_i$, S is replaced by S_T. So: if an intuitionistic tableau-proof starts with

$F\ B \vee C$

$FB,\ FC$

then it is impossible that in the lowest sequents – which are of the form $S,\ TA,\ FA$ – in the tableau-proof of $B \vee C$, TA results from FB and FA results from FC, by application of the rules. Hence, if $\vdash_i' B \vee C$, then $\vdash_i' B$ or $\vdash_i' C$.
The classical variant does not hold; for instance, $\vdash' P \vee \neg P$, but $\nvdash' P$ and $\nvdash' \neg P$.

Solution 8.9. $A \vee B \vdash_i' (A \to B) \to B$, but not $(A \to B) \to B \vdash_i' A \vee B$:

$\underline{T\ A \vee B,\ F\ (A \to B) \to B}$	$T\ (A \to B) \to B,\ F\ A \vee B$
$T\ A \vee B,\ T\ A \to B,\ FB$	$T\ (A \to B) \to B,\ FA,\ FB$
$TA,\ T\ A \to B,\ FB \mid TB,\ T\ A \to B,\ FB$	$\underline{F\ A \to B,\ FA,\ FB} \mid TB,\ FA,\ FB$
$TA,\ FA,\ FB \mid TA,\ TB,\ FB \mid$ closure	$TA,\ FB$
closure closure	no closure

$(A \to B) \to B \vdash_i' \neg\neg(A \vee B)$, but conversely not $\neg\neg(A \vee B) \vdash_i' (A \to B) \to B$:

$\underline{T\ (A \to B) \to B,\ F\ \neg\neg(A \vee B)}$
$\underline{T\ (A \to B) \to B,\ T\ \neg(A \vee B)}$
$T\ (A \to B) \to B,\ F\ A \vee B$
$T\ (A \to B) \to B,\ FA,\ FB$
$\underline{F\ A \to B,\ FA,\ FB} \mid TB,\ FA,\ FB$
$TA,\ FB,\ T\neg(A \vee B) \mid$ closure
$TA,\ FB,\ F\ A \vee B$
$TA,\ FB,\ FA,\ FB$
closure

$\underline{T\ \neg\neg(A \vee B),\ F\ (A \to B) \to B}$
$T\ \neg\neg(A \vee B),\ T\ A \to B,\ FB$
$T\ \neg\neg(A \vee B),\ F\ A,\ FB \mid T\ \neg\neg,\ T\ B,\ FB$
$\underline{F\ \neg(A \vee B),\ FA,\ FB} \mid$ closure
$T\ A \vee B,\ T\ A \to B$
$TA,\ T\ A \to B \mid TB,\ T\ A \to B$
$TA,\ FA \mid TA,\ TB \mid TB,\ FA \mid TB,\ TB$
closure | no closure | no closure | no closure

Solution 8.10. 1) $\neg\neg A \wedge \neg\neg B \vdash_i' \neg\neg(A \wedge B)$ and $\neg\neg(A \wedge B) \vdash_i' \neg\neg A \wedge \neg\neg B$.
2) $\neg\neg A \to \neg\neg B \vdash_i' \neg\neg(A \to B)$ and conversely; $\neg\neg(A \to B) \vdash_i' A \to \neg\neg B$ and conversely.
3) for each formula A, $\vdash_i' \neg\neg\neg A \to \neg A$.
4) It follows from 3) that (if A and B are stable, then) $A \underline{\vee} B$ is stable.
5) $\neg\neg A \vee \neg\neg B \vdash_i' \neg\neg(A \vee B)$, but conversely not $\neg\neg(A \vee B) \vdash_i' \neg\neg A \vee \neg\neg B$.

Solution 8.11. 1) Here is an intuitionistic tableau-proof of $(A \vee \neg A) \to (\neg\neg A \to A)$:

$\underline{F\ (A \vee \neg A) \to (\neg\neg A \to A)}$
$\underline{T\ A \vee \neg A,\ F\ \neg\neg A \to A}$
$\underline{T\ A \vee \neg A,\ T\ \neg\neg A,\ FA}$
$TA,\ T\ \neg\neg A,\ FA \mid T\ \neg A,\ T\ \neg\neg A,\ FA$
closure $\mid T\neg A,\ F\ \neg A,\ FA$
closure

2) $\neg P$ (P atomic) is stable, i.e., $\vdash_i' \neg\neg\neg P \to \neg P$, but $\neg P$ is not decidable, i.e., not $\vdash_i' \neg P \vee \neg\neg P$.

Solution 8.12. 1. If $E = P$ (atomic), then $E^* = \neg\neg P$ and $\vdash_i' \neg\neg\neg\neg P \to \neg\neg P$, i.e., $E^* = \neg\neg P$ is stable. Suppose that A^* and B^* are stable (induction hypothesis).
If $E = A \wedge B$, then $E^* = A^* \wedge B^*$; and by Exercise 8.10 (1) E^* is stable.
If $E = A \to B$, then $E^* = A^* \to B^*$; and by Exercise 8.10 (2) E^* is stable.
If $E = \neg A$, then $E^* = \neg A^*$; and E^* is stable by Exercise 8.10 (3).
If $E = A \vee B$, then $E^* = \neg(\neg A^* \wedge \neg B^*)$; and by Exercise 8.10 (4) E^* is stable.
2. Suppose $A_1, \ldots, A_n \vdash B$ (classically), i.e., there is a schema of the form

$$\begin{array}{c} \underline{\text{axiom } 1, \ldots, \text{axiom } 8, A_1, \ldots, A_n} \\ \hline \\ \underline{C \quad C \to D} \\ D \\ \hline B \end{array}$$

Replace each formula E in this schema by E^*.
(axiom 1)$^* = (A \to (B \to A))^* = A^* \to (B^* \to A^*)$ is again an instance of axiom schema 1. (axiom 5a)$^* = (A \to A \vee B)^* = A^* \to (A \vee B)^* = A^* \to \neg(\neg A^* \wedge \neg B^*)$ is intuitionistically provable.

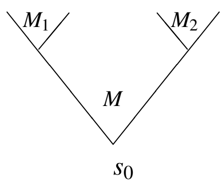
(axiom 6)$^* = ((A \to C) \to ((B \to C) \to (A \vee B \to C)))^* = (A^* \to C^*) \to ((B^* \to C^*) \to (\neg(\neg A^* \wedge \neg B^*) \to C^*))$. Now, $\vdash_i (A^* \to C^*) \to ((B^* \to C^*) \to (\neg(\neg A^* \wedge \neg B^*) \to \neg\neg C^*))$ and, since C^* is stable, $\vdash_i \neg\neg C^* \to C^*$. Therefore, $\vdash_i$ (axiom 6)*.
(axiom 8)$^* = (\neg\neg A \to A)^* = \neg\neg A^* \to A^*$ is intuitionistically provable (since A^* is stable).
Since $(C \to D)^* = C^* \to D^*$, $\dfrac{C^* \quad (C \to D)^*}{D^*}$ is again an application of Modus Ponens. Therefore, the schema above can be transformed into an intuitionistic deduction of B^* from $A_1^*, \ldots, A_n^*$.

Solution 8.13. While classically the formulas $P \to P$ and $P \vee \neg P$ are equivalent and both classically valid (always true), because of the stronger meaning of the $\vee$-connective in intuitionism, the formula $P \vee \neg P$ is intuitionistically no longer valid; however, the formula $P \to P$ is also intuitionistically valid.

Solution 8.14. * Suppose $\Gamma \mid C$ for all formulas C in Γ and let $\Gamma \vdash_i A$. Then either 1. $A \in \Gamma$, or 2. A is an axiom, or 3. there are formulas B and $B \to A$ such that $\Gamma \vdash_i B$ (1), $\Gamma \vdash_i B \to A$ and A is deduced from B and $B \to A$ by Modus Ponens.
In case 1, $\Gamma \mid A$ by hypothesis. In case 2, one easily checks that $\Gamma \mid A$ for each intuitionistic axiom A. In case 3, by induction hypothesis, $\Gamma \mid B$ (2) and $\Gamma \mid B \to A$ (3). From (1) and (2), $\Gamma \Vdash B$ and hence, by (3), $\Gamma \mid A$.
Suppose $H \mid H$ and $H \vdash_i B \vee C$. Then, by the Theorem just proved, $H \mid B \vee C$, i.e., $H \Vdash B$ or $H \Vdash C$. Hence, $H \vdash_i B$ or $H \vdash_i C$.

Solution 8.15. * For $A = P$ (atomic), the theorem is trivial. Induction Hypothesis: $\vdash_i A_1 \rightleftarrows B_1 \vee \ldots \vee B_m$ and $\vdash_i A_2 \rightleftarrows C_1 \vee \ldots \vee C_n$, where each B_i and C_j satisfies the conditions specified.
Case 1: $A = A_1 \vee A_2$; then $\vdash_i A \rightleftarrows B_1 \vee \ldots \vee B_m \vee C_1 \vee \ldots \vee C_n$.
Case 2: $A = A_1 \wedge A_2$; then $\vdash_i A \rightleftarrows (B_1 \wedge C_1) \vee \ldots \vee (B_m \wedge C_1) \vee \ldots \vee (B_m \wedge C_n)$, leaving out those $B_i \wedge C_j$ which are inconsistent.
Case 3: $A = A_1 \to A_2$. If $A_1 \to A_2 \vdash_i A_1$, then $\vdash_i A \rightleftarrows A_2$; hence $\vdash_i A \rightleftarrows C_1 \vee \ldots \vee C_n$. If $A_1 \to A_2 \nvdash_i A_1$, then $\vdash_i A \rightleftarrows B_1 \vee \ldots \vee B_m \to C_1 \vee \ldots \vee C_n$ and hence $\vdash_i A \rightleftarrows (B_1 \to C_1 \vee \ldots \vee C_n) \wedge \ldots \wedge (B_m \to C_1 \vee \ldots \vee C_n)$, where for all k, $1 \leq k \leq m$, $A \nvdash_i B_k$ because we have supposed that $A \nvdash_i A_1$ and therefore $A \nvdash_i B_1 \vee \ldots \vee B_m$. $(B_1 \to C_1 \vee \ldots \vee C_n) \wedge \ldots \wedge (B_m \to C_1 \vee \ldots \vee C_n)$ is consistent since, by hypothesis, $A \nvdash_i A_1$.
Case 4: $A = \neg A_1$; then $\vdash_i A \rightleftarrows \neg B_1 \wedge \ldots \wedge \neg B_m$.
Proof of $A_j \mid A_j$: Let $A_j = P \wedge \neg B \wedge (C \to D)$, where $A_j \nvdash_i C$ and P atomic. To show: $A_j \mid P$ and $A_j \mid \neg B$ and $A_j \mid C \to D$, i.e., $A_j \vdash_i P$ and not $A_j \Vdash B$ and (if $A_j \Vdash C$, then $A_j \mid D$). $A_j \vdash_i P$ is trivial. Because $A_j \vdash_i \neg B$ and A_j is consistent, it follows that not $A_j \Vdash B$. And because of $A_j \nvdash_i C$ it follows that (if $A_j \Vdash C$, then $A_j \mid D$).

Solution 8.16.



a) Let $M_1 = \langle S_1, R_1, \models_1 \rangle$ be an intuitionistic Kripke model such that not $M_1, s_1 \models_1 B$ for some s_1 in S_1. And let $M_2 = \langle S_2, R_2, \models_2 \rangle$ be a Kripke model such that not $M_2, s_2 \models_2 C$ for some s_2 in S_2. Let $M = \langle S, R, \models_i \rangle$ be the following Kripke model:
1) S contains all nodes of S_1 and of S_2 and in addition one extra node s_0.
2) s_0Rs for all s in S_1; s_0Rs for all s in S_2. R restricted to S_1 equals R_1 and R restricted to S_2 equals R_2, i.e., for all s,t in S_1, $sRt := sR_1t$ and for all s,t in S_2, $sRt := sR_2t$.
3) For s in S_1, $s \models_i P := s \models_1 P$ and for s in S_2, $s \models_i P := s \models_2 P$. For all atomic formulas P, by definition, $s_0 \not\models_i P$. One easily checks that for all formulas A, $M, s \models_i A$ iff $M_1, s \models_1 A$, if s in S_1, and $M, s \models_i A$ iff $M_2, s \models_2 A$, if s in S_2.
Now suppose $M, s_0 \models_i B \vee C$. Then $M, s_0 \models_i B$ or $M, s_0 \models_i C$.
Case 1: $M, s_0 \models_i B$; then $M, s_1 \models_i B$ and hence, $M_1, s_1 \models_1 B$; contradiction.
Case 2: $M, s_0 \models_i C$; then $M, s_2 \models_i C$ and hence, $M_2, s_2 \models_2 C$; contradiction.
Therefore, not $M, s_0 \models_i B \vee C$.
b) Suppose $\models_i B \vee C$, not $\models_i B$ and not $\models_i C$. Then there is a Kripke counterexample M_1 to B and a Kripke counterexample M_2 to C. By a) it follows that there is a Kripke counterexample M to $B \vee C$, contradicting $\models B \vee C$.

Solution 8.17. (a) The following schema is an intuitionistic tableau-proof of $(P \to Q) \to (\neg Q \to \neg P)$:

$$\begin{array}{l}
\underline{F\ (P \to Q) \to (\neg Q \to \neg P)} \\
\underline{T\ P \to Q,\ F\ \neg Q \to \neg P} \\
\underline{T\ P \to Q,\ T\ \neg Q,\ F\ \neg P} \\
T\ P \to Q,\ T\ \neg Q,\ TP \\
T\ P \to Q,\ FQ,\ TP \\
FP,\ FQ,\ TP \mid TQ,\ FQ,\ TP
\end{array}$$

Applying our procedure to construct a Kripke counterexample for $(\neg Q \to \neg P) \to (P \to Q)$ we find two different search trees, one of which is open:

$F\ (\neg Q \to \neg P) \to (P \to Q)$	0	
$T\ \neg Q \to \neg P,\ F\ P \to Q$	1	
$T\ \neg Q \to \neg P,\ TP,\ FQ$ $F\ \neg Q,\ TP,\ FQ$	2	P
$TQ,\ TP$	3	P, Q

From this open search tree one can read off a Kripke counterexample to $(\neg Q \to \neg P) \to (P \to Q)$: $M = \langle \{0,1,2,3\}, R, \models_i \rangle$, where R is reflexive and transitive such that $0R1$, $1R2$ and $2R3$; and $2 \models_i P$, $3 \models_i P$ and $3 \models_i Q$. Then $M, 0 \not\models_i (\neg Q \to \neg P) \to (P \to Q)$.
(b) ... (f) are treated similarly.

Solution 8.18. * $M = \langle S, R, \models_i \rangle$ is a Kripke model, for:
1. R is reflexive, i.e., $H \vdash_i H$,
2. R is transitive, i.e., if $H' \vdash_i H$ and $H'' \vdash_i H'$, then $H'' \vdash_i H$,

3. if $H \models_i P$ and HRH', then $H' \models_i P$, i.e., if $H \vdash_i P$ and $H' \vdash_i H$, then $H' \vdash_i P$.
Proof of: $M, H \models_i A$ iff $H \vdash_i A$, for $H \in S$, i.e., $H \mid H$.
1. For $A = P$ (atomic), by definition.
2. Induction hypothesis: $M, H \models_i B$ iff $H \vdash_i B$, and $M, H \models_i C$ iff $H \vdash_i C$.

a) $A = B \wedge C$: $M, H \models_i B \wedge C$ iff $M, H \models_i B$ and $M, H \models_i C$,
(ind. hyp.) iff $H \vdash_i B$ and $H \vdash_i C$,
iff $H \vdash_i B \wedge C$.

b) $A = B \vee C$: $M, H \models_i B \vee C$ iff $M, H \models_i B$ or $M, H \models_i C$,
(ind. hyp.) iff $H \vdash_i B$ or $H \vdash_i C$,
(Exercise 8.14) iff $H \vdash_i B \vee C$.

c) $A = B \rightarrow C$: $M, H \models_i B \rightarrow C$ iff for all H' in S such that HRH',
if $M, H' \models_i B$, then $M, H' \models_i C$.
(ind. hyp.) iff for all H' in S such that $H' \vdash_i H$,
if $H' \vdash_i B$, then $H' \vdash_i C$. (†)

To show: (†) iff $H \vdash_i B \rightarrow C$. i) Suppose $H \vdash_i B \rightarrow C$. Then (†) easily follows. ii) Suppose (†). By Exercise 8.15 $H \wedge B$ is intuitionistically provably equivalent to a disjunction $H_1 \vee \ldots \vee H_n$ such that for all H_j, $1 \leq j \leq n$, $H_j \mid H_j$, i.e., H_j is an element of S. Now $H_j \vdash_i H \wedge B$. So, $H_j \vdash_i H$ and $H_j \vdash_i B$. Hence, by (†), for all j, $1 \leq j \leq n$, $H_j \vdash_i C$. Therefore, by $\vee$-elimination, $H_1 \vee \ldots \vee H_n \vdash_i C$; so, $H \wedge B \vdash_i C$. Consequently, $H \vdash_i B \rightarrow C$.
Now suppose $\models_i A$. One easily checks that $P \rightarrow P \mid P \rightarrow P$, i.e., $P \rightarrow P$ is an element of S. So, $M, P \rightarrow P \models_i A$. Therefore, $P \rightarrow P \vdash_i A$, i.e., $\vdash_i A$.

Solution 8.19. a) $\neg\exists x \in V[\neg A(x)] \rightarrow \forall x \in V[\neg\neg A(x)]$: Suppose $\neg\exists x \in V[\neg A(x)]$, a in V and $\neg A(a)$. Then $\exists x \in V[\neg A(x)]$. Contradiction. Therefore, $\forall x \in V[\neg\neg A(x)]$.
$\forall x \in V[\neg\neg A(x)] \rightarrow \neg\exists x \in V[\neg A(x)]$: Suppose $\forall x \in V[\neg\neg A(x)]$ and $\exists x \in V[\neg A(x)]$. Then for some a in V, both $\neg A(a)$ and $\neg\neg A(a)$. Contradiction. Therefore, $\neg\exists x \in V[\neg A(x)]$.
b) Suppose $\neg\neg\forall x \in V[A(x)]$, a in V and $\neg A(a)$. Then $\neg\forall x \in V[A(x)]$. Contradiction. Therefore $\forall x \in V[\neg\neg A(x)]$.
c) Let $V = \{a\}, A(a) := P \vee \neg P$. Then $\forall x \in V[\neg\neg A(x)]$ iff $\neg\neg(P \vee \neg P)$ and $\forall x \in V[A(x)]$ iff $P \vee \neg P$. Now $\neg\neg(P \vee \neg P)$ is intuitionistically valid, while $P \vee \neg P$ is intuitionistically invalid (see Section 8.1.2).
d1)

$F\ \neg\exists x[\neg A(x)] \rightarrow \forall x[\neg\neg A(x)]$
$T\ \neg\exists x[\neg A(x)],\ F\ \forall x[\neg\neg A(x)]$
$T\ \neg\exists x[\neg A(x)],\ F\ \neg\neg A(a_1)$
$T\ \neg\exists x[\neg A(x)],\ T\ \neg A(a_1)$
$F\ \exists x[\neg A(x)],\ T\ \neg A(a_1)$
$F\ \neg A(a_1),\ T\ \neg A(a_1)$
$TA(a_1),\ T\ \neg A(a_1)$
$TA(a_1),\ FA(a_1)$
closure
Hence, $\vdash_i' \neg\exists x[\neg A(x)] \rightarrow \forall x[\neg\neg A(x)]$

$F\ \forall x[\neg\neg A(x)] \rightarrow \neg\exists x[\neg A(x)]$
$T\ \forall x[\neg\neg A(x)],\ F\ \neg\exists x[\neg A(x)]$
$T\ \forall x[\neg\neg A(x)],\ T\ \exists x[\neg A(x)]$
$T\ \forall x[\neg\neg A(x)],\ T\ \neg A(a_1)$
$T\ \neg\neg A(a_1),\ T\ \neg A(a_1)$
$F\ \neg A(a_1),\ T\ \neg A(a_1)$
$TA(a_1),\ T\ \neg A(a_1)$
$TA(a_1),\ FA(a_1)$
closure
Hence, $\vdash' \forall x[\neg\neg A(x)] \rightarrow \neg\exists x[\neg A(x)]$

d2) By $\exists$-introduction, $\neg\exists x[\neg A(x)], \neg A(a) \vdash \exists x[\neg A(x)]$. Also $\neg\exists x[\neg A(x)], \neg A(a) \vdash \neg\exists x[\neg A(x)]$. Therefore, by $\neg$-introduction, $\neg\exists x[\neg A(x)] \vdash \neg\neg A(a)$. Hence, by $\forall$-introduction, $\neg\exists x[\neg A(x)] \vdash \forall x[\neg\neg A(x)]$.
$\forall x[\neg\neg A(x)], \neg A(a) \vdash \neg A(a) \wedge \neg\neg A(a)$, and hence, by weak negation elimination, $\forall x[\neg\neg A(x)],\ \neg A(a) \vdash B \wedge \neg B$ for any B. Hence, by $\exists$-elimination,
$\forall x[\neg\neg A(x)],\ \exists x[\neg A(x)] \vdash B \wedge \neg B$. So, by $\neg$-introduction, $\forall x[\neg\neg A(x)] \vdash \neg\exists x[\neg A(x)]$.

Solution 8.20. $Z(\exists x[B(x,a_1,\ldots,a_m)]) := \{B(a_k,a_1,\ldots,a_m),\ B(a_1,a_1,\ldots,a_m),\ldots, B(a_m,a_1,\ldots,a_m)\}$, and let $Z(\forall x[B(x,a_1,\ldots,a_m)]) := \{B(a_k,a_1,\ldots,a_m)\}$, where a_k is the first free variable not occurring in $\exists x[B(x,a_1,\ldots,a_m)]$, $\forall x[B(x,a_1,\ldots,a_m)]$ resp. If V is a set of formulas of the form $\exists x[B(x,a_1,\ldots,a_m)]$ or $\forall x[B(x,a_1,\ldots,a_m)]$, then $Z(V)$:= the union of all $Z(C)$ for C in V. Clearly, if V is finite, then $Z(V)$ is finite and the number of elements of $Z(V)$ can easily be estimated from the definitions above.
Now suppose A is a prenex formula $Q^1x_1\ldots Q^nx_n[B]$, where $Q^i = \forall$ or $Q^i = \exists$. Let $V_1 := Z(A)$ and by induction $V_k := Z(V_{k-1}),\ k = 2,\ldots,n$. Note that $\vdash' A$ iff V_1 contains at least one tableau-provable formula. By an easy induction with respect to $k,\ k = 2,\ldots,n$, we find that $\vdash' A$ iff V_k contains at least one tableau-provable formula. Consequently, A is tableau-provable iff V_n contains at least one tableau-provable formula. However, all formulas in V_n are quantifier-free and hence are decidable. And since V_n is finite, we can decide by a finite method whether there is a tableau-provable formula in V_n.

Solution 8.21. Suppose $\vdash_i' \exists x[A(x)]$ and a_k is the only free variable in $A(a_k)$. A tableau-proof of $\exists x[A(x)]$ starts with
$$\begin{array}{c} F\ \exists x[A(x)] \\ FA(a_k) \end{array}$$
and then proceeds to closure. By replacing in a given tableau-proof of $\exists x[A(x)]$ the upper sequent, $F\exists x[A(x)]$, by $F\forall x[A(x)]$, one obtains a tableau-proof of $\forall x[A(x)]$.

Solution 8.22. Let $M = \langle S,\ R,\ U,\ \models_i\rangle$ be a Kripke model (for intuitionistic predicate logic) with constant domain U. $M,s \models_i \forall x[P(x) \vee Q]$:= for all s' in S such that sRs' and for all $n \in U(s')$, $M,s' \models_i P(a)[n]$ or $M,s' \models_i Q$. (1)
$M,s \models_i \forall x[P(x)] \vee Q$:= $M,s \models_i Q$ or for all s' in S such that sRs' and for all $n \in U(s')$, $M,s' \models_i P(a)[n]$. (2)
(1) $\rightarrow$ (2): If $M,s \models_i Q$, we are done. If $M,s \not\models_i Q$, then by (1) for all $n \in U(s)$, $M,s \models_i P(a)[n]$. Now suppose sRs' and $n \in U(s')$; then, since U is constant, $n \in U(s)$ and, since $M,s \models_i P(a)[n]$ and sRs', $M,s' \models_i P(a)[n]$, which was to be shown.

Solution 8.23. Suppose $\models A'$ in $S4$. We want to show that $\models_i A$ (intuitionistically). So, let $M = \langle S,\ R,\ U,\ \models_i\rangle$ be a Kripke model (for intuitionistic predicate logic). Define the Kripke model $M' = \langle S,\ R,\ U,\ \models\rangle$ for $S4$ as follows: $s \models P := s \models_i P$. Then, since R is reflexive and transitive, M' is a Kripke model for the modal logic $S4$ (satisfying the extra condition: if $M',s \models A$ and sRt, then $M',t \models A$).
Claim: $M,s \models_i A$ intuitionistically iff $M',s \models A'$ in $S4$. Now, since $\models A'$ in $S4$, $M' \models A'$ in $S4$ and hence, $M \models_i A$ (intuitionistically).
Proof of claim : for $A = P$ (atomic), $M,s \models_i P$ (intuitionistically) iff

for all t in S, if sRt, then $M,t \models_i P$, iff
for all t in S, if sRt, then $M',t \models P$, iff
$M',s \models \Box P$ in $S4$, iff
$M',s \models P'$ in $S4$.
Induction hypothesis: the claim is correct for B and C. We shall show that the claim also holds for $B \to C$ and for $\forall x[B(x)]$, leaving the other cases to the reader.
$M,s \models_i B \to C$ (intuitionistically) iff for all t in S, if sRt and $M,t \models_i B$, then $M,t \models_i C$. By induction hypothesis, this is equivalent to: for all t in S, if sRt and $M',t \models B'$ in $S4$, then $M',t \models C'$ in $S4$. And this latter expression is equivalent to $M',s \models \Box(B' \to C')$, i.e., $M',s \models (B \to C)'$ in $S4$.
$M,s \models_i \forall x[B(x)]$ iff for all t with sRt and for all n in $U(t)$, $M,t \models_i B(a)[n]$,
(ind. hyp.) iff for all t with sRt and for all n in $U(t)$, $M',t \models B(a)'[n]$ in $S4$,
iff for all t with sRt, $M',t \models \forall x[B(x)']$ in $S4$,
iff $M',s \models \Box\forall x[B(x)']$ in $S4$
iff $M',s \models (\forall x[B(x)])'$ in $S4$.
Conversely, suppose $\models_i A$ intuitionistically. To show: $\models A'$ in $S4$. So, let $M = \langle W, R, U, \models\rangle$ be a Kripke model for $S4$, i.e., R is reflexive and transitive. Define $M_i = \langle W, R, U, \models_i\rangle$ as follows: $w \models_i P :=$ for all w' in W, if wRw', then $w' \models P$. Then M_i is a Kripke model for intuitionistic logic, since from the transitivity of R it follows that: if $w \models_i P$ and wRw', then $w' \models_i P$.
Claim: $M_i,w \models_i A$ (intuitionistically) iff $M,w \models A'$ in $S4$. So, since $\models_i A$ (intuitionistically), $M_i \models_i A$ intuitionistically and hence, $M \models A'$ in $S4$.
Proof of claim : For $A = P$ (atomic), $M_i,w \models_i P$ (intuitionistically) iff $w \models_i P$,
iff for all w' in W, if wRw', then $w \models P$,
iff $M,w \models \Box P$,
iff $M,w \models P'$ in $S4$.
Induction hypothesis: the claim holds for B and C. We shall show that the claim holds for $B \to C$ and for $\forall x[B(x)]$, leaving the other cases to the reader.
$M_i,w \models_i B \to C$ (intuitionistically) := for all w' in W, if wRw' and $M_i,w' \models_i B$, then $M_i,w' \models_i C$. By the induction hypothesis this is equivalent to: for all w' in W, if wRw' and $M,w' \models B'$, then $M,w' \models C'$. And this latter expression is equivalent to $M,w \models \Box(B' \to C')$, in other words, $M,w \models (B \to C)'$ in $S4$.
$M_i,w \models_i \forall x[B(x)]$:= for all w' with wRw' and for all n in $U(w')$, $M_i,w' \models_i B(a)[n]$,
iff (ind. hyp.) for all w' with wRw' and for all n in $U(w')$, $M,w' \models B(a)'[n]$,
iff for all w' with wRw', $M,w' \models \forall x[B(x)']$. And the latter expression is equivalent to $M,w \models \Box\forall x[B(x)']$ and hence to $M,w \models (\forall x[B(x)])'$ in $S4$.

Solution 8.24. Suppose $f : \sigma_{01} \to \mathbb{N}$ were a bijection. Then $\forall\alpha \in \sigma_{01}\ \exists k \in \mathbb{N}\ [k = f(\alpha)]$. By Brouwer's principle, $\forall\alpha \in \sigma_{01}\ \exists m \in \mathbb{N}\ \exists k \in \mathbb{N}\ \forall\beta \in \sigma_{01}\ [\bar{\beta}m = \bar{\alpha}m \to k = f(\beta)]$. Let $\alpha = \underline{0}$. Then there are $m \in \mathbb{N}$ and $k \in \mathbb{N}$ such that

$$\forall\beta \in \sigma_{01}\ [\bar{\beta}m = \bar{\alpha}m = \bar{\underline{0}} \to k = f(\beta)].$$

Now, take β such that $\bar{\beta}m = \bar{\alpha}m = \bar{\underline{0}}m$ and $\beta(m) \neq \alpha(m)$. Then $\beta \neq \alpha$, but $k = f(\beta) = f(\alpha)$. So, f is not injective. Contradiction. The proof for σ_{01mon} is similar. The classical function $f : \sigma_{01mon} \to \mathbb{N}$, defined by $f(\underline{0}) = 0, f(\underline{1}) = 1, f(0\underline{1}) = 2, f(00\underline{1}) = 3$, etc. is intuitionistically not well defined: we cannot determine the value of $f(\alpha)$ for the sequence α defined by $\alpha(n) = 0$ if at stage n I do not have a proof of Goldbach's conjecture G and $\alpha(n) = 1$ if at stage n I do have a proof of G.

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