Chapter 1
Logic; a First Impression

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Abstract In this introductory chapter the topic of the book is explained: distinguishing valid patterns of reasoning from invalid ones. The validity may depend on the meaning of connectives like 'if ..., then ...', 'and', 'or' and 'not', in which case one speaks of propositional logic. But the validity may also depend on the meaning of the quantifiers 'for all' and 'for some', in which case one speaks of predicate logic. If we extend the logical language with symbols for addition and multiplication of natural numbers, Gödel's famous incompleteness theorems show up. In order to have meaning, logical formulae presuppose a universe of discourse, or a set, which may be finite or infinite. In particular infinite sets have peculiar properties. If the validity of a reasoning pattern also depends on the meaning of modalities, like 'necessary' and 'possible', one speaks of modal logic. Modal logic helps to clarify or solve certain issues in the philosophy of language. It turns out that validity of an argument is also dependent on philosophical presuppositions. Changing the philosophical point of view may result in intuitionistic logic. The language of logic may be used as a programming language: Prolog (Programming in Logic); and the theory of sets is the basis for relational databases and the query language SQL; another application of logic is social choice theory. Fallacies and unfair discussion methods are abundantly present in daily discourse and hence deserve attention too.

1.1 General

The study of logic is the study of reasoning. The basic question in this book is what conclusions can be drawn with absolute certainty from a particular set of premisses. To illustrate what we mean by this, let us consider Euclid's geometry.

Euclid (c. 330 B.C.) began his geometry books, called the 'Elements', with a precise formulation of the geometrical axioms (postulates, premisses) on which he wanted to found his geometry. For instance, one of the axioms says that it is possible to draw a straight line through any two points. Next, Euclid used (informal) reason-

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ing to deduce theorems from the geometrical axioms, for instance, the theorem that any triangle which is equiangular also is isosceles.

premisses (postulates, axioms)

reasoning (studied in logic)

conclusion (theorem)

In this book deductive logic is studied and not probabilistic logic which studies the question what conclusions can be drawn from a set of premisses with a certain probability. An example of the latter is, for instance, the question how likely it is that a person gets a certain disease when he has been in touch with other people having the disease.

Logic has a long history: it was studied by the Stoics (see [1, 5, 10, 12]), by Aristotle (see [1, 10, 11]) and by many medieval philosophers (see [1, 2, 10, 13]); the study of logic was greatly advanced by the works of Boole (1847, 1854) [3, 4], Frege (1879) [6, 7] and Russell (1910) [14], becoming a full-fledged discipline with the work of Gödel (1930-1931) [9, 15].

In addition to the term ‘logic’, one also encounters in the literature the expressions ‘mathematical logic’, ‘philosophical logic’ and ‘formal (or symbolic) logic’, which are used to stress one of the many aspects of logic.

1.2 Propositional Logic

Below we give some concrete simple arguments from different fields.

Example 1.1.

a1) If 1 = 2, then I am the Pope of Rome.
I am not the Pope of Rome.
Therefore: not 1 = 2.

a2) If 1 = 2, then I am the Pope of Rome.
Not 1 = 2.
Therefore: I am not the Pope of Rome.

b1) If triangle ABC is equiangular, then it is isosceles.
Triangle ABC is not isosceles.
Therefore: Triangle ABC is not equiangular.

b2) If triangle ABC is equiangular, then it is isosceles.
Triangle ABC is not equiangular.
Therefore: Triangle ABC is not isosceles.

c1) If it snows, then it is cold.
It is not cold.
Therefore: It does not snow.

c2) If it snows, then it is cold.
It does not snow.
Therefore: It is not cold.

Note that all the arguments above consist of two *premisses* and one (putative) *conclusion*. Further note that all arguments a1, b1 and c1 in Example 1.1 have the same structure, namely, the following *pattern of reasoning*:

1. if P_1, then P_2 / not P_2 / therefore: not P_1

$$\frac{\begin{array}{r}P_1 \rightarrow P_2\\ \neg P_2\end{array}}{\neg P_1}$$

Using $\rightarrow$ for 'if ..., then ...' and $\neg$ for 'not', this pattern of reasoning can be represented by the schema to the above right. This pattern of reasoning is called *Modus Tollens*.

The arguments a2, b2 and c2 in Example 1.1 also have the same pattern, namely,

if P_1, then P_2
not P_1
therefore: not P_2

$$\frac{\begin{array}{l}P_1 \rightarrow P_2\\ \neg P_1\end{array}}{\neg P_2}$$

The first pattern of reasoning is *valid*, i.e., it is impossible to replace P_1, P_2 by such propositions that the premisses $P_1 \rightarrow P_2$ and $\neg P_2$ result in true propositions and that at the same time the conclusion $\neg P_1$ results in a false proposition. For suppose P_1, P_2 are interpreted as propositions P_1^* (e.g., it snows) and P_2^* (e.g., it is cold) respectively and suppose that

'if P_1^*, then P_2^*' (if it snows, then it is cold) and 'not P_2^*' (it is not cold) are both true.

Then 'not P_1^*' (it does not snow) must be true too. For suppose that P_1^* (it snows) would be true; then – by the first premiss – P_2^* (it is cold) would be true too. This is a contradiction with the second premiss 'not P_2^*' (it is not cold).

Note that this insight does not depend on the particular choice of P_1^* and P_2^*. P_1^* and P_2^* may be any propositions from number theory, geometry, economics, philosophy, from daily life, and so on.

Concrete arguments which have an underlying pattern of reasoning which is valid are called *correct arguments*. Thus the arguments a1, b1 and c1 in Example 1.1 are correct, since they are particular instances of the valid pattern 1:

$$\frac{\begin{array}{r}P_1 \rightarrow P_2\\ \neg P_2\end{array}}{\neg P_1}$$

We say that $\neg P_1$ is a logical (or valid) consequence of $P_1 \rightarrow P_2$ and $\neg P_2$. Notation: $P_1 \rightarrow P_2, \neg P_2 \models \neg P_1$.

We know that it is impossible for the premisses of a correct argument to be true and at the same time its conclusion to be false. Whether the premisses and the conclusion of a concrete argument are true or false is not the business of the logician, but of the mathematician, the economist, the philosopher, the physicist, and so on, depending on what these propositions are about. The logician is not concerned with the truth or falsity of the axioms of geometry. Given a concrete argument, he is only concerned with the validity or invalidity of the underlying pattern of reasoning and if this is valid, he can only say that *if* the premisses of the concrete argument in question are true, *then* the conclusion must likewise be true.

Warning: If a pattern of reasoning is valid, a concrete argument with that pattern does not imply that the premisses are true, nor that the conclusion is true.

Counterexample Pattern 1 $\begin{array}{l} P_1 \to P_2 \\ \underline{\qquad \neg P_2} \\ \neg P_1 \end{array}$ is a valid pattern of reasoning.

Now take P_1^*: Bill Gates is wealthy.
P_2^*: Bill Gates owns all the gold in Fort Knox.
Then we get the following concrete argument:
If Bill Gates is wealthy, then he owns all the gold in Fort Knox.
Bill Gates does not own all the gold in Fort Knox.
Therefore: Bill Gates is not wealthy.
So, we have a correct argument, since the underlying pattern is valid, with a false conclusion. This is only possible if at least one of the premisses is false. And indeed, the first premiss is actually false. Correctness of a concrete argument means that it is impossible that all the premisses are true and at the same time the conclusion false, in other words: *if* all premisses are true (which actually may not be the case), *then* the conclusion must be true too.

From the definition of validity it follows that a pattern of reasoning is *invalid* if it is possible to interpret $P_1, P_2, \ldots$ in such a way that all premisses result in true propositions while at the same time a false one results from the conclusion. An example of an invalid pattern is the following one:

$$\begin{array}{l} P_1 \to P_2 \\ \underline{\neg P_1 \qquad\quad} \\ \qquad\quad \neg P_2 \end{array}$$

underlying the concrete arguments a2, b2 and c2 in Example 1.1. Taking
P_1^*: Bill Gates owns all the gold in Fort Knox,
P_2^*: Bill Gates is wealthy,
results in the following concrete argument :
If Bill Gates owns all the gold in Fort Knox, then he is wealthy.
Bill Gates does not own all the gold in Fort Knox.
Therefore: Bill Gates is not wealthy.
So, all the premisses are true, while the conclusion is false.

We say that $\neg P_2$ is not a logical (or valid) consequence of $P_1 \to P_2$ and $\neg P_1$. Notation: $P_1 \to P_2, \neg P_1 \not\models \neg P_2$.

Concrete arguments with an underlying pattern of reasoning which is invalid are called *incorrect*. So, the arguments a2, b2 and c2 in Example 1.1 are incorrect.

Warning: A concrete argument with an underlying pattern of reasoning which is invalid does not necessarily imply that the conclusion is false; the conclusion may be true, but in that case the truth of the conclusion does not depend on the truth of the premisses.

Counterexample: The pattern $\dfrac{\begin{array}{l} P_1 \to P_2 \\ \neg P_1 \end{array}}{\neg P_2}$ is an invalid pattern of reasoning.

Taking P_1^*: I own all the gold in Fort Knox,
P_2^*: I am wealthy,
we obtain the following concrete incorrect argument with true premisses and a true conclusion:

If I own all the gold in Fort Knox, then I am wealthy.
I do not own all the gold in Fort Knox.
Therefore: I am not wealthy.

Below is a non exhaustive list of valid patterns of reasoning frequently used in practice:

Example 1.2 (some valid patters of reasoning).

1.	if P_1, then P_2 not P_2 therefore: not P_1	$\dfrac{\begin{array}{l} P_1 \to P_2 \\ \neg P_2 \end{array}}{\neg P_1}$	*Modus Tollens*
2.	if P_1, then P_2 P_1 therefore: P_2	$\dfrac{\begin{array}{l} P_1 \to P_2 \\ P_1 \end{array}}{P_2}$	*Modus Ponens (MP)*
3.	P_1 if and only if (iff) P_2 not P_1 therefore: not P_2	$\dfrac{\begin{array}{l} P_1 \rightleftarrows P_2 \\ \neg P_1 \end{array}}{\neg P_2}$	
4.	not (P_1 and P_2) P_1 therefore: not P_2	$\dfrac{\begin{array}{l} \neg(P_1 \wedge P_2) \\ P_1 \end{array}}{\neg P_2}$	
5.	P_1 or P_2 not P_2 therefore: P_1	$\dfrac{\begin{array}{l} P_1 \vee P_2 \\ \neg P_2 \end{array}}{P_1}$	

We have introduced above $\rightleftarrows$ for 'if and only if (iff)', $\wedge$ for 'and', $\vee$ for the inclusive 'or', i.e., $P_1 \vee P_2$ stands for 'P_1 or P_2 or both P_1 and P_2'. The reader should verify that all patterns in Example 1.2 are valid.

The following two patterns of reasoning are frequently used in practice, although they are invalid:

if P_1, then P_2	$P_1 \to P_2$
not P_1	$\neg P_1$
therefore: not P_2	$\neg P_2$

if P_1, then P_2	$P_1 \to P_2$
P_2	P_2
therefore: P_1	P_1

So, the following concrete arguments are not correct:

> If it rains, then the street becomes wet.
> It does not rain.
> Therefore: The street does not become wet.

> If it is raining, then the street becomes wet.
> The street becomes wet.
> Therefore: It is raining.

It should now be clear that the expressions in patterns of reasoning are built from $P_1, P_2, P_3, \ldots$ using the connectives $\rightleftarrows, \to, \wedge, \vee$ and $\neg$. In fact, we have introduced a new language for representing patterns of reasoning, the alphabet of which consists of the symbols:

$P_1,\ P_2,\ P_3, \ldots$	called atomic formulas
$\rightleftarrows, \to, \wedge, \vee, \neg$	called connectives
(,)	called parentheses.

Of course, $\wedge P_1 P_2 \neg$ is not a well-formed expression of this language. Let us define how the well-formed expressions or formulas of this language are built up.

Formulas:
1. $P_1, P_2, P_3, \ldots$ are formulas. In other words, if P is an atomic formula, then P is a formula.
2. If A and B are formulas, then $(A \rightleftarrows B)$, $(A \to B)$, $(A \wedge B)$ and $(A \vee B)$ are formulas.
3. If A is a formula, then $(\neg A)$ is a formula too.

Example 1.3. P_1, P_3 and P_5 are formulas.
$(\neg P_1)$ and $(P_3 \to P_5)$ are formulas.
$((\neg P_1) \vee (P_3 \to P_5))$ is a formula.

We can minimize the need for parentheses by agreeing that we leave out the most outer parentheses in a formula and that in

$$\rightleftarrows,\ \to,\ \wedge,\ \vee,\ \neg$$

any connective has a higher rank than any connective to the right of it and a lower rank than any connective to the left of it. According to this convention $\neg P_1 \to P_2 \vee P_3$ means $(\neg P_1) \to (P_2 \vee P_3)$, because $\to$ has a higher rank than $\neg$ and $\vee$, and it does not mean $((\neg P_1) \to P_2) \vee P_3$ nor $\neg((P_1 \to P_2) \vee P_3)$. According to the convention just mentioned the expression $\neg P_1 \vee P_3 \to P_5$ stands for the formula $((\neg P_1) \vee P_3) \to P_5$, because $\rightarrow$ has the highest rank and $\vee$ has a higher rank than $\neg$. Notice that the formula $(\neg P_1) \vee (P_3 \rightarrow P_5)$ is a different formula with a quite different meaning.

It is important to notice that the validity or invalidity of the reasoning patterns above does not depend on the content of the P_1, P_2, but solely on the meaning of the connectives $\rightleftarrows$, $\rightarrow$, $\wedge$, $\vee$ and $\neg$. In *propositional logic* one studies the (in)validity of reasoning patterns of which the (in)validity is completely determined by the meaning of the connectives between the propositions in question.

In Chapter 2 a characterization of validity is given both in semantic and in syntactic terms, and it is shown that these two characterizations are equivalent, which gives us confidence that we have given an adequate definition of the notion in question.

In logic we study the validity or invalidity of patterns of reasoning. The expressions in these patterns are formulas of the language specified above. This language is called the *object-language*, because it is the object of study. The language used in studying the object-language is called the *meta-language* or the *observer's language*. In our case the meta-language will be part of English. The situation is similar to the one where an English speaking person is studying Russian, in which case Russian is the object language and English is the meta-language. It is important to keep in mind this distinction between the object-language and the meta-language; otherwise, one may get involved in paradoxes like the antinomy of the liar.

That intuition is not always a reliable guide in judging correctness of a given argument will become clear from a few examples. At the end of this section are a few exercises in which the reader is challenged to judge on intuitive grounds whether the argument given is correct. Although the arguments are simple, they are sufficiently complex to puzzle an untrained intuition. When the reader has finished Chapter 2 he or she will be able to judge the correctness of these arguments with certainty!

Exercise 1.1. Check whether the following argument is correct by translating the propositions in the argument into the language of propositional logic and by determining whether the corresponding pattern of reasoning is valid.
If Socrates did not die of old age [$\neg O$], then the Athenians sentenced him to death [D].
The Athenians did not sentence Socrates to death.
If Socrates died from poison [P], then he did not die of old age.
Therefore: Socrates did not die from poison.

Exercise 1.2. Check whether the following argument is correct.
If the weather is nice [N], then John will come. [J].
The weather is not nice.
Therefore: John will not come,

Exercise 1.3. Check whether the following argument is correct.
John comes [J] if the weather is nice [N].
John comes.
Therefore: the weather is nice.

Exercise 1.4. Check whether the following argument is correct.
John comes [J] *only if* the weather is nice [N].
John comes.
Therefore: the weather is nice.

Exercise 1.5. Check whether the following argument is correct.
It is not the case that John gets promotion [P] and at the same time not a higher salary [$\neg S$].
John does not get promotion or he is not diligent [$\neg D$].
John is diligent.
Therefore, John will not get a higher salary.

1.3 Sets; Finite and Infinite

The quantifiers $\forall$ (for all x) and $\exists$ (for some x) in (the language of) predicate logic are ranging over a certain domain: the set of all persons, the set of all natural numbers, the set of all real numbers, etc. In fact, there are many possible domains, where a domain is just a set of objects. These sets may be finite, like the set consisting of Ann, Bob and Coby, or the set $\{1, 2, 3\}$ consisting of the numbers 1, 2 and 3, but they may be also infinite, like the set $\mathbb{N}$ of all natural numbers. We will study these sets more closely in Chapter 3 with particular attention for the properties of infinite sets. As we shall see, infinite sets have properties quite different from the properties of finite sets. For instance, a proper part of a finite set will be smaller than the original set. But as we shall see in Chapter 3, this property does not hold for infinite sets: a proper part of an infinite set may be equally large as the original set. A simple example is the set $\mathbb{N}_{even} = \{0,2,4,6,\ldots\}$ of the even natural numbers which is a proper subset of the set $\mathbb{N} = \{0,1,2,3,4,5,6,\ldots\}$ of all natural numbers. That these sets are equally large may be seen as follows: there is a one-one correspondence between the elements of both sets.

$$\begin{array}{rcccccc} \mathbb{N}: & 0 & 1 & 2 & 3 & 4 & \ldots \\ & | & | & | & | & | & \ldots \\ \mathbb{N}_{even}: & 0 & 2 & 4 & 6 & 8 & \ldots \end{array}$$

1.4 Predicate Logic

An example of a simple argument which we cannot adequately analyse with the means developed in propositional logic, is the following:

> All men are mortal.
> Socrates is a man.
> Therefore, Socrates is mortal.

If we translate this argument in the formal language of propositional logic, we find as the underlying pattern of reasoning:

$$\frac{\begin{array}{c}P_1\\P_2\end{array}}{P_3}$$

and we know this pattern is invalid since we can substitute true propositions for P_1 and P_2 and at the same time a false one for P_3. On the other hand, it seems to us that the argument above, about Socrates, is correct.

The point is that in the translation of the premisses into P_1 and P_2 and of the conclusion into P_3, the internal structure of the sentences is lost: P_1, P_2 and P_3 are unrelated atomic formulas. But the premisses and the conclusion of the argument are not unrelated; in fact, it is this relationship which causes the argument to be correct. We have to exhibit the internal subject-predicate structure of the premisses and the conclusion in order to make visible that these three sentences are related and in order to see that the underlying pattern of reasoning is valid.

The structure of the argument above is the following pattern:

For all objects x, if x is a person, then x is mortal.	$\forall x[P(x) \to M(x)]$
Socrates is a person.	$P(c)$
Therefore: Socrates is mortal.	$M(c)$

Using $\forall x$ for 'for all x', $P(x)$ for 'x has the property P (to be a Person)', $M(x)$ for 'x has the property M (to be Mortal)' and c for 'Socrates', this pattern of reasoning can be represented by the schema to the above right .

Notice that the following arguments have the same underlying pattern of reasoning:

All philosophers are smart	All natural numbers are positive
John is a philosopher	5 is a natural number
Therefore, John is smart	Therefore, 5 is positive

The pattern just mentioned is *valid*, i.e., it is impossible to choose a domain of individuals and to give to P, M and c appropriate meanings such that from the premisses $\forall x[P(x) \to M(x)]$ and $P(c)$ true propositions result and at the same time from the conclusion $M(c)$ a false proposition.

But, for instance, the pattern

$$\frac{\begin{array}{c}\forall x[P(x) \to M(x)]\\M(c)\end{array}}{P(c)}$$

is *invalid*, since it is possible to choose a domain, to interpret the symbols P, M as predicates P^*, M^* over the domain chosen and to interpret the symbol c as an element c^* in the domain, such that true propositions result from the premisses and a false proposition from the conclusion. For instance, take as domain the set of all persons, let P^* be the predicate 'is a man', M^* the predicate 'is mortal' and let c^* be the element 'Queen Maxima'. Then $\forall x[P(x) \to M(x)]$ yields the true proposition: For every person x, if x is a man, then x is mortal. $M(c)$ yields the true proposition: Queen Maxima is mortal. But $P(c)$ yields the false proposition: Queen Maxima is a man.

Next consider the following elementary argument:

John is ill
Therefore: someone is ill.

In order to exhibit the structure of this argument, we need one more symbol: $\exists x$, for 'there is at least one x such that ...'. Then the underlying pattern of reasoning of this argument is the following:

$$\frac{I(c)}{\exists x[I(x)]}$$

This pattern of reasoning is again valid: it is impossible to take a domain D and to interpret the symbol I as a predicate I^* over D and the symbol c as an individual c^* in D such that a true proposition (c^* has the property I^*) results from the premiss $I(c)$ and at the same time a false proposition (there is at least one individual which has the property I^*) from the conclusion $\exists x[I(x)]$.

Note that the following arguments also have the same (valid) underlying pattern of reasoning and hence are correct.

5 is odd	Peter is rich
Therefore: some natural number is odd	Therefore: someone is rich

In order to be able to exhibit the internal subject-predicate structure of atomic sentences and the mutual relationships between them, we need the following symbols:

SYMBOLS	NAME	MEANING
$x, \ldots$	individual variables	individuals in a given domain
$P, M, I, \ldots$	predicate symbols	predicates over the given domain
$c, \ldots$	individual constants	concrete individuals in the given domain
$\rightleftarrows; \rightarrow; \wedge; \vee, \neg$	connectives	iff; if ..., then ...; and; or; not
$\forall, \exists$	quantifiers	for all; there exists
$[,], (,)$	parentheses	

In fact, we have introduced a new (subject-) predicate language, richer than the former propositional language, in which we can translate the subject-predicate structure of concrete arguments, exhibiting the underlying pattern of reasoning.

Of course, $P\exists\forall\neg$ is not a well-formed expression of this language and we have to define precisely what the well-formed expressions or formulas of this language are. We shall do so in Chapter 4; for the moment it is sufficient to work with a not precisely defined notion of formula.

It turns out that one can select a few elementary steps of reasoning, among which

$$\frac{A \quad A \to B}{B} \text{ called Modus Ponens,} \qquad \frac{A \wedge B}{B}, \qquad \frac{A}{A \vee B}, \qquad \frac{\forall x[A(x)]}{A(t)},$$

such that every valid pattern of reasoning, no matter how complex, can be built up from these elementary steps. This is Gödel's *Completeness Theorem*, 1930.

For instance, the following correct argument can be built up from the elementary steps just specified.

> John loves Jane and John is getting married.
> If John is getting married, then he is looking for another job.
> Hence: John is looking for another job or he does not love Jane.

The underlying pattern of reasoning is:

$$\frac{\begin{array}{c} P_1 \wedge P_2 \\ P_2 \to P_3 \end{array}}{P_3 \vee \neg P_1}$$

And indeed, this pattern can be built up from the elementary steps specified above as follows:

$$\cfrac{\cfrac{\cfrac{\text{premiss}}{P_1 \wedge P_2}}{P_2} \qquad \cfrac{\text{premiss}}{P_2 \to P_3}}{\cfrac{P_3}{P_3 \vee \neg P_1}}$$

And the four elementary steps of reasoning specified above can be supplemented by a few more elementary steps to form what is called Gentzen's [8] system of *Natural Deduction* – to be discussed in Subsection 2.7.2 – such that every correct argument can be simulated by an appropriate combination of the elementary steps in Gentzen's system (1934-5). We shall prove Gödel's completeness theorem in Chapter 2 for propositional logic and in Chapter 4 for predicate logic.

Another example: the argument above about Socrates which has as its underlying pattern of reasoning

$$\frac{\begin{array}{c} \forall x[P(x) \to M(x)] \\ P(c) \end{array}}{M(c)}$$

can be built up from the elementary steps in the system of Natural Deduction as follows:

$$\cfrac{\cfrac{\text{premiss}}{P(c)} \qquad \cfrac{\cfrac{\text{premiss}}{\forall x[P(x) \to M(x)]}}{P(c) \to M(c)}}{M(c)}$$

The schema above is called a logical *deduction* (in the system of Natural Deduction) of $M(c)$ from the premisses $\forall x[P(x) \to M(x)]$ and $P(c)$. We say that $M(c)$ is logically *deducible from* $\forall x[P(x) \to M(x)]$ and $P(c)$, since such a logical deduction exists.

In Chapter 4 a characterization of validity is given both in semantic and in syntactic terms, and it is shown that these two characterizations are equivalent, which gives us confidence that we have given an adequate definition of the notion in question.

Exercise 1.6. Check whether the following argument is correct by translating the propositions in the argument into the language of predicate logic and by determining

whether the corresponding pattern of reasoning is valid.
All gnomes have a beard or a conical cap.
Therefore: all gnomes have a beard or all gnomes have a conical cap.

Exercise 1.7. Check whether the following argument is correct.
All gnomes with a beard have a conical cap.
All gnomes have a beard.
Therefore: all gnomes have a conical cap.

Exercise 1.8. Check whether the following argument is correct.
There is a gnome with a beard.
There is a gnome with a conical cap.
Therefore: there is a gnome with a beard and a conical cap.

Exercise 1.9. Check whether the following argument is correct.
There is at least one gnome such that he has no beard or he has a conical cap.
There is at least one gnome who has a beard.
Therefore: there is at least one gnome who has a conical cap.

1.5 Arithmetic; Gödel's Incompleteness Theorem

In Chapter 2 we shall see that it is possible to fully capture the meaning of the logical connectives in terms of certain logical axioms. For instance, the meaning of the connective $\wedge$ can be fully captured by the following logical axioms: $A \wedge B \to A$, $A \wedge B \to B$ and $A \to (B \to A \wedge B)$. In other words, the propositional connectives can be characterized by appropriate logical axioms. This is expressed by the completeness theorem for propositional logic.

This result can be extended to predicate logic. In Chapter 4 we shall see that the meaning of the quantifiers $\forall$ and $\exists$ may also be fully captured by certain logical axioms. For instance, the meaning of $\forall$ is fully captured by the logical axioms $\forall x[A(x)] \to A(t)$, where t is either an individual variable or an individual constant, and $A(y) \to \forall x[A(x)]$, assuming there are no restrictions on the individual variable y. Gödel's completeness theorem for predicate logic (1930) expresses that the propositional connectives and the quantifiers can be characterized by appropriate logical axioms and rules.

Now, if we add to the logical language symbols $+$ and $\times$ to render addition and multiplication of natural numbers, naturally the question arises whether we may fully capture the meaning of these symbols in terms of certain arithmetical axioms, like $x + 0 = x$ and $x + sy = s(x + y)$, where sy denotes the successor of y. Amazingly, Kurt Gödel [9] proved in 1931 that it is impossible to fully capture the meaning of $+$ and $\times$ by arithmetical axioms. This is his famous Incompleteness theorem. This result has far reaching philosophical consequences.

We shall present Gödel's result and its philosophical implications in Chapter 5.

1.6 Modal Logic

The language of propositional and predicate logic may be further extended with a symbol $\Box$ for modalities, like necessary, obligatory, knowing that, etc. Depending on the precise meaning of the modality one may add several logical axioms for these modalities. For instance, $\Box A \to A$, in case $\Box$ stands for 'necessary' or for 'knowing that'. But for the modality 'obligatory' the axiom $\Box A \to A$ seems to be inappropriate: it is obligatory to stop for a red traffic light, but that does not imply that one actually does so. Since these modalities are used in several philosophical arguments, it is worthwhile to give a logical analysis of them.

By defining $\Diamond A$ by $\neg\Box\neg A$ we get modalities like 'possibly': $\neg A$ is not necessary, in other words, A is possible.

In Chapter 6 we will adapt the notions of validity and deducibility to modal logic and show that these two notions are again equivalent, just as in propositional and predicate logic. However, the notion of validity is now more complicated, since it is given in terms of possible worlds. $\Box A$ (A is necessary, or knowing A) is true in a given world means that A is true in all worlds imaginable from that given world. And $\Diamond A$ (A is possible) is true in a given world means that A is true in at least one world imaginable from that given world.

1.7 Philosophy of Language

In Chapter 7 we shall see that several problems in the philosophy of language are better understood or may be clarified by using the notion of possible world.

For instance, the *de re* - *de dicto* distinction in a sentence like 'it is possible that a republican will win' may be made clear by giving two different logical translations of this sentence:
de re: $\exists x[R(x) \wedge \Diamond W(x)]$: there is an individual x in the actual world w such that x is a Republican in world w and such that there is a world w' (imaginable from the actual world w) in which x wins.
de dicto: $\Diamond\exists x[R(x) \wedge W(x)]$: there is a world w' (imaginable from the actual world w) in which an individual x exists who is a Republican in that world w' and who wins in that world w'.
In the *de re* version the modality $\Diamond$ is within the scope of the existential quantifier $\exists$, while in the *de dicto* version the existential quantifier $\exists$ is within the scope of the modality $\Diamond$.

Another example is the difference between a name like 'Aristotle' and the corresponding description, like 'the most well known student of Plato'. Traditionally these two expressions were identified. But that causes the problem that a sentence like 'Aristotle is the most well known student of Plato' would be nothing more than a logical truth, or, using Kant's terminology, an analytic statement. Kripke proposed to solve this problem by conceiving proper names like 'Aristotle' as a rigid designator, i.e., as referring in all possible worlds to the same object. While the name 'Aristotle' refers in all possible worlds to the same object, also in the world in which he actually was a carpenter instead of a philosopher, the description 'the most well known student of Plato' may refer to different objects in different worlds. The description 'the most well known student of Plato' may help us to pick the proper reference of the name 'Aristotle', but it should not be identified with the name 'Aristotle'.

1.8 Intuitionism and Intuitionistic Logic

A *classical mathematician* studies the properties of mathematical objects like an astronomer, who studies the properties of celestial bodies. From a classical point of view, mathematical objects are like celestial bodies in the sense that they exist independently of us; they are created by God.

An *intuitionist* creates the mathematical objects himself. According to *Brouwer's intuitionism*, mathematical objects, like 5, 7, 12 and $+$, are mental constructions. A proposition about mathematical objects (like $5+7=12$) is true if one has a proof-construction that establishes it. Such a proof is again a mental construction.

> Mathematics is created by a free action, independent of experience [L.E.J. Brouwer, *Collected Works*, Vol. 1, p. 97].

Since, intuitionistically, the truth of a mathematical proposition is established by a proof – which is a particular kind of mental construction –, the meaning of the logical connectives has to be explained in terms of proof-constructions.
A proof of $A \wedge B$ is anything that is a proof of A and of B.
A proof of $A \vee B$ is, in fact, a proof either of A or of B, or yields an effective means, at least in principle, for obtaining a proof of one or other disjunct.
A proof of $A \to B$ is a construction of which we can recognize that, applied to any proof of A, it yields a proof of B. Such a proof is therefore an operation carrying proofs into proofs.
Intuitionists consider $\neg A$ as an abbreviation for $A \to \perp$, postulating that nothing is a proof of $\perp$ (falsity).

It follows that from an intuitionistic point of view it is reckless to assume $A \vee \neg A$. The validity of $A \vee \neg A$ means, intuitionistically, that we have a method adequate in principle to solve any mathematical problem A. However, consider *Goldbach's conjecture*, G, which states that each even number is the sum of two odd primes: $2 = 1+1$, $4 = 3+1$, $6 = 5+1$, $8 = 7+1$, $10 = 7+3$, $12 = 7+5$, $14 = 7+7$, $16 = 13+3$, $18 = 13+5, \ldots$. One can check only finitely many individual instances, while Goldbach's Conjecture is a statement about infinitely many (even) natural numbers. So far neither Goldbach's Conjecture, G, nor its negation, $\neg G$, has been proved. An intuitionist is therefore not in a position to affirm $G \vee \neg G$. A person who claims that he or she can provide a proof either of G or of $\neg G$ is called *reckless*.

Notice that from a classical point of view $A \vee \neg A$ is valid, since A is a statement about mathematical objects created independently of us, for which either A or $\neg A$ holds, although we may not know which one.

In Chapter 8 we will elaborate on Brouwer's intuitionism and see that his different philosophical point of view about the nature of mathematical objects results in a logic which is much more fine-grained, but also more difficult, than classical logic.

1.9 Applications

1.9.1 Programming in Logic: Prolog

Since Gödel's completeness theorem expresses that every valid pattern of reasoning can be built up from a certain small collection of logical rules in a logical proof-system (such as the system of Natural Deduction), the idea to equip a computer with these logical rules is quite natural. If we do so, the computer will be able to simulate reasoning and hence disposes of *Artificial Intelligence*. By adding to such a computer-program a number of data $A_1, \ldots, A_n$, concerning a small and well-described subject, the so-called *knowledge base*, the computer is able to draw conclusions from those data. If $A_1, \ldots, A_n$ represent someone's expertise, one speaks of an *expert system*. And if the knowledge base consists of Euclid's axioms for geometry or Peano's axioms for number theory or of axioms for some other part of mathematics, one speaks of *automated theorem proving*.

It was only in the early 1970's that the idea emerged to use the formal language of logic as a programming language. An example is PROLOG, which stands for PROgramming in LOGic. A logic program is simply a set of formulas (of a particular form) in the language of predicate logic. The formulas below constitute a logic program for kinship relations. The objects are people and there are two binary predicates 'parent of' (p), and 'grandparent of' (g).

- A_1: p(art, bob).
- A_2: p(art, bud).
- A_3: p(bob, cap).
- A_4: p(bud, coe).
- A_5: $g(x,z)$:- $p(x,y),\ p(y,z)$.

'art', 'bob', 'bud', 'cap' and 'coe' are individual constants and A_5 stands for $p(x,y) \wedge p(y,z) \to g(x,z)$. Now if we ask the question

?- g(art, cap)

the answer will be 'yes', corresponding with the fact that g(art, cap) can be logically deduced from the premisses or data $A_1, \ldots, A_5$.

But if we ask the question

?- g(art, amy)

the answer will be 'no', corresponding with the fact that g(art, amy) cannot be logically deduced from $A_1, \ldots, A_5$. Note that this does not mean that $\neg g$(art, amy) logically follows from $A_1, \ldots, A_5$.

And if we ask the question

$$?\text{-}\ g(\text{art}, X)$$

the answer will be $X = \text{cap}$, $X = \text{coe}$.

Once we have observed that data can be translated into formulas in the formal language of logic and that queries concerning the objects in the data – again translated into formulas – can be answered with 'yes' or 'no', depending on whether the putative conclusion can or cannot be logically deduced from the given data, it becomes clear that there is an interesting connection between logic and databases.

In Chapter 9 we shall study more closely how the language of logic may be used as a programming language in the context of artificial intelligence.

1.9.2 Relational Databases

The theory of finite sets is the basis for *relational databases*, which we shall present in Chapter 9. In fact, the query language SQL formulates questions to the database in terms of sets. To illustrate, suppose we have a table P with patients containing their number (nmb), name (nm), address (addr), residence (res) and gender (gen).

	nmb	nm	addr	res	gen
t	t(nmb)	t(nm)	t(addr)	t(res)	t(gen)

Each row in the table, called a tuple t, represents one patient. Mathematically, a tuple t assigns to every attibute nmb, nm, addr, res, gen a value t(nmb), t(nm), t(addr), t(res), t(gen) in a predefined domain. Then

$$\{\ t(\mathsf{nm})\ \mid\ t \in \mathsf{P}\ \mid\ t(\mathsf{res}) = \text{‘Princeton’}\ \wedge\ t(\mathsf{gen}) = \text{‘male’}\ \}$$

is the set of all names of patients in table P who live in Princeton and are male.

This set is generated by the Structured Query Language SQL as follows:
SELECT t.nm
FROM P t
WHERE t.res = ‘Princeton’
AND t.gen = ‘male’

1.9.3 Social Choice Theory

In social choice theory one studies how individual preferences or evaluations should be aggregated to a common (group or social) preference or evaluation respectively. That this is problematic may be demonstrated by the following simple example. Suppose there are nine voters (or judges) who have the following preferences over three candidates or alternatives a, b and c:

4: $a\ b\ c$
3: $b\ c\ a$
2: $c\ b\ a$

That is, the first four voters prefer a to b, b to c and also a to c; similarly for the other voters.

If we apply Plurality Rule (PR) or 'most votes count', only the most preferred candidate is taken into account. So, a has four first votes, b three and c only two. Consequently, the common or social ranking under PR will be: $a\ b\ c$.

If we apply Majority Rule (MR) or pairwise comparison, we see that 3 + 2 = 5 voters, hence a majority, prefer b and c to a; and that 4 + 3 = 7 voters prefer b to c. So, under Majority Rule the common or social ranking will be: $b\ c\ a$.

Many other voting rules exist, which will all lead to different outcomes. But already at this stage we see that the outcome depends on the aggregation rule, rather than on the preferences of the voters.

Another problem is that all familiar voting rules may yield an outcome which is counter-intuitive. For instance, Plurality Rule makes a the winner, while a for a majority of the voters is the least preferred candidate. And Majority Rule in some cases does not even yield a winner, for instance, when there are three voters with the following preferences 1: $a\ b\ c$; 1: $b\ c\ a$ and 1: $c\ a\ b$. So the question arises whether there exists a voting rule that has only nice properties. This question was answered negatively by K. Arrow in 1951: there cannot exist a voting rule, which takes individual preferences as input, that satisfies certain desirable properties among which being non-dictatorial. This *impossibility theorem* has puzzled the social choice community, consisting of political scientists, economists, mathematicians and philosophers, ever since.

However, in 2010 Balinski and Laraki pointed out that the framework of Arrow, in which voters are supposed to give a preference ordering, is ill conceived. Voters should be asked to give evaluations of the candidates, for instance in terms of 'excellent', 'good', 'acceptable', 'poor' and 'reject'. Notice that evaluations are much more informative than preference orderings. Next, Balinksi and Laraki present a voting rule, called *Majority Judgment* (MJ), which takes evaluations of the candidates by the voters as input and yields a social ranking of the candidates as output. This Majority Judgment does satisfy the desired properties.

In Section 9.3 we shall discuss Plurality Rule, Majority Rule and the Borda Rule and show that they all violate one or more of the desired properties. Also a version of Arrow's theorem will be proved. Next we present Balinski and Laraki's Majority Judgment and show that it does satisfy the desired properties.

1.10 Fallacies and Unfair Discussion Methods

For many discussions and meetings it holds that they are led perfectly from a formal, procedural and technical point of view, but that the quality of the in-depth discussion is poor. The cause of poor thinking should be sought in the weakness of human nature, rather than in the restrictions of our intelligence. Among the weaknesses of human nature are ambitions, emotions, prejudices and laziness of thinking. The goal of a discussion is not to be right or to overplay or mislead the other, but to discover the truth or to come to an agreement by common and orderly thinking.

Ideally, an argument consists of carefully specified premisses or assumptions and a conclusion which logically follows from the premisses. Logical correctness of an argument means that *if* the premisses are true, *then* the conclusion must also be true. In Section 1.2 we have already seen that logical correctness of an argument does not mean that the premisses are true, neither that the conclusion is true. We may have a logically correct argument with a false conclusion when at least one of the premisses is false. And a logically incorrect argument may have a conclusion that is true, when its truth is not based on the given premisses but on other grounds. One should also realize that from contradicting premisses one may conclude anything one wants: ex falso sequitur quod libet; a principle popular among many politicians.

In real life premisses and even the conclusion may be tacit, in which case one speaks of *enthymemes*. Premisses may not be explicitly stated for practical reasons or because the speaker is not aware of them himself, but also to mislead the audience.

One may distinguish formal and informal fallacies. A *formal fallacy* is an incorrect argument which may be represented in a formal logical system such as propositional logic. A simple example is: A implies B ($A \to B$) and B; hence A. For instance: if the weather is nice, then John will come. John comes; hence the weather is nice. That this argument is incorrect may become clear from the following example which has exactly the same structure: if Bill Gates owns all the gold in Fort Knox, then he is rich. Bill Gates is rich; hence Bill Gates owns all the gold in Fort Knox. However, a doctor frequently has to reason this way: a patient comes with a certain complaint B that may have several causes A; $A \to B$ and B, so the doctor will start with treating the most likely cause A.

An argument is an *informal fallacy* when the putative conclusion is not supported by the content of the premisses, but is based on the ambitions, emotions, prejudices and/or laziness of thinking of the people involved. In real life, ambitions, emotions, prejudices and laziness of thinking play a major role in argumentation, debating and discussions. A speaker may be too proud to admit that he is wrong, he may be irritated by his opponent and consequently say more than he can justify, he may have prejudices which he does not want to give up and/or he may be too lazy to study an issue carefully and for that reason oversimplify it.

So, in real life discussions and debating it is important that one is aware of all kinds of tricks which are used, consciously or unconsciously, by one's opponent to suggest that you are wrong, while in fact your opponent is wrong. In Chapter 10 we discuss a dozen different fallacies and a dozen unfair discussion methods.

1.11 Solutions

Solution 1.1. The pattern of reasoning is the following one:

$$\begin{array}{ll} (1) & \neg O \to D \\ (2) & \neg D \\ (3) & \underline{P \to \neg O} \\ & \neg P \end{array}$$

This pattern is valid; hence the argument is correct. Suppose (1), (2) and (3) and P. Then by (3) $\neg O$. Then by (1) D, contradicting (2). Therefore, if (1), (2) and (3), then $\neg P$. Note that both the conclusion and the second premiss in this argument are false.

Solution 1.2. The pattern of reasoning is the following one:

$$\begin{array}{ll} (1) & N \to J \\ (2) & \underline{\neg N} \\ & \neg J \end{array}$$

This pattern is invalid and hence the argument is not correct. It may well be that John comes, while the weather is not nice. In that case J is true and hence also the premisses (1) and (2) are true, while the conclusion $\neg J$ is false.
Another counterexample: take for N the proposition 'Bill Gates owns all the gold in Fort Knox' and for J the proposition 'Bill Gates is rich'. Then all premisses are true, while the conclusion is false.

Solution 1.3. The pattern of reasoning is the following one:

$$\begin{array}{ll} (1) & N \to J \\ (2) & \underline{J} \\ & N \end{array}$$

This pattern is equivalent to the former one, since $\neg N \to \neg J$ is equivalent to $J \to N$, and hence invalid.

Solution 1.4. The pattern of reasoning is the following one:

$$\begin{array}{ll} (1) & J \to N \\ (2) & \underline{J} \\ & N \end{array}$$

This pattern is valid and hence the argument is correct. The first premiss may be expressed by $\neg N \to \neg J$ or equivalently by $J \to N$. If both premisses (1) and (2) are true, then the conclusion must be true too.

Solution 1.5. The pattern of reasoning is the following one:

$$\begin{array}{ll} (1) & \neg(P \wedge \neg S) \\ (2) & \neg P \vee \neg D \\ (3) & \underline{D} \\ & \neg S \end{array}$$

This pattern is not valid and hence the argument is incorrect. If P is false and S and D are true, then the premisses are all true, while the conclusion is false.

Solution 1.6. The pattern of reasoning is the following one: $$\frac{\forall x[B(x) \vee C(x)]}{\forall x[B(x)] \vee \forall x[C(x)]}$$
using $B(x)$ for 'x has a beard' and $C(x)$ for 'x has a conical cap'. This pattern is not valid; hence the argument is not correct: taking natural numbers as domain, interpreting $B(x)$ as 'x is even' and $C(x)$ as 'x is odd' yields a true premiss and a false conclusion.

Solution 1.7. The pattern of reasoning is the following one:

$$\begin{array}{l} \forall x[B(x) \to C(x)] \\ \forall x[B(x)] \\ \hline \qquad \forall x[C(x)] \end{array}$$

This pattern is valid; hence the argument is correct: if all objects with the property B also have the property C, and all objects have the property B, then all objects must have the property C, no matter what the objects or what the properties B and C are.

Solution 1.8. The pattern of reasoning is the following one:

$$\begin{array}{l} \exists x[B(x)] \\ \exists x[C(x)] \\ \hline \exists x[B(x) \wedge C(x)] \end{array}$$

This pattern is not valid and hence the argument is not correct: taking natural numbers as domain, interpreting $B(x)$ as 'x is even' and $C(x)$ as 'x is odd', yields true premisses and a false conclusion.

Solution 1.9. The pattern of reasoning is the following one:

$$\begin{array}{l} \exists x[\neg B(x) \vee C(x)] \\ \exists x[B(x)] \\ \hline \qquad \exists x[C(x)] \end{array}$$

This pattern is not valid and hence the argument is not correct: taking natural numbers as domain, interpreting $B(x)$ as 'x is even' and $C(x)$ as 'x is negative', yields true premisses and a false conclusion.

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