Chapter 4
Predicate Logic

H.C.M. (Harrie) de Swart

Abstract In this chapter we extend the language of propositional logic to the one of predicate logic, in which we also can analyse arguments containing subjects and predicates, such as in, for example: All men are mortal; therefore: Socrates is mortal; and in: Socrates is a philosopher; therefore: someone is a philosopher. These simple arguments cannot be adequately dealt with in propositional logic. The semantic notions of logical consequence and logical validity and the syntactic notions of (logical) deducibility and provability are adapted to the language of predicate logic, and again it turns out that these two notions are extensionally equivalent (soundness and completeness).

4.1 Predicate Language

There are many arguments which cannot be analyzed adequately in propositional logic. An example is the following argument:

John is ill
Therefore: someone is ill.

If we translate the premiss and the conclusion into a propositional language, two atomic propositional formulas P_1 and P_2 respectively result. However, P_2 is not a valid consequence of P_1, while the argument above certainly is correct. The point is that P_1 and P_2 are two different atomic formulas not expressing the internal 'subject-predicate structure' of the premiss and the conclusion in the argument above. And it is the similarity in the internal structure of the premiss and the conclusion which is responsible for the correctness of the argument above.

So, we have to enrich the propositional language with symbols to indicate subjects, such as 'John' and 'someone' and symbols to indicate predicates, such as 'is ill'. In propositional logic, treated in Chapter 2, one can only analyze those arguments the correctness of which depends on the meaning of the propositional operations 'if …, then …', 'and', 'or' and 'not'. In *predicate logic*, also called *predicate*

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calculus, one can also analyze arguments the correctness of which depends on the 'subject-predicate structure' of the sentences involved.

With the help of a number of examples we introduce quantifiers, individual variables, constants and terms. Then we pay attention to the translation of English sentences into formulas of predicate logic and consider both intended and non-intended interpretations of these formulas. The scope of a quantifier and free and bound occurrences of a variable in a formula A are defined. A precise definition of the language of predicate logic is given, starting with an alphabet from which formulas can be built by means of connectives and quantifiers.

4.1.1 Quantifiers, Individual Variables and Constants

Below we give a number of examples of atomic propositions, grouping together those which have a similar internal (subject-predicate) structure.

1. Each of the numbers 2, 4, and 6 is even.
   All natural numbers are positive.
   All natural numbers are negative.
   All men are mortal.

The atomic propositions of group 1 all are of the following form:

all objects (of a certain kind) have the property P;
in other words: for each object x, x has the property P.

Notation: $\forall x[P(x)]$.

Here $P(a)$ stands for: a has the property P. In $P(a)$, 'a' is called an *individual variable* (or object variable) to emphasize that a ranges over the domain of individuals (or objects). The variable a indicates an open place, which may be filled by the name of a concrete individual, for instance, 'Socrates'. P(Socrates) then means: Socrates has the property P. $P(x)$ results from $P(a)$ by replacing a by x.

$\forall x$ is read as: for each object x. The symbol $\forall$ is called a *universal quantifier*. (The latin 'quantum' means 'how much'.) One might also use Ax (for All x) or $\bigwedge x$ instead of $\forall x$; the first one because it does not need any special symbol, the second one because of its analogy with $\wedge$ (and). For instance, 'each of the numbers 2, 4 and 6 is even' is equivalent to '2 is even and 4 is even and 6 is even'. However, in the case of an infinite domain as in 'all natural numbers are positive', for instance, the universal quantifier can be represented only by infinitely many conjunctions (0 is positive and 1 is positive and 2 is positive and ...). Such an expression with infinitely many conjunctions is not a formula, since formulas are by definition finite expressions. Therefore, we need quantifiers.

Instead of the variable x one may also use another variable y: $\forall x[P(x)]$ and $\forall y[P(y)]$ have the same meaning! They both mean: all objects (of a certain kind) have the property P.

2. At least one of the numbers 2, 3 and 4 is even.
   There is some natural number x such that $x > 0$.
   Some men are immortal.

The atomic propositions of group 2 are all of the following form:

some objects (of a certain kind) have the property P;
in other words: there is at least one object x such that x has the property P.

Notation: $\exists x[P(x)]$.

$\exists x$ is read as: there is at least one object x such that .... The symbol $\exists$ is called an *existential quantifier*. One might also use Ex (there Exists an x such that) or $\bigvee x$ instead of $\exists x$; the first one again because it does not need any special symbol, the second one because of its analogy with $\vee$ (or). For instance, 'There is some natural number x such that x is even' is analogous to '0 is even or 1 is even or 2 is even or ...'. Again, $\exists y[P(y)]$ and $\exists x[P(x)]$ have exactly the same meaning.

The predicate in an atomic proposition may be built from simpler predicates by means of 'if and only if (iff)', 'if ..., then ...', 'and', 'or' and 'not'. For instance, using $\rightleftarrows$ for 'iff', $\rightarrow$ for 'if ..., then ...', $\wedge$ for 'and', $\vee$ for 'or' and $\neg$ for 'not':
'For each number x, x is even iff x^2 is even' is of the form $\forall x[P(x) \rightleftarrows Q(x)]$.
'All animals having four legs are cows' is of the form $\forall x[P(x) \rightarrow Q(x)]$.
'Some natural numbers are positive and even" is of the form $\exists x[P(x) \wedge Q(x)]$.
'All natural numbers are positive or negative" is of the form $\forall x[P(x) \vee Q(x)]$.
'There is some natural number x such that not $x > 0$' is of the form $\exists x[\neg P(x)]$.

In an atomic proposition more than one quantifier may occur, as is the case in the following examples:
'All natural numbers are equal', or equivalently, 'for every natural number x and for every natural number y, $x = y$' is of the form $\forall x \forall y[R(x,y)]$.
'There are different natural numbers', or equivalently, 'there is a natural number x and there is a natural number y such that $x \neq y$' is of the form $\exists x \exists y[R(x,y)]$.

Here '$R(a,b)$' stands for: a is in the relation R to b. In $R(a,b)$, 'a' and 'b' are individual variables indicating open places which may be filled by the names of concrete individuals, for instance, by 'Janet' and 'Peter' respectively. 'R(Janet, Peter)' then means: Janet is in the relation R to Peter. '$R(x,y)$' results from '$R(a,b)$' by replacing a and b by x and y respectively.

In 'John loves Jane' we call 'John' the *subject* and ' - loves Jane' or 'a loves Jane' the *predicate* of the sentence. In logic we use the expression *predicate* in a more general way than in grammar. In grammar 'a loves Jane' is a *predicate*, but not 'John loves b' or 'a loves b'. In grammar, we call 'John' the *subject* and 'Jane' the *object* of the proposition 'John loves Jane'. In mathematics and in logic, but not in grammar, 'John loves b' and 'a loves b' are also called *predicates*, with one and two arguments respectively; and both 'John' and 'Jane' are called *subjects* of the proposition 'John loves Jane'. Notice that 'a loves Jane' assigns a proposition to each value of a; 'John loves b' assigns a proposition to each value of b and 'a loves b' assigns a proposition to each pair of values of a and b.

'a loves Jane' is a predicate with one argument, also called a *property*; and so is 'John loves b'. But 'a loves b' is a predicate with two arguments, also called a *(binary) relation*. 'a_1 and a_2 are the parents of b' is an example of a 3-ary predicate, also called a *ternary relation*.

3. Every person has a mother; or equivalently: for every person x there is some person y such that x has y as mother.
   For every natural number there is a greater one; or equivalently: for every natural number x there is some natural number y such that $x < y$.

The atomic propositions of group 3 are all of the form:

for every object x there is an object y (possibly depending on x) such that x is in the relation R to y.

Notation: $\forall x \exists y[R(x,y)]$.

4. Someone is the mother of all persons; or equivalently: there is some person y such that for all persons x, x has y as mother.
   There is a greatest natural number; or equivalently: there is some natural number y such that for all natural numbers x, $x < y$.
   There is a least natural number; or equivalently: there is some natural number y such that for all natural numbers x, $y \leq x$.

The atomic propositions of group 4 all are of the form:

there is some object y (independent of any x) such that for all objects x (including y itself), x is in the relation R to y.

Notation: $\exists y \forall x[R(x,y)]$.

From the examples in group 3 and 4 it should become obvious that the reading of $\forall x \exists y[R(x,y)]$ is quite different from the reading of $\exists y \forall x[R(x,y)]$. So, the order of the quantifiers is very important. The following example may clarify the difference: It is true that for every natural number x there is a natural number y such that $x^2 = y$, which is of the form $\forall x \exists y[R(x,y)]$, but it is not true that there is a natural number y such that for *every* natural number x, $x^2 = y$, which is of the form $\exists y \forall x[R(x,y)]$.

In an atomic proposition the names of concrete individuals may occur, as is the case in the following examples.

5. Socrates is a man.
   Socrates is mortal.
   3 is odd.
   4 is even.

The atomic propositions of group 5 are all of the form:

c has the property P.

Notation: $P(c)$.

The letter 'c' is used as the name for some concrete object. Different objects within the same context should be indicated by different names, for instance, $c_1, c_2, \ldots$. We call 'c_1', 'c_2', ... *individual constants*: throughout some context every occurrence of each of them is the name for the same object.

'All natural numbers are greater than or equal to zero' and 'everyone loves Janet' both are of the form $\forall x[R(x,c)]$, where $R(a,c)$ is to be read as: a is in the relation R to c. The symbol 'a' is an *individual variable* and the symbol 'c' is an *individual constant* .

From the atomic propositions considered above one can build composite propositions by means of the propositional operations studied in Chapter 2 on propositional logic. For instance, 'if all natural numbers are even, then all natural numbers are odd' is a composite proposition of the form $\forall x[P(x)] \to \forall x[Q(x)]$, not to be confused with the atomic proposition 'for each natural number x, if x is even, then x is odd', which is of the form $\forall x[P(x) \to Q(x)]$.

Note the difference between:

a) $\forall x[P(x)] \to \forall x[Q(x)]$: if *every* object x has the property P, then also *every* object x has the property Q.
b) $\forall x[P(x) \to Q(x)]$: for each (individual) object x, if x has the property P, then x also has the property Q.

In a) the implication $\to$ is between the two sentences $\forall x[P(x)]$ and $\forall x[Q(x)]$ to form a new sentence $\forall x[P(x)] \to \forall x[Q(x)]$. In b) the implication $\to$ is between the two predicates $P(x)$ and $Q(x)$ to form a new predicate $P(x) \to Q(x)$ and the formula in b) says that every object x has this property $P(x) \to Q(x)$. The formulas in a) and b) have quite different meanings! For instance, 'if all natural numbers are even (which is false), then all natural numbers are odd (which is also false)' is an instance of the formula in a) and is true ($0 \to 0 = 1$), while 'for each natural number x, if x is even, then x is odd' is an instance of the formula in b) and is false.

Similarly, 'if there is an even natural number, then there is a natural number not equal to itself' is a composite proposition of the form $\exists x[P(x)] \to \exists x[Q(x)]$ and false ($1 \to 0 = 0$), not to be confused with the atomic proposition 'there is some natural number x such that *if* x is even, *then* $x \neq x$', which is of the form $\exists x[P(x) \to Q(x)]$ and true, because 'if 3 is even, then $3 \neq 3$' is true ($0 \to 0 = 1$).

4.1.2 Translating English into Predicate Logic, Intended and Non-intended Interpretation

The English sentences 'John is ill' and 'Someone is ill' have the same noun phrase (NP) - verb phrase (VP) syntactic structure, while their translations into the predicate language do not have the same (logical) structure:

$$\begin{array}{rr} \text{John is ill} & I(j), \\ \text{Someone is ill} & \exists x[I(x)]. \end{array}$$

This makes automated translation of English into symbolic logic a non-trivial matter. The following six English sentences also have the same *NP-VP* structure, while their translations into predicate logic have quite different (logical) structures.

English sentences	*Usual translation into logic*
(1) John walks	$W(j)$
(2) Every student walks	$\forall x[S(x) \rightarrow W(x)]$
(3) Some student walks	$\exists x[S(x) \wedge W(x)]$
(4) No student walks	$\neg\exists x[S(x) \wedge W(x)]$
(5) Somebody walks	$\exists x[W(x)]$
(6) Nobody walks	$\neg\exists x[W(x)]$ or $\forall x[\neg W(x)]$

We have translated the sentences (1) - (6) into a formal predicate language the alphabet of which consists of the following symbols with the corresponding intended interpretation:

Symbols	*Intended interpretation*
$x, y, \ldots$	persons
j	John
W; S	is walking; being a student
$\rightleftarrows$, $\rightarrow$, $\wedge$, $\vee$, $\neg$	
$\forall$, $\exists$	
[,], (,)	

The translations of the sentences (1) - (6) are called *formulas* of this formal language. The interpretation of the connectives and the quantifiers has been fixed once and for all in Section 2.2 of Chapter 2 on propositional logic and in Subsection 4.1.1 at the beginning of this section; for this reason these symbols are called *logical symbols*. But the interpretation of the other symbols can be varied and therefore the symbols 'j', 'W' and 'S' are called *non-logical symbols*. Consider, for instance, the following non-intended interpretation:

Symbols	*Example of a non-intended interpretation*
$x, y, \ldots$	natural numbers
j	0
W; S	is even; is odd

Under this (non-intended) interpretation the meanings of the formulas above are as follows:

formula	*Non-intended interpretation, as specified above*
$W(j)$	0 is even;
$\forall x[S(x) \rightarrow W(x)]$	Every odd natural number is even;
$\exists x[S(x) \wedge W(x)]$	Some natural number is both odd and even;
$\neg\exists x[S(x) \wedge W(x)]$	No natural number is both odd and even;
$\exists x[W(x)]$	Some natural number is even;
$\neg\exists x[W(x)]$	No natural number is even.

The translation of the *correct* argument

If every student walks and John is a student, then John walks

into propositional logic would be an *invalid* formula of the form $P \wedge Q \to R$, and hence such a translation is inadequate. However, the translation of this sentence into the predicate language specified above is

$$\forall x[S(x) \to W(x)] \wedge S(j) \to W(j). \qquad (*)$$

Now the reader can easily convince himself that this formula yields a true proposition for each possible interpretation (intended or non-intended): for every domain D, for every unary predicate S^* and W^* over D and for every element j^* in D, if all elements of D with the property S^* have the property W^* and j^* has the property S^*, then j^* also has the property W^*. For instance: if every Soccer player Wins the lottery and John is a Soccer player, then John Wins the lottery; and: if every Son of my father is Wealthy and John is a Son of my father, then John is Wealthy. For this reason the formula $(*)$ is called *valid . The validity of $(*)$ is guaranteed by the fixed meaning of the logical symbols $\forall$, $\to$ and $\wedge$ in this formula.*

Other examples of valid formulas of the formal language under consideration are: $\forall x[S(x) \to S(x)]$, $\forall x[\neg(W(x) \wedge \neg W(x))]$, and
$\forall x[S(x) \to W(x)] \wedge \forall x[S(x)] \to \forall x[W(x)]$.
We will study valid formulas more closely in Section 4.2.

Suppose we want to translate sentences about addition and multiplication of natural numbers into a logical language. Examples of such sentences are:
(1) for any natural number n, $n+0=n$,
(2) for any natural number n, $n \times 0 = 0$,
(3) there is no natural number n such that $n \times n = 2$.
Of course, we might translate these sentences into atomic propositional formulas P_1, P_2 and P_3 of propositional logic, respectively. This suffices, if we want to conclude, for instance, that the sentence ((1) or (2)) logically follows from sentence (1), because $P_1 \vee P_2$ is a valid consequence of P_1. However, if we want to conclude from sentence (1) that $2+0=2$, our translation into propositional formulas is not adequate. The proposition $2+0=2$ should be rendered by a different atomic formula Q and we know from Chapter 2 that Q is not a valid consequence of P_1; on the other hand, the proposition 2 + 0 = 2 does follow from proposition (1). Therefore, a translation into the language of predicate logic, exhibiting the subject-predicate structure of the sentences involved, is needed.

We may take a predicate language with the following non-logical symbols, having the corresponding intended interpretation:

Non-logical symbols	*Intended-interpretation*
0, 1, 2, ...	zero, one, two, ...
$\equiv$	is equal to (=)
A	$A(a,b,c)$: a plus b equals c (Addition)
M	$M(a,b,c)$: a times b equals c (Multiplication)

The symbols 0, 1, 2, ... are individual constants. the symbol $\equiv$ is a binary predicate symbol, i.e., with two arguments, and the symbols A (Addition) and M (Multiplication) are ternary predicate symbols, i.e., with three arguments. The translations of sentences (1), (2) and (3) are now respectively: $\forall x[A(x,0,x)]$, $\forall x[M(x,0,0)]$,

$\neg\exists x[M(x,x,2)]$ and the translation of 2 + 0 = 2 now becomes $A(2,0,2)$, which is a valid consequence of $\forall x[A(x,0,x)]$.

Of course, once having built these formulas one can forget about their origin and consider *non-intended interpretations*, like the following one.

x, y	: persons
0, 1, 2, ...	: John, Mary, Janet, ..., respectively
$a \equiv b$	: a loves b
$A(a,b,c)$	: a and b are the parents of c
$M(a,b,c)$	: a and b are the grandparents of c.

Needless to say that under this non intended interpretation the formula $A(2,0,2)$ yields a false proposition: Janet and John are the parents of Janet.

4.1.3 Scope, Bound and Free Variables

In $\forall x[A(x)]$ and in $\exists x[A(x)]$ we call $A(x)$ the *scope* of the quantifier $\forall x$.
For example, in the expression

$$\exists x[R(a,x) \to S(x,a,b)] \to R(a,b)$$

the scope of the $\exists x$ is the part $R(a,x) \to S(x,a,b)$.

In the expression

$$\forall x \exists y[R(x,y) \to \exists z[S(y,z)]] \to \forall x[\neg R(x,a)]$$

the scope of the first occurrence of the $\forall x$ is the part $\exists y[R(x,y) \to \exists z[S(y,z)]]$, the scope of $\exists y$ is the part $R(x,y) \to \exists z[S(y,z)]$, the scope of $\exists z$ is the part $S(y,z)$ and the scope of the second occurrence of the $\forall x$ is $\neg R(x,a)$.

Similarly, in $\neg A$, $A \rightleftarrows B$, $A \to B$, $A \wedge B$ and $A \vee B$ we call the expression A or pair of expressions A, B the *scope* of the propositional connective in question.

Definition 4.1 (Bound/Free occurrence of a variable in a formula). An occurrence of a variable x in an expression A is said to be *bound* (or as a *bound variable*), if the occurrence is in a quantifier $\forall x$ or $\exists x$ or in the scope of a quantifier $\forall x$ or $\exists x$ (with the same x); otherwise, *free* (or as a *free variable*).

It has turned out that it is convenient to use different letters for free and bound variables:

$a_1, a_2, a_3, \ldots$ for free occurrences only and
$x_1, x_2, x_3, \ldots$ for bound occurrences only.

Example 4.1. In 'a_2 is the mother of a_1' and in '$a_2 > a_1$' both occurrences of a_1 and a_2 are free.
In $\exists x_2[x_2$ is the mother of $a_1]$ (a_1 has a mother) and in $\forall x_2[x_2$ is the mother of $a_1]$ (everyone is a mother of a_1), the occurrence of a_1 is free and both occurrences of x_2 are bound.

In $\forall x_1 \exists x_2 [x_2$ is the mother of $x_1]$ (everyone has a mother) and in $\exists x_1 \forall x_2 [x_2$ is the mother of $x_1]$ (someone has everyone as mother) both occurrences of x_1 and both occurrences of x_2 are bound.
The occurrences of the variables a_1 and a_2 are free in $\exists x_1 \forall x_2 [R(x_1, a_1) \wedge R(x_2, a_2)]$, while both occurrences of the variables x_1 and x_2 are bound in this formula.

A variable a which occurs as a free variable (briefly, occurs free) in A is called a *free variable of* A, and A is then said to contain a as a *free variable* (briefly, to contain a free); and likewise for bound variables.

4.1.4 Alphabet and Formulas

In Subsection 4.1.2 we introduced two different formal predicate languages: one for expressing that certain students walk (John was one of them) and one for expressing certain properties of natural numbers. In the exercises at the end of this section several other predicate languages are introduced. All predicate languages have the individual variables, the connectives and quantifiers in common, they differ only in the choice of the individual constants and predicate symbols, which depends on the context. We do not want to study any particular one of these languages, but we want to study these languages in general, so that any of our results is applicable to each particular language we want to consider.

So, in order to retain flexibility for the applications, we shall assume throughout this chapter that we are dealing with one or another object language in which there is a class of *individual constants*

$$c_1,\ c_2,\ c_3,\ \ldots$$

and a class of *predicate symbols*

$$P_1,\ P_2,\ P_3,\ \ldots$$

where each P_i is supposed to be a different n_i*-place* predicate symbol, i.e., taking n_i arguments ($n_i = 0, 1, 2, \ldots$). By including the possibility that $n_i = 0$, we allow $P_1, P_2, \ldots$ to express atomic *propositions*. Consequently, the predicate calculus extends the propositional calculus. That is, any propositional language can be conceived of as a predicate language: instead of the atomic formula P_i, one can take a 0-ary predicate symbol P_i (with $n_i = 0$).

In Chapter 3 we introduced a formal predicate language for set theory and in Chapter 5 we shall introduce another predicate language for arithmetic, in which we can express properties of natural numbers. In *The Proper Treatment of Quantification in Ordinary English*, R. Montague presented a formal language in which a suitably restricted and regulated part of English or some other natural language can be expressed.

Thus, throughout this chapter our logical predicate language shall consist of the following symbols:

Definition 4.2 (Alphabet of predicate logic).

Symbols	*Name*
$a_1,\ a_2,\ a_3,\ \ldots$	free individual variables
$x_1,\ x_2,\ x_3,\ \ldots$	bound individual variables
$c_1,\ c_2,\ c_3,\ \ldots$	individual constants
$P_1,\ P_2,\ P_3,\ \ldots$	predicate symbols (each P_i is n_i-*ary*)
$\rightleftarrows, \rightarrow, \wedge, \vee, \neg$	connectives
$\forall,\ \exists$	quantifiers
(,), [,]	parentheses

Since the logical predicate language is the object of our study in this chapter, we shall call it the *object-language*. We shall study this language using English as metalanguage, i.e., as the language we use to talk *about* (formulas of) the object language.

In order to prevent writing subscripts and because $\forall x_1[P(x_1)]$ has the same meaning as $\forall x_3[P(x_3)]$, we agree to use x, y, z as (meta)variables over $x_1, x_2, x_3, \ldots$ and simply write $\forall x[P(x)]$ instead of $\forall x_1[P(x_1)], \forall x_2[P(x_2)], \ldots$. The use of the letter x in the expression 'if $A(a)$ is a formula and x is a bound variable, then $\forall x[A(x)]$ is a formula' is similar to the use of the letter n in the expression 'if n is a natural number, then also $n+1$ is natural number'. The letter n itself is not a natural number, but may be replaced by any natural number 0, 1, 2, ... in the expression just mentioned. Similarly, the letter x itself is not a variable, but may be replaced by any variable $x_1, x_2, x_3, \ldots$. So, strictly speaking, the expression $\forall x[P(x)]$ itself is not a formula, but replacing x by x_1 (or $x_2, x_3, \ldots$) and P by P_1 yields a formula $\forall x_1[P_1(x_1)]$, which does belong to the object language.

In a similar way we agree to use the symbols a and b as names for free individual variables $a_1, a_2, \ldots$ in the object language; the symbols c and d as names for individual constants $c_1, c_2, \ldots$ in the object language; and the symbols P, Q, R and S as names for predicate symbols $P_1, P_2, \ldots$ in the object language. Strictly speaking, the symbols a, b, x, y, z, c, d, P, Q, R and S themselves do not belong to the logical predicate language!

Definition 4.3 (Basic Term). A *basic term* is a free individual variable or an individual constant. Later in Definition 4.17 the notion of term will be generalized, allowing it to contain also function symbols.

Definition 4.4 (Atomic formulas). If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are *terms*, then $P(t_1, \ldots, t_n)$ is an *atomic formula*.

Example 4.2. Supposing that S (being a Student) and W (Walking) are unary predicate symbols, that M (having as Mother) is a binary predicate symbol, that a and b stand for any free individual variable $a_1, a_2, \ldots$, and that c and d stand for any individual constant $c_1, c_2, \ldots$, the following expressions are atomic formulas of predicate logic: $S(a)$, $W(a)$, $S(c)$, $W(c)$; $M(a,b)$, $M(a,c)$ (cobi is the Mother of a), $M(c,a)$ (a is the Mother of cobi), $M(c,b)$, $M(c,d)$ (cobi has dora as Mother).

The expression $P(t_1, \ldots, t_n)$ itself is not an atomic formula, but a meta-expression representing any atomic formulas. In particular, the expression $P(a)$ itself is not

an atomic formula, but $P(a_1), P(a_2), P(a_3), \ldots$ are atomic formulas, if P is a unary predicate symbol in the alphabet of our predicate language, which is the object of our study.

Definition 4.5 (Formulas).

a) Each atomic formula is a formula.
b) If A and B are any formulas (either atomic formulas, or composite formulas already constructed), then $(A \rightleftarrows B)$, $(A \to B)$, $(A \wedge B)$, $(A \vee B)$ and $(\neg A)$ are (composite) formulas.
c) If $A(a)$ is any formula in which the free variable a occurs, and x is any bound variable not occurring in $A(a)$, then $\forall x[A(x)]$ and $\exists x[A(x)]$ are (composite) formulas, where $A(x)$ results from $A(a)$ by replacing every occurrence of a in $A(a)$ by x.
d) The only formulas are those given by a), b) and c).

Example 4.3. Supposing that S (being a Student) and W (Walking) are unary predicate symbols in our predicate language, by clause b), $S(a) \to W(a)$ is a formula of our logical predicate language, and by clause c) $\forall x[S(x) \to W(x)]$ is a formula of our predicate language. Supposing that $M(a,b)$ (a has b as Mother) is a binary predicate symbol in our predicate language, by clause c) $\exists y[M(a,y)]$ (a has a Mother) is a formula of our predicate language, and again by clause c), also $\forall x \exists y[M(x,y)]$ (everyone has a Mother) is a formula of our predicate language. And by applying clause b) again, $\forall x[S(x) \to W(x)] \wedge \forall x \exists y[M(x,y)]$ is also a formula of our predicate language.

Strictly speaking, assuming that M is a binary predicate symbol of our predicate language, $\exists y[M(a,y)]$ itself is not a formula of our predicate language, but, for instance, $\exists x_2[M(a_1,x_2)]$ is, expressing that 'a_1 has a mother'. And strictly speaking, $\forall x \exists y[M(x,y]$ itself is not a formula of our predicate language, but $\forall x_1 \exists x_2[M(x_1,x_2)]$ is, expressing that 'everyone has a mother'.

We are using the symbols A, B, C, …, A_1, A_2, A_3, …, from the beginning of the Roman alphabet to stand for any formulas, not necessarily atomic. Such distinct letters as $A, B, C, \ldots$ need not represent distinct formulas in contrast to the symbols $P, Q, R, S, \ldots$ which represent distinct predicate symbols.

Assuming that $A(a)$ is a formula, the expression $\forall x[A(x)]$ itself is, strictly speaking, not a formula, since the letter x is a meta-variable representing any bound variable; but $\forall x[A(x)]$ becomes a formula when the letter x is replaced by any bound variable x_1 or x_2 or x_3 or ….

For instance, supposing again that S (is a Student) is a unary predicate symbol and that M (has as Mother) is a binary predicate symbol of our predicate language, $S(a_1)$ and $M(a_1,a_2)$, are atomic formulas of our predicate language and $\forall x_1[S(x_1)]$ (everyone is a Student) and $\exists x_2[M(a_1,x_2)]$ (a_1 has a mother) are composite formulas of our predicate language. Using, e.g., x_3 instead, we get different formulas $\forall x_3[S(x_3)]$ and $\exists x_3[M(a_1,x_3)]$ which have the same meaning as $\forall x_1[S(x_1)]$ and $\exists x_2[M(a_1,x_2)]$, respectively. This is why the meta-variable x is necessary in clause c) in Definition 4.5; had we written x_1 instead, we would be allowing only $\forall x_1[S(x_1)]$ (but not $\forall x_2[S(x_2)]$, $\forall x_3[S(x_3)]$, etc.) as a formula.

The quantifiers act as unary operators in building formulas, and with our other unary operator $\neg$ are ranked last under the convention for omitting parentheses. Thus, $\forall x A(x) \to B$ means $\forall x[A(x)] \to B$, not $\forall x[A(x) \to B]$.

Definition 4.6 (Closed Formula). A formula A is called *closed* if it contains no free occurrences of variables; otherwise, *open*. A closed formula is also called a *sentence*.

Example 4.4. Supposing that M is a binary predicate symbol of our predicate language, $M(a_1,a_2)$ (a_1 has a_2 as Mother) and $\exists x_2[M(a_1,x_2)]$ (a_1 has a Mother) are open formulas, while $M(c_1,c_2)$ (c_1 has c_2 as Mother), $\exists x_2[M(c_1,x_2)]$ (c_1 has a Mother) and $\forall x_1 \exists x_2[M(x_1,x_2)]$ (everyone has a Mother) are closed formulas.

Since formulas are built up from atomic formulas by successive applications of connectives and quantifiers to formulas already generated before, the following Theorem, called the *induction principle* (for predicate formulas), follows immediately from the definition of formulas. (See also Theorem 2.2.)

Theorem 4.1 (Induction principle for formulas). *Let Φ be a property of formulas, such that a) all atomic formulas have the property Φ,*
b) if A and B have the property Φ, then also $(A \rightleftarrows B)$, $(A \to B)$, $(A \wedge B)$, $(A \vee B)$ and $(\neg A)$ have the property Φ, and
c) if $A(a)$ has the property Φ, x does not occur in $A(a)$ and $A(x)$ results from $A(a)$ by replacing all occurrences of a in $A(a)$ by x, then also $\forall x[A(x)]$ and $\exists x[A(x)]$ have the property Φ.
Then all formulas have the property Φ.

For an application of this induction principle see the proof of Theorem 4.18.

Exercise 4.1. Let $G(a)$ stand for 'a is a girl' and $P(a)$ for 'a is pretty'.

a) Translate each of the following sentences into logical symbolism in an adequate way: (1) Every girl is pretty. (2) Some girl is pretty.
b) Explain why $\forall x[G(x) \wedge P(x)]$ is not a correct representation of the meaning of sentence (1) and why $\exists x[G(x) \to P(x)]$ is not a correct representation of the meaning of sentence (2).

Exercise 4.2. Let M be a binary predicate (relation) symbol with intended interpretation 'is married to', and c and d individual constants with Cod, respectively Diana, as intended interpretation. Translate the following sentences into formulas of predicate logic: 1. Cod is not married to Diana; 2. For all persons x and y, if x is married to y, then y is married to x; 3. Diana is married; 4. There is at least one person who is not married.

Exercise 4.3. Let $A(a,b)$ stand for 'a admires b'. Translate the following two sentences into logical symbolism.
(1) Everyone has someone whom he admires.
(2) There is someone whom everyone admires.
Note that 'everyone admires someone' is ambiguous and can have each of the two readings above.

Exercise 4.4. Let $L(x,y)$ stand for 'x loves y'. Translate the following sentences into logical symbolism.

(1) All persons love each other. (2) Some persons love each other.
(3) Every person loves someone. (4) Someone is loved by everyone.
(5) Everyone is loved by someone. (6) There is a person who loves everyone.

Exercise 4.5. Let $D(a)$ stand for 'a is a Dutchman', $C(a)$ for 'a is a kind of cheese', $W(a)$ for 'a is a kind of wine', $L(a,b)$ for 'a likes b', c for Chip, and d for Donald. Translate the following sentences into logical symbolism in an adequate way.

1. Donald likes all kinds of cheese.
2. Some Dutchmen like all kinds of cheese.
3. Donald likes some kinds of cheese.
4. All Dutchmen like at least one kind of cheese.
5. There is a kind of cheese which is liked by any Dutchman.
6. Chip doesn't like any kind of cheese.
7. All Dutchmen don't like any kind of cheese.
8. All Dutchmen like some kind of cheese and some kind of wine.
9. All Dutchmen who like some kind of cheese, also like some kind of wine.
10. If all Dutchmen like some kind of cheese, then all Dutchmen like some kind of wine.

Exercise 4.6. Consider the predicate language with the following non-logical symbols: the binary predicate symbol $\equiv$ and the individual constants c_1, c_2.

1 Translate the sentences below into this language in an adequate way.
i) The Morning Star is the same as the Evening Star.
ii) Every star identical to the Morning Star, is the same as the Evening Star.

2 For the formulas found in 1, consider the non-intended interpretation:
$\forall x$: for all numbers x, ...; $\exists x$: there is some number x such that ....
$\equiv$: is equal to (=); c_1: 3, c_2: 4.
Are the readings of the formulas found in 1 i) and ii) under this interpretation true or false propositions?

3 Similar question as in 2, but now for the non-intended interpretation:
$\forall x$: for all persons x ...; $\exists x$: there is some person x such that ....
$\equiv$: was older than; c_1: Reagan, c_2: Nixon.

Exercise 4.7. Let $P(a)$ stand for 'a has the property P', and $a \equiv b$ for 'a equals b'. Translate each of the following sentences into logical symbolism, using the binary predicate symbol $\equiv$ for equality.

1. There is *at least one x* which has the property P.
2. There is *at most one x* which has the property P.
3. There is *exactly one x* which has the property P.
4. There are *at least two* objects which have the property P.
5. There are *at most two* objects which have the property P.
6. There are *exactly two* objects which have the property P.

$\exists! x[A(x)]$ is adopted as an abbreviation for the formula expressing *'there is exactly one x such that $P(x)$'* or *'there exists a unique x such that $P(x)$'*.

Exercise 4.8. Translate the following sentences containing the indefinite article 'a' into the language of predicate logic, using the unary predicate symbols C, A, M and W for 'being a Child', 'needs Affection', 'being a Man' and 'to Whistle', respectively: a) A child needs affection. b) A man was whistling. Notice that the indefinite article 'a' or 'an' sometimes has the force of 'all', sometimes of 'some'.

Exercise 4.9. Translate the following sentences containing the word 'any' into the language of predicate logic, using the unary predicate symbols M, O, B and S for 'being Mortal', 'being Older than 150 years', 'celebrating one's Birthday' and 'being Stupid' respectively, and using the propositional formula P for 'there is a party'.
a) For any x, x is mortal.
b) Not for any x, x is older than 150 years.
c) If anyone celebrates his or her birthday, then there is a party.
d) If John was stupid, then anyone is stupid.
Notice that the meaning of 'any' depends on the context. When an any-expression stands by itself, as in sentence a), 'any' has the same logical force as 'all'. But when an any-expression D is put into either of the context $\neg D$, as in sentence b), or $D \to E$, as in sentence c), the meaning of 'any' normally alters from 'all' to 'some'.

Exercise 4.10. Give an interpretation such that $\forall x[P(x) \to Q(x)]$ yields a true proposition, while $\exists x[P(x) \wedge Q(x)]$ yields a false proposition under this interpretation. This shows that from $\forall x[P(x) \to Q(x)]$ one may not conclude that $\exists x[P(x) \wedge Q(x)]$, although one may conclude from it that $\exists x[P(x) \to Q(x)]$.

Exercise 4.11. a) Give an interpretation such that $\forall x[P(x)] \to \forall x[Q(x)]$ yields a true proposition, while $\forall x[P(x) \to Q(x)]$ yields a false proposition under this interpretation. So, from $\forall x[P(x)] \to \forall x[Q(x)]$ one may not conclude that $\forall x[P(x) \to Q(x)]$.
b) Show in a similar way that from $\exists x[P(x) \to Q(x)]$ one may not conclude that $\exists x[P(x)] \to \exists x[Q(x)]$.

4.2 Semantics: Tarski's Truth Definition; Logical (Valid) Consequence

Let A be an atomic formula containing (free occurrences of) variables, a one-place predicate symbol P or a 2-place predicate (or relation) symbol R and individual constants c and d. For instance, $A = P(c)$, $A = P(a)$, $A = R(c,d)$ or $A = R(a,c)$. In order to give a meaning to A, we have to give an interpretation of the symbols occurring in A. Such an *interpretation* M has to specify:

1. a *domain* or *universe of discourse* D; for instance, the set of all men or the set $\mathbb{N}$ of all natural numbers.
2. a unary predicate P^* or a binary predicate R^*, respectively, over the given domain, determining the meaning of the predicate symbols P and R; for instance, assuming the domain is $\mathbb{N}$, $P^*(a)$: a is even, $R^*(a,b)$: $a > b$.

3. elements c^* and d^* in the given domain, determining the meaning of the individual constants c and d.

So, let $M = \langle \mathbb{N};\ P^*,\ R^*;\ c^*,\ d^* \rangle$ be the interpretation with domain $\mathbb{N}$, $P^*(a)$: a is even, $R^*(a,b)$: $a > b$; $c^* = 2$ and $d^* = 3$. Then under interpretation M the formula $P(c)$ yields the proposition $P^*(c^*)$, i.e., 2 is even, which happens to have the truth value 1. Therefore, we say that M is a *model* for the formula $P(c)$, i.e., $P(c)$ yields under interpretation M a true proposition. Notation: $M \models P(c)$.

And under interpretation M the formula $R(c,d)$ yields the proposition $R^*(c^*,d^*)$, i.e., $2 > 3$, which happens to have the truth value 0. Therefore, we say that M is not a *model* for the formula $R(c,d)$, i.e., $R(c,d)$ yields under interpretation M a false proposition. Notation: $M \not\models R(c,d)$.

An interpretation M for a formula A does specify the domain and the meanings of the predicate symbols and individual constants in A, but it does not specify the meaning of the variables that occur free in A. Given an interpretation M for formula A with domain D, a *valuation* v shall give a value in the given domain to the variables occurring free in A. So, let $M = \langle \mathbb{N};\ P^*,\ R^*;\ c^*,\ d^* \rangle$ be the interpretation given above for the formula $P(a)$ or $R(a,c)$ respectively, and let v be the valuation which assigns to the free variable a the value 4, $v(a) = 4$, then under interpretation M and valuation v the formula $P(a)$ yields the proposition $P^*(4)$, i.e., 4 is even, which happens to have the truth value 1. Therefore, we say that interpretation M and valuation v make the formula $P(a)$ true. Notation: $M \models P(a)[v]$ or $M \models P(a)[4]$.

Under the interpretation M just given and valuation v with $v(a) = 4$, the formula $R(a,c)$ yields the proposition $R^*(4,c^*)$, i.e., $4 > 2$, which happens to have the truth value 1. So, interpretation M and valuation v make also the formula $R(a,c)$ true. Notation: $M \models R(a,c)[v]$ or $M \models R(a,c)[4]$.

So, an interpretation M for a formula A together with a valuation v assigns to A a truth value 1 or 0. In the first case we write $M \models A[v]$ and in the second case we write $M \not\models A[v]$.

If A is composed from atomic formulas by means of connectives, the truth tables tell us the truth value of A under a given interpretation and valuation. For instance, if $M = \langle \mathbb{N};\ \text{is even},\ >;\ 2 \rangle$, then $M \models P(a) \wedge R(a,c)[4]$, since '4 is even and $4 > 2$' has truth value $1 \wedge 1 = 1$. But $M \not\models P(a) \wedge R(a,c)[3]$, since '3 is even and $3 > 2$' has truth value $0 \wedge 1 = 0$. And $M \models P(a) \rightarrow R(a,c)[1]$, since 'if 1 is even, then $1 > 2$' has truth value $0 \rightarrow 0 = 1$.

Next, consider the formula $\forall x[P(x)]$.

If we let the individual variable x range over the set of all men and if we interpret the predicate symbol P as 'is mortal', then the atomic proposition 'all men are mortal' results and this proposition has truth value 1. So, for $M = \langle Men; \text{is mortal} \rangle$, M is a model for $\forall x[P(x)]$; notation: $M \models \forall x[P(x)]$. However, if we let the variable x range over the set of all natural numbers and if we interpret the predicate symbol P as 'is even', then the proposition 'all natural numbers are even' results and

this proposition has truth value 0; so, for $M = \langle \mathbb{N}; \text{is even} \rangle$, M is not a model for $\forall x[P(x)]$; notation: $M \not\models \forall x[P(x)]$.

So depending on the interpretation of the individual variable x and the predicate symbol P, a true or false atomic proposition results from the formula $\forall x[P(x)]$:

	$\forall x[P(x)]$
$M = \langle Men; P^* \rangle$ with $P^*(x)$: x is mortal	1
$M = \langle \mathbb{N}; P^* \rangle$ with $P^*(x)$: x is even	0

In the following table for the two formulas $\forall x[P(x)]$ and $\exists x[Q(x)]$ we indicate on the left-hand side an interpretation and on the right-hand side the truth or falsity of the corresponding (atomic) proposition.

	$\forall x[P(x)]$	$\exists x[Q(x)]$
$\mathbb{N}$; $P^*(x)$: $x = x$, $Q^*(x)$: x is even	1	1
Men; $P^*(x)$: x is mortal, $Q^*(x)$: x is immortal	1	0
$\mathbb{N}$; $P^*(x)$: x is even, $Q^*(x)$: x is odd	0	1
Pets; $P^*(x)$: x is a dog, $Q^*(x)$: x is immortal	0	0

Above, we have given two interpretations of the symbols x and P, under which $\forall x[P(x)]$ yields a true proposition ('every natural number is equal to itself' and 'all men are mortal', respectively); and two interpretations under which $\forall x[P(x)]$ yields a false proposition ('all natural numbers are even' and 'all pets are dogs', respectively). So, $\forall x[P(x)]$, although not under all interpretations true, is true under at least one interpretation. For that reason we say that $\forall x[P(x)]$ is *satisfiable*.

'Not all men have black hair' is equivalent to 'there is some man who does not have black hair'. More generally, we see that $\neg\forall x[P(x)]$ (not all objects have the property P) has the same meaning as $\exists x[\neg P(x)]$ (there is some object which does not have the property P), no matter how we interpret the symbols x and P. Hence, we say that $\neg\forall x[P(x)] \rightleftarrows \exists x[\neg P(x)]$ is a *valid* or *always true* formula. So, we shall call a formula A *valid* or *always true* if A yields a true proposition under each possible interpretation of the individual and predicate-symbols which occur in A. Notation: $\models A$. Examples of valid formulas are:

1. $\models \neg\forall x[P(x)] \rightleftarrows \exists x[\neg P(x)]$
2. $\models \neg\exists x[P(x)] \rightleftarrows \forall x[\neg P(x)]$
3. $\models \forall x[P(x)] \rightleftarrows \neg\exists x[\neg P(x)]$
4. $\models \exists x[P(x)] \rightleftarrows \neg\forall x[\neg P(x)]$

In order to see the validity of the formula $A \rightleftarrows A$, we do not have to consider the internal structure of the formula A. However, in order to see the validity of the formula $\neg\forall x[P(x)] \rightleftarrows \exists x[\neg P(x)]$, which is a formula of the form $\neg A \rightleftarrows B$, we do have to consider the internal structure of the subformulas A and B from which this formula has been built. $\neg A \rightleftarrows B$ is not for all formulas A and B valid, but it is valid when A is $\forall x[P(x)]$ and B is $\exists x[\neg P(x)]$.

And we shall call B a *valid* or *logical consequence* of given premisses $A_1, \ldots, A_n$ if every interpretation M and valuation v which make all of the premisses $A_1, \ldots, A_n$ true also make B true. Notation: $A_1, \ldots, A_n \models B$.
For instance, $P(a) \models \exists x[P(x)]$ and $\forall x[P(x) \to Q(x)], P(a) \models Q(a)$.

After this introduction we shall give a precise definition of the notion of $M \models A$, which is *Tarski's truth definition* (1933), and of the notions of (logical) validity and valid (or logical) consequence.

Definition 4.7 (Interpretation). Let A be a formula, containing predicate symbols $P_1, \ldots, P_k$ and individual constants $c_1, \ldots c_l$. An *interpretation* or *structure* for A is a tuple $M = \langle D;\ P_1^*, \ldots, P_k^*;\ c_1^*, \ldots, c_l^* \rangle$, where

1. D is a non-empty set, called the *domain* or *universe of discourse*. All individual variables occurring bound in A are interpreted as ranging over this domain D. For instance, D is the finite set of all men or the infinite set $\mathbb{N}$ of all natural numbers. The requirement that the domain is non-empty is to guarantee that the following formula will be valid: $\forall x[P(x)] \to \exists x[P(x)]$.
2. For each n_i-ary predicate symbol P_i in A, P_i^* is a n_i-ary predicate over D. For instance, if P is a unary and R is a binary predicate symbol in A, and $D = \mathbb{N}$, then $P^*(n)$ might be 'n is even' and $R^*(n,m)$ might be '$n > m$'.
3. For each individual constant c_j in A, c_j^* is a concrete element of D. For instance, if c is an individual constant in A and $D = \mathbb{N}$, then c^* might be 2.

Note that the interpretation of the quantifiers and of the connectives in a formula A has been fixed once and for all in Section 4.1 and in the truth tables for the connectives (see Section 2.2). We are only free to vary the interpretation of the individual variables, the predicate symbols and the individual constants in A.

Given a formula A and an interpretation M for A with domain D, in order to give a meaning to A we still have to interpret the individual variables occurring free in A as elements of D.

Definition 4.8 (Valuation). Let A be a formula and M an interpretation for A with domain D. A *valuation* v for A assigns to each variable occurring free in A an element $v(a)$ in D.

Example 4.5. Let $A = P(a) \wedge R(a,c)$. Then $M = \langle \mathbb{N};\ P^*,\ R^*;\ c^* \rangle$ with $P^*(a) :=$ 'a is even', $R^*(a,b) :=$ '$a > b$' and $c^* = 2$, is an interpretation for A; and v with $v(a) = 4$ is a valuation for A.

Next we shall give *Tarski's truth definition* (1933), which is not a definition of truth, but which defines the notion of $M \models A[v]$, i.e., 'interpretation M and valuation v make A true', or 'under interpretation M and valuation v formula A yields a proposition with truth value 1'.

Definition 4.9 (Tarski's truth definition, 1933). Let A be a formula containing predicate symbols $P_1, \ldots, P_k$ and individual constants $c_1, \ldots, c_l$.
Let $M = \langle D;\ P_1^*, \ldots, P_k^*;\ c_1^*, \ldots, c_l^* \rangle$ be an interpretation for A and let v be a valuation for the variables occurring free in A.

We define $M \models A[v]$ by induction on the build-up of A:

- A is atomic, say $A = P_i(a_1, \ldots, a_k,\ c_1, \ldots, c_l)$.

$$M \models P_i(a_1, \ldots, a_k,\ c_1, \ldots, c_l)\ [v] \text{ iff } P_i^*(v(a_1), \ldots, v(a_k),\ c_1^*, \ldots, c_l^*).$$

For instance, if R is binary predicate symbol, $R^*(a,b)$:= '$a > b$', $c^* = 2$ and $v(a) = 4$, then $M \models R(a,c)\ [v]$ iff $4 > 2$. If $v(a) = 4$, then instead of $M \models R(a,c)\ [v]$ we shall also write $M \models R(a,c)\ [4]$.
Notice that if A contains only the free variables $a_1,\ldots,a_k$, then only the values $v(a_1),\ldots,v(a_k)$ matter in the definition of $M \models A[v]$. In particular, if A contains no free occurrences of variables, then the valuation v in '$M \models A[v]$' does not matter. These properties are preserved throughout the definition of $M \models A[v]$.
Instead of 'not $M \models A[v]$' we shall write: $M \not\models A[v]$. In such a case M is called a *countermodel for* A or a *counterexample to* A.

- $A = B \rightleftarrows C$, $A = B \to C$, $A = B \wedge C$, $A = B \vee C$, $A = \neg B$:

  1. $M \models B \rightleftarrows C\ [v]$ iff ($M \models B[v]$ and $M \models C[v]$) or ($M \not\models B[v]$ and $M \not\models C[v]$).
  2. $M \models B \to C\ [v]$ iff $M \not\models B[v]$ or $M \models C[v]$.
  3. $M \models B \wedge C\ [v]$ iff $M \models B[v]$ and $M \models C[v]$.
  4. $M \models B \vee C\ [v]$ iff $M \models B[v]$ or $M \models C[v]$.
  5. $M \models \neg B\ [v]$ iff $M \not\models B[v]$.

  This definition just follows the truth tables for the connectives given in Section 2.2. This may be easily seen if one realizes that a pair (M,v) consisting of an interpretation M and a valuation v assigns to every formula A a truth value 1 or 0. So, a pair (M,v) corresponds with a line in the truth table and one might write $(M,v)(A) = 1$ iff $M \models A[v]$ and $(M,v)(A) = 0$ iff not $M \models A[v]$. Then, for instance, clause 2 reads as follows: $(M,v)(B \to C) = 1$ iff $(M,v)(B) = 0$ or $(M,v)(C) = 1$.

- $A = \forall x[P(x)]$ or $A = \exists x[Q(x)]$

  In case $A = \forall x[P(x)]$ does not contain any free occurrences of variables,
  $M \models \forall x[P(x)]$ iff for every element d in the domain D of M, $M \models P(a)[d]$.

  For instance, let $M = \langle \mathbb{N};\ \geq 0\rangle$, then $M \models \forall x[P(x)]$ since for every natural number d in $\mathbb{N}$, $M \models P(a)[d]$, i.e., for every natural number d, $d \geq 0$.
  More generally, allowing $A = \forall x[P(x)]$ to contain also free occurrences of variables, $M \models \forall x[P(x)]\ [v]$ iff for every d in the domain D of M, $M \models P(a)[d/v]$, where a is a (new) variable not occurring in $\forall x[P(x)]$ and d/v is the same valuation as v, except that d/v assigns to a the value d.

  In case $A = \exists x[Q(x)]$ does not contain any free variables, $M \models \exists x[Q(x)]$ iff there is at least one element d in the domain D of M, such that $M \models Q(a)[d]$.

  For instance, let $M = \langle \mathbb{N};\ \text{is even}\rangle$, then $M \models \exists x[Q(x)]$ since there is at least one natural number d in $\mathbb{N}$ such that $M \models Q(a)[d]$, i.e., there is a natural number d such that d is even.
  More generally, allowing $A = \exists x[Q(x)]$ to contain also free occurrences of variables, $M \models \exists x[Q(x)]\ [v]$ iff there is an element d in the domain D of M such that $M \models Q(a)[d/v]$, where a is a (new) variable not occurring in $\exists x[Q(x)]$ and d/v is the same valuation as v, except that d/v assigns to a the value d.

This finishes the definition of $M \models A[v]$. Notice that if A contains no free occurrences of variables, the valuation v does not play a role. Now Tarski's notion of $M \models A[v]$ (A

yields a true proposition under interpretation M and valuation v) has been defined, it is straightforward to define satisfiability and validity of a formula A.

Definition 4.10 (Satisfiable). Let A be a formula. A is *satisfiable* := there is an interpretation M for A and a valuation v such that $M \models A[v]$.

Example 4.6. $\forall x[P(x)]$ is satisfiable, since $M = \langle \mathbb{N}; \geq 0\rangle$ makes $\forall x[P(x)]$ true. However, $\forall x[P(x)] \wedge \exists x[\neg P(x)]$ is not satisfiable.

Definition 4.11 (Model). Let A be a formula and let M be an interpretation for A with domain D. M is a *model* of A := for all valuations v assigning elements of D to the variables occurring free in A, $M \models A[v]$. **Notation**: $M \models A$.
Instead of 'M is a model of A', one also says: *M makes A true* or A is *true in M*.
M is called a *countermodel* or *counterexample* for A if M is not a model for A, i.e., not $M \models A$. **Notation**: $M \not\models A$.

Example 4.7. Let $M = \langle \mathbb{N}; =\rangle$. Then $M \models a \equiv a$, since for all $n \in \mathbb{N}$, $n = n$.
Let $M = \langle \mathbb{N};\ \geq;\ 0\rangle$. Then $M \models R(a,c)$, since for all natural numbers n in $\mathbb{N}$, $M \models R(a,c)[n]$, i.e., for all natural numbers n, $n \geq 0$. However, for $M = \langle \mathbb{N};\ \geq;\ 2\rangle$ we have $M \not\models R(a,c)$, since there is a valuation v with $v(a) = 1$ such that $M \not\models R(a,c)[v]$, i.e., it is not the case that $1 \geq 2$.

Definition 4.12 (Closure). Let $A = A(a_1,\ldots,a_k)$ be a formula having $a_1,\ldots,a_k$ as the only free variables and not containing the bound variables $z_1,\ldots,z_k$. Then the *universal closure* of A is by definition the closed formula $\forall z_1 \ldots \forall z_k[A(z_1,\ldots,z_k)]$, where $A(z_1,\ldots,z_k)$ results from $A(a_1,\ldots,a_k)$ by replacing every occurrence of $a_1,\ldots,a_k$ by $z_1,\ldots,z_k$, respectively. **Notation**: $Cl(A)$.

Theorem 4.2. *$M \models A$ iff $M \models Cl(A)$.*

Proof. Evident from the definitions. For instance, for $M = \langle \mathbb{N};\ \geq;\ 0\rangle$, $M \models R(a,c)$ iff $M \models \forall z[R(z,c)]$. □

Since every interpretation M (for a formula A) is a model of some formula B, one often uses the word *model* instead of 'interpretation' or 'structure'. The notion of $M \models A$ is the main notion of *model theory*. However, in *logic* one is not interested in the truth of formulas in individual interpretations M, but in the truth of formulas in *all* interpretations M (of the appropriate kind), in other words, in the validity of formulas.

Definition 4.13 (Validity). A is *valid* or *always true* := for all interpretations M for A, $M \models A$. **Notation**: $\models A$.

Example 4.8. $\models \forall x[R(x,c) \vee \neg R(x,c)]$; $\models \forall x[P(x) \to P(x)]$;
$\models \forall x[P(x) \to Q(x)] \wedge P(c) \to Q(c)$; $\models \neg\forall x[P(x)] \rightleftarrows \exists x[\neg P(x)]$.

Theorem 4.3. 1) $\models \forall x[P(x)] \rightleftarrows \forall y[P(y)]$ *and* 2) $\models \exists x[P(x)] \rightleftarrows \exists y[P(y)]$.

Proof. 1) Let $M = \langle D; P^* \rangle$ be an interpretation. Then $M \models \forall x[P(x)]$ iff $M \models \forall y[P(y)]$, because under interpretation M both formulas express the same proposition: all elements in D have the property P^*. So, every structure $\langle D; P^* \rangle$ is a model of $\forall x[P(x)] \rightleftarrows \forall y[P(y)]$. 2) is shown in a similar way. □

$\forall x \forall y[R(x,y)]$ and $\forall y \forall x[R(x,y)]$ express the same proposition: all objects are in the relation R with each other. Similarly, $\exists x \exists y[R(x,y)]$ and $\exists y \exists x[R(x,y]$ express the same proposition: there are objects which are in the relation R to each other. Therefore:

Theorem 4.4.
$\models \forall x \forall y[R(x,y)] \rightleftarrows \forall y \forall x[R(x,y)]$ *and* $\models \exists x \exists y[R(x,y)] \rightleftarrows \exists y \exists x[R(x,y)]$.

Adapting the definition of 'valid consequence' for propositional logic to predicate logic, we say that B is a *valid (or logical) consequence* of $A_1, \ldots, A_n$, iff every interpretation which makes $A_1, \ldots, A_n$ simultaneously true also makes B true. For instance, $Q(c)$ is a logical consequence of $\forall x[P(x) \to Q(x)]$ and $P(c)$:

$$\forall x[P(x) \to Q(x)],\ P(c) \models Q(c)$$

since every interpretation which makes both $\forall x[P(x) \to Q(x)]$ and $P(c)$ true also makes $Q(c)$ true; in particular, for $M = \langle$Persons; is a man, is mortal; Caspar$\rangle$ we have: if all men are mortal and Caspar is a man, then Caspar is mortal.

Definition 4.14 (Valid (or logical) consequence). B is a *valid (or logical) consequence of* $A_1, \ldots, A_n$:= for every interpretation M and for all valuations v, if $M \models A_1[v]$ and ... and $M \models A_n[v]$, then $M \models B[v]$. **Notation**: $A_1, \ldots, A_n \models B$.

Example 4.9.

1. $\forall x[P(x) \to Q(x)], \exists x[R(x) \wedge \neg Q(x)] \models \exists x[R(x) \wedge \neg P(x)]$. This statement corresponds to Aristotle's syllogism 'Baroco' (see Subsection 4.7.4). For instance, the following argument is of this form:
   All logicians are philosophers.
   There are men who are not philosophers.
   Hence, there are men who are not logicians.
2. $\forall x[P(x) \to \neg Q(x)], \exists x[R(x) \wedge Q(x)] \models \exists x[R(x) \wedge \neg P(x)]$. This statement corresponds to Aristotle's syllogism 'Festino' (see Subsection 4.7.4).
3. $P(a), P(a) \to Q(a) \models Q(a)$

From the definition of $A_1, \ldots, A_n \models B$ it follows immediately that $A_1, \ldots, A_n \not\models B$ (B is not a logical consequence of $A_1, \ldots, A_n$) iff there is an interpretation M and a valuation v which make all of $A_1, \ldots, A_n$ true ($M \models A_1 \wedge \ldots \wedge A_n\ [v]$), but which make B false ($M \not\models B[v]$). Notice that if the formulas $A_1, \ldots, A_n$ and B are all closed, i.e., contain no free occurrences of variables, then the valuation v does not play any role.

Example 4.10. $\neg \forall x[P(x)] \not\models \forall x[\neg P(x)]$, since $M = \langle \mathbb{N}; P^* \rangle$, with $P^*(x) := x$ is even, makes $\neg \forall x[P(x)]$ true ('not all natural numbers are even' has truth value 1), while M makes $\forall x[\neg P(x)]$ false ('all natural numbers are not even' has truth value 0). In Exercise 4.10 we have shown that $\forall x[P(x) \to Q(x)] \not\models \exists x[P(x) \wedge Q(x)]$ and in Exercise 4.11 we have seen that $\forall x[P(x)] \to \forall x[Q(x)] \not\models \forall x[P(x) \to Q(x)]$.

The following theorem generalizes Theorem 2.4 for propositional logic to predicate logic.

Theorem 4.5.

a) $A \models B$	*if and only if (iff)*	$\models A \to B$.
More generally,		
b) $A_1, A_2 \models B$	*if and only if (iff)*	$A_1 \models A_2 \to B$
	if and only if (iff)	$\models A_1 \to (A_2 \to B)$
	if and only if (iff)	$\models A_1 \wedge A_2 \to B$.
Even more generally,		
c) $A_1, \ldots, A_n \models B$	*if and only if (iff)*	$A_1, \ldots, A_{n-1} \models A_n \to B$
	if and only if (iff)	$\models (A_1 \wedge \ldots \wedge A_n) \to B$.

Proof. We shall prove the first statement of b). $A_1, A_2 \models B$:= for every interpretation M and for every valuation v, if $M \models A_1[v]$ and $M \models A_2[v]$, then $M \models B[v]$. (1)
$A_1 \models A_2 \to B$:= for every interpretation M and for every valuation v, if $M \models A_1[v]$, then $M \models A_2 \to B[v]$ (2)
It is easy to see that (1) and (2) mean exactly the same, because $M \models A_2 \to B[v]$ means: if $M \models A_2[v]$, then $M \models B[v]$. □

Notice that $P(a) \not\models \forall x[P(x)]$, because from 'Antoine has property P' we cannot conclude that 'everyone has property P'. More precisely, let $M = \langle \mathbb{N}; \text{is even} \rangle$ and let $v(a) = 2$. Then $M \models P(a)[2]$, but $M \not\models \forall x[P(x)]$. However, the following does hold: if $M \models P(a)$, then $M \models \forall x[P(x)]$. For $M \models P(a)$ means: for every valuation v, $M \models P(a)[v]$, which means the same as: $M \models \forall x[P(x)]$ (see Theorem 4.2).

Corresponding to two possible treatments of the free individual variables in mathematical practice (see below), there are two different notions of 'valid consequence', the one defined in Def. 4.14 and the other to be defined in Def. 4.15 below.

$a^2 - 2a - 3 = 0$ is a *conditional equation*, since it expresses a condition on a. From this condition we should not infer that $2^2 - 2 \cdot 2 - 3 = 0$; however, from $a^2 - 2a - 3 = 0$ we can infer that $(a-3)(a+1) = 0$ and hence that $a = 3$ or $a = -1$. We may say that in these inferences the variable a is *held constant*, since it stands for the same number throughout the deductions. This inference can be written thus:

$$a^2 - 2a - 3 = 0 \to a = 3 \vee a = -1 \text{ or, equivalently, as}$$
$$\forall x[x^2 - 2x - 3 = 0 \ \to \ x = 3 \vee x = -1]. \quad (1)$$

This inference corresponds with our definition of $A \models B$.

However, from $a + b = b + a$ one may conclude that $2 + 3 = 3 + 2$. In the inferences from $a + b = b + a$, the variables a and b are *general* or *allowed to vary*. Using only bound variables, the result of this inference can be written thus:

$$\forall x \forall y[x + y = y + x] \to 2 + 3 = 3 + 2. \quad (2)$$

This inference corresponds with our definition of $A \models^2 B$, as given in Def. 4.15 below.

Note that in (1) parentheses close *after* the $\to$, in (2) *before* the $\to$. Whether we choose to use interpretation (1) or (2) depends on the role the assumptions have in each case we want to infer consequences from assumptions.

Definition 4.15. B is a *valid consequence of* $A_1,\ldots,A_n$ *with all free variables general* := for every structure M, if $M \models A_1$ and … and $M \models A_n$, then $M \models B$.
Notation: $A_1,\ldots,A_n \models^2 B$. So, $A_1,\ldots,A_n \models^2 B$ iff $Cl(A_1),\ldots,Cl(A_n) \models Cl(B)$, where $Cl(B)$ is the universal closure of B.

Theorem 4.6. *If* $A \models B$*, then* $A \models^2 B$*, but in general not conversely.*

Proof. Suppose $A \models B$, i.e., for every interpretation M and for every valuation v, if $M \models A[v]$, then $M \models B[v]$. (*)
To show: $M \models^2 B$. So, suppose that $M \models A$, i.e., for every valuation v, $M \models A[v]$. Then it follows from (*) that for every valuation v, $M \models B[v]$, i.e., $M \models B$.

To establish that in general the converse does not hold, note that $P(a) \models^2 \forall x[P(x)]$, i.e., $\forall x[P(x)] \models \forall x[P(x)]$, but $P(a) \not\models \forall x[P(x)]$, since for $M = \langle \mathbb{N}; \text{is even}\rangle$, $M \models P(a)[2]$ (2 is even), while $M \not\models \forall x[P(x)]$ (not all natural numbers are even). □

Many-sorted and higher-order predicate logic In order to avoid misunderstanding, it should be noted that also for formulas containing two or more quantifiers, like, for instance, $\forall x \exists y[R(x,y)]$, an interpretation contains only *one* (non-empty) domain or set for the bound individual variables of the formula, such that *all* individual variables x, y, etc., are to be interpreted as elements of that one domain. So, in $\forall x \exists y[R(x,y)]$, for instance, we are not allowed to let x range over the set of all Men and y range over the set of all Women; the variables x and y have to be interpreted as elements of the same set, for instance, the set of all persons. The expression 'for every man x there is some woman y such that $R(x,y)$' should be translated into our symbolism by a formula of the form $\forall x[M(x) \to \exists y[W(y) \wedge R(x,y)]]$, where M and W are unary predicate symbols for 'is a man' and 'is a woman' respectively.

The predicate logic we have presented thus far is *one-sorted*, i.e., the language contains only one sort of variables which have to be interpreted as elements of one and the same domain. One might also develop a *two-sorted* predicate logic having two sorts of variables, where the variables of the one sort should be interpreted as elements of a domain D_1 and the variables of the other sort as elements of a domain D_2. This corresponds more closely to mathematical practice, where frequently different sorts of variables are used; for instance, m, n, $p,\ldots$ ranging over natural numbers and x, y, $z,\ldots$ ranging over real numbers. The development of two-sorted predicate logic is similar to that of one-sorted predicate logic. The same holds for predicate logic with more than two sorts of variables.

The predicate calculus we have presented thus far is also *first-order*, i.e., one can only quantify over individuals and not over properties of individuals, nor over properties of properties of individuals, and so on. (For instance, 'being a colour' is a property of the property 'being red' of individuals.) In *second-order* logic, not only quantification over individual variables, $\forall x$, $\exists y$, …, but also quantification over predicate variables is allowed: $\forall P$, $\exists Q,\ldots$. This increases the expressive power of the language considerably. By iteration one can obtain *higher-order* predicate logic.

Exercise 4.12. Let $\mathbb{N}$ be the set of natural numbers and $M = \langle \mathbb{N}, P^*, Q^*, R^*\rangle$ with P^*: is even, Q^*: is odd, R^*: is less than ($<$). Which of the following statements are right

and which are wrong?

$M \models P(a)[2]$	$M \models \exists x[P(x)]$	$M \models \exists x[P(x)] \to \exists x[Q(x)]$	$M \models \forall x \exists y[R(x,y)]$
$M \models P(a)[5]$	$M \models \forall x[P(x)]$	$M \models \exists x[P(x)] \to \forall x[Q(x)]$	$M \models \exists y \forall x[R(x,y)]$
$M \models Q(a)[5]$	$M \models \exists x[Q(x)]$	$M \models \forall x[P(x)] \to \exists x[Q(x)]$	$M \models \exists x[P(x) \to Q(x)]$
$M \models Q(a)[2]$	$M \models \forall x[Q(x)]$	$M \models \forall x[P(x)] \to \forall x[Q(x)]$	$M \models \forall x[P(x) \to Q(x)]$

Exercise 4.13. Which of the following alternatives applies to the following below: (i) not satisfiable, (ii) satisfiable, but not valid, (iii) valid, and hence satisfiable?

1. $\exists x[P(x)] \to \forall x[P(x)]$
2. $\exists x[P(x)] \to \exists x[\neg P(x)]$
3. $\exists x[P(x)] \wedge \forall x[\neg P(x)]$
4. $\forall x[P(x)] \wedge \neg \exists x[P(x)]$
5. $\neg \forall x[P(x)] \to \forall x[\neg P(x)]$
6. $\forall x[\neg P(x)] \to \neg \forall x[P(x)]$
7. $\forall x \exists y[R(x,y)] \wedge \exists x \forall y[\neg R(x,y)]$
8. $\forall x \exists y[R(x,y)] \to \exists y \forall x[R(x,y)]$
9. $\exists x[P(x)] \wedge \exists x[Q(x)] \to \exists x[P(x) \wedge Q(x)]$
10. $\forall x[P(x) \vee Q(x)] \to \exists x[P(x)] \vee \forall x[Q(x)]$

Exercise 4.14 (Kleene [9]). Translate each of the following arguments into the language of predicate logic and establish whether the conclusion logically follows from the premisses. If so, give a proof; if not, give a counterexample.

1. Each politician is a showman. Some showmen are insincere. Therefore, some politicians are insincere.
2. No professors are ignorant. All ignorant people are vain. Therefore, no professors are vain.
3. Only birds have feathers. No mammal is a bird. Therefore, each mammal is featherless.
4. Some masons are not strong. All carpenters are strong. Therefore, some carpenters are not masons.
5. Some plumbers are smart. There are no smart persons who are not careful. Therefore, some plumbers are careful.

Exercise 4.15 (Kleene [9]). The same question as in Exercise 4.14.

1. No animals are immortal. All cats are animals. Therefore, some cats are not immortal.
2. If anyone can solve this problem, some philosopher can solve it. Cabot is a philosopher and cannot solve the problem. Therefore, the problem cannot be solved.
3. Any mathematician can solve this problem if anyone can. Cabot is a mathematician and cannot solve the problem. Therefore, the problem cannot be solved.
4. Some healthy people are fat. No unhealthy people are strong. Therefore, some fat people are not strong.
5. Some students are studious. No student is unqualified. Therefore, some unqualified students are not studious.

Exercise 4.16. Prove or refute: $\forall x[P(x) \to Q(x)] \models \exists x[P(x) \wedge Q(x)]$.

Exercise 4.17. Let $R(a,b)$ stand for 'a is greater than b'. i) Translate the following sentences into the language of predicate logic using the binary predicate symbol R:

(a) For every natural number there is a greater one.
(b) There is no natural number which is greater than all natural numbers.
ii) Let A and B be the translations of (a), (b) respectively. Show that not $A \models B$.
iii) Intuitively, (b) seems to follow from (a). Why does not this contradict $A \not\models B$?
iv) Show that $\forall x \exists y[R(y,x)], \forall x \forall y[R(y,x) \to \neg R(x,y)] \models \neg \exists x \forall y[R(x,y)]$.

Exercise 4.18. Translate the following sentence into the language of predicate logic and show that the resulting formula is always true (valid). Take as domain the set of all men in a certain village and interpret $S(x,y)$ as 'x shaves y': there is no man (in the village) such that he shaves precisely those men (in the village) who do not shave themselves.

Exercise 4.19. Check that the following formulas are valid.
a. $\forall x \exists y[P(x) \to P(y)]$;
b. $\forall y \exists x[P(x) \to P(y)]$;
c. $\exists x \forall y[P(x) \to P(y)]$;
d. $\exists y \forall x[P(x) \to P(y)]$.

Exercise 4.20. Which of the following formulas are valid? Give either a proof or a counterexample.
a. $\forall x \exists y[R(x,y)] \to \exists x \forall y[R(x,y)]$;
b. $\exists x \forall y[R(x,y)] \to \forall x \exists y[R(x,y)]$;
c. $\forall x \exists y[R(x,y)] \to \forall x \exists y[R(y,x)]$;
d. $\exists x \forall y[R(x,y)] \to \exists y \forall x[R(x,y)]$;
e. $\exists x \forall y[R(x,y)] \rightleftarrows \exists y \forall x[R(y,x)]$;
f. $\forall x \exists y[R(x,y)] \rightleftarrows \forall y \exists x[R(y,x)]$.

4.3 Basic Results about Validity and Logical Consequence

Definition 4.16. $A \models\!\!\mid B := A \models B$ and $B \models A$, i.e., for every interpretation M and valuation v, $M \models A\ [v]$ iff $M \models B\ [v]$. This is equivalent to $\models A \rightleftarrows B$.

In what follows it is important to realize that for $M = \langle D; P^* \rangle$ and d an element of D, $M \models P(a)[d]$ is equivalent to saying that the proposition $P^*(d)$ - d has the property P^* - has truth value 1. For instance, for $M = \langle \mathbb{N};$ is odd$\rangle$, $M \models P(a)[3]$ because the proposition '3 is odd' has truth value 1, while $M \not\models P(a)[2]$, because the proposition '2 is odd' has truth value 0.

4.3.1 Quantifiers and Connectives

We start with looking at combinations of the quantifiers $\forall$ and $\exists$, respectively, with negation $\neg$:

Theorem 4.7 (Quantifiers and Negation).
1) $\neg \forall x[P(x)] \models\!\!\mid \exists x[\neg P(x)]$;
2) $\neg \exists x[P(x)] \models\!\!\mid \forall x[\neg P(x)]$.
3) $\neg \forall x[P(x)] \not\models \forall x[\neg P(x)]$, *although conversely,* $\forall x[\neg P(x)] \models \neg \forall x[P(x)]$.
4) $\neg \exists x[P(x)] \models \exists x[\neg P(x)]$, *but conversely,* $\exists x[\neg P(x)] \not\models \neg \exists x[P(x)]$.

Proof. 1) Let $M = \langle D; P^* \rangle$ be an interpretation. Then $M \models \neg\forall x[P(x)]$ (i.e., not all elements in D have the property P^*) iff $M \models \exists x[\neg P(x)]$ (i.e., some element in D does not have the property P^*). So each model M is a model of $\neg\forall x[P(x)] \rightleftarrows \exists x[\neg P(x)]$.
2) is shown in a way similar: there is no element in D which has the property P^* iff all elements in D do not have the property P^*.
3) $M = \langle \mathbb{N}; \text{is odd} \rangle$ is a counterexample, since $M \models \neg\forall x[P(x)]$: the proposition 'not all natural numbers are odd' has truth value 1; but $M \not\models \forall x[\neg P(x)]$: the proposition 'all natural numbers are not odd' has truth value 0.
Conversely, suppose $M = \langle D; P^* \rangle$ is an interpretation and suppose $M \models \forall x[\neg P(x)]$, i.e., all elements in D have the property not-P^*. Then surely not all elements in D have the property P^*, i.e., $M \models \neg\forall x[P(x)]$.
4) Let $M = \langle D; P^* \rangle$ be an interpretation and suppose $M \models \neg\exists x[P(x)]$, i.e., there is no element d in D which has the property P^*, in other words, all elements in D have the property not-P^*. So, since D is non-empty, there is an element in D with the property not-P^*, i.e., $M \models \exists x[\neg P(x)]$.
Conversely, $M = \langle \mathbb{N}; \text{is odd} \rangle$ is a counterexample, for $M \models \exists x[\neg P(x)]$: the proposition 'there is a natural number that is not odd' has truth value 1; but $M \not\models \neg\exists x[P(x)]$: the proposition 'there is no odd natural number' has truth value 0. □

Given a propositional formula A, one might let the variable x range over the lines of the truth table of A, and interpret $P(x)$ as 'formula A is 1 at line x'. Under this interpretation the formula $\neg\forall x P(x)$ yields the proposition 'not in all lines of the truth table A is 1', i.e., $\not\models A$, while the formula $\forall x[\neg P(x)]$ yields the proposition 'in all lines of the truth table A is 0', i.e., $\models \neg A$. Under this interpretation $\neg\forall x[P(x)] \not\models \forall x[\neg P(x)]$ expresses that from $\not\models A$ one may in general not conclude that $\models \neg A$, as we have already seen in Theorem 2.12.

Because the meaning of the universal quantitier $\forall$ is similar to the meaning of the connective $\wedge$, the following theorem is evident:

Theorem 4.8 **($\forall$ and $\wedge$).** $\forall x[P(x)] \wedge \forall x[Q(x)] \models\!\!\!\dashv \forall x[P(x) \wedge Q(x)]$

However, one has to be careful when combining a universal quantifier $\forall$ with the connective $\vee$. Consider the following argument:

Every gnome has a conical cap or is a Quaker.
Therefore: all gnomes have a conical cap or all gnomes are Quakers.

Translating this argument into the language of predicate logic we find:

$$\forall x[P(x) \vee Q(x)] \not\models \forall x[P(x)] \vee \forall x[Q(x)]$$

The following interpretation (or model) is a counterexample: $M = \langle \mathbb{N}; P^*, Q^* \rangle$ with $P^*(x)$: x is even, and $Q^*(x)$: x is odd. Then $M \models \forall x[P(x) \vee Q(x)]$: the proposition 'every natural number is even or odd' has truth value 1. But $M \not\models \forall x[P(x)] \vee \forall x[Q(x)]$: the proposition 'all natural numbers are even or all natural numbers are odd' has truth value 0.

Theorem 4.9 **($\forall$ and $\vee$).** *a)* $\forall x[P(x) \vee Q(x)] \not\models \forall x[P(x)] \vee \forall x[Q(x)]$.
But conversely, b) $\forall x[P(x)] \vee \forall x[Q(x)] \models \forall x[P(x) \vee Q(x)]$.

Proof. We start with an informal proof of b): Suppose all things have the property P or all things have the property Q. If an individual thing has the property P, then it also has the property $P \vee Q$; and similarly, if an individual thing has the property Q, then it also has the property $P \vee Q$. So, in both cases it follows that all things have the property $P \vee Q$, i.e., $\forall x[P(x) \vee Q(x)]$.
More precisely: Suppose $M = \langle D;\ P^*,\ Q^* \rangle$ is an interpretation and $M \models \forall x[P(x)] \vee \forall x[Q(x)]$, i.e., 1) for every thing d in the domain D of M, $M \models P(a)[d]$ or 2) for every thing d in the domain D of M, $M \models Q(a)[d]$. In case 1) it follows from 'if $M \models P(a)[d]$, then $M \models P(a) \vee Q(a)\ [d]$' that for all things d in the domain D of M, $M \models P(a) \vee Q(a)\ [d]$, in other words, $M \models \forall x[P(x) \vee Q(x)]$. In case 2) it follows in a similar way that $M \models \forall x[P(x) \vee Q(x)]$. □

Given propositional formulas A and B, one may let the variable x range over the lines in the truth tables of A, B, interpret $P(x)$ as 'A is 1 in line x' and $Q(x)$ as 'B is 1 in line x'. Under this interpretation $\forall x[P(x) \vee Q(x)]$ yields the proposition 'in all lines of the truth table, A is 1 or B is 1', i.e., $\models A \vee B$. But under this same interpretation $\forall x[P(x)] \vee \forall x[Q(x)]$ yields the proposition 'in all lines of the truth table A is 1 or in all lines of the truth table B is 1', i.e., $\models A$ or $\models B$. Under this interpretation $\forall x[P(x) \vee Q(x)] \not\models \forall x[P(x)] \vee \forall x[Q(x)]$ expresses that in general from $\models A \vee B$ one may not conclude that $\models A$ or $\models B$, as we have already seen in Theorem 2.13.

Because the meaning of the existential quantifier $\exists$ is similar to the meaning of the connective $\vee$, the following theorem is evident:

Theorem 4.10 ($\exists$ and $\vee$). $\exists x[P(x)] \vee \exists x[Q(x)] \models\!\!\!\dashv \exists x[P(x) \vee Q(x)]$.

However, one has to be careful when combining an existential quantifier with the connective $\wedge$. Consider the following argument:

There is gnome who has a conical cap and there is a gnome who is a Quaker.
Therefore: there is a gnome who has a conical cap and is a Quaker.

Translating this argument into the language of predicate logic we find:

$$\exists x[P(x)] \wedge \exists x[Q(x)] \not\models \exists x[P(x) \wedge Q(x)]$$

The following interpretation (or model) is a counterexample: $M = \langle \mathbb{N}; P^*, Q^* \rangle$ with $P^*(x)$: x is even, and $Q^*(x)$: x is odd. Then $M \models \exists x[P(x)] \wedge \exists x[Q(x)]$: the proposition 'there is an even natural number and there is an odd natural number' has truth value 1. But $M \not\models \exists x[P(x) \wedge Q(x)]$: the proposition 'there is natural number that is both even and odd' has truth value 0.

Theorem 4.11 ($\exists$ and $\wedge$). *a)* $\exists x[P(x)] \wedge \exists x[Q(x)] \not\models \exists x[P(x) \wedge Q(x)]$.
But conversely, b) $\exists x[P(x) \wedge Q(x)] \models \exists x[P(x)] \wedge \exists x[Q(x)]$.

Proof. We start with an informal proof of b): Suppose $\exists x[P(x) \wedge Q(x)]$, i.e., there is a thing d such that d has both the property P and the property Q. Then this d has the property P, so $\exists x[P(x)]$; and this same d has the property Q, so $\exists x[Q(x)]$.
More precisely: Suppose $M = \langle D;\ P^*,\ Q^* \rangle$ is a model and $M \models \exists x[P(x) \wedge Q(x)]$, i.e., there is a thing d in the domain D of M such that $M \models P(a) \wedge Q(a)\ [d]$. Then $M \models P(a)[d]$, hence $M \models \exists x[P(x)]$; and $M \models Q(a)[d]$ and hence $M \models \exists x[Q(x)]$. □

Consider the following argument:

If all gnomes have a conical cap, then all gnomes are Quakers.
Therefore: every gnome with a conical cap is a Quaker.

Translating this argument into the language of predicate logic we find:

$$\forall x[P(x)] \to \forall x[Q(x)] \not\models \forall x[P(x) \to Q(x)]$$

The following interpretation (or model) is a counterexample: $M = \langle \mathbb{N}; P^*, Q^* \rangle$ with $P^*(x)$: x is even, and $Q^*(x)$: x is odd. Then $M \models \forall x[P(x)] \to \forall x[Q(x)]$: the proposition 'if all natural numbers are even, then all natural numbers are odd' has truth value $0 \to 0 = 1$. But $M \not\models \forall x[P(x) \to Q(x)]$: the proposition 'for every natural number n, if n is even, then n is odd' has truth value 0.

Theorem 4.12 ($\forall$ and $\to$). *a)* $\forall x[P(x)] \to \forall x[Q(x)] \not\models \forall x[P(x) \to Q(x)]$.
But conversely, b) $\forall x[P(x) \to Q(x)] \models \forall x[P(x)] \to \forall x[Q(x)]$.

Proof. We start with an informal proof of b): Suppose $\forall x[P(x) \to Q(x)]$, i.e., every thing with the property P also has the property Q. Next suppose $\forall x[P(x)]$, i.e., every thing has the property P. Then clearly it follows that every thing has the property Q. More precisely: Suppose $M = \langle D;\ P^*,\ Q^* \rangle$ is a model and $M \models \forall x[P(x) \to Q(x)]$, i.e., for every thing d in the domain D of M, $M \models P(a) \to Q(a)\ [d]$. Suppose next that $M \models \forall x[P(x)]$, i.e., for every thing d in the domain D of M, $M \models P(a)[d]$. Then it clearly follows that for every thing d in the domain D of M, $M \models Q(a)[d]$, in other words, $M \models \forall x[Q(x)]$. □

Given propositional formulas A and B, one may let the variable x range over the lines in the truth tables of A, B, interpret $P(x)$ as 'A is 1 in line x' and $Q(x)$ as 'B is 1 in line x'. Under this interpretation $\forall x[P(x)] \to \forall x[Q(x)]$ yields the proposition 'if A is 1 in all lines of the truth table, then B is 1 in all lines of the truth table', i.e., if $\models A$, then $\models B$. But under this same interpretation $\forall x[P(x) \to Q(x)]$ yields the proposition 'in all lines x of the truth table, if A is 1 at line x, then also B is 1 at line x', i.e., $\models A \to B$. Under this interpretation $\forall x[P(x)] \to \forall x[Q(x)] \not\models \forall x[P(x) \to Q(x)]$ expresses that in general from 'if $\models A$, then $\models B$' one may not conclude that $\models A \to B$, as we have already seen in Theorem 2.11.

Consider the following argument:

There is a gnome such that if he has a conical cap, then he is a Quaker.
There is gnome who has a conical cap.
Therefore: there is a gnome who is a Quaker.

Translating this argument into the language of predicate logic we find:

$$\exists x[P(x) \to Q(x)], \exists x[P(x)] \not\models \exists x[Q(x)]$$

The following interpretation is a counterexample: $M = \langle \mathbb{N}; P^*, Q^* \rangle$ with $P^*(x)$: x is even, and $Q^*(x)$: $x \neq x$. Then $M \models \exists x[P(x) \to Q(x)]$, since $M \models P(a) \to Q(a)\ [3]$: the proposition 'if 3 is even, then $3 \neq 3$' has truth value $0 \to 0 = 1$. Also $M \models \exists x[P(x)]$: the proposition 'there is an even natural number' has truth value 1. But

$M \not\models \exists x[Q(x)]$: the proposition 'there is a natural number which is not equal to itself' has truth value 0.

Theorem 4.13 ($\exists$ and $\rightarrow$). *a)* $\exists x[P(x) \rightarrow Q(x)], \exists x[P(x)] \not\models \exists x[Q(x)]$,
or, equivalently, $\exists x[P(x) \rightarrow Q(x)] \not\models \exists x[P(x)] \rightarrow \exists x[Q(x)]$.
But conversely, b) $\exists x[P(x)] \rightarrow \exists x[Q(x)] \models \exists x[P(x) \rightarrow Q(x)]$.

Proof. We start with an informal proof of b): Suppose $\exists x[P(x)] \rightarrow \exists x[Q(x)]$ (*) and $\neg\exists x[P(x) \rightarrow Q(x)]$. Then $\forall x[\neg\ (P(x) \rightarrow Q(x))]$, i.e., $\forall x[P(x) \wedge \neg Q(x)]$, in other words, $\forall x[P(x)] \wedge \forall x[\neg Q(x)]$. Hence, surely, $\exists x[P(x)]$ and hence by (*) $\exists x[Q(x)]$. Contradiction with $\forall x[\neg Q(x)]$.
More precisely: Suppose $M = \langle D;\ P^*,\ Q^* \rangle$ is a model and $M \models \exists x[P(x)] \rightarrow \exists x[Q(x)]$. (*) Case 1) $M \models \exists x[Q(x)]$, i.e., there is some element d in the domain D of M such that $M \models Q(a)[d]$. Then also $M \models P(a) \rightarrow Q(a)\ [d]$ and hence, $M \models \exists x[P(x) \rightarrow Q(x)]$. Case 2) $M \not\models \exists x[Q(x)]$. Then by (*), $M \not\models \exists x[P(x)]$, i.e., $M \models \forall x[\neg P(x)]$. But then $M \models \forall x[P(x) \rightarrow Q(x)]$, since for every d in the domain D of M, $M \models P(a) \rightarrow Q(a)\ [d]$ $(0 \rightarrow 0 = 1)$. Hence, surely, $M \models \exists x[P(x) \rightarrow Q(x)]$. □

4.3.2 Two different quantifiers

Consider the following argument:

Every gnome has a teacher. .
Therefore: some gnome is the teacher of all gnomes.

Translating this argument into the language of predicate logic, reading $R(x,y)$ as 'x has y as teacher', we find:

$$\forall x \exists y[R(x,y)] \not\models \exists y \forall x[R(x,y)]$$

$M = \langle \mathbb{N};\ < \rangle$ is a counterexample. $M \models \forall x \exists y[R(x,y)]$: the proposition 'for every natural number x there is a larger natural number y' has truth value 1; but $M \not\models \exists y \forall x[R(x,y)]$: the proposition 'there is a natural number y such that all natural numbers x are smaller than y' has truth value 0.

Theorem 4.14 (Interchanging Quantifiers). *a)* $\forall x \exists y[R(x,y)] \not\models \exists y \forall x[R(x,y)]$.
But conversely, b) $\exists y \forall x[R(x,y)] \models \forall x \exists y[R(x,y)]$.

Proof. We start with an informal proof of b): Suppose $\exists y \forall x[R(x,y)]$, i.e., there is a thing d such that each thing x stands in the relation R to d. Then clearly, for every x there is a thing y, namely d, such that x is in relation R to y. For instance, suppose there is someone, say Michael Jackson, such that all persons admire this one person. Then clearly, everyone admires at least one person, namely Michael Jackson.
More precisely: Let $M = \langle D;\ R^* \rangle$ be a model and suppose $M \models \exists y \forall x[R(x,y)]$, i.e., there is some d in the domain D of M such that $M \models \forall x[R(x,b)]\ [d]$. Then clearly, $M \models \forall x \exists y[R(x,y)]$ because for every x one may take $y = d$. □

The following theorem says that a negation in front of a sequence of quantifiers may be pushed inside, provided one changes a universal quantifier $\forall$ into an existential quantifier $\exists$ and an existential quantifier $\exists$ into a universal quantifier $\forall$.

Theorem 4.15 (Negation in front of a sequence of Quantifiers).
1. $\neg\forall x\exists y[R(x,y)] \models\!\!\!\!\dashv \exists x\forall y[\neg R(x,y)]$.
2. $\neg\exists x\forall y[R(x,y)] \models\!\!\!\!\dashv \forall x\exists y[\neg R(x,y)]$.

Proof. 1. Let $M = \langle D; R^* \rangle$ be an interpretation. Then
$M \models \neg\forall x\exists y[R(x,y)]$ iff (by Theorem 4.7, 1)
$M \models \exists x\neg\exists y[R(x,y)]$ iff (by Theorem 4.7, 2)
$M \models \exists x\forall y[\neg R(x,y)]$.
2. $M \models \neg\exists x\forall y[R(x,y)]$ iff $M \models \forall x\neg\forall y[R(x,y)]$ iff $M \models \forall x\exists y[\neg R(x,y)]$. □

Warning Note that 'not $\models A$ (A is not valid)' means that not every interpretation M for A is a model of A, in other words, there is at least one interpretation M that makes A false. In such a case one may in general not conclude that $\models \neg A$, since there may be other interpretations which make A true.

For instance, we have seen in Theorem 4.9 that there are interpretations M which make the formula $\forall x[P(x) \vee Q(x)] \to \forall x[P(x)] \vee \forall x[Q(x)]$ false, but the interpretation $M = \langle \mathbb{N}; \text{is even}, x = x \rangle$ makes $\forall x[P(x) \vee Q(x)] \to \forall x[P(x)] \vee \forall x[Q(x)]$ true. Summarizing, the formula $\forall x[P(x) \vee Q(x)] \to \forall x[P(x)] \vee \forall x[Q(x)]$ yields a false proposition for some interpretations and a true proposition for others. Hence, neither the formula itself nor its negation is valid.

4.3.3 About the axioms and rules for $\forall$ and $\exists$

Later in this chapter the formula $\forall x[A(x)] \to A(t)$, where t is a term, will be chosen as an logical axiom schema for $\forall$, and $A(t) \to \exists x[A(x)]$ as a logical axiom schema for $\exists$. In the next theorem we verify that these formulas are valid or always true.

Theorem 4.16 (Validity of the logical axioms for the Quantifiers).
Let t be a term, and let $A(t)$ result from $A(x)$ by substituting t for all occurrences of x in $A(x)$. Then 1. $\models \forall x[A(x)] \to A(t)$, *and 2.* $\models A(t) \to \exists x[A(x)]$.

Proof. The formula $\forall x[A(x)] \to A(t)$ expresses that if all objects (of a certain kind) have the property A^* and t^* is one these objects, then also t^* has the property A^*. More formally: Let M be a structure with domain D and let v be a valuation assigning values in D to the individual variables occurring free in $A(x)$. We have to show that $M \models \forall x[A(x)] \to A(t)\ [v]$. So, suppose $M \models \forall x[A(x)]\ [v]$, i.e., for all d in D, $M \models A(a)\ [d/v)]$ where a is a free variable not occurring in $A(x)$ and d/v is the same valuation as v except that it assigns d to the variable a. (*)
Now let t^* be the element in D assigned by the valuation v to the term t, i.e., $v(t) = t^*$. Then because of (*), $M \models A(a)\ [t^*/v]$, which is equivalent to $M \models A(t)\ [v]$.

The proof of 2. is similar to the proof of 1. □

Note that if y is a bound variable, $\forall x[A(x)] \to A(y)$ is *not* a formula; and even if it were a formula, in general, not $\models \forall x[A(x)] \to A(y)$. For instance, if $A(x) = \exists y[P(x,y)]$, $\not\models \forall x \exists y[P(x,y)] \to \exists y[P(y,y)]$, for $\langle \mathbb{N}; < \rangle$ is a counterexample to this formula. This demonstrates the usefulness of having two kinds of symbols for free and bound (occurrences of) individual variables. If one uses the same symbols for both free and bound occurrences of individual variables, then $\forall x[A(x)] \to A(y)$ is only valid under the condition that y is *free for* x in $A(x)$, i.e., if any free occurrence of x in $A(x)$ is replaced by an occurrence of y, then the resulting occurrence of y in $A(y)$ should also be free.

In Section 4.4 we shall introduce the following deduction rules for $\forall$ and for $\exists$, assuming that C does not contain the free variable a :

$$\frac{C \to P(a)}{C \to \forall x[P(x)]} \qquad \frac{P(a) \to C}{\exists x[P(x)] \to C}$$

Theorem 4.17 says that these rules are sound in the sense that for any interpretation M, if M makes the premiss true, then M also makes the conclusion true. But the same theorem says that $C \to P(a) \not\models C \to \forall x[P(x)]$ and that $P(a) \to C \not\models \exists x[P(x)] \to C$. Note the difference with the rule Modus Ponens, where we do have $A, A \to B \models B$. In propositional logic we have seen in Theorem 2.11 that '$A \models B$' is a stronger statement than 'if $\models A$, then $\models B$'. This becomes particularly evident in predicate logic, as can be seen from Theorem 4.17 below. For instance, from Theorem 4.17, 1 it follows that 'if $\models P(a)$, then $\models \forall x[P(x)]$' is true, while the stronger statement $P(a) \models \forall x[P(x)]$ is false. Items 2 and 3 of this theorem state that the logical deduction rules for $\forall$ and for $\exists$, to be introduced in Section 4.4, are sound.

Theorem 4.17 (Soundness of the deduction rules for the Quantifiers).
Let $A(a)$ be a formula containing a free variable a and let C be a formula not containing the free variable a. Let M be an interpretation.
1. If $M \models A(a)$, then $M \models \forall x[A(x)]$. But $A(a) \not\models \forall x[A(x)]$.
2. If $M \models C \to A(a)$, then $M \models C \to \forall x[A(x)]$. But $C \to A(a) \not\models C \to \forall x[A(x)]$.
3. If $M \models A(a) \to C$, then $M \models \exists x[A(x)] \to C$. But $A(a) \to C \not\models \exists x[A(x)] \to C$.

Proof. 1) Suppose $M \models A(a)$, i.e., for every d in the domain D of M, $M \models A(a)[d]$. In other words, $M \models \forall x[A(x)]$. On the other hand, let $M = \langle \mathbb{N}; \text{is even} \rangle$. Then $M \models P(a)[2]$ (2 is even), but $M \not\models \forall x[P(x)]$ (not all natural numbers are even). Hence $P(a) \not\models \forall x[P(x)]$.
2) Suppose $M \models C \to A(a)$ and C does not contain the variable a. Then by 1) $M \models \forall x[C \to A(x)]$ and hence, because C does not contain a, $M \models C \to \forall x[A(x)]$. On the other hand, for $C = Q \vee \neg Q$, $C \to A(a)$ is equivalent to $A(a)$ and $C \to \forall x[A(x)]$ is equivalent to $\forall x[A(x)]$. Hence, for $C = Q \vee \neg Q$, $C \to A(a) \models C \to \forall x[A(x)]$ iff $A(a) \models \forall x[A(x)]$, which according to 1) does not hold. Another way to see that $C \to A(a) \not\models C \to \forall x[A(x)]$ does not hold is as follows: from 'if it is September 5, then a(ntoine) has his birthday' one may not conclude 'if it is September 5, then everyone has his birthday'.
3) Suppose $M \models A(a) \to C$ and C does not contain a. That is, for every element d in

the domain of M, $M \models A(a) \to C\ [d]$ (*). We have to show: $M \models \exists x[A(x)] \to C$. So, suppose $M \models \exists x[A(x)]$, i.e., for some d in the domain of M, $M \models A(a)\ [d]$. Hence, because of (*) and because C does not contain a, $M \models C$. On the other hand, from 'if a(ntoine) has his birthday, then it is September 5' one may not conclude that 'if someone has his birthday, then it is September 5'. □

The condition in Theorem 4.17 that C does not contain the free variable a is necessary. To see this, let C be $A(a)$. Then $M \models A(a) \to A(a)$, but in general $M \not\models A(a) \to \forall x[A(x)]$; for instance, the proposition 'if a(ntoine) has his birthday, then everyone has his birthday' is false.

Also $M \models A(a) \to A(a)$, but in general $M \not\models \exists x[A(x)] \to A(a)$; for instance, the proposition 'if there is an even number, then 3 is even' is false.

*4.3.4 Predicate Logic with Function Symbols**

In mathematics, but also in natural language, one frequently uses functions. For instance, the binary function + that assigns to any pair of natural numbers n and m the natural number $n+m$; the unary function *the-mother-of* that assigns to any person a his or her mother: *the-mother-of (a)*. So, it is convenient to extend the predicate language with function symbols:

$$f_1, f_2, f_3, \ldots$$

where each f_i is supposed to be k_i-ary, i.e., taking k_i arguments. Individual constants are then special function symbols, namely, function symbols f_i taking 0 arguments, i.e., $k_i = 0$. An example of a predicate language containing function symbols for addition and multiplication of natural numbers is given in Chapter 5.

With no function symbols present, the only terms - denoting elements of the domain D of a given interpretation M - are free individual variables and individual constants. But with function symbols present, we have to extend the notion of *term*.

Definition 4.17 (Terms). *Terms* are defined (inductively) as follows:
1. Each free individual variable is a term.
2. Each individual constant is a term.
3. If f_i is a k_i-ary function symbol and $t_1, \ldots, t_{k_i}$ are terms, then $f_i(t_1, \ldots, t_{k_i})$ is a term. Note that clause 2 can be treated as a special case of clause 3, taking $k_i = 0$.

Formulas are defined as before (see Definition 4.5), but now allowing the $t_1, \ldots, t_n$ in Definition 4.4 of 'atomic formula' to be any terms, instead of simply any free individual variables or individual constants.

If we extend the predicate language with function symbols, we have to adapt the definition of an interpretation or structure (Definition 4.7) accordingly.

Definition 4.18 (Interpretation). An *interpretation M for the predicate logic with function symbols* is by definition a tuple $\langle D; P_1^*, P_2^*, \ldots\ ; f_1^*, f_2^*, \ldots \rangle$, such that:
1. D is a non-empty set, called the *domain* of M.

2. For any n_i-ary predicate symbol P_i, P_i^* is a n_i-ary predicate over D.
3. For any k_i-ary function symbol f_i, f_i^* is a function that assigns to any k_i tuple of elements of D an element of D.

For instance, if $\oplus$ is a 2-ary function symbol in the predicate language and M is an interpretation with domain $\mathbb{N}$, then the interpretation $\oplus^*$ of $\oplus$ might be the function + that assigns to any pair (or 2-tuple) n,m of natural numbers the natural number $n+m$. If f_i is an individual constant, i.e., $k_i = 0$, then f_i^* is an element of D.

The definitions of $M \models A$ (M is a model of A), $\models B$ (B is valid) and of $A_1,\ldots,A_n \models B$ (B is a valid or logical consequence of $A_1,\ldots,A_n$) are as before, taking into consideration that now all structures M are interpretations for the predicate logic with function symbols.

All results stated for the predicate logic without function symbols also hold for the predicate logic with function symbols, where terms may also contain function symbols in addition to individual variables and constants.

Example 4.11. Let f be a binary (i.e., 2-ary) function symbol, $\equiv$ a binary predicate symbol, and a and b free individual variables. Then $f(a,b)$ and $f(b,a)$ are terms. Let $M = \langle \mathbb{N}; =; +\rangle$ be the model with domain $\mathbb{N}$, interpreting $\equiv$ as = (equality) and f as + (addition). Then $M \models f(a,b) \equiv f(b,a)$, because for all natural numbers n,m, $n+m = m+n$. Also $M \models f(a,b) \equiv a\ [2,0]$, because 2 + 0 = 2, but $M \not\models f(a,b) \equiv a$, because, for instance, $M \not\models f(a,b) \equiv a\ [2,1]$, since $2+1 \neq 2$.

4.3.5 Prenex Form*

Definition 4.19 (Prenex Formula). A formula A is in *prenex (normal) form* if A consists of a (possibly empty) string of quantifiers followed by a formula without quantifiers. We also say that A is a *prenex formula*.

A simple example is the formula $\forall x \forall y \exists z[P(x,y) \wedge Q(y,x) \rightarrow P(z,z)]$. By pulling out quantifiers, we can reduce every formula to a formula in prenex form.

Theorem 4.18 (Prenex Normal Form Theorem). *For every formula A there is a prenex formula B such that $\models A \rightleftarrows B$ (or, equivalently, $A \models\!\!\!\dashv B$).*

Proof. The proof is by induction on the complexity of the formula A (Theorem 4.1).
Induction basis: for an atomic formula $P(t_1,\ldots,t_n)$ the theorem is trivially true.
Induction step for the connectives: suppose $A = B \rightarrow C$, $B \wedge C$, $B \vee C$ or $\neg B$, and B, C are equivalent to prenex formulas B^*, C^* respectively (induction hypothesis). Then $B^* = (Q_1 y_1)\ldots(Q_n y_n)B^1$ and $C^* = (Q'_1 z_1)\ldots(Q'_m z_m)C^1$, where Q_i, Q'_j are quantifiers and B^1, C^1 open. By Theorem 4.3 all bound variables can be chosen distinct. Now A is semantically equivalent to $B^* \rightarrow C^*$, $B^* \wedge C^*$, $B^* \vee C^*$ or $\neg B^*$ respectively. By means of the *prenex operations* (a), (b), (c) and (d) below, we can convert the latter formula into a formula in prenex form.

(a) (1) Replace a part $Qx[B] \to C$ by $Q'x[B \to C]$,
where $Q'x$ is $\forall x$ if Qx is $\exists x$ and $Q'x$ is $\exists x$ if Qx is $\forall x$.
(2) Replace a part $B \to Qx[C]$ by $Qx[B \to C]$.
(b) (1) Replace a part $Qx[B] \wedge C$ by $Qx[B \wedge C]$.
(2) Replace a part $B \wedge Qx[C]$ by $Qx[B \wedge C]$.
(c) (1) Replace a part $Qx[B] \vee C$ by $Qx[B \vee C]$.
(2) Replace a part $B \vee Qx[C]$ by $Qx[B \vee C]$.
(d) Replace a part $\neg Qx[B]$ by $Q'x[\neg B]$ where Q' is as in (a).

It remains to be shown that if E' results from E by a prenex operation, then $\models E \rightleftarrows E'$. But this is straightforward; see Exercise 4.26.

Induction step for the quantifiers: Suppose $A = \forall x[B(x)]$ or $A = \exists x[B(x)]$ and $B(a)$ is equivalent to a prenex formula $B^*(a)$ (induction hypothesis). By Theorem 4.3 we can choose the bound variables in $B^*(a)$ distinct from x. Then $\forall x[B^*(x)]$ and $\exists x[B^*(x)]$ are prenex formulas and $\models A \rightleftarrows \forall x[B^*(x)]$ or $\models A \rightleftarrows \exists x[B^*(x)]$ respectively. □

The prenex normal form theorem states that for every formula A there is a prenex formula B which is equivalent to A. Being prenex, B consists of a finite string of quantifiers followed by a formula C without quantifiers, i.e., $B = Q_1x_1 \ldots Q_nx_n[C]$. According to Theorem 2.18, C is equivalent to a formula C' in conjunctive normal form. So, by combining the prenex normal form theorem and the conjunctive normal form theorem (Theorem 2.18), every formula A is equivalent to a formula of the form

$$Q_1x_1 \ldots Q_nx_n\ [(L_{i_1} \vee \ldots \vee L_{i_k}) \wedge \ldots \wedge (L_{j_1} \vee \ldots \vee L_{j_l})],$$

where each L_n is a literal, i.e., an atomic formula or the negation of an atomic formula. Any *logic program* in the programming language PROLOG, to be treated in Section 9.1, will be a formula of this form with all the quantifiers universal.

In 1936 A. Church and A. Turing proved independently that there is no decision procedure for validity of formulas in predicate logic (see Section 4.5). Nevertheless, there is a decision procedure for formulas in prenex normal form in the prefix of which no existential quantifier precedes any universal quantifier. In the exercises of Section 4.5 some other classes of formulas are given for which a decision procedure is known. Most of these classes consist of formulas having a prenex normal form of a particular type. For more of these results see A. Church [3].

*4.3.6 Skolemization, Clausal Form**

If M is a model with domain D and $M \models \forall x \exists y[P(x,y)]$, then there must be some function $f^* : D \to D$ such that for all $d \in D$, $M \models P(a_1,a_2)[d, f^*(d)]$. This suggests introducing a function symbol f in our language and replacing $\forall x \exists y[P(x,y)]$ by the formula $\forall x[P(x, f(x))]$.

Let A be a formula in prenex normal form, the *Skolem (normal) form* of A is obtained by eliminating all existential quantifiers in (the prefix of) A as fol-

lows: for any expression of the form $\forall x_1 \ldots \forall x_k \exists y[B(x_1,\ldots,x_k,y,\ldots)]$ a new k-ary function symbol f is introduced and the original expression is replaced by $\forall x_1 \ldots \forall x_k[B(x_1,\ldots,x_k,f(x_1,\ldots,x_k),\ldots)]$.

Thus, the Skolem normal forms of $\exists x[P(x)], \forall x \exists y[P(x,y)]$, $\forall x \exists y \forall z[P(x,y,z)]$ and $\forall x \exists y \forall z \exists u[P(x,y,z,u)]$ are $P(c)$, $\forall x[P(x,f(x))]$, $\forall x \forall z[P(x,f(x),z)]$ and $\forall x \forall z[P(x,f(x),z,g(x,z))]$ respectively, where c is a new individual constant and f and g are new function symbols.

Let $Sk(A)$ denote the Skolem normal form of A. Clearly, if $M \models Sk(A)$, then $M \models A$. But not conversely: $\langle \mathbb{N}; \text{is even}; 3 \rangle \models \exists x[P(x)]$, but $\langle \mathbb{N}; \text{is even}; 3 \rangle \not\models P(c)$. And if i is the identity function on $\mathbb{N}$, then $\langle \mathbb{N}; <; i \rangle \models \forall x \exists y[P(x,y)]$, but $\langle \mathbb{N}; <; i \rangle \not\models \forall x[P(x,f(x))]$. However, it is easy to see the following

Theorem 4.19. *1. $Sk(A) \models A$; but not conversely.*
2. $Sk(A)$ is satisfiable iff A is satisfiable.

It follows that if $\models Sk(A)$, then also $\models A$. But the converse does not hold: $\models \forall x \exists y[P(x) \to P(y)]$, but $\not\models \forall x[P(x) \to P(f(x))]$.

Definition 4.20 (Clausal Form). Given any formula A (of first-order predicate logic), the *clausal form* $C(A)$ of A is obtained as follows:
1. construct the prenex (normal) form A' of A; $A' = Q_1 x_1 \ldots Q_n x_n[M]$, where $Q_i = \forall$ or $\exists$ and M is quantifier-free;
2. construct the Skolem (normal) form A^* of A'; $A^* = \forall x_1 \ldots \forall x_k[M^*]$, M^* quantifier-free, but containing $(n-k)$ additional function-symbols;
3. construct the conjunctive normal form $(L_1 \vee \ldots \vee L_{n_1}) \wedge \ldots \wedge (L_k \vee \ldots \vee L_{n_k})$ of M^* (see Theorem 2.18).

Example 4.12. Let $A = \exists x[P(x)] \to \exists x[Q(x)]$. Then $A' = \forall x \exists y[P(x) \to Q(y)]$, $A^* = \forall x[P(x) \to Q(f(x))]$ and $C(A) = \forall x[\neg P(x) \vee Q(f(x))]$.

Theorem 4.20. *For the clausal form $C(A)$ of A the following holds:*
1. $C(A) \models A$. Consequently, if $C(A)$ is valid, then A is valid. But not conversely: $\models \forall x \exists y[P(x) \to P(y)]$, but $\not\models \forall x[\neg P(x) \vee P(f(x))]$.
2. A is satisfiable iff $C(A)$ is satisfiable.
3. The 'complexity' of $C(A)$ is lower than that of A, in the sense that $C(A)$ contains only universal quantifiers and no existential quantifiers that occur in the prenex (normal) form of A.

Automated theorem provers for logic based on *resolution* operate as follows. Given any assumption formulas $A_1,\ldots,A_n$ and given any formula B, they construct $\neg B$ and the clausal forms $C(A_1),\ldots,C(A_n)$ and $C(\neg B)$ of $A_1,\ldots,A_n$ and $\neg B$ respectively. Next they check whether a contradiction can be derived from $C(A_1),\ldots,C(A_n)$ and $C(\neg B)$ by resolution (or otherwise, for instance, by the tableaux-method). If so, then $C(A_1),\ldots,C(A_n)$, $C(\neg B)$ are not simultaneously satisfiable; hence, $A_1,\ldots,A_n$, $\neg B$ are not simultaneously satisfiable and therefore $A_1,\ldots,A_n \models B$. If not, then $C(A_1),\ldots,C(A_n)$, $C(\neg B)$ are simultaneously satisfiable (completeness); hence, $A_1,\ldots,A_n$, $\neg B$ are simultaneously satisfiable and therefore $A_1,\ldots,A_n \not\models B$.

Theorem 4.21. *Any definite logic program (see also Chapter 2 and 9) is actually a formula in clausal form.*

Proof. The structure of any definite *logic program* is by definition the following:

$$\left.\begin{array}{l} P_1 \text{ :- } Q_1, \ldots, Q_{n_1}. \\ \vdots \\ P_k \text{ :- } Q_k, \ldots, Q_{n_k}. \end{array}\right\} \text{I}$$

where P_i and Q_j are atomic formulas.

$$\text{I stands for:} \quad \left.\begin{array}{l} (P_1 \leftarrow Q_1 \wedge \ldots \wedge Q_{n_1}) \ \wedge \\ \quad \vdots \\ \wedge \ (P_k \leftarrow Q_k \wedge \ldots \wedge Q_{n_k}) \end{array}\right\} \text{II}$$

or, equivalently, for:

$$\left.\begin{array}{l} (P_1 \vee \neg Q_1 \vee \ldots \vee \neg Q_{n_1}) \wedge \\ \vdots \\ \wedge (P_k \vee \neg Q_k \vee \ldots \vee \neg Q_{n_k}) \ . \end{array}\right\} \text{III}$$

Remembering that $Cl(A)$ denotes the universal closure of A, III is short for:

$$\left.\begin{array}{l} Cl(P_1 \vee \neg Q_1 \vee \ldots \vee \neg Q_{n_1}) \wedge \\ \vdots \\ \wedge Cl(P_k \vee \neg Q_k \vee \ldots \vee \neg Q_{n_k}) \ . \end{array}\right\} \text{IV}$$

IV, and hence also I is equivalent to a formula $\forall x_1 \ldots \forall x_k[(P_1 \vee \neg Q_1 \vee \ldots \vee \neg Q_{n_1}) \wedge \ldots \wedge (P_k \vee \neg Q_k \vee \ldots \vee \neg Q_{n_k})]$, which is in clausal form.

Exercise 4.21. let P be a unary and Q a 0-ary predicate symbol. Which of the following statements are right? Give either a proof or a counterexample.
1. $\forall x[P(x)] \rightarrow Q \models \forall x[P(x) \rightarrow Q]$ and 2. $\forall x[P(x) \rightarrow Q] \models \forall x[P(x)] \rightarrow Q$.
3. $\exists x[P(x)] \rightarrow Q \models \exists x[P(x) \rightarrow Q]$ and 4. $\exists x[P(x) \rightarrow Q] \models \exists x[P(x)] \rightarrow Q$.

Exercise 4.22. Are the following formulas valid or invalid? Give either a proof or a counterexample.
1. $(\forall x[P(x)] \rightarrow \exists x[Q(x)]) \rightleftarrows \exists x[P(x) \rightarrow Q(x)]$;
2. $(\exists x[P(x)] \rightarrow \forall x[Q(x)]) \rightleftarrows \forall x[P(x) \rightarrow Q(x)]$.

Exercise 4.23. Which of the following statements are right? Give either a proof or a counterexample.
1. $\forall x[P(x) \rightarrow Q(x)] \models (\exists x[P(x)] \rightarrow \exists x[Q(x)])$; and conversely?.
2. $\exists x[P(x) \rightarrow Q(x)] \models (\forall x[P(x)] \rightarrow \forall x[Q(x)])$; and conversely?

Exercise 4.24. Prove or refute: $\forall x \exists y[P(x) \rightarrow Q(y)] \models \exists y \forall x[P(x) \rightarrow Q(y)]$.

Exercise 4.25. (H. Wang) Prove: $\models \exists x \exists y \forall z[(P(x,y) \rightarrow P(y,z) \wedge P(z,z)) \ \wedge (P(x,y) \wedge Q(x,y) \rightarrow Q(x,z) \wedge Q(z,z))]$.

Exercise 4.26. Prove that the formulas in the prenex operations (a), (b), (c) and (d) in the proof of Theorem 4.18 are semantically equivalent.

Exercise 4.27. Following the proof of the prenex normal form theorem (Theorem 4.18) convert each of the following formulas into a formula in prenex form.
1. $\forall x[P(x)] \to \exists x[Q(x)]$; 2. $\exists x[P(x)] \to \forall x[Q(x)]$;
3. $\exists x[P(x,a)] \to \exists x[Q(x) \vee \neg\exists y[R(y)]]$.

Exercise 4.28. Find two prenex normal forms for the formula $\exists x[P(x)] \to \exists x[Q(x)]$. (See also Exercise 4.24)

4.4 Syntax: Provability and Deducibility

In this section we shall generalize the notions of (logical) provability ($\vdash B$) and (logical) deducibility ($A_1,\ldots,A_n \vdash B$), as defined for propositional logic in Section 2.6, to predicate logic.

It turns out that also for (classical) predicate logic one can select a small, finite, number of valid formula schemata, henceforth called (logical) axiom schemata, and rules of inference such that i) precisely all valid formulas can be obtained by finitely many applications of the rules to instances of the axiom schemata and such that ii) for any premisses $A_1,\ldots,A_n$, precisely all valid consequences of $A_1,\ldots,A_n$ can be obtained by finitely many applications of the rules to $A_1,\ldots,A_n$ and to instances of the axiom schemata.

We have selected the following *axiom schemata* and *rules of inference for (classical) predicate logic*:

The axiom schemata $1,\ldots,10b$ for (classical) propositional logic (see Section 2.6), together with the rule of inference Modus Ponens. However, the formulas in these axiom schemata and in applications of the rule Modus Ponens are now understood to be formulas of predicate logic. For the sake of completeness we repeat the axiom schemata for propositional logic and the rule Modus Ponens below:

1. $A \to (B \to A)$; 2. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$
3. $A \to (B \to A \wedge B)$; 4a. $A \wedge B \to A$; 4b. $A \wedge B \to B$
5a. $A \to A \vee B$; 5b. $B \to A \vee B$; 6. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$
7. $(A \to B) \to ((A \to \neg B) \to \neg A)$; 8. $\neg\neg A \to A$
9. $(A \to B) \to ((B \to A) \to (A \rightleftarrows B))$;
10a. $(A \rightleftarrows B) \to (A \to B)$; 10b. $(A \rightleftarrows B) \to (B \to A)$

We add one axiom schema for $\forall$ and one for $\exists$ (compare Theorem 4.16):

$$\text{the } \forall\text{-}\textit{schema } \forall x[A(x)] \to A(t), \text{ and the } \exists\text{-}\textit{schema } A(t) \to \exists x[A(x)],$$

where t is a term, i.e., a free individual variable or an individual constant.

To the rule Modus Ponens (MP), $\dfrac{A \quad A \to B}{B}$, we also add one rule of inference for $\forall$ and one for $\exists$:

$$\text{the } \forall\text{-}rule\ \frac{C \to A(a)}{C \to \forall x[A(x)]} \text{ and the } \exists\text{-}rule\ \frac{A(a) \to C}{\exists x[A(x)] \to C},$$

when C does not contain a (compare Theorem 4.17).

Warning For predicate logic we have three rules of inference:

$$MP\ \frac{A \quad A \to B}{B}, \quad \forall\ \frac{C \to A(a)}{C \to \forall x[A(x)]} \text{ and } \exists\ \frac{A(a) \to C}{\exists x[A(x)] \to C},$$

if C does not contain a. However, there is an important difference between these rules. We do have $A,\ A \to B \models B$ and hence also the weaker statement: if $M \models A$ and $M \models A \to B$, then $M \models B$. But – as we have seen in Theorem 4.17 – we do not have $C \to A(a) \models C \to \forall x[A(x)]$, although we do have the weaker statement: if $M \models C \to A(a)$, then $M \models C \to \forall x[A(x)]$. Similarly for the $\exists$-rule.

The definition of a *(logical) proof of B* is similar to the one for propositional logic, taking into account that we have two more axiom schemata and two more rules of inference.

Definition 4.21 (Proof; Provable). Let B be a formula. A (logical, Hilbert-type) *proof* of B is a finite list of formulas with B as last formula in the list, such that every formula in the list is either an axiom of predicate logic (i.e, an instance of an axiom schema) or obtained by application of one of the rules to formulas earlier in the list.
B is (logically) *provable* := there exists a (logical, Hilbert-type) proof of B.
Notation: $\vdash B$

Example 4.13. $\vdash \forall x[P(x) \to P(x)]$. Below is a (logical) proof of $\forall x[P(x) \to P(x)]$:

$$\text{Ex. 2.12}\ \left\{\begin{array}{l}\dfrac{\text{axiom 1} \quad \text{axiom 2}}{P(a) \to P(a)} \qquad \dfrac{1}{(P(a) \to P(a)) \to (\text{axiom} \to (P(a) \to P(a)))}\end{array}\right.$$

$$MP\ \frac{\text{axiom} \qquad \forall\ \dfrac{\text{axiom} \to (P(a) \to P(a))}{\text{axiom} \to \forall x[P(x) \to P(x)]}}{\forall x[P(x) \to P(x)]}$$

The definition of a (logical) *deduction of B from* $A_1, \ldots, A_n$ in predicate logic is similar to the one for propositional logic. However, in order to prevent that, for instance, one could deduce from $C \to P(a)$ (if it is September 5, then *ad* has his birthday) that $C \to \forall x[P(x)]$ (if it is September 5, then everyone has his birthday), in such a deduction *all free variables of* $A_1, \ldots, A_n$ should be *held constant*, i.e., the $\forall$-rule and the $\exists$-rule may not be applied with respect to a free variable a occurring in $A_1, \ldots, A_n$, except preceding the first occurrence of $A_1, \ldots, A_n$ in the deduction.

Definition 4.22 (Deduction; Deducible).
1. A (logical, Hilbert-type) *deduction* of B *from* $A_1, \ldots, A_n$ (in classical predicate logic) is a finite list $B_1, \ldots, B_k$ of formulas, such that
(a) $B = B_k$ is the last formula in the list, and

(b) each formula in the list is either one of $A_1,\ldots,A_n$, or an axiom of predicate logic (i.e., an instance of one of the axiom schemata), or is obtained by application of one of the rules to formulas preceding it in the list, such that
(c) *all free variables of* $A_1,\ldots,A_n$ *are held constant*, i.e., the $\forall$-rule and the $\exists$-rule are not applied with respect to a free variable a occurring in $A_1,\ldots,A_n$, except preceding the first occurrence of $A_1,\ldots,A_n$ in the deduction.
2. B is (logically) *deducible from* $A_1,\ldots,A_n$:= there exists a (logical, Hilbert-type) deduction of B from $A_1,\ldots,A_n$. **Notation**: $A_1,\ldots,A_n \vdash B$. The symbol $\vdash$ may be read 'yields'. $A_1,\ldots,A_n \nvdash B$ abbreviates: not $A_1,\ldots,A_n \vdash B$.
3. For Γ a (possibly infinite) set of formulas, B is *deducible from* Γ := there is a finite list $A_1,\ldots,A_n$ of formulas in Γ such that $A_1,\ldots,A_n \vdash B$. **Notation**: $\Gamma \vdash B$.

Example 4.14. $\forall x[P(x) \to Q(x)], P(c) \vdash Q(c)$. The following schema is a deduction of $Q(c)$ from $\forall x[P(x) \to Q(x)]$ and $P(c)$.

$$\dfrac{\begin{array}{c}\text{premiss}\\ P(c)\end{array} \qquad \dfrac{\begin{array}{c}\text{premiss}\\ \forall x[P(x) \to Q(x)]\end{array} \qquad \begin{array}{c}\forall\text{-schema}\\ \forall x[P(x) \to Q(x)] \to (P(c) \to Q(c))\end{array}}{P(c) \to Q(c)}\,MP}{Q(c)}\,MP$$

Example 4.15. $\forall x[P(x)] \vdash \exists x[P(x)]$. The following schema is a deduction of $\exists x[P(x)]$ from $\forall x[P(x)]$.

$$\dfrac{\dfrac{\begin{array}{c}\text{premiss}\\ \forall x[P(x)]\end{array} \qquad \begin{array}{c}\forall\text{-schema}\\ \forall x[P(x)] \to P(t)\end{array}}{P(t)}\,MP \qquad \begin{array}{c}\exists\text{-schema}\\ P(t) \to \exists x[P(x)]\end{array}}{\exists x[P(x)]}\,MP$$

Warning: Note that according to our definition the schema $\dfrac{C \to A(a)}{C \to \forall x[A(x)]}$, where the variable a does not occur in C, is not a deduction of $C \to \forall x[A(x)]$ from $C \to A(a)$ (holding all free variables constant), since in this schema the $\forall$-rule is applied with respect to a free variable a occurring in the premiss.

This remark by itself does *not* establish that there is no deduction of $C \to \forall x[A(x)]$ from $C \to A(a)$; it only says that the given schema is not such a deduction. In order to establish that no (other) schema can be a deduction of $C \to \forall x[A(x)]$ from $C \to A(a)$ (holding all free variables constant), we have to prove the generalized soundness theorem:

$$\text{if } A_1,\ldots,A_n \vdash B, \text{ then } A_1,\ldots,A_n \models B.$$

Since we have seen in Theorem 4.17 that $C \to A(a) \not\models C \to \forall x[A(x)]$, it follows by this theorem that $C \to A(a) \nvdash C \to \forall x[A(x)]$, i.e., there is no deduction of $C \to \forall x[A(x)]$ from $C \to A(a)$ (holding all free variables constant).

Sometimes the free variables in the premisses $A_1,\ldots,A_n$ are allowed to vary, such as, for instance, in 'if it rains, then a takes an umbrella', when one means 'for all x, if it rains, then x takes an umbrella' or 'if it rains, then any a takes an umbrella'.

Definition 4.23. B is *deducible from* $A_1,\dots,A_n$ *allowing all free variables to vary* or *with all free variables general* $:= Cl(A_1),\dots,Cl(A_n) \vdash B$. **Notation**: $A_1,\dots,A_n \vdash^2 B$.

Example 4.16. $C \to A(a) \vdash^2 C \to \forall x[A(x)]$, i.e., $\forall x[C \to A(x)] \vdash C \to \forall x[A(x)]$, since the following schema is a deduction of $C \to \forall x[A(x)]$ from $\forall x[C \to A(x)]$ (holding all free variables constant):

$$\dfrac{\dfrac{\overset{\text{premiss}}{\forall x[C \to A(x)]} \qquad \overset{\forall\text{-schema}}{\forall x[C \to A(x)] \to (C \to A(a))}}{C \to A(a)}\ MP}{C \to \forall x[A(x)]}\ \forall$$

where we have chosen the free variable a such that a does not occur in the premiss $\forall x[C \to A(x)]$ and hence not in C.

In a similar way one shows that $A(a) \to C \nvdash \exists x[A(x)] \to C$ (from 'if *ad* has his birthday, then it is May 5' it does not follow that 'if someone has his birthday, then it is May 5) and $A(a) \nvdash \forall x[A(x)]$ (from '*ad* is an Alcoholic', it does not follow that 'everyone is an Alcoholic'), while we do have that $A(a) \to C \vdash^2 \exists x[A(x)] \to C$ (from 'if any a takes an umbrella, then it is raining', it follows that 'if someone takes an umbrella, then it is raining') and $A(a) \vdash^2 \forall x[A(x)]$ (from 'any a has the property A', it follows that 'everything has the property A').

The soundness theorem says that the logical axioms and rules of (classical) predicate logic are sound, i.e., every formula B which may be deduced from given premisses $A_1,\dots,A_n$ by means of the logical axioms and rules is a logical (or valid) consequence of the given premisses.

Theorem 4.22 (Soundness). *a) If* $A_1,\dots,A_n \vdash B$, *then* $A_1,\dots,A_n \models B$.
Hence, in particular, if $\vdash B$, *then* $\models B$.
b) If $A_1,\dots,A_n \vdash^2 B$, *then* $A_1,\dots,A_n \models^2 B$.

Proof. a) Suppose $A_1,\dots,A_n \vdash B$, i.e., there is a finite schema of formulas of the following form:

$$\dfrac{\dfrac{\underline{A_1} \quad \cdots \quad \underline{A_n}}{\quad} \quad \dfrac{\underline{\text{axioms}}}{\quad}}{B}\ MP,\ \forall \text{ or } \exists$$

where the $\forall$- and $\exists$-rule are not applied with respect to a free variable a occurring in $A_1,\dots,A_n$, except preceding the first occurrence of $A_1,\dots,A_n$ in the deduction. (α)
Let M be an interpretation with domain D and let v be a valuation in D of the free individual variables. We have to show: if for all i, $1 \le i \le n$, $M \models A_i[v]$, then $M \models B[v]$. So suppose that for all i, $1 \le i \le n$, $M \models A_i[v]$. (1)
By Theorem 2.7 and Theorem 4.16. all axioms of predicate logic are valid. (2)
For an application of Modus Ponens, note that if $M \models C[v]$ and $M \models C \to D\ [v]$, then $M \models D[v]$. (3)
For an application of the $\forall$- or $\exists$-rule, note that due to the condition (α) stated above, interpretation M and valuation d/v make the premiss true for every d in D and hence M and valuation v make the conclusion of the rule true. (4).

From (1), (2), (3) and (4) it follows that $M \models B[v]$.
(b) follows from (a). □

Without proof we mention that the *deduction theorem* for propositional logic (Theorem 2.24) also holds for (classical) predicate logic: if $\Gamma, A \vdash B$, then also $\Gamma \vdash A \to B$. The introduction and elimination rules for the propositional connectives (Theorem 2.25) may be extended with introduction and elimination rules for the quantifiers.

Theorem 4.23 (Introduction and elimination rules for the quantifiers). *Let a be a free variable, $A(a)$ a formula, t a term and $A(t)$ the result of substituting t for the occurrences of a in $A(a)$. Also let Γ be a list of (zero or more) formulas, and C a formula. Then the following rules hold.*

	INTRODUCTION	*ELIMINATION*
$\forall$	*If $\Gamma \vdash A(a)$, then $\Gamma \vdash \forall x[A(x)]$, provided Γ does not contain a.*	$\forall x[A(x)] \vdash A(t)$
$\exists$	$A(t) \vdash \exists x[A(x)]$	*If $\Gamma, A(a) \vdash C$, then $\Gamma, \exists x[A(x)] \vdash C$, provided Γ and C do not contain a.*

Proof. $\forall$-*introduction*: Let E be an axiom not containing a. Suppose $\Gamma \vdash A(a)$. Then also $\Gamma, E \vdash A(a)$. By the deduction theorem (for predicate logic), $\Gamma \vdash E \to A(a)$. Since by hypothesis Γ and E do not contain a, we can apply the $\forall$-rule; hence $\Gamma \vdash E \to \forall x[A(x)]$. So $\Gamma, E \vdash \forall x[A(x)]$ and since E is an axiom, $\Gamma \vdash \forall x[A(x)]$.
$\forall$-*elimination*: From $\forall x[A(x)]$, by using the $\forall$-axiom, $\forall x[A(x)] \to A(t)$ and Modus Ponens, we may deduce $A(t)$.
$\exists$-*introduction*: From $A(t)$, by using the $\exists$-axiom, $A(t) \to \exists x[A(x)]$, and Modus Ponens, we may deduce $\exists x[A(x)]$.
$\exists$-*elimination* : Suppose $\Gamma, A(a) \vdash C$. Then by the deduction theorem, $\Gamma \vdash A(a) \to C$. Since Γ and C do not contain a, by the $\exists$-rule, $\Gamma \vdash \exists x[A(x)] \to C$. And therefore $\Gamma, \exists x[A(x)] \vdash C$. □

Exercise 4.29.

$$\dfrac{\dfrac{\overset{\text{premiss}}{A(a)} \qquad \overset{1}{A(a) \to (C \to A(a))}}{C \to A(a)}\,MP \qquad \dfrac{\overset{\text{axiom}}{C}}{C \to \forall x[A(x)]}\,\forall}{\forall x[A(x)]}\,MP$$

Is this schema, in which C is an axiom not containing the free variable a, a deduction of $\forall x[A(x)]$ from $A(a)$ (holding all free variables constant)?

Exercise 4.30. Is the following schema a deduction of $\exists y \forall x[P(y,x,a)]$ from $\exists z \forall x[P(b,x,z)]$?

$$\dfrac{\overset{\text{premiss}}{\exists z \forall x[P(b,x,z)]} \qquad \dfrac{\overset{\exists\text{-schema}}{\forall x[P(b,x,a)] \to \exists y \forall x[P(y,x,a)]}}{\exists z \forall x[P(b,x,z)] \to \exists y \forall x[P(y,x,a)]}\,\exists}{\exists y \forall x[P(y,x,a)]}\,MP$$

Exercise 4.31. Is the following schema a deduction of $\exists z \forall y[P(z,y,b)]$ from $\exists x \forall y[P(x,y,b)]$?

$$\dfrac{\overset{\text{premiss}}{\exists x \forall y[P(x,y,b)]} \qquad \dfrac{\overset{\exists\text{-schema}}{\forall y[P(a,y,b)] \to \exists z \forall y[P(z,y,b)]}}{\exists x \forall y[P(x,y,b)] \to \exists z \forall y[P(z,y,b)]}\,\exists}{\exists z \forall y[P(z,y,b)]}\,MP$$

Exercise 4.32. Show that i) $\exists x[A(x)] \vdash \exists z[A(z)]$ and ii) $\forall x[A(x)] \vdash \forall z[A(z)]$.

4.4.1 Natural Deduction

Gentzen's system of *Natural Deduction* for classical predicate logic is obtained by adding to the natural deduction rules for the connectives (see Subsection 2.7.2) the following *introduction and elimination rules for the quantifiers*.

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$\forall I \quad \dfrac{A(a)}{\forall x[A(x)]}$

if the free variable a does not occur in any of the premisses of $A(a)$.

$\exists I \quad \dfrac{A(t)}{\exists x[A(x)]}$

where t is a term

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$\forall E \quad \dfrac{\forall x[A(x)]}{A(t)}$

where t is a term

$\exists E \quad \dfrac{\exists x[A(x)] \qquad \begin{array}{c}[A(a)]\\ \vdots \\ C\end{array}}{C}$

if the variable a in the cancelled formula $A(a)$ does not occur in C or any of the premisses in the righthand derivation.

That the conditions accompanying the rules above are necessary follows immediately from Theorem 4.17. The definition of $\Gamma \vdash_{ND} B$ (*B is deducible from Γ in Gentzen's system of natural deduction*) for classical predicate logic is similar to the one for classical propositional logic (see Definition 2.12), taking into account that now we also have introduction and elimination rules for the quantifiers. And again one easily shows that $\Gamma \vdash B$ iff $\Gamma \vdash_{ND} B$ (compare Theorem 2.26), where the if-part now follows from Theorem 2.25 and Theorem 4.23. Below are some examples of deductions in Gentzen's system of natural deduction.

Example 4.17. $\vdash_{ND} \forall x[A(x) \wedge B(x)] \to \forall x[A(x)] \wedge \forall x[B(x)]$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x[A(x)\wedge B(x)]]^1}{A(a)\wedge B(a)}\,\forall E}{A(a)}\,\wedge E}{\forall x[A(x)]}\,\forall I \qquad \dfrac{\dfrac{\dfrac{[\forall x[A(x)\wedge B(x)]]^1}{A(a)\wedge B(a)}\,\forall E}{B(a)}\,\wedge E}{\forall x[B(x)]}\,\forall I}{\forall x[A(x)]\wedge\forall x[B(x)]}\,\wedge I}{\forall x[A(x)\wedge B(x)]\to\forall x[A(x)]\wedge\forall x[B(x)]}\,\to I \quad (1)
$$

Example 4.18. $\vdash_{ND} (\exists x[A(x)]\to B)\to\forall x[A(x)\to B]$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\exists x[A(x)]\to B]^2 \qquad \dfrac{[A(a)]^1}{\exists x[A(x)]}\,\exists I}{B}\,\to E}{A(a)\to B}\,\to I}{\forall x[A(x)\to B]}\,\forall I}{(\exists x[A(x)]\to B)\to\forall x[A(x)\to B]}\,\to I
\quad (1)\ \ (2)
$$

The reader should be aware again that the logical proofs by natural deduction are very close to our informal way of verifying the formula in question. For instance, how do we prove informally that $(\exists x[A(x)]\to B)\to\forall x[A(x)\to B]$? We suppose that $\exists x[A(x)]\to B$ (2). Then we have to show that $\forall x[A(x)\to B]$. So, we let a be an arbitrary individual and we show that $A(a)\to B$. So, suppose $A(a)$ (1). Then $\exists x[A(x)]$ and hence by (2), B. Therefore $A(a)\to B$ under the assumption of $\exists x[A(x)]\to B$. Since a was arbitrary, it follows that $\forall x[A(x)\to B]$ under the assumption of $\exists x[A(x)]\to B$. Therefore, $(\exists x[A(x)]\to B)\to\forall x[A(x)\to B]$.

Exercise 4.33. Show that $\forall x[A\to B(x)]\vdash_{ND} A\to\forall x[B(x)]$ and $A\to\forall x[B(x)]\vdash_{ND}\forall x[A\to B(x)]$, realizing that a formal proof by natural deduction is very close to an informal proof of the formula in question.

4.4.2 Tableaux

To make this subsection self contained, we repeat some definitions from Section 2.8.

Definition 4.24 (Signed Formula; Sequent). A *signed formula* is any expression of the form $T(A)$ or $F(A)$, where A is a formula. Informally, we may read $T(A)$ as 'A is true' and $F(A)$ as 'A is false'. We frequently write TA and FA instead of $T(A)$ and $F(A)$, respectively. A *sequent* S is any finite set of signed formulas.

Below are the tableaux rules for the propositional connectives.

$T\wedge$	$S,\ T\ B\wedge C$	$F\wedge$	$S,\ F\ B\wedge C$
	$S,\ TB,\ TC$		$S,\ FB \mid S,\ FC$
$T\vee$	$S,\ T\ B\vee C$	$F\vee$	$S,\ F\ B\vee C$
	$S,\ TB \mid S,\ TC$		$S,\ FB,\ FC$
$T\rightarrow$	$S,\ T\ B\rightarrow C$	$F\rightarrow$	$S,\ F\ B\rightarrow C$
	$S,\ FB \mid S,\ TC$		$S,\ TB,\ FC$
$T\neg$	$S,\ T\ \neg B$	$F\neg$	$S,\ F\ \neg B$
	$S,\ FB$		$S,\ TB$

The *tableaux rules* for (classical) predicate logic are the following ones:
1. The T- and F-rules for $\rightarrow$, $\wedge$, $\vee$ and $\neg$ of (classical) propositional logic (see Section 2.8), but now for any formulas of predicate logic.
2. To these, we add T- and F-rules for the quantifiers $\forall$ and $\exists$:

$T\ \exists$	$S,\ T\ \exists x[A(x)]$	$F\ \exists$	$S,\ F\ \exists x[A(x)]$
	$S,\ TA(a)$		$S,\ F\ \exists x[A(x)],\ FA(t)$
a new: a does not occur in $S,\ T\exists x[A(x)]$		t being any term	
$T\forall$	$S,\ T\ \forall x[A(x)]$	$F\forall$	$S,\ F\ \forall x[A(x)]$
	$S,\ T\ \forall x[A(x)],\ TA(t)$		$S,\ FA(a)$
t being any term		a new: a does not occur in $S,\ F\ \forall x[A(x)]$	

The extra condition in the rules $T\ \exists$ and $F\ \forall$ which the free individual variable a has to satisfy can be explained as follows.

If we read the rules downwards as semantic tableaux rules in the sense of E. Beth, interpreting the signed formulas rather than the sequents, the condition on a in the rule $T\ \exists$ and $F\ \forall$ is intuitively clear:
$T\ \exists$: Suppose $T\ \exists x[A(x)]$, i.e., there is at least one object with the property A. This object is not necessarily one of the objects already mentioned before.
$F\ \forall$: Suppose $F\ \forall x[A(x)]$, i.e., not all objects have the property A, or equivalently, at least one object does not have the property A. And again this object is not necessarily one of the objects already mentioned before.

If we read the rules upwards as Gentzen-type rules, interpreting the sequents rather than the signed formulas, rule $T\ \exists$, for instance, taking $S = \{FC\}$, becomes $\frac{FC,\ T\ \exists x[A(x)]}{FC,\ TA(a)}$ and is read as: if $A(a) \rightarrow C$, then $\exists x[A(x)] \rightarrow C$; the condition on a in rule $T\ \exists$ now corresponds to the condition 'a does not occur in C' in Theorem 4.17.

The definitions of '(tableau) branch', 'tableau', '*tableau proof* of B', '*B is tableau-provable*' (notation: $\vdash' B$), a '*tableau-deduction* of B from $A_1,\ldots,A_n$' and of '*B is tableau-deducible* from $A_1,\ldots,A_n$' (notation: $A_1,\ldots,A_n \vdash' B$) are similar to the definitions for propositional logic (see Definition 2.17 and 2.18), allowing that now we have two more T-rules and two more F-rules for the quantifiers.

Definition 4.25 ((Tableau) Branch). (a) A *tableau branch* is a (possibly infinite) set of signed formulas. A branch is *closed* if it contains signed formulas TA and FA

for some formula A. A branch that is not closed is called *open*.
(b) Let $\mathscr{B}$ be a branch and TA, resp. FA, a signed formula occurring in $\mathscr{B}$. TA, resp. FA, is *fulfilled* in $\mathscr{B}$ if (i) A is atomic, or (ii) $\mathscr{B}$ contains the bottom formulas in the application of the corresponding rule to A, and in case of the rules $T\vee$, $F\wedge$ and $T\rightarrow$, $\mathscr{B}$ contains one of the bottom formulas in the application of these rules.
(c) A branch $\mathscr{B}$ is *completed* if $\mathscr{B}$ is closed or every signed formula in $\mathscr{B}$ is fulfilled in $\mathscr{B}$.

Let $\mathscr{B} = \{T\forall x[P(x)],\ldots\}$ be a tableau branch and let $\mathscr{B}_a = \{T\forall x[P(x)], P(a),\ldots\}$. Then the tableau $\mathscr{T}_a = \{\mathscr{B}_a\}$ is called a *one-step expansion* of tableau $\mathscr{T} = \{\mathscr{B}\}$. And if $\mathscr{B} = \{T(P\rightarrow Q),\ldots\}$ is a tableau branch, $\mathscr{B}_1 = \{T(P\rightarrow Q), FP,\ldots\}$ and $\mathscr{B}_2 = \{T(P\rightarrow Q),\ TQ,\ldots\}$, then the tableau $\mathscr{T}' = \{\mathscr{B}_1, \mathscr{B}_2\}$ is called a *one-step expansion* of tableau $\mathscr{T} = \{\mathscr{B}\}$.

Definition 4.26 (Tableau). (a) A set of branches $\mathscr{T}$ is a *tableau* with initial branch $\mathscr{B}_0$ if there is a sequence $\mathscr{T}_0, \mathscr{T}_1, \ldots, \mathscr{T}_n$ such that $\mathscr{T}_0 = \{\mathscr{B}_0\}$, each $\mathscr{T}_{i+1}$ is a one-step expansion of $\mathscr{T}_i$ ($0 \leq i < n$) and $\mathscr{T} = \mathscr{T}_n$.
(b) We say that a branch $\mathscr{B}$ has tableau $\mathscr{T}$ if $\mathscr{T}$ is a tableau with initial branch $\mathscr{B}$.
(c) A tableau $\mathscr{T}$ is *open* if some branch $\mathscr{B}$ in it is open, otherwise $\mathscr{T}$ is *closed*.
(d) A tableau is *completed* if each of its branches is completed, i.e., no application of a tableau rule can change the tableau.

Definition 4.27 (Tableau-deduction; Tableau-proof).
(a) A (logical) *tableau-deduction* of B *from* $A_1,\ldots,A_n$ (in classical predicate logic) is a tableau $\mathscr{T}$ with $\mathscr{B}_0 = \{TA_1,\ldots,TA_n, FB\}$ as initial branch, such that all branches of $\mathscr{T}$ are closed.

In case $n = 0$, i.e., there are no premisses $A_1,\ldots,A_n$, this definition reduces to:
(b) A (logical) *tableau-proof* of B (in classical predicate logic) is a tableau $\mathscr{T}$ with $\mathscr{B}_0 = \{FB\}$ as initial sequent, such that all branches of $\mathscr{T}$ are closed.

Definition 4.28 (Tableau-deducible; Tableau-provable).
(a) B is *tableau-deducible from* $A_1,\ldots,A_n$ (in classical predicate logic) if there exists a tableau-deduction of B from $A_1,\ldots,A_n$. **Notation**: $A_1,\ldots,A_n \vdash' B$.
By $A_1,\ldots,A_n \nvdash' B$ we mean: not $A_1,\ldots,A_n \vdash' B$.
(b) B is *tableau-provable* (in classical predicate logic) if there exists a tableau-proof of B. **Notation**: $\vdash' B$.
(c) For Γ a (possibly infinite) set of formulas, B is *tableau-deducible from* Γ if there exists a finite list $A_1,\ldots,A_n$ of formulas in Γ such that $A_1,\ldots,A_n \vdash' B$.
Notation: $\Gamma \vdash' B$.

Since $S\cup\{T\ \exists x[A(x)]\} = S\cup\{T\ \exists x[A(x)],\ T\ \exists x[A(x)]\}$ and $S\cup\{F\ \forall x[A(x)]\} = S\cup\{F\ \forall x[A(x)],\ F\ \forall x[A(x)]\}$, we have the following two derived rules:

$S,\ T\ \exists x[A(x)]$	and	$S,\ F\ \forall x[A(x)]$
$S,\ T\ \exists x[A(x)], TA(a)$		$S,\ F\ \forall x[A(x)], FA(a)$
provided a is new;		provided a is new.

This enables us to apply the rules $T\ \exists$ and $F\ \forall$ as frequently as we want, each time with a new variable.

Example 4.19. $\vdash' \forall x[\neg P(x)] \to \neg\, \forall x[P(x)]$, since there is a tableau-proof (in classical predicate logic) of $\forall x[\neg P(x)] \to \neg\, \forall x[P(x)]$, i.e., there is a closed tableau $\mathscr{T} = \{\mathscr{B}_5\}$ with $\mathscr{B}_0 = \{F\ \forall x[\neg P(x)] \to \neg\, \forall x[P(x)]\}$ as initial branch, where $\mathscr{B}_5$ consists of all signed formulas in the schema below:

$$\begin{array}{r}
F\ \forall x[\neg P(x)] \to \neg\, \forall x[P(x)] \\
T\ \forall x[\neg P(x)],\ F\ \neg\, \forall x[P(x)] \\
T\ \forall x[\neg P(x)],\ T\ \forall x[P(x)] \\
T\ \neg P(a_1),\ T\ \forall x[\neg P(x)],\ T\ \forall x[P(x)] \\
F\ P(a_1),\ T\ \forall x[\neg P(x)],\ T\ \forall x[P(x)] \\
F\ P(a_1),\ T\ \forall x[\neg P(x)],\ T\ \forall x[P(x)],\ TP(a_1) \\
\text{closure}
\end{array}$$

Example 4.20. not $\vdash' \neg\forall x[P(x)] \to \forall x[\neg P(x)]$. If we try to construct a tableau-proof of $\neg\forall x[P(x)] \to \forall x[\neg P(x)]$, we find the following open, i.e., not closed, and not completed branch consisting of the signed formulas in the schema below:

$$\begin{array}{r}
F\ \neg\forall x[P(x)] \to \forall x[\neg P(x)] \\
T\ \neg\forall x[P(x)],\ F\ \forall x[\neg P(x)] \\
F\ \forall x[P(x)],\ F\ \forall x[\neg P(x)] \\
FP(a_1),\ F\ \forall x[P(x)],\ F\ \forall x[\neg P(x)] \\
FP(a_1),\ F\ \forall x[P(x)],\ F\ \forall x[\neg P(x)],\ F\ \neg P(a_2) \\
FP(a_1),\ F\ \forall x[P(x)],\ F\ \forall x[\neg P(x)],\ TP(a_2) \\
FP(a_3),\ FP(a_1),\ F\ \forall x[P(x)],\ F\ \forall x[\neg P(x)],\ TP(a_2) \\
FP(a_3),\ FP(a_1),\ F\ \forall x[P(x)],\ F\ \forall x[\neg P(x)],\ TP(a_2),\ F\ \neg P(a_4) \\
FP(a_3),\ FP(a_1),\ F\ \forall x[P(x)],\ F\ \forall x[\neg P(x)],\ TP(a_2),\ TP(a_4) \\
\text{and so on}
\end{array}$$

It is clear now that, no matter how far we continue this construction, we will never find a tableau-proof of the formula in question. We may apply the rules in a different order yielding a different open tableau, but not one that is closed. On the contrary, from the resulting open tableau with initial branch $\{F\ \neg\forall x[P(x)]\ \to \forall x[\neg P(x)]\}$, which consists of only one infinitely long, open branch, we can immediately read off a counterexample to $\neg\forall x[P(x)] \to \forall x[\neg P(x)]$ with the set $\mathbb{N}$ of all natural numbers as domain: the natural numbers $1,3,5,\ldots$ do not have the property P, corresponding with the occurrences of $FP(a_1),\ FP(a_3),\ FP(a_5),\ldots$ in the open branch and the natural numbers $2,4,6,\ldots$ do have the property P, corresponding with the occurrences of $TP(a_2),\ TP(a_4),\ TP(a_6),\ldots$ in the (open) branch. So let P^* be the predicate 'is even' over the natural numbers. Then $\langle \mathbb{N}; P^* \rangle$ is a counterexample to $\neg\forall x[P(x)] \to \forall x[\neg P(x)]$, since $\langle \mathbb{N}; P^* \rangle \models \neg\forall x[P(x)]$ (the sentence 'not all natural numbers are even' has truth value 1), while $\langle \mathbb{N}; P^* \rangle \not\models \forall x[\neg P(x)]$ (the sentence 'all natural numbers are not even' has truth value 0).

Note that $\langle \mathbb{N}; P^* \rangle \models P(a_2)$ [2] (the sentence '2 is even' has truth value 1), $\langle \mathbb{N}; P^* \rangle \models P(a_4)$ [4] (the sentence '4 is even' has truth value 1), and so on, corresponding with the occurrence of $TP(a_2),\ TP(a_4),\ldots$ in the open branch above. But $\langle \mathbb{N}; P^* \rangle \not\models P(a_1)$ [1] (the sentence '1 is even' has truth value 0), $\langle \mathbb{N}; P^* \rangle \not\models P(a_3)$ [3] (the sentence '3 is even' has truth value 0), and so on, corresponding with the occurrences of $FP(a_1),\ FP(a_3),\ldots$ in the open branch above.

Example 4.21. $\exists y \forall x[P(x,y)] \vdash' \forall x \exists y[P(x,y)]$, since there is a tableau-deduction of $\forall x \exists y[P(x,y)]$ from $\exists y \forall x[P(x,y)]$, i.e., there is a closed tableau $\mathscr{T} = \{\mathscr{B}_4\}$ starting with the initial branch $\mathscr{B}_0 = \{T\ \exists y \forall x[P(x,y)],\ F\ \forall x \exists y[P(x,y)]\}$, where $\mathscr{B}_4$ is the set consisting of all signed formulas in the schema below.

$$
\begin{array}{r}
T\ \exists y \forall x[P(x,y)],\ F\ \forall x \exists y[P(x,y)] \\
T\ \forall x[P(x,a_1)],\ T\ \exists y \forall x[P(x,y)],\ F\ \forall x \exists y[P(x,y)] \\
T\ \forall x[P(x,a_1)],\ T\ \exists y \forall x[P(x,y)],\ F\ \forall x \exists y[P(x,y)],\ F\ \exists y[P(a_2,y)] \\
TP(a_2,a_1),\ T\ \exists y \forall x[P(x,y)],\ F\ \forall x \exists y[P(x,y)],\ F\ \exists y[P(a_2,y)] \\
TP(a_2,a_1),\ T\ \exists y \forall x[P(x,y)],\ F\ \forall x \exists y[P(x,y)],\ FP(a_2,a_1) \\
\text{closure}
\end{array}
$$

Example 4.22. not $\forall x \exists y[P(x,y)] \vdash' \exists y \forall x[P(x,y)]$. If we try to construct a tableau-deduction of $\exists y \forall x[P(x,y)]$ from $\forall x \exists y[P(x,y)]$, we find the following open and not completed branch consisting of the signed formulas in the schema below, using some obvious abbreviations:

$$
\begin{array}{r}
T\forall x \exists y[P(x,y)],\ F\exists y \forall x[P(x,y)] \\
T\exists y[P(a_1,y)],\ T\forall x \exists y,\ F\exists y \forall x \\
T\exists y[P(a_1,y)],\ T\forall x \exists y,\ F\exists y \forall x,\ F\forall x[P(x,a_1)] \\
TP(a_1,a_2),\ T\forall x \exists y,\ F\exists y \forall x,\ F\forall x[P(x,a_1)] \\
TP(a_1,a_2),\ T\forall x \exists y,\ F\exists y \forall x,\ FP(a_3,a_1) \\
T\exists y[P(a_2,y)],\ TP(a_1,a_2),\ T\forall x \exists y,\ F\exists y \forall x,\ FP(a_3,a_1) \\
T\exists y[P(a_2,y)],\ TP(a_1,a_2),\ T\forall x \exists y,\ F\exists y \forall x,\ FP(a_3,a_1),\ F\forall x[P(x,a_2)] \\
TP(a_2,a_4),\ TP(a_1,a_2),\ T\forall x \exists y,\ F\exists y \forall x,\ FP(a_3,a_1),\ F\forall x[P(x,a_2)] \\
TP(a_2,a_4),\ TP(a_1,a_2),\ T\forall x \exists y,\ F\exists y \forall x,\ FP(a_3,a_1),\ FP(a_5,a_2) \\
\text{and so on}
\end{array}
$$

It is clear that, no matter how far we continue our construction, we will never find a tableau-deduction of $\exists y \forall x[P(x,y)]$ from $\forall x \exists y[P(x,y)]$. (Application of the rules in a different order may result in a different tableau, but not one that is closed.) On the contrary, from the resulting open branch, which is infinitely long, we can immediately read off a counterexample with the set $\mathbb{N}$ of all natural numbers as its domain: for each $n = 1,2,3,\ldots$, $P^*(n,2n)$, corresponding with the occurrences of $TP(a_1,a_2)$, $TP(a_2,a_4)$, $TP(a_3,a_6)$, $\ldots$ in the open branch and for each $n = 1,2,3,\ldots$, not $P^*(2n+1,\ n)$, corresponding with the occurrences of $FP(a_3,a_1)$, $FP(a_5,a_2)$, $FP(a_7,a_3)$, $\ldots$ in the open branch. So, let P^* be the binary predicate over the natural numbers, defined by $P^*(n,m) := m = 2n$. Then $\langle \mathbb{N};\ P^* \rangle \models \forall x \exists y[P(x,y)]$ (the sentence 'for each natural number n there is a natural number m such that $m = 2n$' has truth value 1), but $\langle \mathbb{N};\ P^* \rangle \not\models \exists y \forall x[P(x,y)]$ (the sentence 'there is a natural number m such that for all natural numbers n, $m = 2n$' has truth value 0).

Note that $\langle \mathbb{N};\ P^* \rangle \models \exists y[P(a_1,y)][1]$,
$\langle \mathbb{N};\ P^* \rangle \models P(a_1,a_2)[1,2]$ (the sentence '$2 = 2 \cdot 1$' has truth value 1),
$\langle \mathbb{N};\ P^* \rangle \models \exists y[P(a_2,y)][2]$,
$\langle \mathbb{N};\ P^* \rangle \models P(a_2,a_4)[2,4]$ (the sentence '$4 = 2 \cdot 2$' has truth value 1),
and so on,

corresponding with the occurrences of $T\ \exists y[P(a_1,y)]$, $TP(a_1,a_2)$, $T\ \exists y[P(a_2,y)]$,

$TP(a_2,a_4)$, ... respectively in the open branch in Example 4.22.
But $\langle \mathbb{N};\ P^* \rangle \not\models \forall x[P(x,a_1)][1]$,
$\langle \mathbb{N};\ P^* \rangle \not\models P(a_3,a_1)[3,1]$ (the sentence '$1 = 2 \cdot 3$' has truth value 0),
$\langle \mathbb{N};\ P^* \rangle \not\models \forall x[P(x,a_2)][2]$,
$\langle \mathbb{N};\ P^* \rangle \not\models P(a_5,a_2)[5,2]$ (the sentence '$2 = 2 \cdot 5$' has truth value 0),
and so on,
corresponding with the occurrences of $F\ \forall x[P(x,a_1)]$, $FP(a_3,a_1)$, $F\ \forall x[P(x,a_2)]$, $FP(a_5,a_2)$, ... respectively in the open branch in Example 4.22.

Like in classical propositional logic, the three notions $A_1,\ldots,A_n \vdash' B$, $A_1,\ldots,A_n \vdash B$ and $A_1,\ldots,A_n \models B$ turn out to be equivalent. Remember that the first two of these notions are syntactic, while the latter one is a semantic notion.

Theorem 4.24. *If $A_1,\ldots,A_n \vdash' B$, then $A_1,\ldots,A_n \vdash B$.*

Proof. The proof is a generalization of the proof of the corresponding theorem 2.27 for classical propositional logic, now in addition using the introduction and elimination rules for the quantifiers in Theorem 4.23.

In Theorem 4.22 we have already shown that classical predicate logic is sound.
Theorem (Soundness): if $A_1,\ldots,A \vdash B$, then $A_1,\ldots,A_n \models B$.

So in order to show the equivalence of model theory and proof theory for classical predicate logic, it remains to be shown that the completeness theorem holds:
Theorem (Completeness): if $A_1,\ldots,A_n \models B$, then $A_1,\ldots,A_n \vdash' B$.
We shall do so in Section 4.5.

This is Gödel's *completeness theorem* (1930) for classical predicate logic. The soundness theorem says that we do *not* have *too many* axioms or rules, i.e., every formula which may be deduced from given premisses by our axioms and rules is a logical consequence of those premisses. The completeness theorem, on the other hand, says that we have *enough* axioms and rules (for classical predicate logic), i.e., every formula which is a logical consequence of given premisses can be deduced from those premisses by finitely many applications of our axioms and rules.

In Section 2.6 we already paid attention to the *philosophical meaning of the completeness theorem*. Having the notions of validity and provability for (classical) predicate logic at our disposal, we can add the following observations to our discussion in Section 2.6 (assuming acquaintance with the notions of enumerable set and non-enumerable set, which are treated in Chapter 3).

a) The notions of validity and satisfiability refer to the totality of all structures, which is non-enumerable, while the equivalent proof-theoretic notions of provability and irrefutability (B is *irrefutable* $:= \neg B$ is unprovable) refer only to the enumerable infinity of logical proofs. In other words, in the definition of $\models B$ we have a (universal) quantification over non-enumerably many structures, while in the definition of $\vdash B$ there is an (existential) quantification over only enumerably many logical proofs. So in the completeness theorem a reduction from the non-enumerably to the enumerably infinite is achieved.

b) The proof of Gödel's completeness theorem is more complex than the proofs of other theorems considered thus far. However, from a careful analysis of this proof one can draw some further conclusions which are philosophically interesting. These conclusions are formulated in the Compactness Theorem and the Löwenheim-Skolem Theorem, also to be treated in Section 4.5.

Exercise 4.34. Construct either a tableau-proof of the following formulas or construct a counterexample from an open branch in the tableau.
1. $(\exists x[P(x)] \to \exists x[Q(x)]) \to \exists x[P(x) \to Q(x)]$,
2. $\exists x[P(x) \to Q(x)] \to (\exists x[P(x)] \to \exists x[Q(x)])$, and 3. $\exists x \forall y[P(x) \to P(y)]$.

4.5 Completeness, Compactness and Löwenheim-Skolem

Given formulas B and $A_1, \ldots, A_n$, the tableaux rules suggest a *procedure of searching for a tableau-deduction* of B from $A_1, \ldots, A_n$:
start with $TA_1, \ \ldots, \ TA_n, \ FB$ and apply all the appropriate rules in some definite fixed order, the choice of ordering being unimportant (at least, if we do not care about efficiency); in an application of rule $T \to$ to, for example, $S, \ T \ P \to Q$ we make two branches, one with $S, \ FP$ and the other with $S, \ TQ$ and similarly for applications of the rules $F \wedge$ and $T \vee$. Owing to the rules $F \exists$ and $T \forall$, the systematic search for such a tableau-deduction now does not necessarily come to an end in finitely many steps, because new variables can be introduced again and again, which may or may not cause closure. In the proof of the completeness theorem below we suppose for reasons of simplicity that there are no individual constants in the language. The case in which there are individual constants (and/or function symbols) in the language is treated similarly.

Example 4.23. We wonder whether $\exists x[P(x) \to Q(x)] \vdash' \exists x[P(x)] \to \exists x[Q(x)]$. Start making a tableau with initial branch $\{T \ \exists x[P(x) \to Q(x)], \ F \ \exists x[P(x)] \to \exists x[Q(x)]\}$:

$$T \ \exists x[P(x) \to Q(x)], \ F \ \exists x[P(x)] \to \exists x[Q(x)]$$
$$T \ P(a_1) \to Q(a_1), \ F \ \exists x[P(x)] \to \exists x[Q(x)]$$
$$T \ P(a_1) \to Q(a_1), \ T \ \exists x[P(x)], \ F \ \exists x[Q(x)]$$

Because of rule $T \ \to$ we continue with two branches:

$$FP(a_1), \ T\exists x[P(x)], \ F\exists x[Q(x)] \text{ and } TQ(a_1), \ T\exists x[P(x)], \ F\exists x[Q(x)]$$
$$FP(a_1), T\exists x[P(x)], F\exists x[Q(x)], FQ(a_1) \qquad TQ(a_1), T\exists x[P(x)], F\exists x[Q(x)], FQ(a_1)$$

So, the left – not yet completed – branch is open, while the right branch is closed. For the left branch we may continue with:

$$FP(a_1), \ T\exists x[P(x)], \ F\exists x[Q(x)], \ FQ(a_1)$$
$$FP(a_1), \ TP(a_2), \ T\exists x[P(x)], \ F\exists x[Q(x)], \ FQ(a_1)$$
$$FP(a_1), \ TP(a_2), \ T\exists x[P(x)], \ F\exists x[Q(x)], \ FQ(a_1), \ FQ(a_2)$$

By *ad hoc* considerations we can see that the left-most branch will never close, no matter how many more tableaux rules we apply. From this open branch we

can construct a counterexample $M = \langle \mathbb{N};\ P^*,\ Q^* \rangle$ with the natural numbers as domain: by definition, $M \not\models P(a)[1]$, corresponding to the occurrence of $FP(a_1)$ in the left branch; $M \models P(a)[n]$ iff $n > 1$, corresponding to the occurrence of $TP(a_2)$, $TP(a_3), \ldots$ in the left branch; and $M \not\models Q(a)[n]$ for all $n \in \mathbb{N}$, corresponding to the occurrence of $FQ(a_1)$, $FQ(a_2), \ldots$ in the left branch.

Then $M \models \exists x[P(x) \to Q(x)]$, corresponding to the occurrence of $T\ \exists x[P(x) \to Q(x)]$ in the left branch, since $M \models P(a) \to Q(a)[1]$; also $M \models \exists x[P(x)]$, corresponding to the occurrence of $T\ \exists x[P(x)]$ in the left branch, since $M \models P(a)[2]$, but $M \not\models \exists x[Q(x)]$, corresponding to the occurrence of $F\ \exists x[Q(x)]$ in the left branch, since there is no natural number n with $M \models Q(a)[n]$.

Like in propositional logic, any completed open tableau branch τ in a tableau with initial branch $\{TA_1, \ldots, TA_n,\ FB\}$ yields a model M_τ (see Definition 4.29) of $A_1, \ldots, A_n$ in which B does not hold, showing that $A_1, \ldots, A_n \not\models B$ (see Lemma 4.1) and if all branches in a tableau with initial branch $\{TA_1, \ldots, A_n,\ FB\}$ are closed, then $A_1, \ldots, A_n \vdash' B$ (see Lemma 4.2).

Definition 4.29 (Model M_τ). Let τ be a completed open tableau branch. Then $M_\tau = \langle D;\ P_1^*,\ P_2^*, \ldots \rangle$ is the model defined by (1) D is the set of all natural numbers i such that a_i occurs in τ, and (2) $P_i^*(n_1, \ldots, n_i)$ iff $T\ P_i(a_{n_1}, \ldots, a_{n_i})$ occurs in τ. From the construction of M_τ it follows that one can always take $D = \mathbb{N}$.

Lemma 4.1. *Let τ be a completed open tableau branch. Then for each formula $E(a_{n_1}, \ldots, a_{n_k})$:*
a) if $TE(a_{n_1}, \ldots, a_{n_k})$ occurs in τ, then $M_\tau \models E(a_{n_1}, \ldots, a_{n_k})[n_1, \ldots, n_k]$,
b) if $FE(a_{n_1}, \ldots, a_{n_k})$ occurs in τ, then $M_\tau \not\models E(a_{n_1}, \ldots, a_{n_k})[n_1, \ldots, n_k]$.

Proof. The proof is by induction on the construction of E and generalizes the proof of the completeness theorem for classical propositional logic in Chapter 2.
Basic step: $E = P(a_{n_1}, \ldots, a_{n_k})$ is atomic. a) If $T\ P(a_{n_1}, \ldots, a_{n_k})$ occurs in τ, then by Definition 4.29, $M_\tau \models P(a_{n_1}, \ldots, a_{n_k})\ [n_1, \ldots, n_k]$. b) If $F\ P(a_{n_1}, \ldots, a_{n_k})$ occurs in τ, then – since τ is open – $T\ P(a_{n_1}, \ldots, a_{n_k})$ does not occur in τ and hence, by Definition 4.29, $M_\tau \not\models P(a_{n_1}, \ldots, a_{n_k})\ [n_1, \ldots, n_k]$.
Induction step. Suppose that a) and b) have been shown for C and D (induction hypothesis). We want to prove a) and b) for $C \wedge D$, $C \vee D$, $C \to D$ and $\neg C$.

If $E = C \wedge D$ and $T\ C \wedge D$ occurs in τ, then - because τ is completed - both TC and TD occur in τ. Hence, by the induction hypothesis, $M_\tau \models C\ [n_1, \ldots, n_k]$ and $M_\tau \models D\ [n_1, \ldots, n_k]$. So, $M_\tau \models C \wedge D\ [n_1, \ldots, n_k]$. If $E = C \wedge D$ and $F\ C \wedge D$ occurs in τ, then - because τ is completed - FC occurs in τ or TD occur in τ. Hence, by the induction hypothesis, $M_\tau \not\models C\ [n_1, \ldots, n_k]$ or $M_\tau \not\models D\ [n_1, \ldots, n_k]$. So, $M_\tau \not\models C \wedge D\ [n_1, \ldots, n_k]$.

The cases $E = C \vee D$, $E = C \to D$ and $E = \neg C$ are treated similarly. Next suppose that a) and b) have been shown for $A(a_i, a_{n_1}, \ldots, a_{n_k})$ (induction hypothesis). We want to prove a) and b) for $\forall x[A(x, a_{n_1}, \ldots, a_{n_k})]$ and for $\exists x[A(x, a_{n_1}, \ldots, a_{n_k})]$.

If $E = \forall x[A(x, a_{n_1}, \ldots, a_{n_k})]$ and TE occurs in τ, then - because τ is completed - $T\ A(a_i, a_{n_1}, \ldots, a_{n_k})$ occurs in τ for every $i \in D$. Hence, by the induction hypothesis, $M_\tau \models A(a_i, a_{n_1}, \ldots, a_{n_k})[i, n_1, \ldots, n_k]$ for every $i \in D$. So, $M_\tau \models$

$\forall x[A(x,a_{n_1},\ldots,a_{n_k})][n_1,\ldots,n_k]$. If $E = \forall x[A(x,a_{n_1},\ldots,a_{n_k})]$ and FE occurs in τ, then - because τ is completed - $F\ A(a_i,a_{n_1},\ldots,a_{n_k})$ occurs in τ for some $i \in D$ with a_i new. Hence, by the induction hypothesis, $M_\tau \not\models A(a_i,a_{n_1},\ldots,a_{n_k})[i,n_1,\ldots,n_k]$ for some $i \in D$. So, $M_\tau \not\models \forall x[A(x,a_{n_1},\ldots,a_{n_k})][n_1,\ldots,n_k]$.

The case $E = \exists x[A(x,a_{n_1},\ldots,a_{n_k})]$ is treated similarly. □

Lemma 4.2. *If all branches in a tableau with initial branch $\{TA_1,\ldots,TA_n,FB\}$ are closed, then $A_1,\ldots,A_n \vdash' B$.*

Proof. Suppose all branches in a tableau with initial sequent $\{TA_1,\ldots,TA_n,FB\}$ are closed. Then for some natural number k they all close in less than k steps. For if not, then (by König's lemma 1926; see Exercise 4.43) there would be an open infinite tableau branch starting with $\{TA_1,\ldots,TA_n,FB\}$. The finitely many (closed) tableau branches together yield a tableau-deduction of B from $A_1,\ldots,A_n$. □

Lemma 4.1 and Lemma 4.2 together yield the completeness theorem for classical predicate logic.

Theorem 4.25 (Completeness of classical predicate logic).
a) If $A_1,\ldots,A_n \models B$, then $A_1,\ldots,A_n \vdash' B$. In particular, if $n = 0$:
b) If $\models B$, then $\vdash' B$.

Proof. a) Suppose $A_1,\ldots,A_n \models B$. Apply the procedure of searching for a tableau-deduction of B from $A_1,\ldots,A_n$. Let $\mathscr{T}$ be the resulting completed tableau. Let $a_{n_1},\ldots,a_{n_k}$ be the free variables occurring in $A_1,\ldots,A_n,B$. If there were an open tableau branch τ in $\mathscr{T}$, then by Lemma 4.1 for all $i = 1,\ldots,n$, $M_\tau \models A_i[n_1,\ldots,n_k]$ and $M_\tau \not\models B[n_1,\ldots,n_k]$, contradicting $A_1,\ldots,A_n \models B$. Therefore, all branches in tableau $\mathscr{T}$ with initial branch $\{TA_1,\ldots,TA_n,FB\}$ are not open, i.e., not not closed and hence closed. So, by Lemma 4.2, $A_1,\ldots,A_n \vdash' B$.
b) is a special case ($n = 0$) of a). □

The proof of the completeness theorem for (classical) predicate logic given above is close to Gödel's original proof, 1930; see van Heijenoort [6] for Gödel's proof. Another interesting completeness proof has been given by Henkin in [7].

4.5.1 Undecidability

In contrast to the case of propositional logic, the construction of a completed tableau with initial branch $\{TA_1,\ldots,TA_n,FB\}$ will in general not end after finitely many steps, because the quantifier rules $T\ \forall$ and $F\ \exists$ may be applied again and again. Although for many concrete formulas B we can make a decision about their being valid (provable) by constructing a completed tableau with $\{FB\}$ as initial branch by *ad hoc* considerations about the growth of the tableau branches – as in the examples 4.20 and 4.22: $\neg\forall x[P(x)] \to \forall x[\neg P(x)]$, resp. $\forall x\exists y[P(x,y)] \to \exists y\forall x[P(x,y)]$ – , constructing a completed tableau with initial branch $\{FB\}$ does not give a uniform

decision procedure for validity (provability) in predicate logic. It is not the case that given *any* formula B, our construction of a completed tableau with initial branch $\{FB\}$ will tell us after a finite number of steps (which, given B, can be determined in advance) whether B is tableau-provable (and hence valid) or not.

A *positive test* for validity (provability) is a mechanical test such that for each formula B, B is valid (provable) iff the test applied to input B gives a positive answer in finitely many steps. And a *negative test* for validity is a mechanical test such that for each formula B, B is invalid (not provable) iff the test applied to input B gives a negative answer in finitely many steps. A *decision procedure* for validity is now simultaneously both a positive and a negative test. Conversely, if one has both a positive and a negative test (possibly different) for validity, then one can obtain a decision procedure by applying the steps of both tests alternately to the input formula B. $(*)$

Note that constructing a completed tableau with initial branch $\{FB\}$ clearly is a positive test for validity: if it is applied to a valid formula B, the construction will come to an end after finitely many steps and provide a tableau-deduction of B. But our procedure does not give a negative test for validity: if it is applied to a non-valid formula B, the procedure may run forever without presenting an answer, because the rules $T\ \forall$ and $F\ \exists$ may be applied again and again. In 1936 A. Church (see Kleene [9], Section 45) and A. Turing [15] proved independently that there is no decision procedure for validity (provability) in classical predicate logic.

Theorem 4.26 (Church-Turing: Predicate logic is undecidable). *There is no decision procedure for validity (provability) in (classical) predicate logic.*

This theorem not only says that constructing a completed tableau with initial branch $\{FB\}$ does not give a decision procedure for validity of an arbitrary formula B, but also that no other decision procedure can exist. In other words, classical predicate logic is *undecidable*.

From the Church-Turing Theorem and remark $(*)$ above it follows that there can be no negative test for validity, since constructing a completed tableau with initial branch $\{FB\}$ is a positive test for validity.

And since A is not satisfiable if and only if $\neg A$ is valid, it follows that there is a negative test for satisfiability, but no positive test for satisfiability.

In the exercises we will consider some particular classes of formulas B, for which one can determine a natural number N such that constructing a completed tableau with initial branch $\{FB\}$ with only N applications of the $T\ \forall$ and $F\ \exists$-rules, provides a decision procedure for formulas in the class; for each formula B in the class it yields in finitely many steps either a tableau-proof of B or the conclusion that no tableau-deduction of B exists. So, for formulas B in the given class it holds that if no tableau-proof of B is found after N applications of the $T\forall$ and $F\exists$-rules in the construction of a completed tableau with initial branch $\{FB\}$, then there is no tableau-proof of B at all.

4.5.2 Compactness and Löwenheim-Skolem Theorems

Definition 4.30 (Validity and Satisfiability in a given Domain). Let D be a non-empty domain and B a formula.
a) B is *valid in* D := $M \models B\ [v]$ for all models M with domain D and for each valuation v in D.
b) B is *satisfiable in* D := there is at least one model M with domain D and at least one valuation v in D such that $M \models B\ [v]$.
c) A class Γ of formulas is *simultaneously satisfiable in* D := there is a model M with domain D and a valuation v in D such that $M \models B\ [v]$ for all B in Γ.

On close inspection we have shown in the proof of the Completeness Theorem 4.25 much more than is stated in the formulation of this theorem itself.

Theorem 4.27 (Löwenheim, 1915). *a) If not $\vdash' B$, then B is not valid in an enumerable domain ($\mathbb{N}$ or a finite subset of $\mathbb{N}$).*
b) Löwenheim's theorem: if a formula B is satisfiable in any non-empty domain, then it is satisfiable in an enumerable domain ($\mathbb{N}$ or a finite subset of $\mathbb{N}$).
c) If $A_1, \ldots, A_n$ are simultaneously satisfiable in any non-empty domain, then they are simultaneously satisfiable in an enumerable domain ($\mathbb{N}$ or a finite subset of $\mathbb{N}$).

Proof. a) Suppose not $\vdash' B$. Then, by Lemma 4.2, not all branches in a completed tableau with initial branch $\{FB\}$ are closed. Hence, there is an open branch τ in such a tableau. Let $a_{n_1}, \ldots, a_{n_k}$ be the free variables occurring in B. Then, by Lemma 4.1, since FB occurs in τ, $M_\tau \not\models B[n_1, \ldots, n_k]$. And M_τ is a model with $\mathbb{N}$ or a finite subset of $\mathbb{N}$ as domain (see Definition 4.29).
b) Suppose B is satisfiable (in some non-empty domain). Then not $\vdash' \neg B$. Let $a_{n_1}, \ldots, a_{n_k}$ be the free variables occurring in $\neg B$. Then by the proof of a), $M_\tau \not\models \neg B[n_1, \ldots, n_k]$, i.e., $M_\tau \models B[n_1, \ldots, n_k]$. And the domain of M_τ is $\mathbb{N}$ or a finite subset of $\mathbb{N}$. c) Follows from b) taking $B = A_1 \wedge \ldots \wedge A_n$. □

Theorem 4.28 (Compactness; Skolem). *Let $A_0,\ A_1,\ A_2, \ldots$ be an infinite list of formulas. a) Compactness (Gödel 1930): If, for each natural number k, $A_0, \ldots, A_k$ are simultaneously satisfiable, then $A_0,\ A_1,\ A_2, \ldots$ are simultaneously satisfiable in an enumerable domain ($\mathbb{N}$ or a finite subset of $\mathbb{N}$).*
b) Skolem's (1920) generalization of Löwenheim (1915): If $A_0,\ A_1,\ A_2, \ldots$ are simultaneously satisfiable, then $A_0,\ A_1,\ A_2, \ldots$ are simultaneously satisfiable in an enumerable domain ($\mathbb{N}$ or a finite subset of $\mathbb{N}$).

Proof. b) follows immediately from a). To prove a), suppose that for each natural number k, $A_0, \ldots, A_k$ are simultaneously satisfiable in some non-empty domain D. Construct a tableau with initial branch $\{TA_0, TA_1, TA_2, \ldots, F(P \wedge \neg P)\}$ by admitting step by step more and more assumption formulas A_i in the tableau construction (see also Kleene [9], Section 50).

Now suppose that all branches in this tableau would close. Then (by König's Lemma, see Exercise 4.43) there is a natural number m such that they all close in less than m steps. Let $A_0, \ldots, A_k$ be all formulas A_i occurring in the finitely

many closed branches. Then $A_0,\ldots,A_k \vdash' P \wedge \neg P$, contradicting the hypothesis of (a). So, there must be at least one open branch τ in the tableau with initial branch $\{TA_0, TA_1, TA_2, \ldots, F(P \wedge \neg P)\}$. Then, by Lemma 4.1, M_τ is a model with domain $\mathbb{N}$ (or a finite subset of $\mathbb{N}$) such that for all $i = 0,1,2,\ldots$, $M_\tau \models A_i[v]$ for some valuation v in the domain of M_τ. □

Let Γ be a (possibly infinite) set of sentences (closed formulas). We say that M is a *model* of Γ if M is a model of all sentences in Γ. We can specialize Theorem 4.28 to the case that Γ is an infinite list $A_0,\ A_1,\ A_2,\ldots$ of *closed* formulas (sentences).

Corollary 4.1. *Let Γ be a (possibly infinite) set of sentences (closed formulas).*
a) Compactness theorem: If each finite subset of Γ has a model, then Γ has a model.
b) Downward Löwenheim-Skolem theorem: If Γ has a model, then Γ has an enumerable model.

It is important to realize that the downward Löwenheim-Skolem theorem is due to the fact that first-order predicate languages contain only enumerably many symbols.

In addition to the 'downward-' there is also an 'upward-' Löwenheim-Skolem theorem, saying that under certain conditions, if Γ has a model, then it has arbitrarily large models. The interested reader is referred to van Dalen [4].

Once one has proved the completeness theorem, 'if $\Gamma \models B$, then $\Gamma \vdash B$', also for an infinite set Γ of premisses, the compactness theorem is an immediate consequence of it. The argument goes as follows: By definition, $\Gamma \vdash B$ iff there is a finite subset Γ' of Γ such that $\Gamma' \vdash B$. Therefore, using soundness and completeness, $\Gamma \models B$ iff there is a finite subset Γ' of Γ such that $\Gamma' \models B$. Taking for $B = P \wedge \neg P$, we obtain by contraposition: $\Gamma \not\models P \wedge \neg P$ iff for each finite subset Γ' of Γ, $\Gamma' \not\models P \wedge \neg P$. Or equivalently, Γ has a model iff each finite subset Γ' of Γ has a model.

For historical details concerning the Löwenheim-Skolem Theorem the reader is referred to van Heijenoort [6].

On the Meaning of the Compactness and Löwenheim-Skolem Theorem
From Theorem 4.28 b) it follows immediately that there can be no class Γ of formulas of first-order predicate logic such that Γ is simultaneously satisfiable in D iff D has non-enumerably many elements. Therefore, the expression 'having non-enumerably many elements' cannot be formulated in a first-order predicate language; in other words, *'non-enumerable' is not a first-order property*.

The Löwenheim-Skolem Theorem points out that *the expressive power of first-order predicate languages is restricted*: 'being non-enumerable' cannot be formulated in first-order logic. On the other hand, 'having infinitely many elements' is a first-order property; see Exercise 4.35.

As is explained in Exercise 4.35, there are classes $\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ldots$ of formulas such that for each $n \in \mathbb{N}$, Γ_n is simultaneously satisfiable in D iff D contains at least n elements. So, the expressions 'having at least one element', 'having at least two elements', and so on, can all be formulated in an appropriate first-order predicate language. However, below we shall prove that 'having finitely many elements' cannot be formulated in a first-order predicate language.

Theorem 4.29. *There is no class Γ of formulas such that Γ is simultaneously satisfiable in D iff D contains finitely many elements.* $(*)$

Proof. Suppose there was a class Γ of formulas such that $(*)$ holds. Now consider $\Delta := \Gamma \cup \Gamma_1 \cup \Gamma_2 \cup \ldots$, where for each n, Γ_n is a class of formulas expressing that there are at least n elements (see above). Then each finite subset of Δ is simultaneously satisfiable. So, by Theorem 4.28, Δ is simultaneously satisfiable in $\mathbb{N}$ or in a finite subset of $\mathbb{N}$. But by virtue of the formulas in Δ, Δ cannot be satisfiable in a finite subset of $\mathbb{N}$. So,5 Δ is simultaneously satisfiable in $\mathbb{N}$. Therefore, Γ is simultaneously satisfiable in $\mathbb{N}$. Contradiction with $(*)$. □

Summarizing *The Löwenheim-Skolem Theorems point out that the expressive power of first-order predicate languages is restricted: 'finite' and 'non-enumerable' are not first-order properties.*

4.5.3 Second-order Logic

It is interesting to note that the notions of 'finite' and of 'non-enumerable', which are not first-order properties (see Subsection 4.5.2), can be formulated in *second-order logic*. In second-order logic one is allowed to quantify not only over individual variables, but also over function variables and predicate variables. A second-order formula is a formula that contains at least one occurrence of a function or predicate variable. Here are some examples of second-order formulas, using x, y, z as individual variables, u as a function variable and X as a (unary) predicate variable:

Example 4.24. $\exists u \forall x[u(x) = x]$: there exists an identity function;
$\forall x \forall y \exists X[X(x) \wedge X(y)]$: every two individuals share some property;
$a = b \rightleftarrows \forall X[X(a) \rightleftarrows X(b)]$: a and b are equal iff they have the same properties (Leibniz' Law).

Now let Inf be the second-order sentence

$$\exists z \exists u[\forall x[z \neq u(x)] \wedge \forall x \forall y[x \neq y \rightarrow u(x) \neq u(y)]].$$

Inf is true in an interpretation with domain D iff there is an injective function (u) with domain D whose range is a proper subset of D ($z \notin \mathrm{Ran}(u)$). So, Inf is true in an interpretation iff the domain is Dedekind infinite (see Exercise 3.33). Consequently, 'being finite' can be expressed by the second-order formula $\neg Inf$.

Let En be the second-order sentence

$$\exists z \exists u \forall X[X(z) \wedge \forall x[X(x) \rightarrow X(u(x))] \rightarrow \forall x[X(x)]].$$

En is true in an interpretation iff the domain of the interpretation is enumerable (see Exercise 4.38). Consequently, 'non-enumerable' can be expressed by the second-order formula $\neg En$.

$\neg En$ is true in an interpretation iff the domain of the interpretation is non-enumerable. Since such interpretations exist, $\neg En$ is satisfiable. But $\neg En$ is not

satisfiable in any enumerable domain. Therefore: *the Löwenheim-Skolem theorem fails for second-order logic*. The compactness theorem and many other properties of first-order logic also fail for second-order logic. See Chapter 5 and Boolos [2], Chapter 22.

4.5.4 Skolem's Paradox

Below we shall work out the astonishing and philosophically interesting consequence of the Löwenheim-Skolem theorem known as *Skolem's paradox*.

Let Γ be the set of (closed) axioms of some axiomatic set theory formulated in a first-order predicate language, for instance, $\Gamma = ZF$ (Zermelo-Fraenkel set theory; see Chapter 3). It is generally believed that such a Γ is consistent, in other words, that there is some model M which makes all axioms in Γ true. But then it follows from the Löwenheim-Skolem theorem that Γ has an enumerable model. There are only enumerably many 'sets' in this model. (1)

On the other hand, we know that Cantor's theorem (Corollary 3.6) is deducible from Γ, saying that the set $P(\mathbb{N})$ of all subsets of $\mathbb{N}$ is not enumerable. (2)

(1) and (2) together constitute what is called *Skolem's paradox* (1922-3).

Skolem's paradox is not an antinomy (or real paradox), but rather a veridical (or truth-telling) paradox (see Section 2.10). It tells us the (astonishing) truth that there is an *enumerable model* which makes all axioms in Γ true, although it follows from Γ that there are *non-enumerably many sets*. How is this possible? How can we explain this phenomenon?

The set of all subsets in the model is indeed enumerable and therefore there is a bijective mapping from it to the set of natural numbers. But this mapping is *not* in the model; so it does not make invalid the theorem of set theory which states that there is no bijective mapping from the set $P(\mathbb{N})$ of all subsets of $\mathbb{N}$ to $\mathbb{N}$. More precisely: Let Γ be the axioms of set theory formulated in a first order predicate language. Then

$$\Gamma \vdash \neg\exists x[x \text{ is a bijection from } P(\mathbb{N}) \text{ to } \mathbb{N}].$$

Now let M be a countable model of Γ. Then

$$M \models \neg\exists x[x \text{ is a bijection from } P(\mathbb{N}) \text{ to } \mathbb{N}],$$

i.e., there is no bijection (in the sense of M) *in* M from the set $P(\mathbb{N})^M$ in M to the set $\mathbb{N}^M$ in M. This does not exclude that there is a bijection *outside of* M (i.e., being not a set or object of the model M) from the set $P(\mathbb{N})^M$ in M to the set $\mathbb{N}^M$ in M.

Skolem's 'paradox' may be further clarified by the following two observations.

1. From the axioms of set theory it follows that there are non-enumerably many subsets of a given infinite set. But given some set, we can actually *define* only enumerably many subsets of it. So, it is not that surprising that there is an enumerable model of the axioms of set theory.

2. Skolem's 'paradox' is the result of an application of the Löwenheim-Skolem theorem which was a by-product of the completeness theorem. These theorems are the result of considerations *about* the formal system as a whole and are not deducible within the formal system itself. This explains the possibility that looking from the outside to a formal system for set theory, the collection of all subsets of a given infinite set may be enumerable, while at the same time this collection is non-enumerable within the formal system itself.

We finish this section with a quotation from van Heijenoort [6], pp. 290-291:

> For Skolem the discrepancy between an intuitive set-theoretic notion and its formal counterpart leads to the 'relativity' of set-theoretic notions. Thus, two sets are equivalent if there exists a one-to-one mapping of the first onto the second; but this mapping is itself a collection of ordered pairs of elements. If, in a formalized set theory, this collection exists as a set, the two given sets are equivalent in the theory; if it does not, the sets are not equivalent in the theory and, when one set is that of the natural numbers as defined in the theory, the other becomes 'nondenumerable'. The existence of such a 'relativity' is sometimes referred to as the Löwenheim-Skolem paradox. But, of course, it is not a paradox in the sense of an antinomy; it is a novel and unexpected feature of formal systems.

Exercise 4.35. a) Show that $P(a_1), \neg P(a_2)$ are simultaneously satisfiable in D iff D contains at least two elements.
b) Find a class Γ of formulas such that Γ is simultaneously satisfiable in D iff D contains at least three elements.
c) Show that $\forall x[\neg P(x,x)]$, $\forall x \forall y \forall z[P(x,y) \wedge P(y,z) \to P(x,z)]$, $\forall x \exists y[P(x,y)]$ are simultaneously satisfiable in D iff D contains at least denumerably many elements.
d) Show that it is impossible to find a class Γ of formulas such that Γ is satisfiable in D iff D has non-enumerably many elements.
Thus if one attempts to characterize a mathematical structure by means of a set of axioms formulated in first order predicate logic, one is in a certain sense doomed to failure if that structure involves a non-enumerable infinity of elements.

Exercise 4.36. Let Γ be a (possibly infinite) set of formulas. Γ is *consistent* := for no formula B, $\Gamma \vdash B$ and $\Gamma \vdash \neg B$. Note that Γ is consistent iff there is at least one formula C such that $\Gamma \nvdash C$. Supposing that Γ is a finite set of formulas, show that Γ is consistent iff Γ has a model with an enumerable domain. (Skolem, 1922; see van Heijenoort [6], p. 293.)

Exercise 4.37. Using Skolem's result (1922) that Γ is consistent iff Γ has a model with an enumerable domain (see Exercise 4.36), prove the completeness theorem: if $A_1, \ldots, A_n \models B$, then $A_1 \ldots, A_n \vdash B$. Skolem himself did not make this step from his result (1922) to the completeness theorem (K. Gödel, 1930) for philosophical reasons. The notions of validity and valid consequence contain a universal quantification over all non-enumerably many structures and for that reason the idea of formulating the completeness theorem did not even occur to Skolem.

Exercise 4.38. Let En be the second-order formula

$$\exists z \exists u \forall X[X(z) \wedge \forall x[X(x) \to X(u(x))] \to \forall x[X(x)]].$$

Prove: En is true in an interpretation with domain D iff D is enumerable.

Exercise 4.39. Prove that there is a decision procedure (to test tableau-provability) for the class of formulas having a prenex normal form such that, in the prefix, no existential quantifier precedes any universal quantifier.

Exercise 4.40. A *monadic formula* contains by definition only unary (monadic) predicate symbols. Prove that each monadic formula is equivalent to a truth-functional composition of formulas of the form $\forall x[B(x)]$ and $\exists x[B(x)]$, where B does not contain any quantifiers.

Exercise 4.41. Prove that there is a decision procedure (to test tableau-provability) for the class of monadic formulas (see Exercise 4.40).

Exercise 4.42. Prove that there is a decision procedure (to test tableau-provability) for the class of formulas having a prenex normal form $\exists x \forall y[M]$, where M is the matrix, containing no individual variables except x and y and containing only one binary predicate symbol P. Similarly, if M contains two binary predicate symbols.

Solutions of the decision problem for special classes of more complex formulas can be found in Church [3], Section 46.

Exercise 4.43. Let $\mathbb{N}^*$ be the set of all n-tuples $(k_1,\ldots,k_n)$ of natural numbers, $n \in \mathbb{N}$. Let $s = (k_1,\ldots,k_n) \in \mathbb{N}^*$ and $t = (l_1,\ldots,l_m) \in \mathbb{N}^*$. Then the *concatenation* of s and t, denoted by st, is the $(n+m)$-tuple $(k_1,\ldots,k_n,\ l_1,\ldots,l_m)$. And s is a *prefix* of t := there is a tuple $s' \in \mathbb{N}^*$ such that $t = ss'$.

Let T be a subset of $\mathbb{N}^*$. T is a *tree* := a) for each $t \in T$, every prefix of t is also in T, and b) for each $t \in T$, and for every $i \in \mathbb{N}$, if $t(i) \in T$, then for every $j \leq i$, $t(j)$ is also in T. The elements of a tree T are called *nodes*. For $s,t \in T$, t is an *immediate successor* of s := for some $i \in \mathbb{N}$, $t = s(i)$. A *path* in T is a finite or infinite sequence $s_0, s_1, \ldots$ of nodes in T, starting with the empty tuple $()$, i.e., $s_0 = ()$, and such that each node s_{i+1} is an immediate successor of the preceding node s_i.

König's Lemma: Let T be a tree such that each node in T has only finitely many immediate successors. If there are arbitrarily long finite paths in T, then there is an infinite path in T. Prove König's Lemma and show that the lemma need not hold for trees in which some node has infinitely many immediate successors.

4.6 Predicate Logic with Equality

The predicate logic with equality arises from predicate logic (without equality) by giving one of the binary predicate symbols, say $\equiv$, special treatment. That is, in the predicate logic with equality one allows only interpretations (structures) which interpret $\equiv$ as equality (=); no other interpretations of $\equiv$ are allowed.

Definition 4.31 (Interpretation). Let M be an interpretation. M is an *interpretation* or model *for the predicate logic with equality* := M interprets $\equiv$ as $=$ (equality).

Here $\equiv$ is a particular binary predicate symbol in the predicate language, while $=$ is the name of the equality relation, which is a mathematical object. For convenience, one frequently writes = instead of the logical predicate symbol $\equiv$, in which case the sign = is used both as a symbol in the predicate language and as a symbol denoting mathematical equality, which is the interpretation of the predicate symbol $\equiv$.

Definition 4.32 (Validity). $\models A$ (A is *valid in the predicate logic with equality*) := for all interpretations M for the predicate logic with equality, $M \models A$.
'$A_1, \ldots, A_m \models B$' in the predicate logic with equality is defined similarly.

A Hilbert-type proof system for the predicate logic with equality is obtained by adding to the axiom schemata and rules of inference for the predicate logic without equality the following formulas as further axioms:

$\forall x[x \equiv x]$ $\qquad$ $\forall x \forall y[x \equiv y \to (P(\ldots,x,\ldots) \to P(\ldots,y,\ldots))]$

$\forall x \forall y \forall z[x \equiv y \to (x \equiv z \to y \equiv z)]$ $\qquad$ $\forall x \forall y[x \equiv y \to f(\ldots,x,\ldots) \equiv f(\ldots,y,\ldots)]$

One easily sees that these axioms are valid in the predicate logic *with* equality. Of course, they are not valid (in the predicate logic without equality): taking $\mathbb{N}$ as domain and interpreting $\equiv$ as $<$ (is less than), a false proposition results from $\forall x[x \equiv x]$. The provability and deducibility results, in particular the deduction theorem, already established for the predicate logic (without equality) all hold also for the predicate logic with equality.

Theorem 4.30. *In the predicate logic with equality:*

$\vdash \forall x[x \equiv x]$ $\qquad$ $r \equiv s \vdash t \equiv r \rightleftarrows t \equiv s$

$\vdash \forall x \forall y[x \equiv y \to y \equiv x]$ $\qquad$ $r \equiv s \vdash r \equiv t \rightleftarrows s \equiv t$

$\vdash \forall x \forall y \forall z[x \equiv y \wedge y \equiv z \to x \equiv z]$ $\qquad$ $r \equiv s \vdash P(\ldots,r,\ldots) \rightleftarrows P(\ldots,s,\ldots)$

(where r, s and t are terms) $\qquad$ $r \equiv s \vdash f(\ldots,r,\ldots) \equiv f(\ldots,s,\ldots)$

Proof.
1) $\forall x[x \equiv x]$ is an axiom and hence provable in the predicate logic *with equality*.
2) To show that $\vdash \forall x \forall y[x \equiv y \to y \equiv x]$ in the predicate logic with equality, suppose $a_1 \equiv a_2$. From the second equality axiom: $a_1 \equiv a_2 \to (a_1 \equiv a_1 \to a_2 \equiv a_1)$. By Modus Ponens: $a_1 \equiv a_1 \to a_2 \equiv a_1$. From the first equality axiom: $a_1 \equiv a_1$. Applying Modus Ponens: $a_2 \equiv a_1$. Therefore, $a_1 \equiv a_2 \to a_2 \equiv a_1$. Therefore, $\forall x \forall y[x \equiv y \to y \equiv x]$ is provable in the predicate logic with equality.
3) To show that $r \equiv s \vdash P(\ldots,r,\ldots) \rightleftarrows P(\ldots,s,\ldots)$ in the predicate logic *with equality*, assume $r \equiv s$ and assume $P(\ldots,r,\ldots)$. From the third equality axiom: $r \equiv s \to (P(\ldots,r,\ldots) \to P(\ldots,s,\ldots))$. Then by two applications of Modus Ponens, $P(\ldots,s,\ldots)$. Conversely, assume $P(\ldots,s,\ldots)$. We have already shown that $\vdash \forall x \forall y[x \equiv y \to y \equiv x]$. So $\vdash r \equiv s \to s \equiv r$. Assuming $r \equiv s$, by Modus Ponens $s \equiv r$. Then from the third equality axiom, $P(\ldots,r,\ldots)$.
4) The other cases are similar. □

Another equivalent proof system for the predicate logic with equality is obtained by adding to the axiom schemata and rules of inference for predicate logic the axiom $\forall x[x \equiv x]$ and the axiom schema $\forall x \forall y[x \equiv y \to (A(x) \to A(y))]$.

Theorem 4.31. *Let M be a model (structure) for the predicate logic (without equality) with domain D, such that M satisfies the equality axioms. Let R be the interpretation in M of $\equiv$. Then R is an equivalence relation on D (see Subsection 3.4.3). And there is a model M' such that*
1. the domain of M' is the quotient set D/R (see Subsection 3.4.3),
2. M' interprets the binary predicate symbol $\equiv$ as the equality relation $=$; hence M' is a model for the predicate logic with equality, and
3. for any formula $A = A(a_1,\ldots,a_n)$, $M \models A\ [d_1,\ldots,d_n]$ iff $M' \models A\ [[d_1]_R,\ldots,[d_n]_R]$, where for $d \in D$, $[d]_R$ is the equivalence class of d with respect to R.

Proof. Let $M = \langle D; P_1^M,\ P_2^M,\ldots; f_1^M, f_2^M,\ldots\rangle$ be a structure (for the predicate logic without equality), which satisfies the equality axioms. Then M also satisfies the formulas in Theorem 4.30, since these formulas are deducible from the equality axioms in the predicate logic (without equality). Let R be $\equiv^M$, i.e., the interpretation in M of the binary predicate symbol $\equiv$; R is not necessarily the equality relation. Since M satisfies the first three formulas in Theorem 4.30, $\forall x[x \equiv x]$, $\forall x\forall y[x \equiv y \to y \equiv x]$, $\forall x\forall y\forall z[x \equiv y \wedge y \equiv z \to x \equiv z]$, it follows that the relation R on D is reflexive, symmetric and transitive. In other words, R is an equivalence relation on D. As explained in Subsection 3.4.3, any equivalence relation on D separates D into disjoint non-empty equivalence classes. For $d \in D$, let $[d]_R$ be the equivalence class of d with respect to R, i.e.,

$$[d]_R := \{d' \in D \mid R(d,d')\} = \{d' \in D \mid M \models a_1 \equiv a_2\ [d,d']\}.$$

Define the model M' as follows: a) the domain of M' is the quotient set D/R of all equivalence classes $[d]_R$ with $d \in D$;
b) for any n-ary predicate symbol P, $P^{M'}([d_1]_R,\ldots,[d_n]_R) := P^M(d_1,\ldots,d_n)$;
c) for any n-ary function symbol f, $f^{M'}([d_1]_R,\ldots,[d_n]_R) := [f^M(d_1,\ldots,d_n)]_R$.
By Theorem 4.30, M satisfies in particular: $r \equiv s \to (P(\ldots,r,\ldots) \rightleftarrows P(\ldots,s,\ldots))$ and $r \equiv s \to f(\ldots,r,\ldots) \equiv f(\ldots,s,\ldots)$. Consequently, the definitions of $P^{M'}$ and $f^{M'}$, given above, are correct, i.e., if $[d_1]_R = [e_1]_R$ and $\ldots$ $[d_n]_R = [e_n]_R$, then $P^M(d_1,\ldots,d_n)$ iff $P^M(e_1,\ldots,e_n)$ and $f^M(d_1,\ldots,d_n)Rf^M(e_1,\ldots,e_n)$.

M' interprets the binary predicate symbol $\equiv$ by the equality relation $=$ on D/R. For $\equiv^{M'}([d_1]_R,[d_2]_R) := \ \equiv^M(d_1,d_2)$; but $\equiv^M(d_1,d_2)$ iff $R(d_1,d_2)$; hence $\equiv^M(d_1,d_2)$ iff $[d_1]_R = [d_2]_R$. Therefore $\equiv^{M'}$ is the equality relation on D/R. By straightforward induction it follows from the definition of M' that $M \models A\ [d_1,\ldots,d_n]$ iff $M' \models A\ [[d_1]_R,\ldots,[d_n]_R]$. □

It is straightforward to check that the *soundness theorem* holds for the predicate logic with equality: if $\Gamma \vdash B$ in the predicate logic with equality, then $\Gamma \models B$ in the predicate logic with equality. Making use of Theorem 4.31 one can easily see that from the completeness theorem, the compactness theorem and the Löwenheim-Skolem theorem for the predicate logic (without equality) similar theorems follow for the predicate logic with equality.

Theorem 4.32. *Let Γ be a possibly infinite set of closed formulas, and let B be a closed formula.*

a) **Completeness for the predicate logic with equality***: if $\Gamma \models B$ in the predicate logic with equality, then $\Gamma \vdash' B$ in the predicate logic with equality.*
b) **Compactness for the predicate logic with equality***: Γ has a model (in the predicate logic with equality) if and only if every finite subset of Γ has a model (in the predicate logic with equality).*
c) **Downward Löwenheim-Skolem for the predicate logic with equality***: If Γ is simultaneously satisfiable in the predicate logic with equality, then Γ is simultaneously satisfiable in an enumerable domain in the predicate logic with equality.*

Proof. a) Let $\Gamma = \{A_1,\ldots,A_n\}$. The construction of a complete tableau $\mathscr{T}$ with initial branch $\{TA_1,\ldots,TA_n,FB\}$ in the predicate logic with equality starts with $\{TA_1,\ldots,TA_n,\ TE_1,\ldots,TE_m,FB\}$, where $E_1,\ldots,E_m$ are the equality axioms for the predicate and function symbols occurring in $A_1,\ldots,A_n,B$. If all branches in $\mathscr{T}$ close, then by Lemma 4.2, $A_1,\ldots,A_n,E_1,\ldots,E_m \vdash' B$, i.e., $A_1,\ldots,A_n \vdash' B$ in the predicate logic with equality. If τ is an open branch in $\mathscr{T}$, then by Lemma 4.1 the model M_τ (see Definition 4.29) makes all of $A_1,\ldots,A_n,\ E_1,\ldots,E_m$ true and B false. Then by Theorem 4.31 there is a model M'_τ for the predicate logic with equality which makes $A_1,\ldots,A_n$ true and B false. Therefore, if $A_1,\ldots,A_n \models B$ in the predicate logic with equality, there can be no open branch starting with $TA_1,\ldots,TA_n,\ TE_1,\ldots,TE_m,\ FB$; in other words, in that case all such branches will close and hence $A_1,\ldots,A_n \vdash' B$ in the predicate logic with equality. In case that Γ contains infinitely many sentences, the construction of a complete tableau has to be adapted, such that at each step one more assumption formula in Γ is taken into consideration.
b) Because only a finite number of sentences in Γ can be used in a formal deduction, it follows that $\Gamma \vdash B$ iff there is a finite subset Γ' of Γ such that $\Gamma' \vdash B$. From the soundness and completeness theorems it follows that $\Gamma \models B$ iff for some finite subset Γ' of Γ, $\Gamma' \models B$. Taking for B the formula $\exists x[x \not\equiv x]$ and noting that $\exists x[x \not\equiv x]$ is not true in any structure, the result follows by contraposition.
c) Let M be a model for the predicate logic with equality, which makes all formulas in Γ simultaneously true. Construct a complete tableau $\mathscr{T}$ with initial branch $\{TA_1,TA_2,\ldots,TE_1,\ldots,TE_m,F(P\wedge\neg P)\}$ where $\{A_1,A_2,\ldots\} = \Gamma$ and $E_1,\ldots,E_m$ are the equality axioms for the predicate and function symbols occurring in Γ. If all branches would close, then $\Gamma \vdash' P\wedge\neg P$ in the predicate logic with equality, and hence $M \models P\wedge\neg P$. Contradiction. Therefore, there is at least one open branch τ in $\mathscr{T}$, which by Lemma 4.1 yields a model M_τ that satisfies (simultaneously) all formulas in Γ and the equality axioms. Then by Theorem 4.31 there is a model M'_τ for the predicate logic with equality which simultaneously satisfies all formulas in Γ. Since the domain D of M_τ is enumerable and since the domain of M'_τ is D modulo R for some equivalence relation R, the domain of M'_τ is also enumerable. □

Warning For instance $M = \langle\{0\};\ =\rangle$ is a model of the formula $\exists x\forall y[x \equiv y]$ in the predicate logic with equality. So, by the downward Löwenheim-Skolem theorem for the predicate logic with equality (Theorem 4.32) the formula $\exists x\forall y[x \equiv y]$ has a model with an enumerable domain. Notice that this domain cannot be $\mathbb{N}$ or any other denumerable domain, since in the predicate logic with equality $\exists x\forall y[x \equiv y]$

expresses that there is exactly one element in the domain. Of course, in the predicate logic without equality, $\exists x \forall y[x \equiv y]$ does have a model with $\mathbb{N}$ as domain, for instance, $M = \langle \mathbb{N};\ \leq \rangle$ is a model of $\exists x \forall y[x \equiv y]$.

For applications of the Compactness theorem in mathematics see Exercises 4.44, 4.45 and 4.46.

Exercise 4.44. * The *elementary theory of fields*, designated by FL, has as non-logical symbols the constants 0, 1 and -1 and the binary function symbols $+$ and $\cdot$. The non-logical axioms of FL are:

$\forall x \forall y \forall z[(x+y)+z = x+(y+z)]$ $\quad$ $\forall x[x+0 = x]$

$\forall x[x+(-1 \cdot x) = 0]$ $\quad$ $\forall x \forall y[x+y = y+x]$

$\forall x \forall y \forall z[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$ $\quad$ $\forall x[x \cdot 1 = x]$

$\forall x[x \neq 0 \rightarrow \exists y[x \cdot y = 1]]$ $\quad$ $\forall x \forall y[x \cdot y = y \cdot x]$

$\forall x \forall y \forall z[x \cdot (y+z) = (x \cdot y)+(x \cdot z)]$ $\quad$ $0 \neq 1$.

The models of FL are just the fields. Let A_n be the formula $1+1+\ldots+1 = 0$, where there are n occurrences of 1 on the left. By adding the non-logical axioms $\neg A_2,\ \neg A_3,\ \ldots,\ \neg A_{n-1},\ A_n$, we get the elementary theory $FL(n)$ of fields of characteristic n ($n \geq 2$). To get the elementary theory $FL(0)$ of fields of characteristic 0, we add all of the $\neg A_n$ as non-logical axioms. Prove the following assertions:

1. If $FL(n)$ is consistent, then $n = 0$ or n is prime. Hint: use the mathematical fact that the characteristic of a field is 0 or a prime number.
2. If $FL(0) \models B$, then there is an n_0 such that for every $n \geq n_0$, $FL(n) \models B$. Hint: use the compactness theorem.
3. We cannot replace the infinite number of non-logical axioms we added to FL to get $FL(0)$ by a finite number. Hint: use 2.
4. There is no extension of FL whose models are just the finite fields.

Exercise 4.45. * **Theorem**: Let R be a partial ordering on V. Then there is a complete partial ordering R' on V such that $R \subseteq R'$. Prove this theorem for finite sets V using mathematical induction on the number of elements of V and, using the compactness theorem, prove this theorem also for infinite sets.

Exercise 4.46. * Let $B = \langle V,\ \sqcap,\ \sqcup,\ \neg,\ 0,\ 1 \rangle$ be a Boolean algebra. An element v in V is called an *atom* (of B) if $v \neq 0$ and for all $y \in V$, if $y \leq v$ (i.e., $y \sqcap v = y$), then $y = 0$ or $y = v$. B is *atomic* := for all x in V, if $x > 0$, then there is a y in V such that y is an atom of B and $y \leq x$. Let $AT_B := \{v \in V \mid v \text{ is an atom of } B\}$.

1. Prove that every atomic Boolean algebra B is isomorphic to a subalgebra of a set-algebra $\langle P(W),\ \cap,\ \cup,\ C_W,\ \emptyset,\ W \rangle$. Hint: consider $f : V \rightarrow P(AT_B)$, defined by $f(w) := \{v \in V \mid v \text{ is an atom of } B \text{ and } v \leq w\}$.
2. Using the compactness theorem, prove that every Boolean algebra can be embedded in an atomic Boolean algebra. Hint: use the mathematical fact that the smallest Boolean algebra, generated by finitely many elements, is finite and hence atomic.

4.7 About the Relation of Logic with other Disciplines

4.7.1 Logic and Philosophy of Language

4.7.1.1 Definite Descriptions

Both Russell (1872-1970) and Wittgenstein (1889-1951), for different sets of reasons, rejected Frege's [5] distinction between sense (Sinn) and reference (Bedeutung) (see Chapter 7). Frege's analysis of a sentence like 'The king of France is bald' would be that this sentence lacks a truth value (reference, Bedeutung), because the subject expression has no reference, but that the lack of a truth value does not render the sentence meaningless, since this sentence does have a sense (Sinn). Russell, having already rejected Frege's theory of sense and reference, explains how sentences like this one can be meaningful, while there is nothing for the proposition, expressed by the sentence, to be about. Russell claims in [14] that the sentence in question appears to be in subject-predicate form, but is not really so. Its grammatical form is misleading as to its logical form. Russell's analysis of 'The king of France is bald' is as follows:

$\exists x$ [x is king of France $\wedge$ x is bald $\wedge$ $\forall y$ [y is king of France $\rightarrow y = x$]], or
equivalently, but shorter $\exists x$ [x is bald $\wedge$ $\forall y$ [y is king of France $\rightleftarrows y = x$]].

And since there is no king of France, this sentence is false.

Russell analyzed 'The king of France is bald' as no simple subject-predicate statement but as a far more complicated one, in which two different quantified variables occur. In Russell's theory, the deep structure of such statements is very different from what their surface grammar suggests. Russell does not give an explicit definition enabling one to replace a definite description by an equivalent one wherever it appears, but a *contextual definition*, which enables one to replace sentences containing definite descriptions by equivalent sentences not containing definite descriptions. Russell used the following 'iota'-notation:

$\iota x A(x)$ the unique x with property A, and
$C(\iota x A(x))$ the unique x with property A has property C
as shorthand for $\exists x[A(x) \wedge C(x) \wedge \forall y[A(y) \rightarrow y = x]]$.

Where the condition C is complex, the iota notation is ambiguous. Russell's simple example is well known:

$\neg B(\iota x F(x))$ The king of France is not bald.

Here the ambiguity of the iota notation corresponds to an ambiguity in the English, between these two:
1. $\neg(B(\iota x F(x)))$, i.e., $\neg\exists x[F(x) \wedge B(x) \wedge \forall y[F(y) \rightarrow y = x]]$: there is no object x such that x is king of France and x is bald and x is the only king of France. And this happens to be true.
2. $(\neg B)(\iota x F(x))$, i.e., $\exists x[F(x) \wedge (\neg B)(x) \wedge \forall y[F(y) \rightarrow y = x]]$: there is some object x such that x is king of France and x is not bald and x is the only king of France. And this happens to be false; so we have $\neg((\neg B)(\iota x F(x)))$.

Note that this latter expression is not equivalent to $B(\iota x F(x))$, i.e., $\exists x[F(x) \wedge B(x) \wedge \forall y[F(y) \rightarrow y = x]]$ (the king of France is bald): $\neg((\neg B)(\iota x F(x)))$ is true, while $B(\iota x F(x))$ is false. In Russell's jargon, the definite description $\iota x F(x)$ has *narrow scope* in version 1 and *wide scope* in version 2.

A less confusing notation for definite descriptions would result by treating them as a kind of quantifier: $(Ix)(F(x), B(x))$ instead of $B(\iota x F(x))$. Then the sentence in version 1, $\neg(B(\iota x F(x)))$, would be rendered by $\neg(Ix)(F(x), B(x))$, and the sentence in version 2, $(\neg B)(\iota x F(x))$, by $(Ix)(F(x), \neg B(x))$. While it was somewhat strange to have both, $\neg(B(\iota x F(x)))$ and $\neg((\neg B)(\iota x F(x)))$ in the new notation this would become $\neg(Ix)(F(x), B(x))$ and $\neg(Ix)(F(x), \neg B(x))$, which looks similar to $\neg\forall x[A(x)]$ and $\neg\forall x[\neg A(x)]$. which does not look like a contradiction at all.

4.7.1.2 Analytic-Synthetic

Immanuel Kant in his *Critique of Pure Reason* [8] makes a distinction between analytic and synthetic judgments. Kant calls a judgment *analytic* if its predicate is contained (though covertly) in the subject, in other words, the predicate adds nothing to the conception of the subject. Kant gives 'All bodies are extended (Alle Körper sind ausgedehnt)' as an example of an analytic judgment; I need not go beyond the conception of *body* in order to find extension connected with it. If a judgment is not analytic, Kant calls it *synthetic*. So, a synthetic judgment adds to our conception of the subject a predicate which was not contained in it, and which no analysis could ever have discovered therein. Kant mentions 'All bodies are heavy (Alle Körper sind schwer)' as an example of a synthetic judgment.

Kant makes in [8] also a distinction between *a priori* knowledge and *a posteriori* knowledge. A priori knowledge is knowledge existing altogether independent of experience, while a posteriori knowledge is empirical knowledge, which has its sources in experience.

Sometimes one speaks of *logically necessary* truths instead of analytic truths and of *logically contingent* truths instead of synthetic truths, to be distinguished from physically necessary truths (truths which physically could not be otherwise, true in all physically possible worlds). The distinction between necessary and contingent truth is a *metaphysical* one, while the distinction between a priori and a posteriori truth is an *epistemic* one. Although these – the metaphysical and the epistemological – are certainly different distinctions, it is controversial whether they coincide in extension, that is, whether all and only necessary truths are *a priori* and all and only contingent truths are *a posteriori*.

In [8] Kant stresses that mathematical judgments are both a priori and synthetic. 'Proper mathematical propositions are always judgments *a priori*, and not empirical, because they carry along with them the conception of necessity, which cannot be given by experience'. Why are mathematical judgments synthetic? Kant considers the proposition $7 + 5 = 12$ as an example. 'The conception of twelve is by no means obtained by merely cogitating the union of seven and five; and we may analyse our conception of such a possible sum as long as we will, still we shall never

discover in it the notion of twelve'. We must go beyond this conception of $7+5$ and have recourse to an intuition which corresponds to counting using our fingers: first take seven fingers, next five fingers extra, and then by starting to count right from the beginning we arrive at the number twelve.

7:	1	1	1	1	1	1	1					
5:								1	1	1	1	1
7 + 5:	1	1	1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9	10	11	12

'Arithmetical propositions are therefore always synthetic, of which we may become more clearly convinced by trying large numbers'. Geometrical propositions are also synthetic. As an example Kant gives 'A straight line between two points is the shortest', and explains 'For my conception of *straight* contains no notion of *quantity*, but is merely *qualitative*. The conception of the *shortest* is therefore wholly an addition, and by no analysis can it be extracted from our conception of a straight line'.

In more modern terminology, following roughly a 'Fregean' account of analyticity, one would define a proposition A to be *analytic* iff either
(i) A is an instance of a logically valid formula; e.g., 'No unmarried man is married' has the logical form $\neg\exists x[\neg P(x) \wedge P(x)]$, which is a valid formula, or
(ii) A is reducible to an instance of a logically valid formula by substitution of synonyms for synonyms; e.g., 'No bachelor is married'.

W.V. Quine [13] is sceptical of the analytic/synthetic distinction. Quine argues as follows. In order to define the notion of analyticity we used the notion of synonymy in clause (ii) above. However, if one tries to explain this latter notion, one has to take recourse to other notions which directly or indirectly will have to be explained in terms of analyticity.

4.7.2 Logic and Philosophy of Science

It is an old problem to draw the line between scientifically meaningful and meaningless statements. Consider the following quotation, taken from Hume's *Enquiry Concerning Human Understanding*.

> When we run over libraries, persuaded of these principles, what havoc must we make? If we take in our hand any volume; of divinity or school metaphysics, for instance; let us ask, *Does it contain any abstract reasoning concerning quantity of number*? No. *Does it contain any experimental reasoning concerning matter of fact and existence*? No. Commit it then to the flames: for it can contain nothing but sophistry and illusion". (David Hume, 1711-1776)

As we learn from A.J. Ayer [1], the quotation above is a good formulation of the positivist's position. In the 1930's the adjective *logical* was added, resulting in the term *Logical Positivism*, which underscored the successes of modern logic and the expectation that the new logical discoveries would be very fruitful for philosophy. This logical positivism was typical of the *Vienna Circle*, a group of philosophers (among them Moritz Schlick, Rudolf Carnap and Otto Neurath), scientists and mathematicians (among them Karl Menger and Kurt Gödel). According to A.J. Ayer [1],

Einstein, Russell and Wittgenstein had a clear kinship to the Vienna Circle and had a great influence upon it.

In order to draw a sharp distinction between scientifically meaningful statements and scientifically meaningless statements the *verification principle* was formulated: only those statements are scientifically meaningful which can be verified in principle; in other words, the meaning of a proposition is its method of verification. However, a proposition like 'all ravens are black', which has as logical form $\forall x[R(x) \to B(x)]$, cannot be verified due to the universal quantifier, $\forall$; at the same time we consider this proposition to be (scientifically) meaningful.

On the other hand, the proposition 'all ravens are black' can be conclusively falsified, since its negation 'not all ravens are black', being of the form $\neg\forall x[R(x) \to B(x)]$, is logically equivalent to 'some raven is not black', which has the logical form $\exists x[R(x) \wedge \neg B(x)]$, and hence can be verified. For this reason the *falsification principle* was formulated: only those statements are scientifically meaningful which can be falsified in principle. This principle seems to be more in conformity with scientific practice: hypotheses are set up and rejected as soon as experimental results force us to do so. However, Otto Neurath himself soon realized that a slightly more complex proposition, like 'all men are mortal', which has the logical form $\forall x\exists y[R(x,y)]$ (for every person there is a moment of time such that ...), can neither be verified (due to the universal quantifier, $\forall$) nor falsified, since its negation 'not all men are mortal', being of the form $\neg\forall x\exists y[R(x,y)]$, is equivalent to 'some men are immortal', which has the logical form $\exists x\forall y[\neg R(x,y)]$, and hence – again due to the universal quantifier – cannot be verified.

Falsification of $\forall x\exists y[R(x,y)]$ is equivalent to verification of $\neg\forall x\exists y[R(x,y)]$, i.e., verification of $\exists x\forall y[\neg R(x,y)]$, which is not possible in principle due to the universal quantifier. At the same time we want to consider a statement like 'all men are mortal' as (scientifically) meaningful. Therefore, we have to give up not only the verification principle, but also the falsification principle. This was already realized by Otto Neurath during his stay (1938-39) in the Netherlands (oral communication by Johan J. de Iongh).

Summarizing: statements of the form $\forall x\exists y[R(x,y)]$ cannot be verified due to the universal quantifier $\forall$ and cannot be falsified due to the existential quantifier $\exists$.

Instead of the verification or falsification principle, a weaker criterion was formulated, called the *confirmation principle*: a statement is scientifically meaningful if and only if it is to some degree possible to confirm or disconfirm it. One way to confirm (increase the degree of credibility of) universal generalizations like 'all ravens are black' is to find things that are both ravens and black, and one way to disconfirm this proposition is to find things that are ravens but not black. The problem with this confirmation principle is that 'all ravens are black', $\forall x[R(x) \to B(x)]$, is logically equivalent to 'all non-black things are non-ravens', $\forall x[\neg B(x) \to \neg R(x)]$, and according to the confirmation principle, the latter proposition is confirmed by observations of non-black non-ravens; thus observations of brown shoes, white chalk, etc., would confirm the proposition 'all ravens are black'. Various attempts have been made to give the verification principle, in this weaker form, a precise expression, but the results have not been altogether satisfactory. For instance, a solution might be found

by replacing the material implication $\rightarrow$ in $\forall x[R(x) \rightarrow B(x)]$ by the counterfactual implication $\Box\!\!\rightarrow$ (see Chapter 6), for $\forall x[A(x) \Box\!\!\rightarrow B(x)]$ is not logically equivalent to $\forall x[\neg B(x) \Box\!\!\rightarrow \neg A(x)]$.

4.7.3 Logic and Artificial Intelligence; Prolog

As already mentioned in Chapter 2, the language of logic can be used to represent knowledge. And the language of predicate logic is a richer tool than the propositional language. Suppose, for instance, that someone knows the following:

(1)	john is a parent of bob	(2)	john is a parent of claudia
(3)	john is male	(4)	bob is male
(5)	claudia is female	(6)	x is brother of y if x and y have a parent in common and x is male.

Introducing a predicate language containing the individual constants j, b and c, the unary predicate symbols 'male' and 'female' and the binary predicate symbols 'parent' and 'brother', (1) to (6) can be represented by the following formulas:

(1a)	parent(j,b).	(2a)	parent(j,c).
(3a)	male(j).	(4a)	male(b).
(5a)	female(c).	(6a)	brother$(x,y) \leftarrow$ parent$(z,x) \wedge$ parent$(z,y) \wedge$ male(x).

Note that (1) to (6) cannot be adequately formulated in a propositional language. In the programming language Prolog, to be treated in Section 9.1, these formulas are rendered as follows:

(1b)	parent(j,b).	(2b)	parent(j,c).
(3b)	male(j).	(4b)	male(b).
(5b)	female(c).	(6b)	brother(X,Y) :- parent(Z,X), parent(Z,Y), male(X).

(1b) to (6b) constitute what is called a *logic program*; (1b) to (5b) are called a *fact* and (6b) is called a *rule* in the logic program.

In (6a) and (6b) all variables are understood to be quantified universally. So (6a) is short for

$$\forall x \forall y \forall z\ [\text{parent}(z,x) \wedge \text{parent}\ (z,y) \wedge \text{male}(x) \rightarrow \text{brother}(x,y)]$$

which is equivalent to

$$\forall x \forall y\ [\ \exists z\ [\ \text{parent}(z,x) \wedge \text{parent}(z,y)\ ] \wedge \text{male}(x) \rightarrow \text{brother}(x,y)].$$

(1b), ..., (6b), taken together, can be considered to form a *knowledge base* from which new knowledge can be obtained by logical reasoning. The programming language Prolog has a built-in inference mechanism. When provided with the database consisting of (1b), ..., (6b), Prolog will give the following answers to the following questions, respectively:

?-brother(b,c). Answer: yes (corresponding with the fact that 'brother(b,c)' is a valid consequence of the given database).

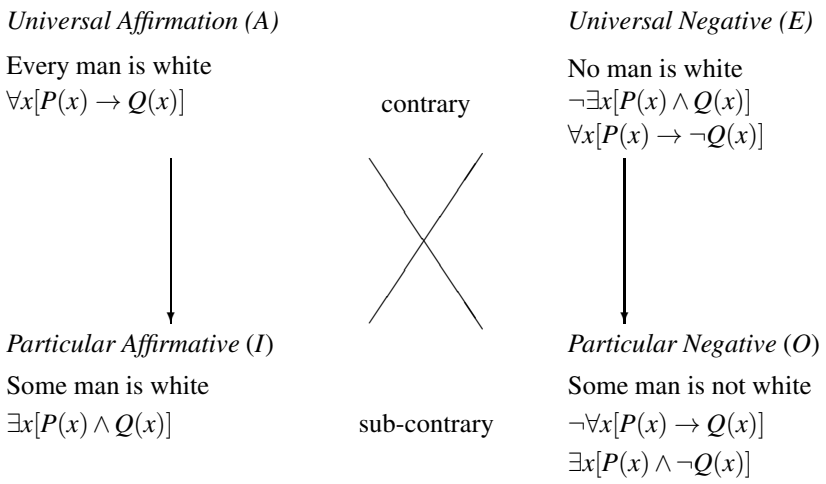
?-brother(c,b). Answer: no (corresponding with the fact that 'brother(c,b)' is not a valid consequence of the given database).

?-brother(X,c). (For which X, brother(X,c) ?) Answer: X = bob.

4.7.4 Aristotle's Organon

While Stoic Logic was primarily concerned with propositions, Aristotle's logic (see [10, 11, 12] was mainly concerned with predicate logic, at least with a (small) part of it. After Aristotle's death in 322 B.C. his students grouped together a number of his treatises on reasoning. This collection was called the *Organon*, or instrument of science. Its two best known contributions to logic are described below.

The doctrine of the square of opposition This doctrine occurs in one of the earlier works of the Organon, the *Peri Hermeneias* (On Exposition), also known under its Latin name, *De Interpretatione*. Because there is a *practical* interest in the winning of arguments, it is important to know what statements are opposed to each other and in what ways. However, the only statements considered are of the form 'P is Q' and 'P is not Q' with a universal or existential quantification. The doctrine can be summarized in the following figure, called the *square of opposition*. Neither the square of opposition itself nor the vowels A,E,I and O, by which the four types have been distinguished since the Middle Ages, occur in Aristotle's work.



Two statements are *contradictory* when they cannot both be true and cannot both be false. Two statements are *contrary* when they cannot both be true, but may both be false. Note that Aristotle here assumes implicitly that $\exists x[P(x)]$ is true.

Later logicians have said the two particular statements are *subaltern* to the universal statements under which they occur in the figure, and *sub-contrary* to each other. Again assuming that $\exists x[P(x)]$ is true, subcontraries cannot both be false, al-

though they may both be true. Aristotle also assumes that each universal statement entails its subaltern, which again means that Aristotle is assuming implicitly the truth of $\exists x[P(x)]$.

Syllogisms In the *Prior Analytics*, one of the later works of the Organon, there is a *theoretical* interest in valid reasoning. However, Aristotle was only concerned with arguments of a particular form, called syllogisms. A *syllogism* is an argument consisting of two premisses and one conclusion, where the two premisses relate the terms of the conclusion to a third term, called the middle. For instance,

$$\models \underset{(A)}{\forall x[P(x) \to Q(x)]} \wedge \underset{(A)}{\forall x[Q(x) \to R(x)]} \to \underset{(A)}{\forall x[P(x) \to R(x)]}$$

corresponds to Aristotle's syllogism *b A r b A r A*. In this example, Q is the middle term, since it relates the terms P and R of the conclusion. Below is another example:

$$\models \underset{(E)}{\forall x[P(x) \to \neg Q(x)]} \wedge \underset{(I)}{\exists x[R(x) \wedge P(x)]} \to \underset{(O)}{\exists x[R(x) \wedge \neg Q(x)]}$$

corresponds to Aristotle's syllogism *f E r I O*. In this example, P is the middle term.

4.8 Solutions

Solution 4.1. a) (1) $\forall x[G(x) \to P(x)]$; (2) $\exists x[G(x) \wedge P(x)]$.
b) $\forall x[G(x) \wedge P(x)]$ says among other things that $\forall x[G(x)]$ (every individual is a girl), which is not implied by 'every girl is pretty'. 'Some girl is pretty', rendered by (2), implies that there is at least one girl ($\exists x[G(x)]$), who in addition is pretty. However, this is not implied by $\exists x[G(x) \to P(x)]$, which says that there is some individual x such that *if* x is a girl, *then* x is pretty.

Solution 4.2. 1. $\neg M(c,d)$; 2. $\forall x \forall y[M(x,y) \to M(y,x)]$; 3. $\exists x[M(x,d)]$;
4. $\exists x \forall y[\neg M(x,y)]$.

Solution 4.3. (1) $\forall x \exists y[A(x,y)]$; (2) $\exists y \forall x[A(x,y)]$.

Solution 4.4.
(1) $\forall x \forall y[L(x,y)]$: for all objects x and y, x is in the relation L to y.
(2) $\exists x \exists y[L(x,y)]$: there are objects x and y such that x is in the relation L to y.
(3) $\forall x \exists y[L(x,y)]$: for every object x there is at least one object y (possibly depending on x) such that x is in the relation L to y.
(4) $\exists y \forall x[L(x,y)]$: there is an object y such that for all x, x is in the relation L to y.
(5) $\forall y \exists x[L(x,y)]$: for every object y there is an object x (possibly depending on y) such that x is in the relation L to y. Interchanging x and y, this formula is equivalent to $\forall x \exists y[L(y,x)]$.
(6) $\exists x \forall y[L(x,y)]$: there is an object x such that for all objects y, x is in relation L to y. Interchanging x and y, this formula is equivalent to $\exists y \forall x[L(y,x)]$.

Solution 4.5. 1. $\forall x[C(x) \to L(d,x)]$ 2. $\exists y[D(y) \wedge \forall x[C(x) \to L(y,x)]]$
3. $\exists x[C(x) \wedge L(d,x)]$ 4. $\forall y[D(y) \to \exists x[C(x) \wedge L(y,x)]]$
5. $\exists x[C(x) \wedge \forall y[D(y) \to L(y,x)]]$
6. $\neg\exists x[C(x) \wedge L(c,x)]$, or, equivalently, $\forall x[C(x) \to \neg L(c,x)]$
7. $\forall y[D(y) \to \neg\exists x[C(x) \wedge L(y,x)]]$ or $\forall y\forall x[D(y) \wedge C(x) \to \neg L(y,x)]$
8. $\forall y[D(y) \to \exists x[C(x) \wedge L(y,x)] \wedge \exists x[W(x) \wedge L(y,x)]]$
9. $\forall y[D(y) \wedge \exists x[C(x) \wedge L(y,x)] \to \exists x[W(x) \wedge L(y,x)]]$
10. $\forall y[D(y) \to \exists x[C(x) \wedge L(y,x)]] \to \forall y[D(y) \to \exists x[W(x) \wedge L(y,x)]]$

Solution 4.6. 1. i) $c_1 \equiv c_2$; ii) $\forall x[x \equiv c_1 \to x \equiv c_2]$.
2. i) '3 = 4' is false; ii) 'all numbers equal to 3 are equal to 4' is false.
3. i) 'Reagan was older than Nixon' is true; ii) 'all persons older than Reagan are older than Nixon' is true.

Solution 4.7.
1. $\exists x[P(x)]$, or equivalently, $\exists x\forall y[y \equiv x \to P(y)]$
2. $\exists x\forall y[P(y) \to y \equiv x]$
3. $\exists x\forall y[P(y) \rightleftarrows y \equiv x]$
4. $\exists x\exists y[\neg(x \equiv y) \wedge P(x) \wedge P(y)]$, or equivalently, $\exists x\exists y[x \not\equiv y \wedge \forall z[z \equiv x \vee z \equiv y \to P(z)]]$, or equivalently, $\exists x\exists y\forall z[x \not\equiv y \wedge (z \equiv x \vee z \equiv y \to P(z))]$.
5. $\exists x\exists y\forall z[x \not\equiv y \wedge (P(z) \to z \equiv x \vee z \equiv y)]$
6. $\exists x\exists y\forall z[x \not\equiv y \wedge (P(z) \rightleftarrows z \equiv x \vee z \equiv y)]$

Solution 4.8. a) $\forall x[C(x) \to A(x)]$; b) $\exists x[M(x) \wedge W(x)]$

Solution 4.9. a) $\forall x[M(x)]$.
b) $\neg\exists x[O(x)]$, or, equivalently, $\forall x[\neg O(x)]$.
c) $\exists x[B(x)] \to P$, or, equivalently, $\forall x[B(x) \to P]$.
d) $S(j) \to \forall x[S(x)]$ or, equivalently, $\forall x[S(j) \to S(x)]$.

Solution 4.10. 'For any natural number n, if $n \neq n$, then $n \neq n$' is a true sentence of the form $\forall x[P(x) \to Q(x)]$, but 'there is a natural number n such that $n \neq n$ and $n \neq n$' is a false sentence of the form $\exists x[P(x) \wedge Q(x)]$. If we assume that a domain is by definition non-empty, i.e., contains at least one element, then it follows from $\forall x[A(x)]$ that $\exists x[A(x)]$.

Solution 4.11.
a) 'If all natural numbers are even, then all natural numbers are odd' is a true ($0 \to 0 = 1$) sentence of the form $\forall x[P(x)] \to \forall x[Q(x)]$, but 'for each natural number n, if n is even, then n is odd' is a false sentence of the form $\forall x[P(x) \to Q(x)]$.
b) 'There is a natural number n such that *if* n is even, *then* $n \neq n$' is a true sentence of the form $\exists x[P(x) \to Q(x)]$, since, for instance, 'if 3 is even, then $3 \neq 3$' is true ($0 \to 0 = 1$). But 'if there is a natural number n such that n is even, then there is also a natural number n such that $n \neq n$' is a false ($1 \to 0 = 0$) sentence of the form $\exists x[P(x)] \to \exists x[Q(x)]$.

Solution 4.12. For $M = \langle \mathbb{N}, P^*, Q^*, R^* \rangle$ with P^*: is even, Q^*: is odd, R^*: is less than ($<$) we have:

$M \models P(a)[2]$ (2 is even) $M \models \exists x[P(x)]$ (there is an even natural number)
$M \not\models P(a)[5]$ (not: 5 is even) $M \not\models \forall x[P(x)]$ (not: all natural numbers are even)
$M \models Q(a)[5]$ (5 is odd) $M \models \exists x[Q(x)]$ (there is an odd natural number)
$M \not\models Q(a)[2]$ (not: 2 is odd) $M \not\models \forall x[Q(x)]$ (not: all natural numbers are odd)

$M \models \exists x[P(x)] \to \exists x[Q(x)]$ (true: $1 \to 1 = 1$)
$M \not\models \exists x[P(x)] \to \forall x[Q(x)]$ (false: $1 \to 0 = 0$)
$M \models \forall x[P(x)] \to \exists x[Q(x)]$ (true: $0 \to 1 = 1$)
$M \models \forall x[P(x)] \to \forall x[Q(x)]$ (true: $0 \to 0 = 1$)
$M \models \forall x \exists y[R(x,y)]$ (true: for each natural number x there is a natural number y such that $x < y$). But $M \not\models \exists y \forall x[R(x,y)]$ (false: there is a natural number y such that for each natural number x, $x < y$).
$M \models \exists x[P(x) \to Q(x)]$ (true: there is a natural number x such that *if* x is even, *then* x is odd; since 'if 3 is even, then 3 is odd' has truth value 1).
$M \not\models \forall x[P(x) \to Q(x)]$ (false: for all natural numbers x, *if* x is even, *then* x is odd; since 'if 2 is even, then 2 is odd' has truth value 0).

Solution 4.13. 1. $\exists x[P(x)] \to \forall x[P(x)]$ is satisfiable: $M = \langle \mathbb{N};\ x = x \rangle$ is a model; but the formula is not valid: $M = \langle \mathbb{N};\ \text{is odd} \rangle$ is a countermodel.
2. $\exists x[P(x)] \to \exists x[\neg P(x)]$ is satisfiable: $M = \langle \mathbb{N};\ \text{is odd} \rangle$ is a model: but the formula is not valid: $M = \langle \mathbb{N};\ x = x \rangle$ is a countermodel.
3. $\exists x[P(x)] \wedge \forall x[\neg P(x)]$ is not satisfiable, since by Theorem 4.7 $\forall x[\neg P(x)]$ means the same as $\neg \exists x[P(x)]$.
4. $\forall x[P(x)] \wedge \neg \exists x[P(x)]$ is not satisfiable, since by Theorem 4.7 $\neg \exists x[P(x)]$ means the same as $\forall x[\neg P(x)]$.
5. $\neg \forall x[P(x)] \to \forall x[\neg P(x)]$ is satisfiable: $M = \langle \mathbb{N};\ \text{is negative} \rangle$ is a model; but the formula is not valid: $M = \langle \mathbb{N};\ \text{is odd} \rangle$ is a countermodel.
6. $\forall x[\neg P(x)] \to \neg \forall x[P(x)]$ is valid and hence satisfiable.
7. $\forall x \exists y[R(x,y)] \wedge \exists x \forall y[\neg R(x,y)]$ is not satisfiable, since $\exists x \forall y[\neg R(x,y)]$ has the same meaning as $\neg \forall x \exists y[R(x,y)]$ (see Theorem 4.15).
8. $\forall x \exists y[R(x,y)] \to \exists y \forall x[R(x,y)]$ is satisfiable: $M = \langle \mathbb{N};\ x \geq y \rangle$ is a model; but the formula is not valid: $M = \langle \mathbb{N};\ x \leq y \rangle$ is a countermodel.
9. $\exists x[P(x)] \wedge \exists x[Q(x)] \to \exists x[P(x) \wedge Q(x)]$ is satisfiable: $M = \langle \mathbb{N};\ \text{is even},\ x = x \rangle$ is a model; but the formula is not valid: $M = \langle \mathbb{N};\ \text{is even},\ \text{is odd} \rangle$ is a countermodel.
10. $\forall x[P(x) \vee Q(x)] \to \exists x[P(x)] \vee \forall x[Q(x)]$ is valid and hence satisfiable: in case that $\neg \exists x[P(x)]$, it follows from $\forall x[P(x) \vee Q(x)]$ that $\forall x[Q(x)]$.

Solution 4.14. 1. Not $\forall x[P(x) \to S(x)], \exists x[S(x) \wedge I(x)] \models \exists x[P(x) \wedge I(x)]$. Counterexample: $M = \langle \mathbb{N};\ P^*,\ S^*,\ I^* \rangle$ with $P^*(x)$: x is even, $S^*(x)$: $x = x$ and $I^*(x)$: x is odd.
2. Not $\neg \exists x[P(x) \wedge I(x)], \forall x[I(x) \to V(x)] \models \neg \exists x[P(x) \wedge V(x)]$. Counterexample: $M = \langle \mathbb{N};\ P^*,\ V^*,\ I^* \rangle$ with $P^*(x)$: x is even, $V^*(x)$: x is even, and $I^*(x)$: $x \neq x$.
3. $\forall x[F(x) \to B(x)], \neg \exists x[M(x) \wedge B(x)] \models \forall x[M(x) \to \neg F(x)]$. Proof: Suppose the two premisses are true under an interpretation M. To show: $M \models \forall x[M(x) \to \neg F(x)]$. So, suppose $M \models M(a)[d]$ for an arbitrary element d in the domain of M. Then by the second premiss, $M \models \neg B(a)[d]$ and hence, by the first premiss, $M \models \neg F(a)[d]$.
4. Not $\exists x[M(x) \wedge \neg S(x)], \forall x[C(x) \to S(x)] \models \exists x[C(x) \wedge \neg M(x)]$. Counterexample:

$M = \langle \mathbb{N};\ M^*,\ S^*,\ C^* \rangle$ with $M^*(x)$: x is even, $S^*(x)$: x is odd, and $C^*(x)$: $x \neq x$.
5. $\exists x[P(x) \wedge S(x)], \neg\exists x[S(x) \wedge \neg C(x)] \models \exists x[P(x) \wedge C(x)]$. Proof: the second premiss is equivalent to $\forall x[S(x) \to C(x)]$ and the first premiss says that some plumber is smart; so, by the second premiss, this plumber will also be careful.

Solution 4.15. 1. $\neg\exists x[A(x) \wedge I(x)], \forall x[C(x) \to A(x)] \not\models \exists x[C(x) \wedge \neg I(x)]$.
Counterexample: $M = \langle \mathbb{N};\ C^*(x) : x \neq x,\ A^*(x) : x \text{ is even},\ I^*(x) : x \text{ is odd} \rangle$.
2. $\exists x[S(x)] \to \exists x[P(x) \wedge S(x)],\ P(c) \wedge \neg S(c) \not\models \neg\exists x[S(x)]$.
Counterexample: $M = \langle \mathbb{N};\ S^*(x) : x \text{ is even},\ P^*(x) : x = x;\ c^* : 3 \rangle$.
3. $\forall x[M(x) \wedge \exists y[S(y)] \to S(x)],\ M(c) \wedge \neg S(c) \models \neg\exists x[S(x)]$.
Proof: Let $M = \langle D;\ M^*,\ S^*;\ c^* \rangle$ be a model of the two premisses. Then it is in particular a model of $M(c) \wedge \exists y[S(y)] \to S(c)$ (1) and of $M(c) \wedge \neg S(c)$ (2). Now suppose M were a model of $\exists x[S(x)]$. Then by (2) it would be a model of $M(c) \wedge \exists y[S(y)]$. So, by (1), M would be a model of $S(c)$, contradicting (2). Hence, M is a model of $\neg\exists x[S(x)]$.
4. $\exists x[H(x) \wedge F(x)],\ \neg\exists x[\neg H(x) \wedge S(x)] \not\models \exists x[F(x) \wedge \neg S(x)]$.
Counterexample: $M = \langle \mathbb{N};\ H^*(x) : x = x,\ F^*(x) : x \text{ is even},\ S^*(x) : x = x \rangle$.
5. $\exists x[S_1(x) \wedge S_2(x)], \neg\exists x[S_1(x) \wedge U(x)] \not\models \exists x[S_1(x) \wedge U(x) \wedge \neg S_2(x)]$.
Counterexample: $M = \langle \mathbb{N};\ S_1^*(x) : x \text{ is even},\ S_2^*(x) : x \text{ is even},\ U^*(x) : x \neq x \rangle$.

Solution 4.16. $\forall x[P(x) \to Q(x)] \not\models \exists x[P(x) \wedge Q(x)]$. $M = \langle \mathbb{N};\ P^*,\ Q^* \rangle$, with $P^*(x) = Q^*(x) := x \neq x$, is a counterexample: $M \models \forall x[P(x) \to Q(x)]$ (for every natural number x, if $x \neq x$, then $x \neq x$), but $M \not\models \exists x[P(x) \wedge Q(x)]$ (it is not the case that there is a natural number x such that $x \neq x$).

Solution 4.17. i) (a) $\forall x\exists y[R(y,x)]$; (b) $\neg\exists x\forall y[R(x,y)]$.
ii) $\forall x\exists y[R(y,x)] \not\models \neg\exists x\forall y[R(x,y)]$. Counterexample: Let $M = \langle \mathbb{N}; \leq \rangle$. Then $M \models \forall x\exists y[R(y,x)]$ (for every natural number n there is a natural number m such that $m \leq n$). But $M \not\models \neg\exists x\forall y[R(x,y)]$, since $M \models \exists x\forall y[R(x,y)]$ (there is a natural number n, namely 0, such that for all natural numbers m, $n \leq m$.
iii) Concluding (b) from (a) we use tacitly that if $m > n$, then not $n > m$. Suppose (a) and not (b). So, there is a natural number n greater than all natural numbers. From (a) it follows that there is a natural number m such that $m > n$ (1). However, by the choice of n, $n > m$ (2). However, (1) and (2) contradict $m > n \to \neg(n > m)$.
iv) To show: $\forall x\exists y[R(y,x)], \forall x\forall y[R(y,x) \to \neg R(x,y)] \models \neg\exists x\forall y[R(x,y)]$ So, suppose $M \models \forall x\exists y[R(y,x)]$, $M \models \forall x\forall y[R(y,x) \to \neg R(x,y)]$ and $M \models \exists x\forall y[R(x,y)]$. Then for some d_1 in the domain of M, $M \models \forall y[R(a_1,y)][d_1]$. Since $M \models \forall x\exists y[R(y,x)]$ it follows that $M \models \exists y[R(y,a_1)][d_1]$ and therefore for some d_2 in the domain of M, $M \models R(a_2,a_1)[d_2,d_1]$ (1). From $M \models \forall y[R(a_1,y)][d_1]$ it follows that also $M \models R(a_1,a_2)[d_1,d_2]$ (2). But (1) and (2) contradict that $M \models \forall x\forall y[R(y,x) \to \neg R(x,y)]$.

Solution 4.18. $\models \neg\exists y\forall x[S(y,x) \rightleftarrows \neg S(x,x)]$.
Proof: Suppose that $M \models \exists y\forall x[S(y,x) \rightleftarrows \neg S(x,x)]$. Then there is some element d in the domain D of M such that $M \models \forall x[S(a,x) \rightleftarrows \neg S(x,x)]\ [d]$, where a is a new free variable. Hence, in particular, $M \models S(a,a) \rightleftarrows \neg S(a,a)\ [d]$, i.e., $M \models S(a,a)[d]$ iff $M \models \neg S(a,a)[d]$. Contradiction. So, for every interpretation M, $M \models \neg\exists y\forall x[S(y,x) \rightleftarrows \neg S(x,x)]$.

Solution 4.19. Let $M = \langle D; P^* \rangle$ be an interpretation.
a) $M \models \forall x[P(x) \to P(x)]$; hence, $M \models \forall x \exists y[P(x) \to P(y)]$.
b) $M \models \forall y[P(y) \to P(y)]$; hence, $M \models \forall y \exists x[P(x) \to P(y)]$.
c) To show: for any interpretation M, $M \models \exists x \forall y[P(x) \to P(y)]$. Case 1: $M \models \exists x[\neg P(x)]$, i.e., there is a d in the domain of M such that $M \models \neg P(a)\ [d]$. But then $M \models P(a) \to P(b)\ [d, d']$ for any valuation d' of the free variable b ($0 \to 0/1 = 1$). Therefore, $M \models \exists x \forall y[P(x) \to P(y)]$. Case 2: $M \models \neg \exists x[\neg P(x)]$, i.e., $M \models \forall x[P(x)]$, i.e., all objects in the domain of M have the property P^*. Then $M \models \forall x \forall y[P(x) \to P(y)]$ ($1 \to 1 = 1$). Hence, in particular, $M \models \exists x \forall y[P(x) \to P(y)]$.
d) To show: for any interpretation M, $M \models \exists y \forall x[P(x) \to P(y)]$. Case 1: $M \models \exists y[P(y)]$, i.e., there is some d in the domain of M such that $M \models P(b)[d]$. But then $M \models \forall x[P(x) \to P(b)]\ [d]$ ($0/1 \to 1 = 1$) and hence $M \models \exists y \forall x[P(x) \to P(y)]$. Case 2: $M \models \neg \exists y[P(y)]$, i.e., $M \models \forall y[\neg P(y)]$, that is, no element in the domain of M has the property P^*. But then $M \models \forall y \forall x[P(x) \to P(y)]$ ($0 \to 0 = 1$) and hence, in particular, $M \models \exists y \forall x[P(x) \to P(y)]$.

Solution 4.20. a) The formula $\forall x \exists y[R(x,y)] \to \exists x \forall y[R(x,y)]$ contains a transition from $\exists y$ to $\forall y$ and hence cannot be valid. Let $M = \langle \mathbb{N};\ R^* \rangle$ with $R^*(d_1, d_2) := d_2$ is even (and $d_1 = d_1$). Then $M \models \forall x \exists y[R(x,y)]$ (there is some natural number which is even), but $M \not\models \exists x \forall y[R(x,y)]$ (it is not the case that all natural numbers are even).
b) The formula $\exists x \forall y[R(x,y)] \to \forall x \exists y[R(x,y)]$ contains a transition from $\exists x$ to $\forall x$ and hence cannot be valid. $M = \langle \mathbb{N};\ R^* \rangle$, with $R^*(d_1, d_2) := d_1$ is even (and $d_2 = d_2$), is a counterexample.
c) Let $M = \langle \mathbb{N};\ < \rangle$. Then $M \models \forall x \exists y[R(x,y)]$ (for every natural number x there is a greater one y), but $M \not\models \forall x \exists y[R(y,x)]$ (it is not the case that for every natural number x there is a smaller one y; there is no natural number less than 0).
d) The formula $\exists x \forall y[R(x,y)] \to \exists y \forall x[R(x,y)]$ contains again a transition from $\exists x$ to $\forall x$ and hence cannot be valid. See the counterexample in b).
e) and f) The right and left part of $\rightleftarrows$ express the same proposition; only the variables x and y have been interchanged.

Solution 4.21. 1. Let $M = \langle \mathbb{N};\ \text{is even}, 0 = 1 \rangle$. Then $M \models \forall x[P(x)] \to Q$: the proposition 'if all natural numbers are even, then $0 = 1$' has truth value $0 \to 0 = 1$. But $M \not\models \forall x[P(x) \to Q]$: it is not the case that for every natural number n, if n is even, then $0 = 1$ (for instance, 'if 2 is even, then $0 = 1$' has truth value $1 \to 0 = 0$). So, M is a counterexample.
2. Of course, $\forall x[P(x) \to Q] \models \exists x[P(x) \to Q]$. And $\exists x[P(x) \to Q] \models\!\!\mid \forall x[P(x)] \to Q$. Hence, $\forall x[P(x) \to Q] \models \forall x[P(x)] \to Q$.
3. $\exists x[P(x)] \to Q \models \exists x[P(x) \to Q]$ follows from $\exists x[P(x)] \to Q \models\!\!\mid \forall x[P(x) \to Q]$.
4. Let $M = \langle \mathbb{N};\ \text{is even}, 0 = 1 \rangle$. Then $M \models \exists x[P(x) \to Q]$: the proposition 'there is a natural number n such that if n is even, then $0 = 1$' has truth value 1 (for instance, 'if 3 is even, then $0 = 1$' has truth value $0 \to 0 = 1$). But $M \not\models \exists x[P(x)] \to Q$: the proposition 'if there is an even natural number, then $0 = 1$' has truth value $1 \to 0 = 0$.

Solution 4.22. 1. Suppose $M \models \forall x[P(x)] \to \exists x[Q(x)]$ and $M \not\models \exists x[P(x) \to Q(x)]$. Then $M \models \neg \exists x[P(x) \to Q(x)]$, i.e., $M \models \forall x[\neg(P(x) \to Q(x))]$. So, $M \models \forall x[P(x) \wedge$

$\neg Q(x)]$; therefore, $M \models \forall x[P(x)]$ and $M \models \forall x[\neg Q(x)]$. Contradiction with $M \models \forall x[P(x)] \to \exists x[Q(x)]$. So, $\forall x[P(x)] \to \exists x[Q(x)]) \models \exists x[P(x) \to Q(x)]$.

Conversely, suppose $M \models \exists x[P(x) \to Q(x)]$ and $M \models \forall x[P(x)]$. Then for some element d in the domain of M, $M \models P(a) \to Q(a)\ [d]$, where a does not occur in $\exists x[P(x) \to Q(x)]$. Since $M \models \forall x[P(x)]$, $M \models P(a)[d]$. So, $M \models Q(a)[d]$. Therefore, $M \models \exists x[Q(x)]$. This shows that also $\exists x[P(x) \to Q(x)] \models \forall x[P(x)] \to \exists x[Q(x)]$.
2. $\exists x[P(x)] \to \forall x[Q(x)]) \models \forall x[P(x) \to Q(x)]$, but $M = \langle \mathbb{N};$ is even, is even$\rangle$ is a counterexample to the converse formula, $\forall x[P(x) \to Q(x)] \to (\exists x[P(x)] \to \forall x[Q(x)])$.

Solution 4.23. 1. $\forall x[P(x) \to Q(x)] \models \exists x[P(x)] \to \exists x[Q(x)]$.
For suppose $M \models \forall x[P(x) \to Q(x)]$ (i) and $M \models \exists x[P(x)]$. Then for some element d in the domain of M, $M \models P(a)[d]$, where a is a new variable. From (i) it follows that $M \models Q(a)[d]$. Hence, $M \models \exists x[Q(x)]$.

However, conversely, $M = \langle \mathbb{N};$ is even, is odd$\rangle$ makes $\exists x[P(x)] \to \exists x[Q(x)]$ true: the proposition 'if there is an even natural number, then there is an odd natural number' has truth value $1 \to 1 = 1$. But $M \not\models \forall x[P(x) \to Q(x)]$: the proposition 'every even natural number is odd' has truth value 0.
2. $\exists x[P(x) \to Q(x)] \not\models \forall x[P(x)] \to \forall x[Q(x)]$. For $M = \langle \mathbb{N};\ x = x,$ is even$\rangle$ is a counterexample. $M \models \exists x[P(x) \to Q(x)]$: the proposition 'there is some natural number n such that if $n = n$, then n is even' has truth value 1; for instance, 'if $2 = 2$, then 2 is even' has truth value $1 \to 1 = 1$. Also $M \models \forall x[P(x)]$. But $M \not\models \forall x[Q(x)]$.

However, conversely, we do have $\forall x[P(x)] \to \forall x[Q(x)] \models \exists x[P(x) \to Q(x)]$. For suppose $M \models \forall x[P(x)] \to \forall x[Q(x)]$ (i) and $M \not\models \exists x[P(x) \to Q(x)]$. Then $M \models \neg\exists x[P(x) \to Q(x)]$, i.e., $M \models \forall x[\neg(P(x) \to Q(x))]$. So, $M \models \forall x[P(x) \wedge \neg Q(x)]$. Therefore, $M \models \forall x[P(x)]$ and $M \models \forall x[\neg Q(x)]$. Contradiction with (i).

Solution 4.24. $\forall x\exists y[P(x) \to Q(y)] \models \exists y\forall x[P(x) \to Q(y)]$. For suppose $M \models \forall x\exists y[P(x) \to Q(y)]$ (i) and $M \models \neg\exists y\forall x[P(x) \to Q(y)]$. Then $M \models \forall y\exists x[P(x) \wedge \neg Q(y)]$. So, $M \models \exists x[P(x)]$ and $M \models \forall y[\neg Q(y)]$. Hence, for some element d in the domain of M, $M \models P(a)[d]$, where a is a new free variable. From (i) it follows that $M \models \exists y[P(a) \to Q(y)]\ [d]$. So, $M \models \exists y[Q(y)]$. Contradiction with $M \models \forall y[\neg Q(y)]$.

Solution 4.25. Let W (Wang) be the formula in question and let M be a model.
Case 1: $M \models \exists x\exists y[\neg P(x,y)]$. Then $M \models W$.
Case 2: $M \not\models \exists x\exists y[\neg P(x,y)]$. Then $M \models \forall x\forall y[P(x,y)]$.
Subcase 2a): $M \models \exists x\exists y[\neg Q(x,y)]$. Then $M \models W$.
Subcase 2b): $M \not\models \exists x\exists y[\neg Q(x,y)]$. Then $M \models \forall x\forall y[Q(x,y)]$. Consequently, $M \models W$.

Solution 4.26. (a)(1) (i) To show: $\forall x[B(x)] \to C \models \exists x[B(x) \to C]$. Suppose $M \models \forall x[B(x)] \to C\ [v]$ and $M \not\models \exists x[B(x) \to C]\ [v]$. Then $M \models \neg\exists x[B(x) \to C]\ [v]$, i.e., $M \models \forall x[\neg(B(x) \to C)]\ [v]$. So, $M \models \forall x[B(x) \wedge \neg C]\ [v]$; therefore, $M \models \forall x[B(x)]\ [v]$ and $M \models \neg C\ [v]$. Contradiction with $M \models \forall x[B(x)] \to C\ [v]$.
Next we show: $\exists x[B(x) \to C] \models \forall x[B(x)] \to C$. So, suppose $M \models \exists x[B(x) \to C]\ [v]$. Then for some element d in the domain D of M, $M \models B(a) \to C\ [d/v]$, where a is new. Now suppose that $M \models \forall x[B(x)]\ [v]$. Then $M \models B(a)\ [d/v]$. So, $M \models C\ [v]$.
(a)(1) (ii) To show: $\exists x[B(x)] \to C \models \forall x[B(x) \to C]$. Suppose $M \models \exists x[B(x)] \to C\ [v]$

and $M \not\models \forall x[B(x) \to C]\ [v]$. Then $M \models \neg\forall x[B(x) \to C]\ [v]$, i.e., $M \models \exists x[\neg(B(x) \to C)]\ [v]$. So, $M \models \exists x[B(x) \wedge \neg C]\ [v]$; therefore, $M \models \exists x[B(x)]\ [v]$ and $M \models \neg C\ [v]$. Contradiction with $M \models \exists x[B(x)] \to C\ [v]$.
Next we show: $\forall x[B(x) \to C] \models \exists x[B(x)] \to C$. So, suppose $M \models \forall x[B(x) \to C]$ [v] and $M \models \exists x[B(x)]\ [v]$. Then for some element d in the domain D of M, $M \models B(a)\ [d/v]$, where a is new. Since $M \models \forall x[B(x) \to C]\ [v]$, it follows that $M \models B(a) \to C\ [d/v]$. So, $M \models C\ [v]$.
The validity of the other formulas is shown similarly.

Solution 4.27.

$$\begin{aligned}
\models\ \forall x[P(x)] \to \exists x[Q(x)] &\rightleftarrows \forall x[P(x)] \to \exists y[Q(y)] \\
&\rightleftarrows \exists x[P(x) \to \exists y[Q(y)]] \\
&\rightleftarrows \exists x \exists y[P(x) \to Q(y)]. \\
\models\ \exists x[P(x)] \to \forall x[Q(x)] &\rightleftarrows \exists x[P(x)] \to \forall y[Q(y)] \\
&\rightleftarrows \forall x[P(x) \to \forall y[Q(y)]] \\
&\rightleftarrows \forall x \forall y[P(x) \to Q(y)]. \\
\models\ \exists x[P(x,a)] \to \exists x[Q(x) \vee \neg\exists y[R(y)]] &\rightleftarrows \exists x[P(x,a)] \to \exists z[Q(z) \vee \forall y[\neg R(y)]] \\
&\rightleftarrows \exists x[P(x,a)] \to \exists z \forall y[Q(z) \vee \neg R(y)] \\
&\rightleftarrows \forall x[P(x,a) \to \exists z \forall y[Q(z) \vee \neg R(y)]] \\
&\rightleftarrows \forall x \exists z[P(x,a) \to \forall y[Q(z) \vee \neg R(y)]] \\
&\rightleftarrows \forall x \exists z \forall y[P(x,a) \to Q(z) \vee \neg R(y)].
\end{aligned}$$

Solution 4.28.

$$\begin{aligned}
\models\ \exists x[P(x)] \to \exists x[Q(x)] &\rightleftarrows \exists x[P(x)] \to \exists y[Q(y)] \\
\text{Theorem 4.18 (a)(1)} &\rightleftarrows \forall x[P(x) \to \exists y[Q(y)]] \\
\text{Theorem 4.18 (a)(2)} &\rightleftarrows \forall x \exists y[P(x) \to Q(y)]. \\
\models\ \exists x[P(x)] \to \exists x[Q(x)] &\rightleftarrows \exists x[P(x)] \to \exists y[Q(y)] \\
\text{Theorem 4.18 (a)(2)} &\rightleftarrows \exists y[\exists x[P(x)] \to Q(y)] \\
\text{Theorem 4.18 (a)(1)} &\rightleftarrows \exists y \forall x[P(x) \to Q(y)].
\end{aligned}$$

Solution 4.29. No, since the $\forall$-rule is applied with respect to the free variable a occurring in the premiss $A(a)$.

Solution 4.30. No, since the $\exists$-rule is applied to a formula of the form $A(a) \to C$ where C does contain a.

Solution 4.31. Yes.

Solution 4.32.

i)

$$\dfrac{\underset{\exists x[A(x)]}{\text{premiss}} \qquad \dfrac{\overset{\exists\text{-schema}}{A(a) \to \exists z[A(z)]}}{\exists x[A(x)] \to \exists z[A(z)]}\ \exists}{\exists z[A(z)]}\ MP$$

ii)

$$\dfrac{\underset{\forall x[A(x)]}{\text{premiss}} \qquad \dfrac{\overset{\forall\text{-schema}}{\forall x[A(x)] \to A(a)}}{\forall x[A(x)] \to \forall z[A(z)]}\ \forall}{\forall z[A(z)]}\ MP$$

Solution 4.33.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{premiss}}{\forall x[A \to B(x)]} \qquad [A]^1}{A \to B(a)}}{B(a)}}{\forall x[B(x)]}}{A \to \forall x[B(x)]}\ (1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{premiss}}{A \to \forall x[B(x)]} \qquad [A]^1}{\forall x[B(x)]}}{B(a)}}{A \to B(a)}}{\forall x[A \to B(x)]}\ (1)$$

Solution 4.34. 1. A tableau-proof of $(\exists x[P(x)] \to \exists x[Q(x)]) \to \exists x[P(x) \to Q(x)]$:

$$\begin{array}{c}
F\ (\exists x[P(x)] \to \exists x[Q(x)]) \to \exists x[P(x) \to Q(x)] \\
T\ \exists x[P(x)] \to \exists x[Q(x)],\ F\ \exists x[P(x) \to Q(x)] \\
\begin{array}{ll}
F\ \exists x[P(x)],\ F\ \exists x[P(x) \to Q(x)] & \mid\ T\ \exists x[Q(x)],\ F\ \exists x[P(x) \to Q(x)] \\
FP(a),\ F\ \exists x[P(x) \to Q(x)] & \mid\ TQ(a),\ F\ \exists x[P(x) \to Q(x)] \\
FP(a),\ F\ P(a) \to Q(a) & \mid\ TQ(a),\ F\ P(a) \to Q(a) \\
FP(a), TP(a), FQ(a) & \mid\ TQ(a), TP(a), FQ(a)
\end{array}
\end{array}$$

2. The schema below shows an open, not completed, branch in a tableau with initial branch $\{F\ \exists x[P(x) \to Q(x)] \to (\exists x[P(x)] \to \exists x[Q(x)])\}$:

$$\begin{array}{c}
F\ \exists x[P(x) \to Q(x)] \to (\exists x[P(x)] \to \exists x[Q(x)]) \\
T\ \exists x[P(x) \to Q(x)],\ F\ \exists x[P(x)] \to \exists x[Q(x)] \\
T\ \exists x[P(x) \to Q(x)],\ T\ \exists x[P(x)],\ F\ \exists x[Q(x)] \\
T\exists x[\ \to\ ],\ T\exists x,\ F\exists x,\ T\ P(a_1) \to Q(a_1) \\
T\exists x[\ \to\ ],\ T\exists x,\ F\exists x,\ FP(a_1) \\
T\exists x[\ \to\ ],\ T\exists x,\ F\exists x,\ FP(a_1),\ TP(a_2) \\
T\exists x[\ \to\ ],\ T\exists x,\ F\exists x,\ FP(a_1),\ TP(a_2),\ FQ(a_1) \\
T\exists x[\ \to\ ],\ T\exists x,\ F\exists x,\ FP(a_1),\ TP(a_2),\ FQ(a_1),\ FQ(a_2) \\
T\ P(a_3) \to Q(a_3),\ T\exists x,\ F\exists x,\ FP(a_1),\ TP(a_2),\ FQ(a_1),\ FQ(a_2) \\
FP(a_3),\ T\exists x,\ F\exists x,\ FP(a_1),\ TP(a_2),\ FQ(a_1),\ FQ(a_2) \\
FP(a_3),\ TP(a_4),\ F\exists x,\ FP(a_1),\ TP(a_2),\ FQ(a_1),\ FQ(a_2) \\
\text{and so on,}
\end{array}$$

where we have used some obvious abbreviations.

From this open branch we can read off a counterexample to the formula in question having as domain the set $\mathbb{N}$ of all natural numbers: the even natural numbers have the property P^*, corresponding to the occurrences of $TP(a_2)$, $TP(a_4)$, ..., the odd natural numbers do not have the property P^*, corresponding to the occurrences of $FP(a_1)$, $FP(a_3), \ldots$ and all natural numbers have the property not-Q, corresponding to the occurrences of $FQ(a_1)$, $FQ(a_2)$, $FQ(a_3), \ldots$; take for P^* the predicate 'is even' and let $Q(a)$ be interpreted as $a \neq a$. Under this interpretation there results a true proposition from $\exists x[P(x) \to Q(x)]$, but a false proposition from $\exists x[P(x)] \to \exists x[Q(x)]$.

P.S. After application of rule $T\exists$ to $T\ \exists x[P(x) \to Q(x)]$ and to $T\ \exists x[P(x)]$ respectively, one may delete the occurrences of these signed formulas. If one does so, one finds a counterexample with a finite domain.

3. The schema below is a tableau-proof of $\exists x \forall y[P(x) \to P(y)]$:

$$F\ \exists x\forall y[P(x) \to P(y)]$$
$$F\ \exists x\forall y[P(x) \to P(y)],\ F\ \forall y[P(a_1) \to P(y)]$$
$$F\ \exists x\forall y[P(x) \to P(y)],\ F\ \forall y[P(a_1) \to P(y)],\ F\ P(a_1) \to P(a_2)$$
$$F\ \exists x\forall y,\ F\ \forall y[P(a_2) \to P(y)],\ F\ \forall y[P(a_1) \to P(y)],\ F\ P(a_1) \to P(a_2)$$
$$F\ \exists x\forall y,\ F\ P(a_2) \to P(a_3),\ F\ \forall y[P(a_1) \to P(y)],\ F\ P(a_1) \to P(a_2)$$
$$F\ \exists x\forall y,\ TP(a_2),\ FP(a_3),\ F\ \forall y[P(a_1) \to P(y)],\ TP(a_1),\ FP(a_2)$$

where we have used some obvious abbreviations.

Solution 4.35. a) trivial. b) $\{P(a_1),\ \neg P(a_2),\ \neg P(a_3),\ Q(a_2),\ \neg Q(a_3)\}$ is simultaneously satisfiable in D iff D contains at least three elements.
c) Suppose $M = \langle D;\ P^*\rangle$ is a model of the three formulas in question. Let d_1 be an element in D. Since $M \models \forall x\exists y[P(x,y)]$, there must be some element d_2 in D such that $M \models P(a_1,a_2)[d_1,d_2]$. Since $M \models \forall x[\neg P(x,x)]$, it follows that $d_2 \neq d_1$. From $M \models \forall x\exists y[P(x,y)]$ we conclude that $M \models P(a_2,a_3)[d_2,d_3]$ for some d_3 in D. Again $d_3 \neq d_2$, since $M \models \forall x[\neg P(x,x)]$. But also $d_3 \neq d_1$, since from $M \models \forall x\forall y\forall z[P(x,y) \wedge P(y,z) \to P(x,z)]$ it follows that $M \models P(a_1,a_3)[d_1,d_3]$ and $M \models \forall x[\neg P(x,x)]$. From $M \models \forall x\exists y[P(x,y)]$ it follows that there must be some element d_4 in D such that $M \models P(a_3,a_4)[d_3,d_4]$ and we can again show that $d_4 \neq d_3$, $d_4 \neq d_2$ and $d_4 \neq d_1$. By induction one shows that D contains at least denumerably many elements. Conversely, let $d_0,d_1,d_2,\ldots$ be denumerably many elements in D. Define $P^*(d_i,d_j)$ iff $i < j$. Then $\langle D;\ P^*\rangle$ is a model of the formulas in question.
d) By the Lőwenheim-Skolem Theorem.

Solution 4.36. Let Γ be consistent and $A \in \Gamma$. Then $\Gamma \vdash A$; hence, $\Gamma \nvdash \neg A$. So, there is at least one formula C such that $\Gamma \nvdash C$. Conversely, suppose $\Gamma \nvdash C$ and for some formula B both $\Gamma \vdash B$ and $\Gamma \vdash \neg B$; then $\Gamma \vdash B \wedge \neg B$ and hence $\Gamma \vdash C$ for any formula C; contradiction.
Let $\Gamma = A_1,\ldots,A_n$ and suppose Γ is consistent. Then for some formula C, $\Gamma \nvdash C$, i.e., $A_1,\ldots A_n \nvdash C$. So, by the completeness theorem, $A_1,\ldots A_n \not\models C$, i.e., there is a model M such that $M \models A_1 \wedge \ldots \wedge A_n\ [n_1,\ldots,n_k]$ and $M \not\models C\ [n_1,\ldots,n_k]$, if $a_{n_1},\ldots,a_{n_k}$ are the free variables in $A_1,\ldots,A_n,C$. So, Γ is satisfiable and hence, by Löwenheim's Theorem 4.27, Γ has an enumerable model. Conversely, if Γ has an enumerable model, then for no formula B, $\Gamma \vdash B$ and $\Gamma \vdash \neg B$.

Solution 4.37. Suppose $A_1,\ldots,A_n \models B$ and $A_1,\ldots,A_n \nvdash B$. Then $A_1,\ldots,A_n,\neg B \nvdash B$ and therefore $\Gamma = A_1,\ldots,A_n,\neg B$ is consistent. Using Skolem's result, formulated in Exercise 4.36, $A_1,\ldots,A_n,\neg B$ has an enumerable model, contradicting $A_1,\ldots,A_n \models B$. Therefore, $A_1,\ldots,A_n \vdash B$.

Solution 4.38. Let M be an interpretation for second-order logic with domain D. $M \models En$ iff there are $d \in D$ and $f : D \to D$ such that for all $V \subseteq D$, if 1. $d \in V$, and 2. for all $x \in D$, if $x \in V$, then $f(x) \in V$, then for all $x \in D$, $x \in V$.
Suppose $M \models En$. Take $V' := \{d, f(d), f(f(d)),\ldots\}$. Then V' satisfies 1. and 2. Therefore $D \subseteq V'$. Hence, D is enumerable. Conversely, suppose D is finite or $D = \{d_0,d_1,d_2,\ldots\}$. Then En is true in any interpretation with domain D.

Solution 4.39. Let B be of the form $\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m [A(x_1,\ldots,x_n,\ y_1,\ldots,y_m)]$ with A quantifier-free. For reasons of simplicity we suppose that $n = 2$ and $m = 1$. Then developing a completed tableau with initial branch $\{F\ \forall x_1 \forall x_2 \exists y[A(x_1,x_2,y)]\}$ goes as follows:

$$\begin{array}{rl}
F\ \forall x_1 \forall x_2 \exists y[A(x_1,x_2,y)] & \\
F\ \forall x_2 \exists y[A(a_1,x_2,y)] & (a_1 \text{ new}) \\
F\ \exists y[A(a_1,a_2,y)] & (a_2 \text{ new}) \\
FA(a_1,a_2,a_1),\ F\ \exists y[A(a_1,a_2,y)] & \\
FA(a_1,a_2,a_1),\ FA(a_1,a_2,a_2),\ \ F\ \exists y[A(a_1,a_2,y)] & \\
FA(a_1,a_2,a_1),\ FA(a_1,a_2,a_2),\ \ FA(a_1,a_2,a_3),\ F\ \exists y[A(a_1,a_2,y)] & \\
\vdots & \text{(propositional rules)}
\end{array}$$

Further applications of rule $F\exists$ do not make sense. If $FA(a_1,a_2,a_1),\ FA(a_1,a_2,a_2)$, $FA(a_1,a_2,a_3)$ with a_3 new does not provide a closed tableau (which is decidable), then there is no deduction of B.

Solution 4.40. In order to show that every monadic formula is equivalent to a truth-functional composition of formulas of the form $\forall x[B(x)]$ and $\exists x[B(x)]$, where B is quantifier-free, we proceed as follows. Let $A = Q_1 x_1 \ldots Q_n x_n [M]$ be a monadic formula in prenex normal form.
For instance, let $A_0 := \exists y \forall x[(\neg P(x) \vee Q(y)) \wedge (P(y) \vee \neg Q(x))]$.
STEP 1: a) If $Q_n = \forall$, replace M by its conjunctive normal form; if $Q_n = \exists$, replace M by its disjunctive normal form. (See Theorem 4.7.)
b) Replace $\forall x_n[C \wedge D]$ by $\forall x_n[C] \wedge \forall x_n[D]$; and replace $\exists x_n[C \vee D]$ by $\exists x_n[C] \vee \exists x_n[D]$ respectively. Applying step 1 b) to A_0 yields

$$A_0' := \exists y[\ \forall x[\neg P(x) \vee Q(y)] \wedge \forall x[P(y) \vee \neg Q(x)]\ ].$$

c) In the result of step 1b) replace expressions of the form $\forall x[E \vee F]$ by $\forall x[E] \vee F$, if x does not occur in F; and replace expressions of the form $\exists x[E \wedge F]$ by $\exists x[E] \wedge F$, if x does not occur in F. Applying step 1 c) to A_0' yields

$$A_1 := \exists y[\ (\forall x[\neg P(x)] \vee Q(y)) \wedge (P(y) \vee \forall x[\neg Q(x)])\ ].$$

d) Remove vacuous occurrences of quantifiers.
STEP $k+1$ ($k < n$): similar to step 1 with $n-k$ instead of n. Below we present the results of the different substeps of step 2 in the case of our example.
2a): $\exists y[\ (\forall x[\neg P(x)] \wedge P(y))\ \vee\ (Q(y) \wedge P(y)) \vee$
$(\forall x[\neg P(x)] \wedge \forall x[\neg Q(x)]) \vee (Q(y) \wedge \forall x[\neg Q(x)])]$
2b): $\exists y[\ \forall x[\neg P(x)] \wedge P(y)]\ \vee\ \exists y[Q(y) \wedge P(y)] \vee$
$\exists y[\ \forall x[\neg P(x)] \wedge \forall x[\neg Q(x)]\ ]\ \vee\ \exists y[\ Q(y) \wedge \forall x[\neg Q(x)]\ ]$
2c) and d): $(\forall x[\neg P(x)] \wedge \exists y[P(y)])\ \vee\ \exists y[Q(y) \wedge P(y)] \vee (\forall x[\ \neg P(x)] \wedge \forall x[\neg Q(x)])\ \vee$ $(\exists y[Q(y)] \wedge \forall x[\neg Q(x)])$.

Solution 4.41. Let A be a monadic formula. Let C be a truth-functional composition of formulas of the form $\forall x[B(x)]$ and $\exists x[B(x)]$, B quantifier-free, such that C is equivalent to A (see Exercise 4.40). Starting with FC and applying the propositional rules we find a sequent of the form

$$F\forall x[B_1(x)], T\exists x[B_2(x)],\ \ldots, T\forall x[B_3(x)], F\exists x[B_4(x)].$$

Next apply all $F\forall$- and $T\exists$-rules, yielding

$$FB_1(a_1),\ TB_2(a_2),\ \ldots, T\forall x[B_3(x)], F\exists x[B_4(x)]\ (a_1, a_2 \text{ new}).$$

For each $T\forall$- and $F\exists$-formula in this sequent finitely many applications of the corresponding rules suffice to find a closed tableau, if there is any. If $FB_1(a_1), TB_2(a_2)$, $\ldots$, $TB_3(a_1)$, $TB_3(a_2)$, $TB_3(a_3)$, $FB_4(a_1), FB_4(a_2)$, $FB_4(a_3)$, $FB_4(a_4)$, where a_3 and a_4 are new, does not yield a closed tableau (which is decidable), then there is no closed tableau for the original formula.

Solution 4.42. Let B be a formula of the form $\exists x\forall y[M(x,y)]$, where M is quantifier-free, and suppose that the only predicate symbol appearing in M is a binary predicate symbol P. Our systematic search for a formal deduction of B starts as follows:

$$\begin{array}{l} F\ \exists x\forall y[M(x,y)] \\ F\ \forall y[M(a_1,y)], F\ \exists x\forall y[M(x,y)] \\ F\ M(a_1,a_2), F\ \exists x\forall y[M(x,y)] \\ \qquad\qquad \vdots \text{ (propositional rules)} \end{array}$$

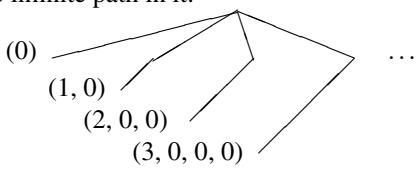
The propositional rules applied to $F\ M(a_1,a_2)$ may give rise to signed atomic formulas of the form $P(a_1,a_1)$, $P(a_1,a_2)$, $P(a_2,a_1)$ and $P(a_2,a_2)$. Several branches may result, each containing the expression $F\ \exists x\forall y[M(x,y)]$. One more application of rule $F\exists$ yields at each branch:

$$\begin{array}{l} F\ \forall y[M(a_2,y)], F\ \exists x\forall y[M(x,y)] \\ F\ M(a_2,a_3),\ F\ \exists x\forall y[M(x,y)] \\ \qquad\qquad \vdots \text{ (propositional rules)} \end{array}$$

The propositional rules applied to $F\ M(a_2,a_3)$ may give rise to signed atomic formulas of the form $P(a_2,a_2)$, $P(a_2,a_3)$, $P(a_3,a_2)$ and $P(a_3,a_3)$. So, the only way closure can result from interaction of $F\ M(a_1,a_2)$ and $F\ M(a_2,a_3)$ is via $P(a_2,a_2)$. Applying rule $F\exists$ more than two times does not make sense: if not all branches are closed after two applications of rule $F\exists$, there is no deduction of $\exists x\forall y[M(x,y)]$. If M contains n binary predicate symbols, one has to allow 2^n applications of rule $F\exists$ in order that the construction of a completed tableau provides a decision procedure.

Solution 4.43. Suppose T is a tree such that each node in T has only finitely many immediate successors. For s a node in T, let $\Phi(s)$:= there are arbitrarily long finite paths going through s. Let s_0 be the empty tuple ().
(1) $\Phi(s_0)$, by hypothesis.
(2) If $\Phi(s)$, then there is an immediate successor t of s such that $\Phi(t)$.
From (1) and (2) it follows that starting with (), we are thus always able to pick a next node with the property Φ, ad infinitum, yielding an infinite path in T.
Proof of (2): Let $s(0),\ldots,s(k)$ be the immediate successors of s. If the paths through $s(0),\ldots,s(k)$ were no longer than $l_0,\ldots,l_k$ respectively, then all paths through s would be no longer than $\max(l_0,\ldots,l_k)$. □

In the following tree the empty node () has infinitely many successors and it has arbitrarily long finite paths, but there is no infinite path in it:



Solution 4.44. 1. Suppose $FL(n)$ is consistent. Then $FL(n)$ is satisfiable. Let M be a model of $FL(n)$. Then M is a field of characteristic n. Hence, $n = 0$ or n is prime.
2. If $FL(0) \models B$, then there is an n_0 such that for every $n \geq n_0$, $FL(n) \models B$. *Proof*: Suppose $FL(0) \models B$. Then it follows from the compactness theorem that there is a finite subset Γ' of $FL(0)$ such that $\Gamma' \models B$. Choose n_0 larger than all the n such that $\neg A_n$ occurs in Γ'.
3. We cannot replace the infinite number of non-logical axioms we added to FL to get $FL(0)$ by a finite number. *Proof*: Suppose we could. Let B be the conjunction of these non-logical axioms. B would be true in fields of characteristic 0 but in no other fields. $FL(0) \models B$. Choosing n_0 as in assertion 2, we would conclude that there are no fields of characteristic greater than n_0, which is absurd.
4. There is no extension Γ of FL whose models are just the finite fields. *Proof*: Suppose we had such an extension Γ. Let B_n be a formula which expresses that there are at least n individuals; for example, B_3 is $\exists x \exists y \exists z[x \neq y \wedge x \neq z \wedge y \neq z]$. Let Δ be obtained from Γ by adding all the B_n as non-logical axioms. Then Δ has no model. Then it follows from the compactness theorem that there is a finite subset Δ' of Δ which has no model. Choose n_0 larger than all the n such that B_n occurs in Δ' and choose a finite field M having more than n_0 elements. Then M is a model of Δ'. Contradiction.

Solution 4.45. 1. Let R be a partial ordering on V and V finite. Let n be the number of elements of V. If $n = 1$, then the proof is trivial. Suppose the induction hypothesis and let V have $n+1$ elements. V has a minimal element, say v_0. Then $V - \{v_0\}$ is partially ordered by $R\lceil V - \{v_0\}$. By the induction hypothesis there is a complete partial ordering R_1 on $V - \{v_0\}$ such that $R \subseteq R_1$. Let $R' := R_1 \cup \{(v_0, v) \mid v \in V\}$. Then R' is a complete partial ordering on V such that $R \subseteq R'$.
2. Let R be a partial ordering on V and V infinite. Consider a language containing a binary predicate symbol $\leq$ and an individual constant c_v for each $v \in V$. Let Γ be the following set of sentences: If $v_1 R v_2$, then Γ contains $c_{v_1} \leq c_{v_2}$.
If $v_1 \neq v_2$, then Γ contains $\neg(c_{v_1} \equiv c_{v_2})$.
$\forall x[\, x \leq x\,]$, $\forall x, y[\, x \leq y \wedge y \leq x \rightarrow x \equiv y\,]$, $\forall x, y, z[\, x \leq y \wedge y \leq z \rightarrow x \leq z\,]$
$\forall x, y[\, x \leq y \vee y \leq x\,]$
If $M = \langle D;\ R\rangle$ is a model of Γ, then R yields a complete partial ordering on V. By the compactness theorem, it suffices to prove that every finite subset of Γ has a model. So, let Γ' be a finite subset of Γ and let $V' := \{v \in V \mid c_v \text{ occurs in } \Gamma'\}$. $R\lceil V'$ is a partial ordering on V' and V' is finite. Therefore, there is a complete partial ordering on V' which contains $R\lceil V'$. That is, Γ' has a model.

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