Confounding in General Factorial Experiments

14

14.1 Introduction

In Chap. 13, incomplete block designs for 2^p factorial experiments were obtained by confounding one or more interaction contrasts with block contrasts. In this chapter, we extend the idea of confounding to encompass experiments in which some or all factors have more than two levels. We will code the levels of an m-level factor as $0, 1, \ldots, m-1$.

In Sect. 14.2, we consider single-replicate 3^p experiments arranged in $b = 3^s$ blocks of size $k = 3^{p-s}$. The techniques used in designing these types of experiment can be adapted for m^p experiments in m^s blocks of size m^{p-s} where m is a prime number.

Pseudofactors are introduced in Sect. 14.3 to facilitate confounding in symmetric 4^p experiments and asymmetric 2^p4^q experiments. Then, in Sect. 14.4, we consider asymmetric experiments involving factors or pseudofactors at both two and three levels, allowing us to look at more complicated situations where the treatment factors have a mixture of 2, 3, 4, and 6 levels.

Analysis of a two-replicate 3^3 experiment with partial confounding is illustrated using the SAS and R software packages in Sects. 14.5 and 14.6, respectively.

14.2 Confounding with Factors at Three Levels

14.2.1 Contrasts

In a factorial experiment where all treatment factors have 3 levels, each main effect has 2 degrees of freedom associated with it, each two-factor interaction has $2 \times 2 = 4$ degrees of freedom, etc. (see rule 3 of Sect. 7.3). Therefore, we can find 2 orthogonal contrasts to measure each main effect, 4 orthogonal contrasts to measure each two-factor interaction, and so on.

In a 3^2 experiment, for example, two orthogonal contrasts measuring the main effect of each of factors A and B are the linear and quadratic trend contrasts. Similarly, four orthogonal trend contrasts $A_L B_L$, $A_L B_Q$, $A_Q B_L$, and $A_Q B_Q$ (see Sect. 6.3) measuring the interaction are reproduced in Table 14.1. A different set of four orthogonal contrasts, labeled in pairs as $(AB; A^2B^2)$ and $(AB^2; A^2B)$, is also shown in Table 14.1. Although this second set of contrasts is less useful than the set of trend contrasts in measuring details of the interaction for quantitative factors, it will prove extremely useful for confounding purposes. The reader is asked to verify that *any* contrasts that measure the main effects of A and B are orthogonal to all the contrasts in Table 14.1 measuring the interaction.

A. Dean et al., *Design and Analysis of Experiments*,
Springer Texts in Statistics, DOI 10.1007/978-3-319-52250-0_14

Table 14.1 Sets of orthogonal contrasts measuring the interaction in a 3^2 experiment

TC	$A_L B_L$	$A_L B_Q$	$A_Q B_L$	$A_Q B_Q$	$(AB; A^2B^2)$		$(AB^2; A^2B)$	
00	1	-1	-1	1	-1	1	-1	1
01	0	2	0	-2	0	-2	1	1
02	-1	-1	1	1	1	1	0	-2
10	0	0	2	-2	0	-2	0	-2
11	0	0	0	4	1	1	-1	1
12	0	0	-2	-2	-1	1	1	1
20	-1	1	-1	1	1	1	1	1
21	0	-2	0	-2	-1	1	0	-2
22	1	1	1	1	0	-2	-1	1

Table 14.2 Groups of treatment combinations corresponding to orthogonal interaction contrasts in a 3^2 experiment

$(AB; A^2B^2)$			$(AB^2; A^2B)$		
00^*	$01^\dagger$	02^+	00^*	$01^\dagger$	02^+
$10^\dagger$	11^+	12^*	10^+	11^*	$12^\dagger$
20^+	21^*	$22^\dagger$	$20^\dagger$	21^+	22^*

Notice that the pair of contrasts labeled $(AB; A^2B^2)$ are two orthogonal contrasts that compare the three groups of treatment combinations (00, 12, 21) and (01, 10, 22) and (02, 11, 20). Any linear combination of this pair of contrasts is also a contrast between these three groups of treatment combinations. We have illustrated these groups of treatment combinations in the left-hand side of Table 14.2, where treatment combinations with the same superscript are in the same group. Notice that each group contains one treatment combination from each row and each column, making sure that each level of each factor is represented once in each group.

Similarly, the pair of contrasts labeled $(AB^2; A^2B)$ comprise two orthogonal contrasts that compare the three groups of treatment combinations (00, 11, 22) and (01, 12, 20) and (02, 10, 21). Any linear combination of this pair of contrasts is also a contrast between these three groups of treatment combinations. The groups are illustrated in the right-hand side of Table 14.2 and also have the property that each group contains one treatment combination from each row and each column.

The reason for the labeling $(AB; A^2B^2)$ and $(AB^2; A^2B)$ is to match the contrasts with the equation method of confounding in Sect. 14.2.3. The contrast names themselves have little meaning, except to acknowledge that each contrast belongs to the AB interaction and, as will be seen, each pair corresponds to a set of equations that partitions the treatment combinations into the three groups represented in Table 14.2.

Many texts list only one of the two labels in each pair, since each is the square of the other. For example, when the exponents are reduced modulo 3, then $A^2B = (AB^2)^2$. The convention is then to list AB^2 rather than A^2B, for example, since the leading exponent is one. However, we will list both labels to aid in identifying a complete set of confounded contrasts in designs with more than three blocks.

14.2.2 Confounding Using Contrasts

In this section we consider the division of treatment combinations into blocks by deliberately confounding negligible contrasts, as in Sect. 13.3.2 for 2^p experiments. For 3^p experiments, we look at designs with 3^s blocks of size 3^{p-s}, starting with 3 blocks of size 3^{p-1}. For a design with $b = 3$ blocks, two degrees of freedom are used to measure the block differences. Therefore, in a single-replicate design, we must confound a pair of treatment contrasts.

As a simple example, we start with an experiment with two factors A and B in which the interaction is known to be negligible. We will attempt to use two of the interaction contrasts shown in Table 14.1 to divide the treatment combinations into 3 blocks. A pair of trend contrasts, such as $A_L B_Q$ and $A_Q B_Q$ cannot be used to give blocks of equal size, since the values of the coefficients do not fall into 3 groups of 3. However, the pair of contrasts labeled $(AB; A^2B^2)$ have three pairs of coefficients $(-1,\ 1)$, $(0, -2)$, and $(1,\ 1)$ each of which appear three times. If we use these as a guide to dividing the treatment combinations into blocks, we obtain the design in Table 14.3.

Any contrast that is orthogonal to the two confounded contrasts can be estimated without requiring block adjustments. Estimable contrasts include all contrasts measuring the main effects of A and B and the remaining two interaction contrasts labeled $(A^2B; AB^2)$ and linear combinations of these. The trend contrasts in Table 14.1 are not orthogonal to any of the AB, A^2B^2, AB^2, A^2B contrasts, so they do not fall into either the confounded or the estimable category. They are *partly confounded*. In general, interaction trend contrasts can be estimated completely only when no contrasts from the interaction are confounded.

In the present example, the interaction has four degrees of freedom. Two are used to measure blocks. The other two correspond to two estimable contrasts, which are negligible and provide two degrees of freedom to measure σ^2.

If the contrasts labeled $(A^2B; AB^2)$ in Table 14.1 were used instead of the contrasts labeled $(AB; A^2B^2)$ to provide three blocks, the design of Table 14.4 would result. This has the same properties as the design in Table 14.3 in that all main-effect contrasts are estimable and there are two estimable contrasts $(AB; A^2B^2)$ remaining in the interaction. Neither design is better than the other, and a choice can be made at random. Block design randomization should be carried out before the design is used.

As we saw in 2^p experiments, there is a correspondence between the contrasts used for confounding, the contrast names, and the equation method of confounding. In the next section we show how to obtain the design of Table 14.3 by the equation method.

14.2.3 Confounding Using Equations

3^p Experiments in Three Blocks

The design in Table 14.3, which was obtained by confounding the two interaction contrasts labeled $(AB; A^2B^2)$ in Table 14.1, can be obtained by an equation method similar to that of Sect. 13.4. Notice that in Block I the digits of the three treatment combinations add to 0 or 3. In Block II they add to 1 or

Table 14.3 3^2 experiment in 3 blocks of 3, confounding $(AB; A^2B^2)$

Block	Contrast coefficients	Treatment combinations		
I	$(-1,\ 1)$	00	12	21
II	$(0, -2)$	01	10	22
III	$(1,\ 1)$	02	11	20

Table 14.4 3^2 experiment in 3 blocks of 3, confounding $(A^2B; AB^2)$

Block	Contrast coefficients	Treatment combinations		
I	$(-1,\ 1)$	00	11	22
II	$(1,\ 1)$	01	12	20
III	$(0, -2)$	02	10	21

4, and in Block III they add to 2. Now that both factors have three levels, we work modulo 3, which means that we subtract 3 from the sum of the digits until we obtain one of 0, 1, or 2, or equivalently, we take the remainder on division by 3. Writing the treatment combinations as $a_1 a_2$, the blocks can be defined by the confounding equations

$$\begin{aligned}&\text{Block I:} \quad \text{Treatment combinations with } L = a_1 + a_2 = 0 \pmod 3\,,\\ &\text{Block II:} \quad \text{Treatment combinations with } L = a_1 + a_2 = 1 \pmod 3\,,\\ &\text{Block III:} \text{ Treatment combinations with } L = a_1 + a_2 = 2 \pmod 3\,.\end{aligned}$$

Equivalently, the same three blocks can be obtained if the equations are multiplied by 2; that is,

$$\begin{aligned}&\text{Block I:} \quad \text{Treatment combinations with } 2L = 2a_1 + 2a_2 = 0 \pmod 3\,,\\ &\text{Block II:} \quad \text{Treatment combinations with } 2L = 2a_1 + 2a_2 = 2 \pmod 3\,,\\ &\text{Block III:} \text{ Treatment combinations with } 2L = 2a_1 + 2a_2 = 1 \pmod 3\,.\end{aligned}$$

Thus, if the contrasts labeled (AB; A^2B^2) in Table 14.1 are confounded with blocks, the treatment combinations in the three blocks satisfy both

$$L = a_1 + a_2 = 0,\ 1,\ \text{or } 2 \text{ (mod 3)},$$

and

$$2L = 2a_1 + 2a_2 = 0,\ 2,\ \text{or } 1 \text{ (mod 3)}.$$

Alternatively, if the contrasts labeled (AB^2; A^2B) are to be confounded, the equations

$$L = a_1 + 2a_2 = 0,\ 1,\ \text{or } 2 \text{ (mod 3)}$$

and, multiplying by 2,

$$2L = 2a_1 + a_2 = 0,\ 2,\ \text{or } 1 \text{ (mod 3)}$$

will produce the design in Table 14.4. Notice that the coefficients in the confounding equations correspond to the exponents in the contrast names. A set of equations defines the same set of blocks when it is multiplied by 2. Therefore, the confounded contrast names always come in pairs—one name being the square of the other—$(AB^2)^2 = A^2B^4 = A^2B$, reducing exponents (mod 3).

In general, in a 3^p experiment, if the equations

$$L = z_1 a_1 + z_2 a_2 + \cdots + z_p a_p = 0,\ 1, \text{or } 2 \text{ (mod 3)}$$

are used to produce three blocks, two contrasts will be confounded that can be labeled ($A^{z_1}B^{z_2} \cdots P^{z_p}$; $A^{2z_1}B^{2z_2} \cdots P^{2z_p}$), where z_i is 1 or 2 if the factor is present in the interaction, and 0 if it is not, and where the exponent is reduced modulo 3. For example, in a 3^5 experiment, the equations

$$L = a_1 + 2a_2 + a_4 = 0,\ 1,\ \text{or } 2 \text{ (mod 3)}$$

will give 3 blocks of size 3^4 confounding AB^2D and $A^2B^4D^2 = A^2BD^2$, which represent two contrasts from the three-factor interaction ABD. It is rarely of importance to identify exactly what the contrasts look like (they are any pair of orthogonal contrasts between the groups of treatment combinations in the three blocks). What is important is the knowledge that the confounded contrasts belong to a particular interaction and, therefore, that all other main-effect and interaction contrasts are estimable.

3^p Experiments in Nine Blocks

The equation method of confounding can be used to produce $b = 9 = 3^2$ blocks of size 3^{p-2} in a 3^p experiment by selecting two pairs of contrasts to be confounded. If the pair $(A^{z_1}B^{z_2}\cdots P^{z_p};\ A^{2z_1}B^{2z_2}\cdots P^{2z_p})$ is chosen for confounding together with the pair $(A^{y_1}B^{y_2}\cdots P^{y_p};\ A^{2y_1}B^{2y_2}\cdots P^{2y_p})$, the $b = 9$ blocks are produced from the nine possible pairs of values of the two equations

$$L_1 = z_1a_1 + z_2a_2 + \cdots + z_pa_p = 0,\ 1,\ \text{or } 2 \ (\text{mod } 3),$$

$$L_2 = y_1a_1 + y_2a_2 + \cdots + y_pa_p = 0,\ 1,\ \text{or } 2 \ (\text{mod } 3).$$

The $b - 1 = 8$ confounded contrasts are the two pairs originally chosen, together with all possible products. This is most conveniently set out as a table. The selected pairs of contrasts are written in the first row and first column. The table is then filled out by multiplication, and the exponents are reduced modulo 3, as follows:

	$A^{y_1}B^{y_2}\cdots P^{y_p}$	$A^{2y_1}B^{2y_2}\cdots P^{2y_p}$
$A^{z_1}B^{z_2}\cdots P^{z_p}$	$A^{z_1+y_1}B^{z_2+y_2}\cdots P^{z_p+y_p}$	$A^{z_1+2y_1}B^{z_2+2y_2}\cdots P^{z_p+2y_p}$
$A^{2z_1}B^{2z_2}\cdots P^{2z_p}$	$A^{2z_1+y_1}B^{2z_2+y_2}\cdots P^{2z_p+y_p}$	$A^{2z_1+2y_1}B^{2z_2+2y_2}\cdots P^{2z_p+2y_p}$

If $b = 3^s$ blocks of size 3^{p-s} are required, then s independent pairs of contrast names need to be chosen for confounding. All possible products determine the entire set of $b - 1 = 3^s - 1$ confounded contrasts.

Example 14.2.1 3^4 `experiment in 9 blocks of size 9`

Suppose that a 3^4 experiment, with factors A, B, C, D, is to be run in $b = 9$ blocks of size 9. Further, suppose that the only interactions thought to be important are the 2-factor interactions and, therefore, these should not be confounded. Now $b = 3^2$ blocks are required, so 2 pairs of contrasts should be chosen for confounding. The $ABCD$ interaction has 16 degrees of freedom, so we can find 16 orthogonal contrasts and label them in pairs as

$$\begin{array}{ll} (ABCD;\ A^2B^2C^2D^2), & (AB^2CD;\ A^2BC^2D^2), \\ (ABCD^2;\ A^2B^2C^2D), & (AB^2CD^2;\ A^2BC^2D), \\ (ABC^2D;\ A^2B^2CD^2), & (AB^2C^2D;\ A^2BCD^2), \\ (ABC^2D^2;\ A^2B^2CD), & (AB^2C^2D^2;\ A^2BCD). \end{array}$$

Selecting two pairs of contrasts from the 4-factor interaction for confounding contrasts is not a good choice. For example, if $(ABCD^2;\ A^2B^2C^2D)$ and $(ABCD;\ A^2B^2C^2D^2)$ were chosen, the set of eight confounded degrees of freedom would be

	$ABCD^2$	$A^2B^2C^2D$
$ABCD$	$A^2B^2C^2$	D^2
$A^2B^2C^2D^2$	D	ABC

Table 14.5 3^4 experiment in 3^2 blocks of 9; confounding (ABD; $A^2B^2D^2$), (BCD^2; B^2C^2D), (AB^2C; A^2BC^2), and (AC^2D^2; A^2CD)

Block	L_1, L_2	Treatment combinations
I	0,0	0000 0112 0221 1022 1101 1210 2011 2120 2202
II	1,2	0001 0110 0222 1020 1102 1211 2012 2121 2200
III	2,1	0002 0111 0220 1021 1100 1212 2010 2122 2201
IV	0,1	0010 0122 0201 1002 1111 1220 2021 2100 2212
V	1,0	0011 0120 0202 1000 1112 1221 2022 2101 2210
VI	2,2	0012 0121 0200 1001 1110 1222 2020 2102 2211
VII	1,1	0100 0212 0021 1122 1201 1010 2111 2220 2002
VIII	2,0	0101 0210 0022 1120 1202 1011 2112 2221 2000
IX	0,2	0102 0211 0020 1121 1200 1012 2110 2222 2001

and we can see that two orthogonal contrasts in the main-effect D would also be confounded. All possible selections of two pairs of contrasts from the $ABCD$ interaction will confound either a main effect or a two-factor interaction. However, in this example, the three-factor interactions are also thought to be negligible, so one possible choice is to confound (ABD; $A^2B^2D^2$) together with (BCD^2; B^2C^2D). This gives the following set of eight confounded degrees of freedom.

$$\begin{array}{lll} & BCD^2 & B^2C^2D \\ ABD & AB^2C & AC^2D^2 \\ A^2B^2D^2 & A^2CD & A^2BC \end{array}$$

Thus, each 3-factor interaction (which has 8 degrees of freedom) has two orthogonal contrasts confounded with blocks and six estimable contrasts, which are assumed to be negligible. This means that there are 24 degrees of freedom from the 3-factor interactions and a further 16 degrees of freedom from the $ABCD$ interaction available for estimating σ^2. The design is obtained by using the linear functions L_1 and L_2, corresponding to the selected confounded contrasts ABD and BCD^2 as follows. For each treatment combination, compute the values of L_1 and L_2 modulo 3:

$$\begin{aligned} L_1 &= a_1 + a_2 \phantom{{}+a_3} + a_4 = 0,\ 1, \text{ or } 2 \pmod 3. \\ L_2 &= \phantom{a_1 + {}} a_2 + a_3 + 2a_4 = 0,\ 1, \text{ or } 2 \pmod 3. \end{aligned}$$

The design is given in Table 14.5, and it can be verified that the nine blocks are obtained from the nine possible pairs of values of L_1 and L_2. □

14.2.4 A Real Experiment—Dye Experiment

An experiment is described in the book *Design and Analysis of Industrial Experiments*, edited by O. L. Davies, that investigates three reactants (the base material and two inorganic materials, called here M and N) in the manufacture of a cotton dyestuff. The three factors of interest in the experiment were the concentration of M in the free water in the reaction mixture (factor A at three equally spaced levels), the volume of free water in the reaction mixture (factor B at three equally spaced levels), and the concentration of N in the free water in the reaction mixture (factor C at three equally spaced levels).

Table 14.6 Data for dye experiment

Block I		Block II		Block III	
TC	Volume	TC	Volume	TC	Volume
000	74	020	69	010	13
021	130	011	46	001	112
012	56	002	71	022	125
110	110	100	211	120	199
101	166	121	220	111	218
122	227	112	216	102	201
220	195	210	147	200	74
211	146	201	47	221	198
202	90	222	164	212	102

Source Data adapted from *The Design and Analysis of Industrial Experiments*, Second edition, 1979. Editor O. L. Davies. Published by Longman Group Limited.

Although it was possible to control the conditions in the laboratory fairly accurately, the experimenters divided the treatment combinations into blocks of size 9. This was done as a safeguard against time trends, because the time required to complete the investigation was reasonably long. The experiment involved $r = 2$ replications of each treatment combination, but here we will analyze only the first replicate.

The observations were the volumes of dyestuff resulting from the chemical reactions and are shown in Table 14.6. Looking at the treatment combinations (TC) listed in Block I, we can see that they all satisfy the confounding equation $a_1 + 2a_2 + 2a_3 = 0$ (mod 3). Consequently, the experimenters have confounded two contrasts from the 3-factor interaction, which we can label as (AB^2C^2; A^2BC). Since there are only three blocks, these are the only two contrasts confounded. If the 3-factor interaction can be assumed to be negligible, the remaining six degrees of freedom can be used to measure the error variability. The analysis of variance table is shown in Table 14.7. The sum of squares for testing that the main effect of A (averaged over the levels of the other factors) can be calculated either by using the formulae of Chap. 7 or by adding together the sums of squares for two orthogonal contrasts. For example, rule 4 of Sect. 7.3 gives

$$\begin{aligned} ssA &= 9\sum_{i=1}^{3}\overline{y}_{.i..}^{2} - 27\,\overline{y}_{....}^{2} \\ &= 9(5980.44 + 38{,}590.42 + 16{,}698.38) - 27(18{,}045.44) \\ &= 64{,}196.222. \end{aligned}$$

Two orthogonal contrasts for A are the linear and quadratic contrasts. From Table A.2, the coefficients for the (nonnormalized) linear contrast are $(-1,\ 0,\ 1)$, and those for the quadratic contrast are $(1, -2,\ 1)$. The least squares estimates for these two contrasts are

$$\hat{A}_{\mathrm{L}} = (-\overline{y}_{.0..} + \overline{y}_{.2..}) = 51.889$$

and

$$\hat{A}_{\mathrm{Q}} = (\overline{y}_{.0..} - 2\overline{y}_{.1..} + \overline{y}_{.2..}) = -186.333.$$

Table 14.7 Analysis of variance for the dye experiment

Source of variation	Degrees of freedom	Sum of squares	Mean square	Ratio	p-values
Block	2	182.00			
A	2	64,196.22	32,098.11	26.60	0.0010
A_{L}	1	12,116.06	12,116.06	10.04	0.0194
A_{Q}	1	52,080.17	52,080.17	43.16	0.0006
B	2	16,857.56	8,428.78	6.98	0.0271
B_{L}	1	12,853.39	12,853.39	10.65	0.0172
B_{Q}	1	4,004.17	4,004.17	3.32	0.1184
C	2	2,334.89	1,167.44	0.97	0.4324
C_{L}	1	1,422.22	1,422.22	1.18	0.3193
C_{Q}	1	912.67	912.67	0.76	0.4179
AB	4	12,512.89	3,128.22	2.59	0.1428
AC	4	4,044.89	1,011.22	0.84	0.5481
BC	4	2,698.89	674.72	0.56	0.7015
Error	6	7,240.67	1,206.78		
Total	26	110, 068.00			

To normalize the contrasts, one would divide $\hat{A}_{\mathrm{L}}$ by $\sqrt{\Sigma c_i^2/(rbc)} = \sqrt{2/9}$ and divide $\hat{A}_{\mathrm{Q}}$ by $\sqrt{\Sigma c_i^2/(rbc)} = \sqrt{6/9}$.

The sum of squares for testing the hypothesis that the linear contrast for A is negligible is the square of the normalized contrast estimate,

$$ss(A_{\mathrm{L}}) = \frac{(-\overline{y}_{.0..} + \overline{y}_{.2..})^2}{2/9} = \frac{(51.889)^2}{2/9} = 12{,}116.06;$$

the sum of squares for testing the hypothesis that the quadratic contrast for A is negligible is

$$ss(A_{\mathrm{Q}}) = \frac{(\overline{y}_{.0..} - 2\overline{y}_{.1..} + \overline{y}_{.2..})^2}{6/9} = \frac{(-186.333)^2}{6/9} = 52{,}080.17;$$

and we see that

$$ss(A_{\mathrm{L}}) + ss(A_{\mathrm{Q}}) = 12{,}116.06 + 52{,}080.17 = 64{,}196.23 = ssA.$$

The other sums of squares in Table 14.7 can be obtained in a similar way.

For testing the hypotheses that the three main effects and the three 2-factor interactions are negligible at individual significance levels $\alpha^* = 0.01$ (an overall level significance level of $\alpha \leq 0.06$), we would compare the ratios in the analysis of variance table (Table 14.7) with the critical values from the F-distribution ($F_{2,6,0.01} = 10.9$ for the main effects and $F_{4,6,0.01} = 9.15$ for the 2-factor interactions), and we would reject only the hypothesis that the main effect of A is negligible. Plots for the average response due to A and B are shown in Fig. 14.1. We can see from the plot of the A average responses that, as the levels of the concentration of inorganic material M in the free water increase, the volumes of dyestuff first increase and then begin to decrease. We might expect to see both a significant linear trend and a significant quadratic trend. Testing the two hypotheses that the linear trend in A is negligible

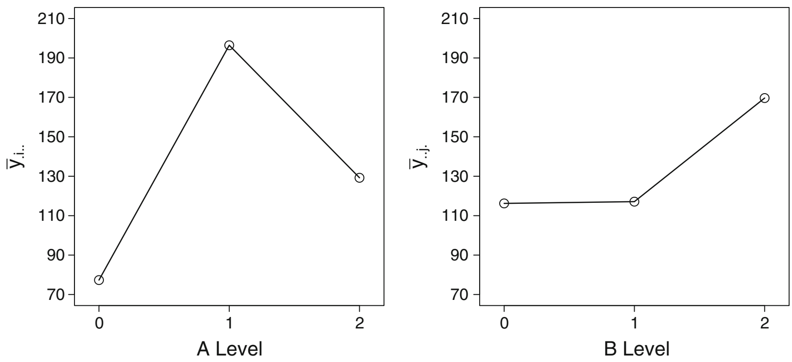

Fig. 14.1 Main-effect plots for the dye experiment

and the quadratic trend in A is negligible, each at level 0.005 (to give an overall significance level of $\alpha^* \leq 0.01$), we have

$$ss(A_L)/msE = 10.04 < F_{1,6,0.005} = 18.6$$

and

$$ss(A_Q)/msE = 43.16 > F_{1,6,0.005} = 18.6,$$

and we conclude that there is a quadratic trend in the levels of A, and that the turning point is towards the center of the range of levels investigated (otherwise, the linear trend would also have been significantly different from zero). Since the objective of the experiment was to boost the volume of dyestuff produced, the results of the experiment suggest that further investigation around the second concentration of inorganic material M might be wise. Although the hypothesis of no effect of B was not rejected, Fig. 14.1 suggests that further experimentation with higher volumes of free water in the reaction mixture is worth consideration.

The above method of testing these two hypotheses uses Bonferroni's method of combining significance levels. An alternative method is to use Scheffé's method of multiple comparisons and test the two hypotheses simultaneously at level 0.01. Since

$$ss(A_L)/msE = 10.04 < 2F_{2,6,0.01} = 21.8$$

and

$$ss(A_Q)/msE = 43.16 > 2F_{2,6,0.01} = 21.8,$$

we arrive at the same conclusion. The more powerful method here is the first since

$$F_{1,6,0.005} < 2F_{2,6,0.01} .$$

14.2.5 Plans for Confounded 3^p Experiments

At the end of the chapter, Table 14.20 gives suggested confounding schemes for 3^p experiments in blocks of size 3, 9, or 27. As illustrated in Sect. 13.6, if the design in the table confounds important contrasts in the experiment, then a relabeling of treatment factors should be attempted.

14.3 Designing Using Pseudofactors

14.3.1 Confounding in 4^p Experiments

A treatment factor F with four levels coded 0, 1, 2, 3 can be represented by two factors F_1 and F_2 each having two levels coded 0, 1. The levels of F_1 and F_2 taken together correspond to the levels of the original factor F. One possible correspondence is given below:

F	F_1	F_2
0	0	0
1	0	1
2	1	0
3	1	1

The factors F_1 and F_2 are called *pseudofactors*. All factors in a 4^p experiment can be represented by pseudofactors. Thus, a 4^p experiment in 4^s blocks of size 4^{p-s} can be represented as a 2^{2p} experiment in 2^{2s} blocks of size $2^{2(p-s)}$. The techniques of confounding in a 2^{2p} experiment as discussed in Chap. 13 can therefore be used. The only difference is that an interaction of pseudofactors of the form $F_1G_1G_2$, say, does not represent a 3-factor interaction. It represents one of nine orthogonal contrasts measuring the two-factor interaction, FG. Similarly, F_1F_2 does not represent a contrast in a two-factor interaction. It represents one of three orthogonal contrasts measuring the main effect of factor F.

Example 14.3.1 4^2 `experiment in 4 blocks of size 4`

Consider a 4^2 experiment with two factors F and G to be run in 4 blocks of size 4. The main effects are to be estimated, but the interaction is thought to be negligible. If F and G are represented by pseudofactors F_1, F_2, G_1, G_2 having two levels each, we can consult Table 13.29 hoping to find a suitable 2^4 experiment in 4 blocks of size 4.

In Table 13.29, we find a design that confounds AC, ABD, and BCD. If we make the correspondence $F_1 = A$, $F_2 = B$, $G_1 = C$, $G_2 = D$, then the design confounds F_1G_1, $F_1F_2G_2$, $F_2G_1G_2$, all three of which belong to the interaction of F and G. All main-effect contrasts of F and G are orthogonal to all interaction contrasts and can therefore be estimated without adjustment for blocks. The design is shown in Table 14.8, with blocks corresponding to combinations of values of $L_1 = a_1 + a_3$ (mod 2) and $L_2 = a_1 + a_2 + a_4$ (mod 2).

If we make a different correspondence, say $F_1 = A$, $F_2 = D$, $G_1 = B$, $G_2 = C$, then a slightly different design is obtained, this time confounding F_1G_2, $F_1F_2G_1$, and $F_2G_1G_2$, which again belong to the interaction of F and G. There is no particular reason to prefer one design over the other. However, a third correspondence, $F_1 = A$, $F_2 = C$, $G_1 = B$, $G_2 = D$, would not be good, since it confounds F_1F_2, $F_1G_1G_2$, $F_2G_1G_2$, and this includes one degree of freedom F_1F_2 from the main effect of F. □

Table 14.8 4^2 experiment in 4 blocks of 4, confounding three degrees of freedom (F_1G_1, $F_1F_2G_2$, $F_2G_1G_2$) from FG

Block	L_1, L_2	Pseudofactors F_1, F_2, G_1, G_2	Factors F, G
I	0,0	0000 0101 1011 1110	00 11 23 32
II	0,1	0001 0100 1010 1111	01 10 22 33
III	1,0	0010 0111 1001 1100	02 13 21 30
IV	1,1	0011 0110 1000 1101	03 12 20 31

Since two-level pseudofactors are being used, block sizes need only be a power of two, not necessarily a power of four.

14.3.2 Confounding in $2^p \times 4^q$ Experiments

Since factors with 4 levels can be written in terms of pseudofactors having 2 levels each, a $2^p \times 4^q$ experiment can be written in terms of pseudofactors as a $2^{(p+2q)}$ experiment, and no new techniques are needed.

Example 14.3.2 $2^3 \times 4$ `experiment in 4 blocks of size 8`

Suppose that a $2^3 \times 4$ experiment with factors F, G, H, and J is to be run in 4 blocks of size 8. This could be designed using pseudofactors by selecting a design for a 2^5 experiment in 4 blocks from Table 13.29. A design is shown that confounds ABE, CDE, and $ABCD$. If we let the combination of levels of A and C represent the levels of the 4-level factor $J_1J_2 = J$ with the representation $00 = 0$, $01 = 1$, $10 = 2$, $11 = 3$, and let the levels of B, D, and E respectively represent the levels of F, G, and H, we obtain the design of Table 14.9, that confounds one contrast from each of the 3-factor interactions FHJ, GHJ, and FGJ. All main effects and 2-factor interactions can be estimated. There are 10 degrees of freedom available for estimating σ^2. These come from the two unconfounded degrees of freedom from each of FHJ, GHJ, and FGJ and the one degree of freedom from FGH and the three from $FGHJ$. □

14.4 Designing Confounded Asymmetric Experiments

A factorial experiment is called an asymmetric experiment when the treatment factors do not all have the same number of levels. For example, $2^2 \times 4^2$, $2^5 \times 3$, $2^2 \times 3^2 \times 4^2$, and 3×6 experiments are all asymmetric experiments. We have already discussed the design of asymmetric $2^p \times 4^q$ experiments in Sect. 14.3.2. We used pseudofactors for the factors with four levels, thus allowing the symmetric designs for 2^{p+2q} experiments to be used. We can use this idea only when the numbers of levels of all factors are powers of the same prime number. For all of the other examples mentioned above, the use of pseudofactors would transform the experiment into a $2^p \times 3^q$ experiment. Consequently, we concentrate on this type of situation in this section.

Since 2 and 3 are relatively prime, the only type of design that can be constructed using the equation method will confound contrasts within the two symmetric parts of the experiment. Consequently, to obtain a design for a $2^p \times 3^q$ experiment in $2^s \times 3^t$ blocks of size $2^{p-s} \times 3^{q-t}$, we combine a design for a 2^p experiment in 2^s blocks with a design for a 3^q experiment in 3^t blocks, using the idea of

Table 14.9 $2^3 \times 4$ experiment in 4 blocks of 8, using pseudofactors and confounding one degree of freedom from interactions FGJ, GHJ, and FHJ

Blocks	Treatment combinations (Factors F, G, H, J)							
I	0000	1002	0101	1103	0013	1011	0112	1110
II	0010	1012	0111	1113	0003	1001	0102	1100
III	0100	1102	0001	1003	0113	1110	0012	1010
IV	0110	1112	0011	1013	0103	1101	0002	1000

a *crossed array* as illustrated in the following example. The total number of blocks created in the combined design is always the product of the numbers of blocks in the original two designs. Likewise, the block sizes in the combined design are products of the block sizes in the original two designs. The confounded contrasts in the combined design are those confounded in the separate designs together with those indicated by all possible products of contrast names.

Example 14.4.1 $2^2 \times 3^2$ `experiment in 6 blocks of size 6`

Suppose that a $2^2 \times 3^2$ experiment is to be run in 6 blocks of size 6. We label the two 2-level factors as A and B and the two 3-level factors as C and D. Since the design must confound within the two symmetric parts of the experiment, one contrast from A, B, or AB must be confounded to divide the 2^2 treatment combinations into two blocks, and one pair of contrasts from C, D, or CD must be confounded to divide the 3^2 treatment combinations into three blocks. The confounded contrasts in the combined design are those confounded in the separate designs together with their products.

For example, we could combine the two designs in Table 14.10. The design labeled d_1 is for a 2^2 experiment in two blocks of size 2 confounding AB, with treatment combinations (TC) grouped into blocks determined by the two values of $L_1 = a_1 + a_2$ (mod 2). The design labeled d_2 is for a 3^2 experiment in three blocks of size 3 confounding the pair of contrasts $(CD^2;\ C^2D)$, with blocks determined by the three values of $L_2 = a_3 + 2a_4$ (mod 3). The combined array in Table 14.11 divides the treatment combinations into blocks according to the six combinations of values of L_1 and L_2.

A quick way to obtain the design with six blocks is to combine each of the 2 blocks of d_1 with each of the 3 blocks of d_2. For example, to combine the first blocks of d_1 and d_2, each of the combinations 00 and 11 in block I_1 of d_1 is combined with each of the combinations 00, 11, and 22 in block I_2 of d_2 to give the treatment combinations 0000, 1100, 0011, 1111, 0022, 1122 in the first block of the combined design. The other blocks of the combined design are obtained in a similar way.

The $b - 1 = 5$ confounded contrasts are those corresponding to the original confounding schemes, namely the contrast AB and the pair of contrasts represented by $(CD^2;\ C^2D)$, together with the pair of contrasts represented by the products of these labels—namely $(ABCD^2;\ ABC^2D)$. □

Table 14.10 Design d_1 for a 2^2 experiment confounding AB and design d_2 for a 3^2 experiment confounding $(CD^2;\ C^2D)$

Design	L_1	Block	TC	Design	L_2	Block	TC
d_1	0	I_1	00 11	d_2	0	I_2	00 11 22
	1	II_1	01 10		1	II_2	02 10 21
					2	III_2	01 12 20

Table 14.11 $2^2 \times 3^2$ experiment in 6 blocks of 6, confounding AB, $(CD^2;\ C^2D)$, $(ABCD^2;\ ABC^2D)$

L_1, L_2	Block combinations	Treatment combinations
0,0	$\text{I}_1,\ \text{I}_2 \ \rightarrow\ \text{I}$	0000 0011 0022 1100 1111 1122
0,1	$\text{I}_1,\ \text{II}_2 \ \rightarrow\ \text{II}$	0002 0010 0021 1102 1110 1121
0,2	$\text{I}_1,\ \text{III}_2 \ \rightarrow\ \text{III}$	0001 0012 0020 1101 1112 1120
1,0	$\text{II}_1,\ \text{I}_2 \ \rightarrow\ \text{IV}$	0100 0111 0122 1000 1011 1022
1,1	$\text{II}_1,\ \text{II}_2 \ \rightarrow\ \text{V}$	0102 0110 0121 1002 1010 1021
1,2	$\text{II}_1,\ \text{III}_2 \ \rightarrow\ \text{VI}$	0101 0112 0120 1001 1012 1020

Example 14.4.2 $4 \times 6 \times 3$ `experiment in 6 blocks of size 12`

Suppose that a $4 \times 6 \times 3$ experiment with factors F, G, H is to be run in 6 blocks of size 12. If we use the pseudofactor labels F_1, F_2, G_1, G_2, and H, then the factors F_1, F_2, and G_1 are in the 2^3 pseudofactor experiment and G_2 and H are in the 3^2 pseudofactor experiment. In the 2^3 experiment, we confound $F_1F_2G_1$ to give the two blocks of the design d_1 of Table 14.12, and in the 3^2 experiment, we confound the pair of contrasts $(G_2H;\, G_2^2H^2)$ to give the three blocks of the design d_2. Combining each treatment combination in design d_1 with those in d_2 gives the design in Table 14.13, which has $b = 6$ blocks of size 12. The $b-1 = 5$ confounded degrees of freedom correspond to the original three confounded contrasts together with their products, that is, $F_1F_2G_1$, $(G_2H;\, G_2^2H^2)$, and $(F_1F_2G_1G_2H;\, F_1F_2G_1G_2^2H^2)$.

Translating back to the original factors, we can see that one degree of freedom from the interaction FG is confounded, together with two degrees of freedom from each of GH and FGH. This means that all contrasts from the three main effects and also from the interaction FH can be estimated.

If we take the mapping of pseudofactor levels to factor levels as follows, then the design of Table 14.13 is as shown in Table 14.14:

Table 14.12 Design d_1 for a 2^3 experiment confounding $F_1F_2G_1$ and design d_2 for a 3^2 experiment confounding $(G_2H;\, G_2^2H^2)$

Design	Block	Treatment combinations	Design	Block	Treatment combinations
d_1	I_1	000 011 101 110	d_2	I_2	00 12 21
	II_1	001 010 100 111		II_2	01 10 22
				III_2	02 11 20

Table 14.13 $4 \times 6 \times 3$ experiment in 6 blocks of size 12 using psuedofactors, confounding one degree of freedom from FG and two degrees of freedom from each of GH and FGH

Block combinations		Pseudofactor combinations					
I_1, I_2	$\to$ I	00000	00012	00021	01100	01112	01121
		10100	10112	10121	11000	11012	11021
I_1, II_2	$\to$ II	00001	00010	00022	01101	01110	01122
		10101	10110	10122	11001	11010	11022
I_1, III_2	$\to$ III	00002	00011	00020	01102	01111	01123
		10102	10111	10120	11002	11011	11020
II_1, I_2	$\to$ IV	00100	00112	00121	01000	01012	01021
		10000	10012	10021	11100	11112	11121
II_1, II_2	$\to$ V	00101	00110	00122	01001	01010	01022
		10001	10010	10022	11101	11110	11122
II_1, III_2	$\to$ VI	00102	00111	00120	01002	01011	01023
		10002	10011	10020	11102	11111	11120

Table 14.14 $4 \times 6 \times 3$ experiment in 6 blocks of size 12, confounding one degree of freedom from FG and two degrees of freedom from each of GH and FGH

Block	Treatment combinations
I	000 012 021 130 142 151 230 242 251 300 312 321
II	001 010 022 131 140 152 231 240 252 301 310 322
III	002 011 020 132 141 150 232 241 250 302 311 320
IV	030 042 051 100 112 121 200 212 221 330 342 351
V	031 040 052 101 110 122 201 210 222 331 340 352
VI	032 041 050 102 111 120 202 211 220 332 341 350

F	F_1	F_2	G	G_1	G_2
0	0	0	0	0	0
1	0	1	1	0	1
2	1	0	2	0	2
3	1	1	3	1	0
			4	1	1
			5	1	2

□

14.5 Using SAS Software

In this section we illustrate the use of the SAS software in analyzing a two-replicate factorial experiment with partial confounding. This we do via an example. The analysis is straightforward, as was illustrated in the previous chapter. Along with the correct analysis, we also fit an incorrect model—one without block effects—to illustrate the effect of partial confounding on the analysis.

Example 14.5.1 `Dye experiment, continued`

The dye experiment was described in Sect. 14.2.4, where part of the data was analyzed as though it came from a single-replicate confounded experiment. In fact, in the original experiment, the design was a partially confounded design made up of two single-replicate 3^3 designs with different confounding schemes. The three factors of interest in the experiment were the concentration of inorganic material M in the free water in the reaction mixture (factor A at three equally spaced levels), the volume of free water in the reaction mixture (factor B at three equally spaced levels), and the concentration of inorganic material N in the free water in the reaction mixture (factor C at three equally spaced levels). The observations were the volumes of dyestuff resulting from the chemical reactions and are shown in Table 14.15 together with the design (prior to randomization). The contrasts (AB^2C^2; A^2BC) are confounded in the first set of three blocks and estimable in the second set, whereas the contrasts (ABC^2; A^2B^2C) are confounded in the second set of three blocks and estimable in the first set.

Since no contrast is completely confounded, no terms need be omitted from the model. Table 14.16 shows the SAS input statements for analyzing this experiment with partial confounding. The statements are exactly as they would be for a replicated experiment with three factors and no confounding. A second run of `PROC GLM` with no block parameter in the model is included for illustration purposes to show the effect of the partial confounding.

Table 14.15 Data for the dye experiment

Block I		Block II		Block III	
TC	Volume	TC	Volume	TC	Volume
000	74	020	69	010	13
021	130	011	46	001	112
012	56	002	71	022	125
110	110	100	211	120	199
101	166	121	220	111	218
122	227	112	216	102	201
220	195	210	147	200	74
211	146	201	47	221	198
202	90	222	164	212	102
Block IV		Block V		Block VI	
TC	Volume	TC	Volume	TC	Volume
000	85	010	12	020	115
011	52	021	107	001	148
022	70	002	75	012	47
120	164	100	184	110	145
101	288	111	204	121	142
112	239	122	265	102	216
210	104	220	183	200	75
221	165	201	65	211	124
202	60	212	70	222	114

Source Data adapted from *The Design and Analysis of Industrial Experiments*. Second edition, 1979. Editor O. L. Davies published by Longman Group Ltd.

Table 14.16 SAS program for the dye experiment

```
DATA DYE; INPUT BLK A B C Y;
  LINES;
  1 0 0 0  74
  1 0 2 1 130
  1 0 1 2  56
  1 1 1 0 110
  : : : :  :
  6 2 2 2 114
* Analysis of variance -- correct, with block effect; PROC GLM;
  CLASS  BLK A B C ;
  MODEL Y = BLK A B C A*B A*C B*C A*B*C ;
* Analysis of variance -- without block effect, for comparison;
PROC GLM;
  CLASS  A B C;
  MODEL Y =  A B C A*B A*C B*C A*B*C;
```

Fig. 14.2 Correct analysis of variance for the dye experiment

Results Viewer - SAS Output

The GLM Procedure
Dependent Variable: Y

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	31	221034.7963	7130.1547	8.25	<.0001
Error	22	19010.8519	864.1296		
Corrected Total	53	240045.6481			

Source	DF	Type III SS	Mean Square	F Value	Pr > F
BLK	5	2027.6481	405.5296	0.47	0.7950
A	2	140999.7037	70499.8519	81.58	<.0001
B	2	19447.8148	9723.9074	11.25	0.0004
C	2	4934.4815	2467.2407	2.86	0.0790
A*B	4	27922.6296	6980.6574	8.08	0.0004
A*C	4	13043.6296	3260.9074	3.77	0.0175
B*C	4	2913.1852	728.2963	0.84	0.5130
A*B*C	8	10794.8148	1349.3519	1.56	0.1935

Fig. 14.3 Incorrect analysis of variance, omitting the blocking factor to show the effect of partial confounding

Results Viewer - sashtml.htm

The GLM Procedure
Dependent Variable: Y

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	26	219007.1481	8423.3519	10.81	<.0001
Error	27	21038.5000	779.2037		
Corrected Total	53	240045.6481			

Source	DF	Type III SS	Mean Square	F Value	Pr > F
A	2	140999.7037	70499.8519	90.48	<.0001
B	2	19447.8148	9723.9074	12.48	0.0001
C	2	4934.4815	2467.2407	3.17	0.0582
A*B	4	27922.6296	6980.6574	8.96	<.0001
A*C	4	13043.6296	3260.9074	4.18	0.0092
B*C	4	2913.1852	728.2963	0.93	0.4588
A*B*C	8	9745.7037	1218.2130	1.56	0.1826

All contrasts from the main-effects and 2-factor interactions are orthogonal to the block contrasts and can be estimated without adjustment for blocks. Consequently, the sums of squares for these terms are the same whether or not the block parameter is in the model. This can be verified by comparing the Type III sums of squares for the two runs of `PROC GLM` shown in Figs. 14.2 and 14.3. Comparing these two analysis of variance tables, we can see that inclusion of the block parameter in the model changes the sum of squares for the three-factor interaction, since the three-factor interaction is partially confounded with blocks. The degrees of freedom for the three-factor interaction remain at 8, as all 8 orthogonal contrasts can be estimated from some portion of the data.

The analysis of variance table (Fig. 14.2) provides *no* evidence that certain contrasts are partially confounded. However, partially confounded contrasts are estimated with larger variance due to the adjustment for blocks. As a result, for the corresponding effects, confidence intervals are wider and tests are less powerful. □

14.6 Using R Software

In this section we illustrate the use of the R software in analyzing a two-replicate factorial experiment with partial confounding. This we do via an example. The analysis is straightforward, as was illustrated in the previous chapter. Along with the correct analysis, we also fit an incorrect model—one without block effects—to illustrate the effect of partial confounding on the analysis.

Example 14.6.1 `Dye experiment, continued`

The dye experiment was described in Sect. 14.2.4, where part of the data was analyzed as though it came from a single-replicate confounded experiment. In fact, in the original experiment, the design was a partially confounded design made up of two single-replicate 3^3 designs with different confounding schemes. The three factors of interest in the experiment were the concentration of inorganic material M in the free water in the reaction mixture (factor A at three equally spaced levels), the volume of free water in the reaction mixture (factor B at three equally spaced levels), and the concentration of inorganic material N in the free water in the reaction mixture (factor C at three equally spaced levels). The observations were the volumes of dyestuff resulting from the chemical reactions and are shown in Table 14.15 (p. 487) together with the design (prior to randomization). The contrasts (AB^2C^2; A^2BC) are confounded in the first set of three blocks and estimable in the second set, whereas the contrasts (ABC^2; A^2B^2C) are confounded in the second set of three blocks and estimable in the first set.

Since no contrast is completely confounded, no terms need be omitted from the model. Table 14.17 shows the R commands and output for analyzing this experiment with partial confounding. The statements are exactly as they would be for a replicated experiment with three factors and no confounding. A second call of the linear models function `lm` with no blocking factor in the model is included for illustration purposes to show the effect of the partial confounding.

All contrasts from the main-effects and 2-factor interactions are orthogonal to the block contrasts and can be estimated without adjustment for blocks. Consequently, the sums of squares for these terms are the same whether or not the block parameter is in the model. This can be verified by comparing the Type I sums of squares for the two models fit in Table 14.17—the first model with block effects entered first, so for which factorial effects are adjusted for block effects, and the second model without block effects. Inclusion of the block parameter in the model changes the sum of squares for the three-factor interaction, since the three-factor interaction is partially confounded with blocks. The degrees of freedom for the three-factor interaction remain at 8, as all 8 orthogonal contrasts can be estimated from some portion of the data.

The analysis of variance table (the first in Table 14.17) provides no evidence that certain contrasts are partially confounded. However, partially confounded contrasts are estimated with larger variance due to the adjustment for blocks. As a result, for the corresponding effects, confidence intervals (not shown) are wider and tests are less powerful. □

Table 14.17 R program and output for the dye experiment

```
> dye.data = read.table("data/dye.txt", header=T)
> head(dye.data, 3)

  Blk A B C   y
1   1 0 0 0  74
2   1 0 2 1 130
3   1 0 1 2  56

> # Create factor variables
> dye.data = within(dye.data,
+              {fBlk = factor(Blk); fA = factor(A);
+                 fB = factor(B);   fC = factor(C) })
> # Analysis of variance
> model1 = lm(y ~ fBlk + fA*fB*fC, data=dye.data)
> anova(model1)

Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value  Pr(>F)
fBlk       5    979     196    0.23 0.94703
fA         2 141000   70500   81.58 6.7e-11
fB         2  19448    9724   11.25 0.00043
fC         2   4934    2467    2.86 0.07899
fA:fB      4  27923    6981    8.08 0.00036
fA:fC      4  13044    3261    3.77 0.01749
fB:fC      4   2913     728    0.84 0.51300
fA:fB:fC   8  10795    1349    1.56 0.19353
Residuals 22  19011     864

> # ANOVA without block effects for comparison
> model1 = lm(y ~ fA*fB*fC, data=dye.data)
> anova(model1)

Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value  Pr(>F)
fA         2 141000   70500   90.48 1.1e-12
fB         2  19448    9724   12.48 0.00015
fC         2   4934    2467    3.17 0.05816
fA:fB      4  27923    6981    8.96 9.7e-05
fA:fC      4  13044    3261    4.18 0.00915
fB:fC      4   2913     728    0.93 0.45877
fA:fB:fC   8   9746    1218    1.56 0.18257
Residuals 27  21039     779
```

Exercises

1. Suggest a confounding scheme for a 3^5 experiment in 9 blocks of size 27 if all 2-factor interactions and the 3-factor interaction ABE are to estimated.

2. Suggest a confounding scheme for a 3^5 experiment in 27 blocks of size 9 if all 2-factor interactions and the 3-factor interaction ABE are to be estimated.

3. **Dye experiment, continued**

   (a) For the dye experiment of Sect. 14.2.4, check that the variances of the errors appear to be equal for the different levels of the three factors. Check also that the assumption of normality of the error variables is reasonable.
   (b) Calculate the normalized contrast estimate for Linear $A \times$ Linear B, using the method outlined in Sect. 14.2.4.
   (c) Compute the sum of squares for testing the hypothesis that the Linear $A \times$ Linear B contrast is negligible, using the method outlined in Sect. 14.2.4.
   (d) Test the hypothesis that the Linear $A \times$ Linear B contrast is negligible, using an individual significance level of 0.01.
   (e) Draw an interaction plot for AC and verify that the interaction appears to be negligible.
   (f) Assuming that the contrasts were preplanned, calculate confidence intervals for the pairwise differences in yields due to the three different levels of each of A, B and C. State your overall confidence level.

4. **Dye experiment, continued**

   The experimenters who ran the dye experiment were interested in the linear and quadratic components of the main effects and interactions. Analyze the experiment accordingly. What information have you gathered about the levels of the factors if high yield is of importance?

5. A set of hypothetical data is given in Table 14.18 for a partially confounded 3^2 experiment in 6 blocks of 3. The design is made up of two single-replicate designs: The first confounds the contrasts $(AB; A^2B^2)$ from the interaction, while the second confounds the contrasts $(AB^2; A^2B)$.

   (a) By hand, write out the estimates of the linear and quadratic contrasts for the main effects and their associated variances.
   (b) Using the contrast estimates in part (a), calculate the sums of squares for A and B.

Table 14.18 Partially confounded 3^2 experiment in $b = 6$ blocks of $k = 3$. Hypothetical data are shown in parentheses with corresponding treatment combinations

Replicate	Block	Treatment combinations (Response)		
1	I	00 (53)	12 (59)	21 (80)
Confounds $(AB; A^2B^2)$	II	01 (66)	10 (71)	22 (78)
	III	02 (69)	11 (91)	20 (92)
2	IV	00 (46)	11 (62)	22 (58)
Confounds $(AB^2; A^2B)$	V	01 (65)	12 (61)	20 (76)
	VI	02 (34)	10 (50)	21 (66)

(c) Calculate the least squares estimates of a pair of orthogonal contrasts for (AB^2; A^2B) from the first replicate and the estimates of a pair of orthogonal contrasts for (AB; A^2B^2) from the second replicate. Using these contrast estimates, calculate the sum of squares for the AB interaction (adjusted for blocks).

(d) Prepare an analysis of variance table. Test any hypotheses that you think are of interest and state your conclusions about the two factors and their interaction.

(e) Check your analysis in part (d) using a computer program.

6. **Sugar beet experiment**

F. Yates, in a 1935 paper published in a supplement to the *Journal of the Royal Statistical Society*, describes an agricultural experiment on the yield of sugar beet. The three factors of interest were three standard fertilizers, nitrogen, phosphate, and potassium (factors N, P, and K) each at three equally spaced levels. The experimental field was divided into $b = 3$ blocks and each block subdivided into $k = 9$ 0.1 acre plots. The experiment was designed so that the contrasts (NP^2K; N^2PK^2) were confounded with blocks. The randomized design and yields of sugar beet are shown in Table 14.19.

(a) Prepare an analysis of variance table for the data, assuming that the three-factor interaction is negligible.

(b) Investigate the linear and quadratic trends of the main effects and the two-factor interactions. Yates assumed in his analysis that the only important contrast for each two factor interaction was the linear×linear contrast. Is this assumption supported by your analysis?

(c) Draw any plots that help to illustrate the important features of the analysis.

7. **Example 14.3.2, continued**

In Example 14.3.2, p. 483, we showed one way of associating design factors F, G, H, and J of a $2^3 \times 4$ factorial experiment to the 2-level pseudofactors A–E of a specific design from Table 13.29.

Table 14.19 Data for the sugar beet experiment

Block I		Block II		Block III	
Levels of N, P, K	Yield	Levels of N, P, K	Yield	Levels of N, P, K	Yield
211	2575	121	2599	202	2189
120	2472	220	2517	020	2093
200	2411	022	2411	210	2354
002	2403	110	2252	111	2268
010	2220	212	2381	001	1926
021	2252	201	2067	122	2152
101	2295	102	2021	221	2349
112	2362	011	1953	012	2025
222	2434	000	1989	100	2106

Source Yates (1935).

There are 10 different ways to make this association (since there are 10 ways of selecting two of A–E to represent J_1 and J_2).

(a) Investigate the confounding schemes for each of the ten possible associations. Specifically, for each association, determine the number of contrasts confounded for each effect, and compare the results.
(b) State under which circumstances you would recommend each design.

8. Consider a $2^2 \times 3^2$ design confounding AB, $(CD^2; C^2D)$, and $(ABCD^2; ABC^2D)$.

(a) Give the design—namely, list the treatment combinations block by block.
(b) Describe how to randomize the design.
(c) Give a set of five orthogonal treatment contrasts that are confounded with blocks.

9. Suggest a confounding scheme for a $2^3 \times 3^3$ experiment in 12 blocks of size 18. Under what circumstances would the design be useful? Write out two blocks of the design.
10. Suggest a confounding scheme for a $2^2 \times 3^2 \times 4$ experiment in 12 blocks of size 12. Under what circumstances would the design be useful? Write out two blocks of the design.
11. Suggest a confounding scheme for a $2^2 \times 3^2 \times 6$ experiment in 9 blocks of size 24. Under what circumstances would the design be useful? Explain how to find the blocks of the design.
12. Suggest a confounding scheme for a $2^2 \times 3^2 \times 6$ experiment in 12 blocks of size 18. Under what circumstances would the design be useful? Write out two blocks of the design.

Table 14.20 Confounding schemes for 3^p experiments in $b = 3^s$ blocks of size $k = 3^{p-s}$. For each design, s independent generators are underlined, and s corresponding equations are given. To obtain Block I of a design, list all k combinations of the first a_i's shown, then use the equations modulo 3 to complete each treatment combination

3^p	b	k	Confounded contrasts	Block I
3^2	3	3	$(\underline{AB}; A^2B^2)$	a_1 $a_2 = 2a_1$
3^3	3	9	$(\underline{ABC^2}; A^2B^2C)$	a_1, a_2 $a_3 = a_1 + a_2$
3^3	9	3	$(\underline{AB^2}; A^2B)$, $(\underline{AC}; A^2C^2)$, $(BC; B^2C^2)$, $(ABC^2; A^2B^2C)$	a_1 $a_2 = a_1$ $a_3 = 2a_1$
3^4	3	27	$(\underline{ABCD^2}; A^2B^2C^2D)$	a_1, a_2, a_3 $a_4 = a_1 + a_2 + a_3$
3^4	9	9	$(\underline{AB^2C}; A^2BC^2)$, $(\underline{ABD}; A^2B^2D^2)$, $(AC^2D^2; A^2CD)$, $(BCD^2; B^2C^2D)$	a_1, a_2 $a_3 = 2a_1 + a_2$ $a_4 = 2a_1 + 2a_2$
3^5	9	27	$(\underline{ABCD^2}; A^2B^2C^2D)$, $(\underline{AB^2E^2}; A^2BE)$ $(AC^2DE; A^2CD^2E^2)$, $(BC^2DE^2; B^2CD^2E)$	a_1, a_2, a_3 $a_4 = a_1 + a_2 + a_3$ $a_5 = a_1 + 2a_2$