Dynamics of rigid Bodies

5

Chapter 5

Up to now we have discussed idealized bodies where their spatial extension could be neglected and they were therefore adequately described by the model of a point mass. We have investigated their motion under the influence of forces and have presented besides Newton's laws fundamental conservation laws for linear momentum, energy and angular momentum.

All phenomena found in nature which are due to the spatial extension of bodies demand for their explanation an extension of our model. Besides the translation of point masses, discussed so far, we have to take into account the fact that extended bodies can also rotate around fixed or free axes.

At first, we will restrict ourselves to the motion of free extended bodies under the influence of forces. The motion of single volume elements of an extended body against each other, that results in a deformation of the body will be discussed in the next chapter. Such still idealized extended bodies that do not change their form, are called **rigid bodies**.
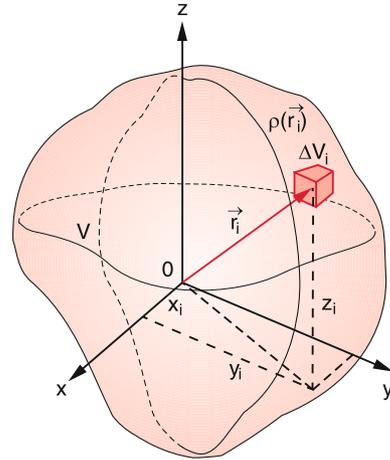
5.1 The Model of a Rigid Body

We can partition an extended rigid body with volume V and total mass M into many small volume elements ΔV_i with masses Δm_i which are rigidly bound together (Fig. 5.1). We can regard theses mass elements Δm_i as point masses and treat them according to the rules discussed in Chap. 2.

The total body can then be composed of these volume elements:

$$V = \sum_{i=1}^{N} \Delta V_i , \qquad M = \sum_{i=1}^{N} \Delta m_i .$$

We name the ratio

$$\varrho = \Delta m / \Delta V ; \qquad [\varrho] = \text{kg/m}^3 \qquad (5.1)$$

the mass density of the volume element ΔV. The total mass can then be expressed as

$$M = \sum_{i=1}^{N} \varrho_i \Delta V_i . \qquad (5.2)$$

If the volume elements ΔV become smaller and smaller, their number N correspondingly larger, the sums converge for the limiting case $\Delta V \to 0$ to volume integrals [5.1]

$$V = \lim_{\substack{\Delta V_i \to 0 \\ N \to \infty}} \sum_{i=1}^{N} \Delta V_i = \int_V dV ;$$

$$M = \int_V \varrho dV , \qquad (5.3)$$

where the volume integral stands for the three-dimensional integral

$$V = \int_{z_1}^{z_2} \left[\int_{y_1}^{y_2} \left(\int_{x_1}^{x_2} dx \right) dy \right] dz \qquad (5.4)$$

Figure 5.1 Partition of a spatially extended body into small volume elements ΔV_i

for the example of a cuboid, while for a spherical volume with radius R and a volume element $dV = r^2 \cdot \sin \vartheta \cdot dr \cdot d\vartheta \cdot d\varphi$ (see Sect. 13.2) the integral can be written as

$$V = \int_{r=0}^{R} \int_{\vartheta=0}^{\pi} \int_{\varphi=0}^{2\pi} r^2 \, \sin \vartheta \, dr \, d\vartheta \, d\varphi . \qquad (5.5)$$

The mass density $\varrho(x, y, z)$ can generally depend on the location (x, y, z). For homogeneous bodies ϱ is constant for all points of the body and we can extract ϱ out of the integral. The mass M of the body can then be expressed as

$$M = \varrho \int_V dV = \varrho V . \qquad (5.6)$$

5.2 Center of Mass

As has been shown in the previous chapter the position vector $\boldsymbol{r}_S$ of the CM of a system with N particles at the positions $\boldsymbol{r}_i$ (Fig. 5.2) is

$$\boldsymbol{r}_S = \frac{\sum_{i=1}^{N} \boldsymbol{r}_i \Delta m_i}{\sum_{i=1}^{N} \Delta m_i}$$

$$= \frac{1}{M} \sum_{i=1}^{N} \boldsymbol{r}_i \varrho(\boldsymbol{r}_i) \Delta V_i . \qquad (5.7)$$

For the limiting case $\Delta V \to 0$ and $N \to \infty$ this becomes

$$\boldsymbol{r}_S = \frac{1}{M} \int_V \boldsymbol{r} \, dm$$

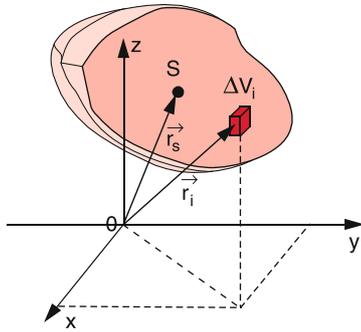$$= \frac{1}{M} \int_V \boldsymbol{r} \varrho(\boldsymbol{r}) \, dV . \qquad (5.8a)$$

Figure 5.2 Definition of the Center of Mass of an extended body

This corresponds to the three equations for the components

$$x_S = \frac{1}{M} \int_V x \varrho(x, y, z)\, dV\ ,$$

$$y_S = \frac{1}{M} \int_V y \varrho(x, y, z)\, dV\ ,$$

$$z_S = \frac{1}{M} \int_V z \varrho(x, y, z)\, dV\ .$$

For homogeneous bodies ($\varrho = $ const) (5.8a) simplifies to

$$\mathbf{r}_S = \frac{1}{V} \int_V \mathbf{r}\, dV\ . \tag{5.8b}$$

Example

Center of Mass of a homogeneous hemisphere.

If the center of the sphere is at the origin ($x = y = z = 0$) (Fig. 5.3) symmetry arguments require $x_S = y_S = 0$. For $\varrho = $ const we obtain from (5.8b)

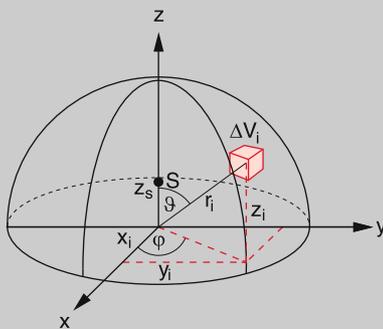$$z_S = \frac{1}{M} \int_V z \varrho\, dV = \frac{1}{V} \int_V z\, dV\ .$$

Figure 5.3 Center of Mass of a hemisphere

This becomes with $z = r \cdot \cos \vartheta$ and $dV = r^2 dr \cdot \sin \vartheta\, d\vartheta\, d\varphi$

$$z_S = \frac{1}{V} \int_{r=0}^{R} \int_{\vartheta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} r^3 \cos \vartheta \sin \vartheta\, dr\, d\vartheta\, d\varphi \tag{5.9}$$

$$= \frac{3}{8} R\ .$$

◄

5.3 Motion of a Rigid Body

The center points P_i of the volume elements dV_i are defined by their position vectors $\mathbf{r}_i$, the CM by $\mathbf{r}_S$. The vector

$$\mathbf{r}_{iS} = \mathbf{r}_i - \mathbf{r}_S$$

points from the center of mass S to the point P_i (Fig. 5.4). The vector

$$d\mathbf{r}_{iS}/dt = \mathbf{v}_{iS} = \mathbf{v}_i - \mathbf{v}_S \tag{5.10}$$

gives the relative velocity of P_i with respect to the CM.

In a rigid body all distances are fixed, i.e. $|\mathbf{r}_{iS}| = $ const. Differentiation of $r_{iS}^2 = $ const gives

$$2\mathbf{r}_{iS} \cdot \mathbf{v}_S = 0\ ,$$

which implies that the vector of the relative velocity is perpendicular to the position vector. This can be written as (see Sect. 2.4)

$$\mathbf{v}_{iS} = (\boldsymbol{\omega} \times \mathbf{r}_{iS})\ , \tag{5.11}$$

where ω is the angular velocity of P_i rotating about an axis through the CM perpendicular to the velocity vector $\mathbf{v}_{iS}$.

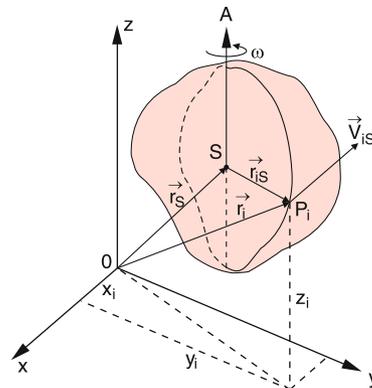**Figure 5.4** The motion of a rigid spatially extended body

Chapter 5

For the general motion of the rigid body the velocity of the point P_i

$$v_i = v_S + (\omega \times r_{iS}) \tag{5.12}$$

can be composed of two contributions: The translational velocity v_S of the CM and the rotation $(\omega \times r_{iS})$ of P_i around the CM. Since the consideration discussed above is valid for an arbitrary point P_i we can make the general statement:

> The motion of an extended rigid body can always be composed of the translation of its CM and a rotation of the body about its CM.

Note: The rotational axis is not necessarily constant but can change its direction in the course of time, even when no external forces act onto the body (see Sect. 5.7).

The Eq. 5.10 and 5.11 are based on the condition $r_{ik}^2 = $ const for a rigid body. They are no longer valid, if deformations of the body occur, because then vibrational motions of P_i against the CM can be present as additional movements.

The complete characterization of the motion of a free rigid body demands 6 time-dependent parameters: The position coordinates

$$r_S(t) = \{x_S(t), y_S(t), z_S(t)\}$$

for the description of the CM-motion and three angular coordinates for the description of the rotation of the rigid body about its CM.

> The free rigid body has six degrees of freedom for its motion.

If one point of the body (for example the CM) is kept fixed the body can still rotate about this point but cannot perform a translation. The number of degrees of freedom then reduces to three, namely the three rotational degrees of freedom. If the body rotates around a fixed axis, only one degree of freedom is left, namely the one-dimensional rotation described by the angle φ.

5.4 Forces and Couple of Forces

While a force F acting on a point mass is unambiguously defined when its magnitude and direction is given, for forces acting on an extended body the point of origin P has to be added (Fig. 5.5).

We will investigate the change of motion which an extended body suffers under the action of a force $F(P_i)$ with its origin in an arbitrary point P_i. We can simplify the treatment when we add two antiparallel forces $F_2(S)$ and $F_3(S) = -F_2(S)$ with equal magnitude which both act on the center of mass S and therefore do not affect the motion of the body, because they act
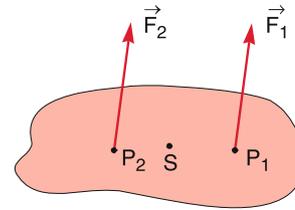

Figure 5.5 The two forces F_1 and F_2 have equal magnitudes but different points of application P_1 and P_2. They effect different motions of the body

on the same point S and since $F_2(S) + F_3(S) = 0$ they cancel each other.

Now we combine the two antiparallel forces F_1 and F_3 with equal magnitude (Fig. 5.6) which form a **couple of forces**, but regard at first the remaining single force F_2, which acts on the center of mass S. This force causes a translation of the CM. The couple of forces brings about a torque

$$D_S = (r_{iS} \times F_1) , \tag{5.13a}$$

referred to the center of mass S. Since $F_1 + F_3 = 0$ this couple of forces does not cause an acceleration of the CM. It induces, however, a rotation of the body around S. Summarizing we can make the general statement:

> A force F acting on an arbitrary point $P \neq S$ of an extended body causes an acceleration of the CM and a rotation of the body about the center of mass S.
>
> An extended rigid body initially at rest suffers an accelerated translation of its center of mass S and a rotation around S when a force F acts on a point $P \neq S$.

In this chapter we will investigate such motions in more detail.

At first we will restrict the treatment to the special case where the body rotates around a space-fixed axis. The motion is then restricted to a rotation which has only one degree of freedom.
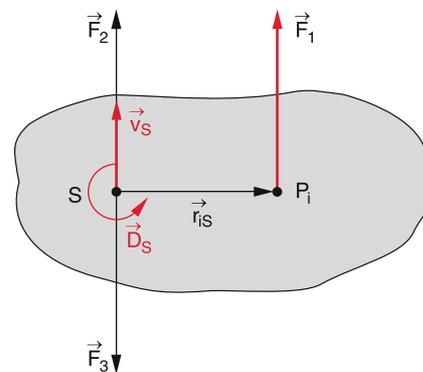

Figure 5.6 Decomposition of a force F_1 into a couple of forces $F_1 F_3$ and a force F_2 that attacks at the Center of Mass
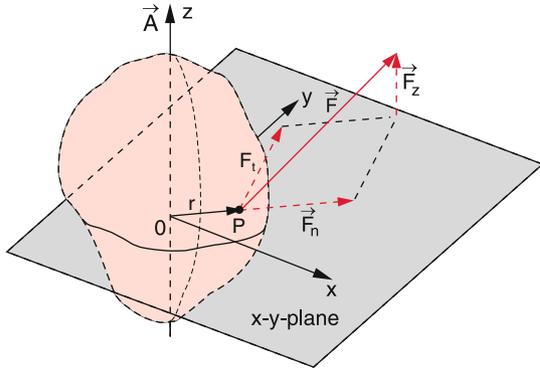
Figure 5.7 Rotation of a rigid body about a fixed axis A, induced by a force F attacking at the point P. Decomposition of the projection of F onto the x-y-plane into a normal and a tangential component

We choose the direction of the rotation axis A as the z-direction and the x-y-plane through the point $P(r)$ where the force F acts on the body, which has the distance r from the rotation axis (Fig. 5.7). We can then decompose the force into the three components $F_z \parallel A$, the radial component $F_r \parallel r$ and the tangential component $F_t \perp r$ and $\perp A$. F_z is perpendicular to the x-y-plane and the other two components are in the x-y-plane.

The torque exerted by the force F onto the body is

$$D = (r \times F) = (r \times F_t) + (r \times F_z) \, ,$$

since $(r \times F_n) = 0$.

The first term causes a torque about the z-axis, and therefore an acceleration of the rotation about the z-axis. The second term would change the direction of the rotation axis. If this axis is fixed by axle bearings the torque only acts on the bearings and does not lead to a change of motion.

If the rotation axis intersects the center of mass S (Fig. 5.8) which we choose as the origin of our coordinate system, the torque exerted by the weight $F_w = M \cdot g$ of the body is zero, as can be seen by the following derivation:

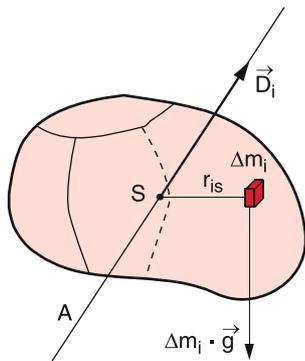The torque with respect to the rotation axis caused by the gravitational force $\Delta m_i \cdot g$ on the mass element Δm_i is $D_i =$

Figure 5.8 Contribution of the mass element Δm to the torque about an axis through the Center of Mass, due to the weight of Δm

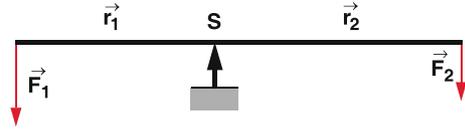**Figure 5.9** Principle of beam balance

$(r_{iS} \times \Delta m_i \cdot g)$. The torque exerted by the weight of the whole body is then

$$D = \int_V (r \times g) \, dm = -g \times \int_V r \, dm$$

$$= -(g \times Mr_S) = 0 \, , \tag{5.13b}$$

because the center of mass S is the origin and therefore is $r_S = 0$.

> If a body can rotate around an axis through the CM it is always at equilibrium, independent on the space orientation of the axis because the torque exerted by its weight is always zero.

All beam balances are based on this principle (Fig. 5.9). The balance is at equilibrium if $\sum D_i = 0$, which means

$$r_1 \times F_1 + r_2 \times F_2 = 0 \, .$$

This is the equilibrium condition for a balance as two-armed lever.

5.5   Rotational Inertia and Rotational Energy

We consider an extended body which rotates about a fixed axis A with the angular velocity ω (Fig. 5.10). The mass element Δm_i with the distance $r_{i\perp} = |r_i|$ from the axis A has the velocity $v_i = r_i \cdot \omega$. Its kinetic energy is then

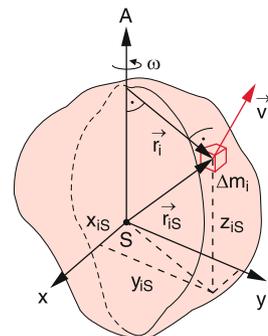$$E_{\text{kin}}(\Delta m_i) = \tfrac{1}{2}\Delta m_i v_i^2 = \tfrac{1}{2}\Delta m_i r_i^2 \omega^2 \, .$$

Figure 5.10 Definition of moment of inertia

Chapter 5

The summation over all mass elements gives the total rotational energy of the body

$$E_{\text{rot}} = \lim_{\substack{N \to \infty \\ \Delta m_i \to 0}} \left(\frac{1}{2} \sum_{i=1}^{N} \Delta m_i r_{i\perp}^2 \omega^2 \right) = \frac{1}{2} \omega^2 \int r_\perp^2 \, dm \ . \quad (5.14)$$

The expression

$$I \stackrel{\text{Def}}{=} \int_V r_\perp^2 \, dm = \int_V r_\perp^2 \varrho \, dV \qquad (5.15)$$

is called the *rotational inertia* (often also *moment of inertia*) of the rotating body referred to the axis A. With this definition we obtain for the rotational energy

$$E_{\text{rot}} = \tfrac{1}{2} I \omega^2 \ . \qquad (5.16a)$$

The angular momentum of Δm_i with respect to the axis A is

$$\boldsymbol{L}_i(\Delta m_i) = \boldsymbol{r}_{i\perp} \times (\Delta m_i \boldsymbol{v}_i) = r_{i\perp}^2 \Delta m_i \boldsymbol{\omega} \ , \qquad (5.17a)$$

which gives the total angular momentum of the body as

$$\boldsymbol{L} = I \cdot \boldsymbol{\omega} \ . \qquad (5.17b)$$

Replacing in (5.16a) ω^2 by L^2/I^2 we obtain for the rotational energy

$$E_{\text{rot}} = \frac{1}{2} I \omega^2 = \frac{L^2}{2I} \ . \qquad (5.16b)$$

The rotational inertia I is a measure for the mass distribution in an extended body relative to the rotational axis. For geometrically simple bodies with homogeneous mass distribution $\varrho = \text{const}$ the rotational inertia I can be readily calculated, as is illustrated in the following examples. For bodies with a complex geometrical structure I has to be measured (see below).

If the rotational axis A intersects the center of mass $S(r = 0)$, the rotational inertia can be written as

$$I_S = \varrho \cdot \int r^2 \, dV \ .$$

The rotational inertia is always defined with respect to a definite rotational axis and depends on the location of this axis with relative to the CM.

5.5.1 The Parallel Axis Theorem (Steiner's Theorem)

If a body rotates about an axis B which is parallel to the axis A through the CM, the rotational inertia with respect to the axis B
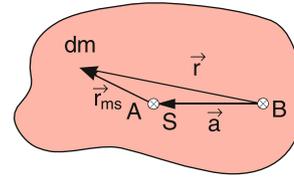

Figure 5.11 Steiner's rule: The drawing shows the plane through dm, perpendicular to the axis A

can be readily calculated, if it is known with respect to the axis A. If the distance between the two axes is a (Fig. 5.11) we can write

$$I_B = \int r^2 dm = \int (r_{mS} + a)^2 dm$$

$$= \int r_{mS}^2 dm + 2a \int r_{mS} dm + a^2 \int dm \ .$$

According to (5.8) is $\int r_{mS} dm = R_S \cdot M = 0$, because the center of mass S is the origin of the coordinate system and therefore is $R_{\text{CM}} = R_S = 0$.

We then obtain

$$\boldsymbol{I}_B = \boldsymbol{I}_S + \boldsymbol{a}^2 \boldsymbol{M} \ . \qquad (5.18)$$

Equation 5.18 is called the *parallel axis theorem* or *Steiner's theorem*. It states:

The inertial moment of a body rotating around an axis B is equal to the sum of the inertial moment with respect to an axis A through the center of mass S with a distance a from the axis B plus the moment of inertia of the total mass M concentrated in S with respect to B.

This illustrates that it is sufficient to determine the rotational inertia with respect to an axis A through S. With (5.18) we can then obtain the rotational inertia with respect to any axis parallel to A.

In the following we will give examples for the calculation of I for homogeneous bodies with different geometrical structures.

Example

1. *Thin disc.* We assume the height h in the z-direction is small compared to the extension of the body in the x- and y-directions.
   a) Rotational axis in the z-direction:

   $$I_z = \varrho \cdot \int (x^2 + y^2) dV \ .$$

   b) Rotational axis in the x-direction:

   $$I_x = \varrho \cdot \int (y^2 + z^2) dV \approx \varrho \cdot \int y^2 dV \ ,$$

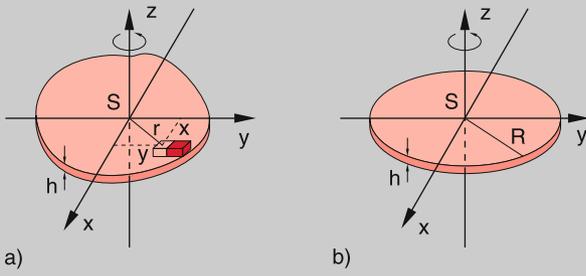   because $|z| \le h/2 \ll y_{\text{max}}$.

Figure 5.12 Moment of inertia of a thin disc with arbitrary shape (**a**), circular disc (**b**)

c) Rotational axis in the y-direction:

$$I_y = \varrho \cdot \int (x^2 + z^2) \mathrm{d}V \approx \varrho \cdot \int x^2 \mathrm{d}V \ .$$

This shows that with the approximation $z \ll x, y$ one obtains

$$I_z \approx I_x + I_y \ . \tag{5.19}$$

For plane bodies (for example a triatomic molecule) the rotational inertia for the rotation around an axis through the CM perpendicular to the plane is equal to the sum of the two other moments of inertia.

For the case of a thin circular disc we obaqin from (5.19) because of the rotational symmetry (Fig. 5.12b)

$$I_x = I_y = \tfrac{1}{2} I_z \ .$$

For the homogeneous circular disc with radius R it is not difficult to calculate I_z:

$$I_z = \varrho \cdot \int r^2 \mathrm{d}V = 2\pi h \varrho \int r^3 \mathrm{d}r = \varrho \cdot h \cdot \pi \cdot R^4/2 \ ,$$

because $\mathrm{d}V = 2\pi r \cdot h \cdot \mathrm{d}r$. With $M = \varrho \cdot \pi \cdot R^2 \cdot h$ this gives

$$I_z = \tfrac{1}{2} M R^2 \ . \tag{5.20a}$$

2. *Hollow cylinder* with height h, outer radius R and wall thickness $d \ll R$ (Fig. 5.13). Rotation about the z-axis as symmetry axis:

$$I_z = \varrho \int_V r^2 \mathrm{d}V = 2\pi h \varrho \int_{R-d}^{R} r^3 \, \mathrm{d}r \ ,$$

with $\mathrm{d}V = 2\pi \cdot R \cdot h \cdot \mathrm{d}r$ and $\mathrm{d} \ll R$ one obtains

$$I_z = h \cdot \varrho \cdot \pi \left[R^4 - (R-d)^4 \right] \approx 2\pi \varrho h R^3 d \ .$$

This gives with $M = 2\pi \cdot r \cdot \varrho \cdot d \cdot h$
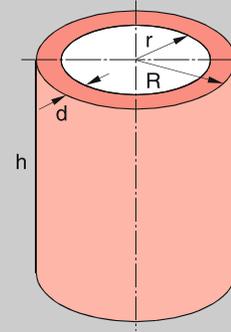
$$I_z = M \cdot R^2 \ . \tag{5.20b}$$

Figure 5.13 Moment of inertia of a hollow cylinder

Remark. A simpler derivation starts with the relation for the total mass of the hollow cylinder with outer radius r_2 and inner radius r_1:

$$M = \pi \cdot \varrho \cdot h \left(r_2^2 - r_1^2 \right) \ .$$

$$\begin{aligned}
I_z &= \int r^2 \, \mathrm{d}m = 2\pi \varrho \cdot h \int r^3 \mathrm{d}r = \tfrac{1}{2}\pi \varrho \cdot h \left(r_2^4 - r_1^4 \right) \\
&= \tfrac{1}{2}\pi \varrho \cdot h \left(r_2^2 - r_1^2 \right) \cdot \left(r_2^2 + r_1^2 \right) \\
&\approx \tfrac{1}{2} M \cdot 2R^2 = MR^2 \ .
\end{aligned}$$

3. *Full cylinder* with radius R and height h.

$$I_z = 2\pi \varrho h \int_0^R r^3 \, \mathrm{d}r = \frac{\pi}{2} h \varrho R^4 \tag{5.20c}$$

$$= \frac{M}{2} R^2 \ ,$$

which, of course, concurs with (5.20a).

4. *Thin rod* (length $L \gg$ diameter (Fig. 5.14))
   a) Rotation about the vertical axis $\boldsymbol{a}$ through the center of mass S.

$$I_S = \varrho \int x^2 \, \mathrm{d}V = \varrho A \int_{-L/2}^{+L/2} x^2 \, \mathrm{d}x \tag{5.21a}$$

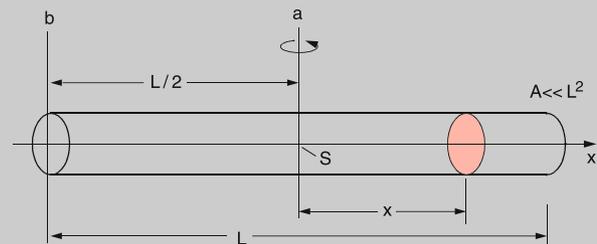$$= \frac{1}{12} \varrho A L^3 = \frac{1}{12} M L^2 \ .$$

Figure 5.14 Rotation of a thin rod with arbitrary cross section about an axis a through the Center of Mass and about an axis b at one end of the rod

Chapter 5

b) Rotation about the vertical axis b through an end point of the rod. According to the parallel axis theorem (5.18) the moment of inertia is

$$I_b = I_S + M \left(\frac{L}{2}\right)^2$$

$$= \frac{1}{12}ML^2 + \frac{1}{4}ML^2 = \frac{1}{3}ML^2 . \tag{5.21b}$$

This result could have been obtained also directly from

$$I_b = \varrho A \int_0^L x^2 \, dx = \frac{1}{3}\varrho A L^3 = \frac{1}{3}ML^2 .$$

5. *Diatomic molecule.* Because of the small electron mass ($m_e \approx 1/1836m_p$) the electrons do not contribute essentially to the moment of inertia when the molecule rotates around an axis A through the CM perpendicular to the inter-nuclear axis (Fig. 5.15). Because the diameter of the nuclei ($d \approx 10^{-14}$ m) is very small compared with the inter-nuclear distance R ($\approx 10^{-10}$ m) we can treat the nuclei as point masses and obtain

$$I_{SA} = m_1 r_1^2 + m_2 r_2^2 . \tag{5.22a}$$

With the inter-nuclear distance $R = r_1 + r_2$ and the reduced mass $\mu = m_1 \cdot m_2/(m_1 + m_2)$ (5.22a) becomes with $r_1/r_2 = m_2/m_1$

$$I_{SA} = \mu \cdot R^2 . \tag{5.22b}$$

When the molecule rotates around its inter-nuclear axis B the nuclei lie on the rotational axis and do not contribute to the moment of inertia. Now the electrons provide the major contribution. Because of the small electron mass the moment of inertia is now very small and the rotational energy

$$E_{\mathrm{rot}} = L^2/2I_B$$

for a given angular momentum L becomes much larger than for the rotation around A (see Chap. 11 and Vol. 3).
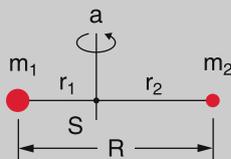
Figure 5.15 Moment of inertia of a diatomic molecule
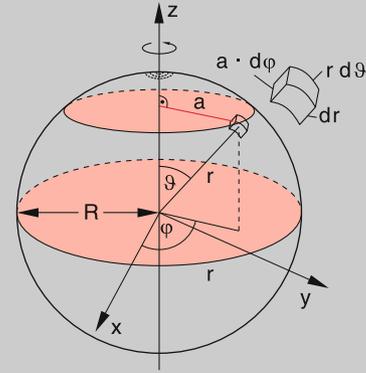
Figure 5.16 Derivation of the moment of inertia of a sphere

6. *Moment of inertia for a homogeneous sphere.* Because of the spherical symmetry the moment of inertia is independent of the direction of the rotational axis through the center of the sphere. The moment of inertia of a mass element Δm with a distance a from the rotation axis (Fig. 5.16) is $dI = a^2 dm$. For the whole sphere we obtain

$$I_S = \varrho \int a^2 dV$$

with $dV = r^2 \sin \vartheta \, dr d\vartheta d\varphi$ and $a = r \sin \vartheta$.

$$I_S = \varrho \int_{r=0}^{R} \int_{\vartheta=0}^{\pi} \int_{\varphi=0}^{2\pi} r^4 \sin^3 \vartheta \, dr \, d\vartheta \, d\varphi$$

$$= \frac{1}{5}\varrho R^5 2\pi \int_{\vartheta=0}^{\pi} \sin^3 \vartheta \, d\vartheta \tag{5.23}$$

$$= \frac{2}{5}\varrho R^2 \frac{4}{3}\pi R^3 = \frac{2}{5}MR^2 .$$

◀

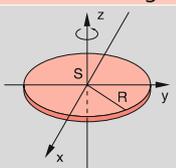These examples with their rotational inertia are compiled in Tab. 5.1.

5.6 Equation of Motion for the Rotation of a Rigid Body

For the rotation of a rigid body around a space-fixed axis the angular momentum of a mass element Δm_i is:

$$L_i = (r_{i\perp} \times p_i) = \Delta m_i (r_{i\perp} \times v_i) = \Delta m_i r_{i\perp}^2 \omega , \tag{5.24}$$

where the velocity v_i is perpendicular to the rotation axis (pointing into the direction of ω which is the z-direction) and to the radius r.

Table 5.1 Moments of inertia of some symmetric bodies that rotate about a symmetry axis

Geometrical figure	Realization	Moment of inertia
	Thin disc	$\frac{1}{2}MR^2$
	Hollow cylinder with thin wall	MR^2
	Full cylinder	$\frac{1}{2}MR^2$
	Thin rod $L \gg r$	$\frac{1}{12}ML^2$
	Homogeneous sphere	$\frac{2}{5}MR^2$
	Hollow sphere with thin wall	$\frac{2}{3}MR^2$
	Cuboid	$I_x = \frac{1}{12}M(b^2 + c^2)$ $I_y = \frac{1}{12}M(a^2 + c^2)$ $I_z = \frac{1}{12}M(a^2 + b^2)$
	Diatomic molecule	$I = \frac{m_1 m_2}{m_1+m_2} \cdot R^2$
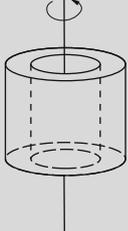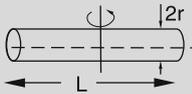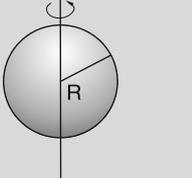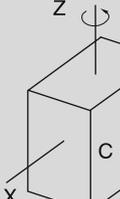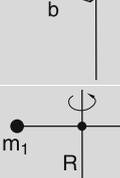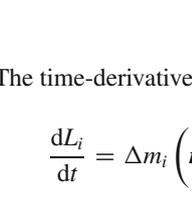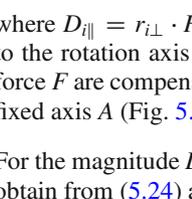
The time-derivative of (5.24) is

$$\frac{dL_i}{dt} = \Delta m_i \left(\boldsymbol{r}_{i\perp} \times \frac{d\boldsymbol{v}_i}{dt} \right) = (\boldsymbol{r}_{i\perp} \times \boldsymbol{F}_t) = \boldsymbol{D}_{i\parallel} , \quad (5.25)$$

where $D_{i\parallel} = r_{i\perp} \cdot F_i$ is the component of the torque $\boldsymbol{D}_i$ parallel to the rotation axis A. The other components F_z and F_r of the force F are compensated by elastic forces of the mounting of the fixed axis A (Fig. 5.17).

For the magnitude $D_i = |D_{i\parallel}|$ of the not compensated torque we obtain from (5.24) and (5.25)

$$D_i = \Delta m_i r_{i\perp}^2 \frac{d\omega}{dt} . \quad (5.26)$$
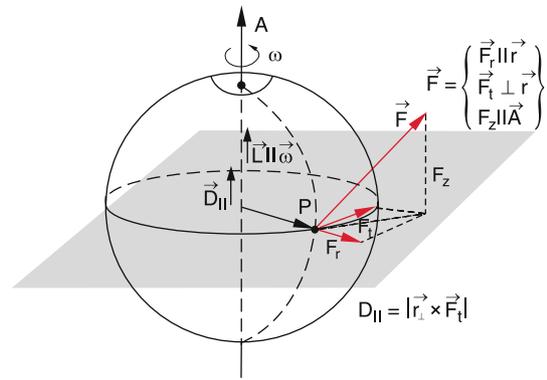
Figure 5.17 Torque acting on the rotation about a fixed axis induced by a force *F* attacking at the point *P*

The integration overall mass elements yields

$$D = I \cdot \frac{d\omega}{dt} = I \cdot \frac{d^2\varphi}{dt^2} , \quad (5.27)$$

where φ is the angle of *r* against the *x*-axis.

5.6.1 Rotation About an Axis for a Constant Torque

For a constant torque which does not change with time the integration of (5.27) yields the equation of rotation analogous to (2.6) for the translation of a body

$$\varphi = \frac{D}{2I}t^2 + At + B . \quad (5.28a)$$

The integration constants A and B are specified by the initial conditions $\varphi(0) = \varphi_0$ and $d\varphi(0)/dt = \omega_0$. This gives for (5.28a)

$$\varphi = \frac{D}{2I}t^2 + \omega_0 t + \varphi_0 . \quad (5.28b)$$

Example

1. A full cylinder, a hollow cylinder and a ball with equal masses M and equal radii r role down an inclined plane. All three bodies start at the same time. Who will win the race? The question can be answered experimentally as demonstration experiment during the lecture and arises always astonishment.
   Solution: The rotation takes place around the momentary rotation axis which is the contact line between the body and the plane (Fig. 5.18). The torque acting on the body is $D = M \cdot g \cdot \sin\alpha$ where α is the inclination angel of the plane. The rotational inertia is

according to the parallel axis theorem $I = I_S + Mr^2$. Equation 5.27 then becomes

$$Mgr \sin \alpha = (I_S + Mr^2)\, \dot{\omega} \, . \qquad (5.29)$$

The translational acceleration $a = \mathrm{d}^2s/\mathrm{d}t^2$ of the center of mass S is equal to the acceleration $r \cdot \mathrm{d}\omega/\mathrm{d}t$ of the perimeter which rolls on the inclined plane. This gives the relation

$$\frac{\mathrm{d}^2s}{\mathrm{d}t^2} = r\dot{\omega} = r\frac{Mgr \sin \alpha}{I_S + Mr^2}$$
$$= \frac{g \sin \alpha}{1 + I_S/Mr^2} = a \, . \qquad (5.30)$$

Compare this with the acceleration of a body which slides frictionless down the plane without rolling. In this case the acceleration is $a_t = g \cdot \sin \alpha$.

For the rolling body part of the potential energy is converted into rotational energy and only the rest is available for translational energy. The translational acceleration is reduced by the factor $b = (1 + I_S/Mr^2)$, which depends on the moment of inertia I_S of the rolling body. The race is therefore won by the body with the smallest moment of inertia. According to Sect. 5.5.1 these are:

Ball: $b = 7/5 \to a = 5/7 \cdot g \cdot \sin \alpha$,
Full cylinder: $b = 3/2 \to a = 2/3 \cdot g \cdot \sin \alpha$,
Hollow cylinder: $b = 2 \quad \to a = 1/2 \cdot g \cdot \sin \alpha$.

Therefore the ball wins the race barely before the full cylinder, while the hollow cylinder arrives last. It is instructive to consider the situation from another point of view: When the body has travelled the distance s from the starting point on the inclined plane the loss of potential energy is $\Delta E_{\mathrm{pot}} = M \cdot g \cdot h = M \cdot g \cdot s \cdot \sin \alpha$ which is converted into kinetic energy $E_{\mathrm{kin}} = E_{\mathrm{trans}} + E_{\mathrm{rot}} = \frac{1}{2}(Mv^2 + \omega^2 I_S) = \frac{1}{2}Mv^2(1 + I_S/Mv^2)$. This gives for the translational velocity

$$v^2 = \frac{2gs \sin \alpha}{1 + I_S/Mr^2} \, .$$

Differentiation yields with $\mathrm{d}(v^2)/\mathrm{d}t = 2 \cdot v \cdot a$ the result (5.30) for the acceleration a.
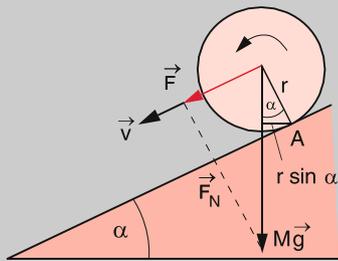

Figure 5.18 Rolling cylinder on an inclined plane

2. *Maxwell's Wheel.* A cylindrical disc with radius R, mass M and rotational inertia $I_S = \frac{1}{2}MR^2$ is centered on a thin axis through S with radius $r \ll R$. The disc hangs on a strand which is wrapped around the axis (Fig. 5.19). The mass of the axis should be negligible compared with the mass M of the disc. When the wheel is released it will roll down on the strand under the influence of the torque $\boldsymbol{D} = \boldsymbol{r} \times M\boldsymbol{g}$ and will move down with the acceleration

$$a = r\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \frac{rD}{I}$$
$$= \frac{r^2Mg}{\frac{1}{2}MR^2 + Mr^2}$$
$$= \frac{g}{1 + R^2/2r^2} \, .$$

The acceleration g is therefore reduced by the factor $(1 + \frac{1}{2}R^2/r^2)$. This allows to observe this small acceleration when performing the experiment. After the CM of the wheel has travelled the distance h the total potential energy Mgh has been converted into kinetic energy $Mgh = E_{\mathrm{kin}} = E_{\mathrm{trans}} + E_{\mathrm{rot}}$ where

$$E_{\mathrm{trans}} = \frac{1}{2}Mv_{\mathrm{trans}}^2 = \frac{1}{2}Mr^2\omega^2$$
$$= Mgh\frac{2r^2}{R^2 + 2r^2}$$

and

$$E_{\mathrm{rot}} = \frac{1}{2}I\omega^2$$
$$= Mgh\frac{R^2}{R^2 + 2r^2} \, .$$

Hint: The result is obtained from the relations $r^2\omega^2 = v_t^2$ and $v_t^2 = g^2T^2/(1 + \frac{1}{2}R^2/r^2)^2$ with the fall time $T = [(2h/g)(1 + \frac{1}{2}R^2/r^2)]^{1/2}$. By far the larger fraction $1/(1 + 2r^2/R^2)$ of the total energy is converted into rotational energy.

At the lowest point of its path where the strand is completely unwound, the wheel continues to rotate in the same sense (why?) and the strand winds up again, which causes the wheel to rise nearly up to the starting point. Because of frictional losses it does not completely reach its original starting heights.
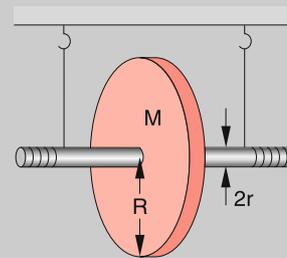

Figure 5.19 Maxwell's wheel

5.6.2   Measurements of rotational inertia; Rotary Oscillations About a Fixed Axis

The experimental determination of inertial moments for bodies with arbitrary form uses a rotary table consisting of a circular disc with a concentric axis which can turn in fixed ball bearings (Fig. 5.20). A coil spring with one end attached to the axis and the other end to the mounting is bent by the turn of the table and causes by its tension a restoring torque which is proportional to the displacement angle φ from the equilibrium position $\varphi = 0$ (see Sect. 6.2)

$$D = -D_\mathrm{r} \cdot \varphi \ . \tag{5.31}$$

The proportionality factor D_r is called *torsional rigidity*. Its value depends on the rigidity of the spring. The equation of motion (5.27) for this case is

$$I_0 \ddot{\varphi} = -D_\mathrm{r}\varphi \ , \tag{5.32a}$$

where I_0 is the inertial moment of the rotary table. We have neglected any friction. The solution of the differential equation (5.32a) is with the initial condition $\varphi(0) = 0$

$$\varphi = a \sin\left(\sqrt{D_\mathrm{r}/I_0}\, t\right) \ . \tag{5.32b}$$

Once deflected from its equilibrium position the rotary table performs a harmonic oscillation with the oscillation period

$$T_0 = 2\pi \sqrt{I_0/D_\mathrm{r}} \ . \tag{5.32c}$$

If a circular disc with known mass M and radius R is placed concentrically on the table, the moment of inertia increases to $I = I_0 + \frac{1}{2}MR^2$ and the oscillation period becomes

$$T_1 = 2\pi \sqrt{\left(I_0 + \tfrac{1}{2}MR^2\right)/D_\mathrm{r}} \ . \tag{5.32d}$$

From the difference $T_1^2 - T_0^2 = 2\pi^2 \mu R^2/D_\mathrm{r}$ the torsional rigidity D_r can be determined. Now a body A with arbitrary form can be placed on the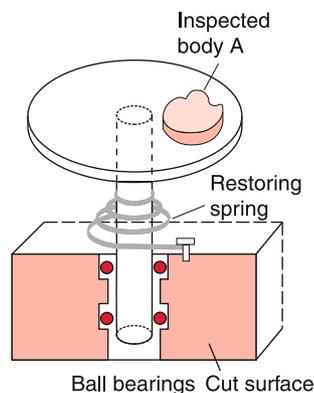 table. The total moment of inertia I_A now depends on the location of A with respect to the center of the table. The measured oscillation period

$$T = 2\pi \sqrt{(I_0 + I_A)/D_\mathrm{r}} \tag{5.32e}$$

allows the determination of I_A. With the parallel axis theorem (5.18) the moment of inertia I_S of A with respect to its center of mass S is $I_S = I_A - Ma^2$, where a is the distance between the center of the rotary table and the CM of A.

5.6.3   Comparison Between Translation and Rotation

Table 5.2 shows a comparison between corresponding quantities for the description of translation of a point mass and rotation of an extended body. Note the analogous notation for momentum, angular momentum, energy and power, if the mass m is replaced by the moment of inertia I.

5.7   Rotation About Free Axes; Spinning Top

Up to now we have discussed only rotations of rigid bodies about space-fixed axes. Even for the example of the cylinder rolling down the inclined plane the direction of the rotational axis remained constant although it performed a translation.

In the present section we will deal with the more complex situation that a body can rotate about a free axis, which might change its direction in space. We will treat at first the case that no external forces act on the body and then discuss the cases where external torques are present.

Such rigid bodies rotating about free axes are called *spinning tops* or *gyroscopes*.

Figure 5.20   Turntable with cut through the ball bearing

Table 5.2   Comparison of corresponding quantities for rotation and translation

Translation	Rotation
Length L	Angle φ
Mass m	Moment of inertia I
Velocity $\boldsymbol{v}$	Angular velocity $\boldsymbol{\omega}$
Momentum $\boldsymbol{p} = m \cdot \boldsymbol{v}$	Angular momentum $\boldsymbol{L} = I \cdot \boldsymbol{\omega}$
Force $\boldsymbol{F}$	Torque $\boldsymbol{D} = \boldsymbol{r} \times \boldsymbol{F}$
$\boldsymbol{F} = \dfrac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$	$\boldsymbol{D} = \dfrac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t}$
$E_\mathrm{kin} = \dfrac{m}{2}v^2$	$E_\mathrm{rot} = \dfrac{I}{2}\omega^2$
Restoring force	Restoring torque
$\boldsymbol{F} = -D \cdot \boldsymbol{x}$	$D = -D_\mathrm{r} \cdot \varphi$
Period of a linear oszillation	Period of torsional oszillation
$T = 2\pi \sqrt{m/D}$	$T = 2\pi \sqrt{I/D_\mathrm{r}}$

For the general motion one must take into account the translation of the CM (which can be always treated separately) and the rotation around the CM. If the motion is discussed in the CM-system where the CM is at rest, one has to regard only the rotation about the CM. We will see, that the space-orientation of free axes generally changes with time and the motion of an arbitrary point of the rigid body might perform a complicated trajectory.

In order to calculate the motion about free axes we have to determine the dependence of the moment of inertia on the direction of the rotation axis, which, however, always should intersect the CM.

5.7.1 Inertial Tensor and Inertial Ellipsoid

When a rigid body rotates with the angular velocity ω around an axis through the center of mass S with arbitrary space orientation (Fig. 5.21) the mass element Δm_i moving with the velocity $v_i = \omega \times r_i$ has the angular momentum

$$L_i = \Delta m_i \, (r_i \times v_i) = \Delta m_i \, (r_i \times (\omega \times r_i)) \; , \qquad (5.33a)$$

using the vector relation (see Sect. 13.1.5.4)

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C \; ,$$

this can be transformed into

$$L_i = \Delta m_i \left[(r_i^2 \cdot \omega) - (r_i \cdot \omega) \, r_i \right] \; . \qquad (5.33b)$$

The total angular momentum of the rigid body is then obtained by integration over all mass elements. This gives

$$L = \int (r^2 \omega - (r \cdot \omega) \, r) \, \mathrm{d}m \; . \qquad (5.34a)$$

This vector equation corresponds to the three equations for the components

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \; , \end{aligned} \qquad (5.34b)$$
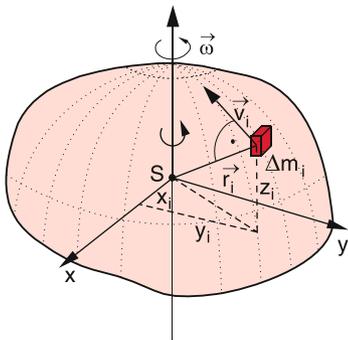

Figure 5.21 Rotation of a body about an arbitrary axis through the Center of Mass

where the coefficients I_{ik} are abbreviations for the expressions

$$\begin{aligned} I_{xx} &= \int (r^2 - x^2) \, \mathrm{d}m \\[4pt] I_{xy} &= I_{yx} = - \int xy \, \mathrm{d}m \\[4pt] I_{yy} &= \int (r^2 - y^2) \, \mathrm{d}m \\[4pt] I_{yz} &= I_{zy} = - \int yz \, \mathrm{d}m \\[4pt] I_{zz} &= \int (r^2 - z^2) \, \mathrm{d}m \\[4pt] I_{xz} &= I_{zx} = - \int xz \, \mathrm{d}m \; . \end{aligned} \qquad (5.35a)$$

Equation 5.34b can be readily checked when inserting the relations $r^2 = x^2 + y^2 + z^2$ and $r \cdot \omega = x\omega_x + y\omega_y + z\omega_z$ into (5.34a) and using (5.35a). The components I_{ik} can be written in form of the matrix

$$\widetilde{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \; , \qquad (5.35b)$$

which allows to write Eq. 5.34b in the vector form

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \; . \qquad (5.34c)$$

This can be shortened to

$$L = \widetilde{I} \cdot \omega \; . \qquad (5.34d)$$

The mathematical term for I is a *tensor of rank two*, which is called **inertial tensor**. The diagonal elements of I give the moments of inertia for rotation axes in the direction of the coordinate axes x, y, z.

To illustrate the advantage of introducing this inertial tensor we will at first calculate the rotational energy of the body for a rotation about an arbitrary axis ω. For a mass element Δm_i (Fig. 5.21) the rotational energy is

$$\begin{aligned} \tfrac{1}{2}\Delta m_i v_i^2 &= \tfrac{1}{2}\Delta m_i \, (\omega \times r_i) \, (\omega \times r_i) \\ &= \tfrac{1}{2}\Delta m_i \left[\omega^2 r_i^2 - (\omega \cdot r_i)^2 \right] \; , \end{aligned}$$

where the right hand side follows from the vector relation $(A \times B) \cdot (A \times B) = A^2 B^2 - (A \cdot B)^2$.

The spatial integration over all mass elements gives the rotational energy of the whole rigid body

$$E_{rot} = \frac{\omega^2}{2} \int r^2 dm - \frac{1}{2} \int (\boldsymbol{\omega} \cdot \boldsymbol{r})^2 dm$$

$$= \frac{\omega_x^2 + \omega_y^2 + \omega_z^2}{2} \int (x^2 + y^2 + z^2) dm$$

$$- \frac{1}{2} \int [\omega_x x + \omega_y y + \omega_z z]^2 dm \qquad (5.36)$$

$$= \frac{1}{2} [\omega_x^2 I_{xx} + \omega_y^2 I_{yy} + \omega_z^2 I_{zz}]$$

$$+ \omega_x \omega_y I_{xy} + \omega_x \omega_z I_{xz} + \omega_y \omega_z I_{yz} ,$$

where the definitions (5.35b) have been used. Within the tensor notation (5.36) can be written as

$$E_{rot} = \tfrac{1}{2} \boldsymbol{\omega}^T \cdot \tilde{\boldsymbol{I}} \cdot \boldsymbol{\omega} ,$$

which explicitly means

$$E_{rot} = \frac{1}{2} (\omega_x \omega_y \omega_z) \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} .$$

> This shows that for arbitrary orientations of the rotation axis all elements of the inertial tensor can contribute to the rotational energy.

When the rotation axis $\boldsymbol{\omega}$ forms the angles α, β, γ with the coordinate axes the components of $\boldsymbol{\omega}$ are

$$\omega_x = \omega \cdot \cos\alpha , \quad \omega_y = \omega \cdot \cos\beta , \quad \omega_z = \omega \cdot \cos\gamma .$$

When the rotational energy is written in the form of Eq. 5.16 as

$$E_{rot} = \tfrac{1}{2} I \omega^2 ,$$

the comparison with (5.36) yields for the scalar moment of inertia

$$I = \cos^2\alpha\, I_{xx} + \cos^2\beta\, I_{yy} + \cos^2\gamma\, I_{zz}$$
$$+ 2\cos\alpha \cos\beta\, I_{xy} + 2\cos\alpha \cos\gamma\, I_{xz} \qquad (5.37a)$$
$$+ 2\cos\beta \cos\gamma\, I_{yz} .$$

When we introduce a vector R in the direction of the rotation axis with the components $x = R \cdot \cos\alpha$; $y = R \cdot \cos\beta$; $z = R \cdot \cos\gamma$ Eq. 5.37a can be written as

$$R^2 I = x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz}$$
$$+ 2xy I_{xy} + 2xz I_{xz} + 2yz I_{yz} . \qquad (5.37b)$$

This is a quadratic equation in x, y, and z with constant coefficients I_{ik}. All points (x, y, z) for which $R^2 \cdot I = \text{const}$ are located on an ellipsoid because (5.37) describes for $R^2 \cdot I = k = \text{const}$
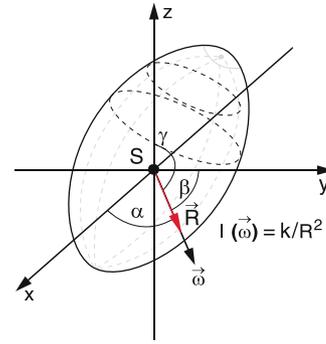

Figure 5.22 Inertial ellipsoid

an ellipsoid, with axes which depend on the coefficients I_{ik}. Since $I \propto M \cdot R_m^2$ the constant $k = M \cdot R_m^4$ has the dimension $[k] = \text{kg} \cdot \text{m}^4$. Its value depends on the mass M of the rigid body and the mass distribution relative to the center of mass S which is expressed by a mean distance R_m.

The moment of inertia $I_\omega = k/R^2$ for a rotation about an arbitrary axis $\boldsymbol{\omega} = \{\omega_x, \omega_y, \omega_z\}$ is proportional to $1/R^2$ where R is the distance from the center of the ellipsoid to its surface (Fig. 5.22). With this notation one can say that the scalar value I of the moment of inertia as a function of the spatial orientation (α, β, γ) of the rotation axis represents the **inertial ellipsoid**.

5.7.2 Principal Moments of Inertia

We introduce a coordinate system (ξ, η, ζ) which is generated by three orthogonal vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ with axes which point into the directions of the principal axes $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ of the inertial ellipsoid (Fig. 5.23). Their magnitude is normalized when dividing by $\sqrt{k}$. In this coordinate system the ellipsoid equation (5.37) becomes with $R^2 \cdot I = 1$

$$\xi^2 I_a + \eta^2 I_b + \zeta^2 I_c = 1 . \qquad (5.38)$$

In this *principal axis coordinate system* all off-diagonal elements I_{ik} with $i \neq k$ of the inertial tensor I are zero and the
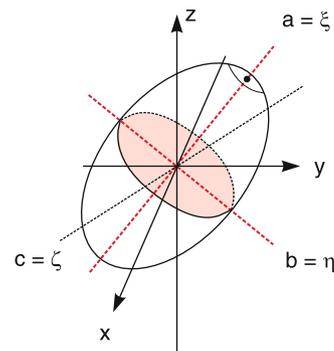

Figure 5.23 Definition of principal axes of inertia

tensor becomes a diagonal tensor

$$\widetilde{I} = \begin{bmatrix} I_a & 0 & 0 \\ 0 & I_b & 0 \\ 0 & 0 & I_c \end{bmatrix} . \tag{5.39}$$

Mathematically such a principal axes transformation can be performed by a diagonalization of the corresponding matrix [5.2]. The principal inertia moments I_a, I_b, I_c (i.e. the moments of inertia for rotations about the principal axes a, b, c) are the solutions of the determinant equation

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0 . \tag{5.40}$$

Note, that generally the principal moments of inertia do not concur with the elements I_{xx}, I_{yy}, I_{zz}, because all elements of the tensor can change under the principal axes transformation.

According to international agreements [5.3] the assignment of the principal moments follows the definition:

$$I_a \leq I_b \leq I_c .$$

The moment of inertia for a rotation about an arbitrary axis with direction angles α, β, φ, against the x, y, z, axis is (Fig. 5.24)

$$I = I_a \cos^2 \alpha + I_b \cos^2 \beta + I_c \cos^2 \gamma . \tag{5.41}$$

This equation corresponds to (5.37a) since all off-diagonal elements are zero. The principal axes transformation has made the expression for the general moment of inertia I simpler.
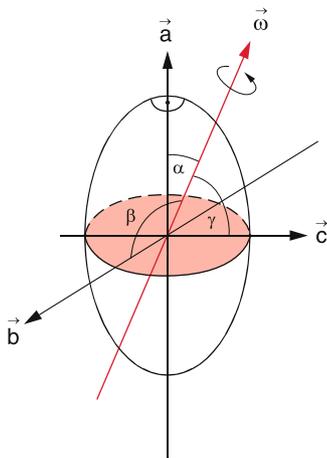

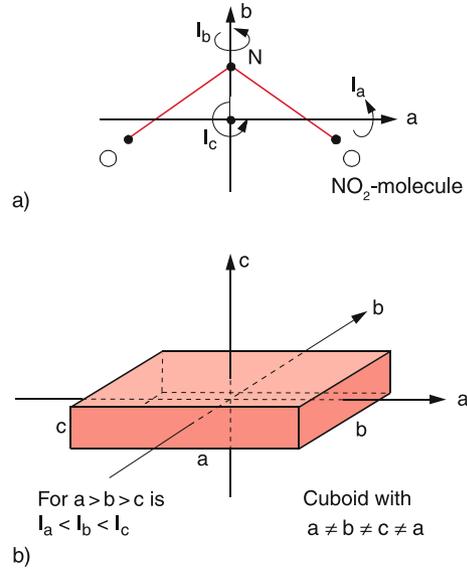**Figure 5.24** Inertial moment about an arbitrary axis

Figure 5.25 Examples of asymmetric tops

With the principal moments of inertia the angular momentum and the rotational energy can be written as

$$L = \{L_a, L_b, L_c\} = \{\omega_a I_a, \omega_b I_b, \omega_c I_c\} , \tag{5.42}$$

$$\begin{aligned} E_{\rm rot} &= \frac{1}{2} \left(\omega_a^2 I_a + \omega_b^2 I_b + \omega_c^2 I_c \right) \\ &= \frac{L_a^2}{2I_a} + \frac{L_b^2}{2I_b} + \frac{L_c^2}{2I_c} . \end{aligned} \tag{5.43}$$

If all three principal moments are different ($I_a \neq I_b \neq I_c \neq I_a$) the body is called an **asymmetric top**.

Example: A cuboid with three different side lengths a, b, c (Fig. 5.25b) or the NO_2 molecule (Fig. 5.25a).

If two principal moments of inertia are equal the body is called a **symmetric top**.

Example: All bodies with rotational symmetry (circular cylinder linear molecules but also quadratic cuboids).

Every body with rotational symmetry is a symmetric top, but a symmetric top has not necessarily a rotational symmetry (for example a quadratic post). The inertial ellipsoid of a symmetric top is, however, always rotationally symmetric.

We distinguish between

- *Prolate symmetric tops* (Fig. 5.26a) with $I_a < I_b = I_c$. The inertial ellipsoid is a stretched rotational ellipsoid where the diameter along the symmetry axis z is larger than the diameter in the x-y-plane (Fig. 5.27a).
- *Oblate symmetric tops* (Fig. 5.26b) with $I_a = I_b < I_c$. The inertial ellipsoid is a squeezed rotational ellipsoid (disc, Fig. 5.27b).
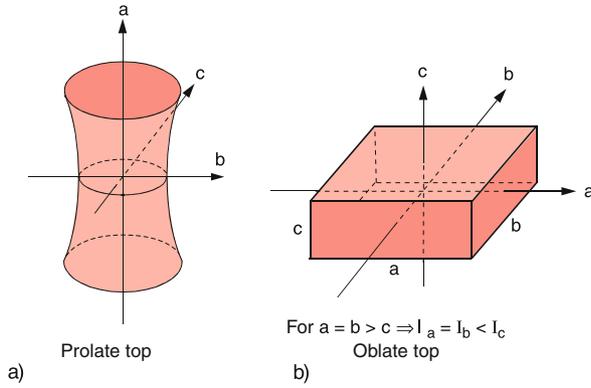
Figure 5.26 Examples of symmetric tops: **a** prolate and **b** oblate symmetric top
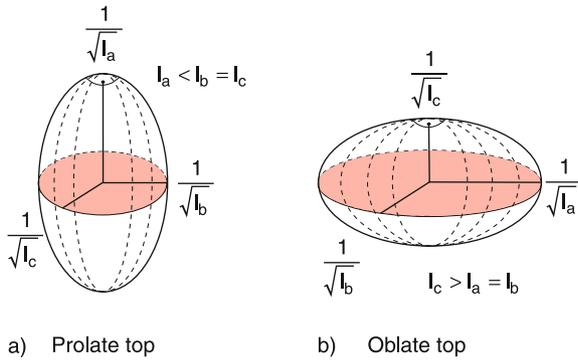

Figure 5.27 Inertial ellipsoids of **a** prolate and **b** oblate symmetric top

> For an asymmetric top the angular momentum *L* and the rotational axis $\omega = \{\omega_x, \omega_y, \omega_z\}$ generally point into different directions, because the components I_x, I_y, I_z, in Eq. 5.42 are different, except if the body rotates aboud one of its principal axes.

When all three principal moments of inertia are equal, the body is a spherical top, because in this case its inertial ellipsoid is a sphere.

Examples: A ball or a cube.

5.7.3   Free Rotational axes

The Eq. 5.42 and Fig. 5.28 give the following important information: Angular momentum *L* and rotational axis ω point for all bodies with free axes (where the rotation axis is not fixed by mountings) *only then* into the same direction if at least one of the following conditions is fulfilled.

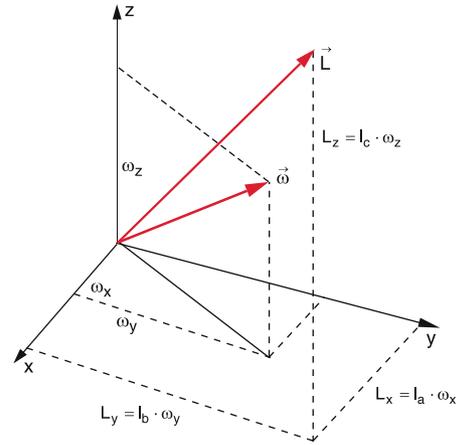■ $I_a = I_b = I_c$ (spherical top) or

Figure 5.28 Angular momentum *L* and rotational axis are generally not parallel. This is illustrated in a (x, y, z) coordinate system that coincides with the principal axes (a, b, c)

■ the body rotates about one of its principal inertial axes, which implies that only one of the three components ω_x, ω_y, ω_z is not zero.
■ For a symmetric top is $L \parallel \omega$ if the body rotates around an arbitrary axis perpendicular to the symmetry axis.

Since without external torque the angular momentum *L* is constant and has a constant orientation in space the body has in these three cases a space fixed rotational axis and rotates around this constant axis with constant angular velocity ω. Its rotational motion is then identical to the rotation about axes with fixed mountings (see Sect. 5.6)

The principal axes of a body are therefore also called *free axes* because the body can freely rotate about them even if they are not fixed by mountings.

The experiment shows, however, that a stable rotation is only possible about the axes of the smallest and the largest moment of inertia. For the rotation about a free axis of the intermediate moment of inertia any tiny perturbation makes the motion instable and the body finally flips into a rotation about one of the other two principal axes.

Examples

1. A cuboid with $I_a < I_b < I_c$ is suspended by a thread (Fig. 5.29) and can be induced to rotations about the thread by a small motor which twists the thread. The cuboid rotates stable if the thread direction coincides with the axis of the inertial moments I_a or I_c. If it is suspended in a way that the thread direction coincides with the axis of I_b the cuboid flips for faster rotations into a rotation around the axis b, as shown in Fig. 5.29c), it rotates then no longer about the thread but around the dashed line in Fig. 5.29c), which is a free axis.
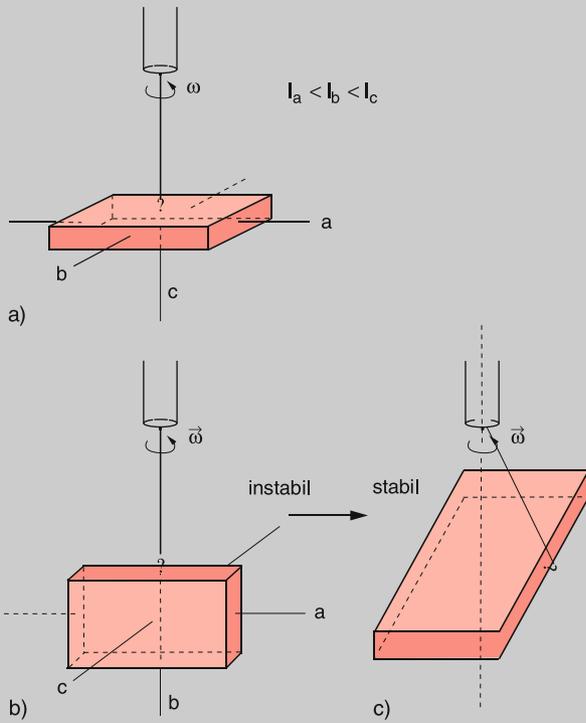
Chapter 5

Figure 5.29 Rotation of a cuboid about free axes: **a** stable rotation about the axis of maximum moment of inertia; **b** instable rotation about the axis of median moment of inertia, which jumps into a rotation about the axis c of maximum moment of inertia (**c**)

2. A closed chain hangs on a thread and is induced to rotations by a motor (Fig. 5.30). Due to the centrifugal force the chain widens to a circle which orientates itself in a horizontal plane, because in this position the rotation takes place about the axis of the maximum inertial moment and therefore the rotational energy $E_{rot}L^2/2I$ becomes a minimum. In this stable rotational mode the rotation axis does not coincide with the direction of the thread.
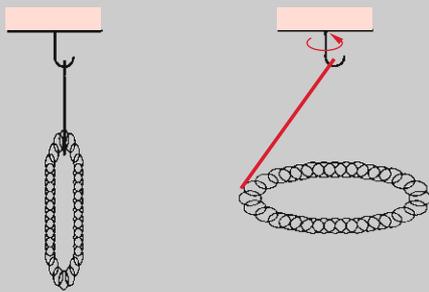

Figure 5.30 Rotation of a chain about the axis of maximum moment of inertia

3. A thrown discus flies stable as long as the rotation proceeds about the symmetry axis (axis of the maximum moment of inertia) (Fig. 5.31).
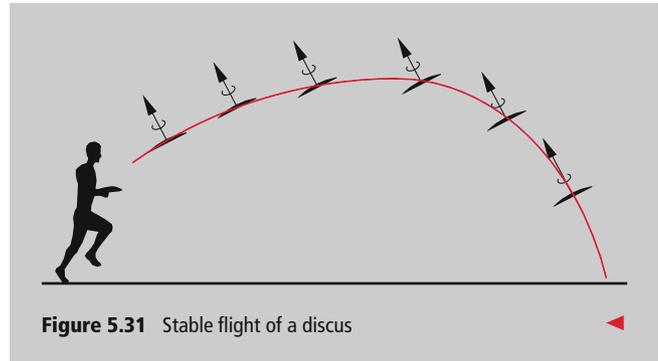

Figure 5.31 Stable flight of a discus ◀

5.7.4 Euler's Equations

For an arbitrary orientation of the rotation axis angular momentum L and rotation axis ω are no longer parallel. The motion of the body is now more complicated. In order to investigate this motion as seen by an observer sitting in a space-fixed inertial coordinate system S, we have to describe it in this system S.

The time derivative dL/dt of the angular momentum is equal to the torque D acting on the body.

$$\left(\frac{dL}{dt}\right)_S = D \ . \tag{5.44}$$

A coordinate system K where the axes are the principal axes of the body, which is therefore rigidly connected to the rotating body rotates with the angular velocity ω against the system S. In this system the time derivative of L is (see Sect. 3.3.2)

$$\left(\frac{dL}{dt}\right)_K = \left(\frac{dL}{dt}\right)_S - (\omega \times L) \ , \tag{5.45}$$

which gives the vector equation for the torque D

$$D = \left(\frac{dL}{dt}\right)_K + (\omega \times L) \ . \tag{5.46}$$

This equation corresponds formally to (3.14) if we replace L by r. **Note**, that in (5.46) dL/dt is the derivative of L in the body fixed principal axes system K, while ω is the angular velocity in the space-fixed system S. If (5.46) is written for the three components in the direction of the principal axes one obtains for example for the axis a the relation

$$\begin{aligned}
D_a &= \left(\frac{dL}{dt}\right)_a + (\omega \times L)_a \\
&= \frac{d}{dt}(I_a\omega_a) + (\omega_b L_c - \omega_c L_b) \\
&= I_a\frac{d\omega_a}{dt} + \omega_b I_c\omega_c - \omega_c I_b\omega_b \ ,
\end{aligned}$$

where D_a is the component of the torque in the direction of the principal axis a.

Similar equations can be derived for the other two components. This leads to the **Euler-equations**

$$I_a \frac{d\omega_a}{dt} + (I_c - I_b)\,\omega_c\omega_b = D_a$$

$$I_b \frac{d\omega_b}{dt} + (I_a - I_c)\,\omega_a\omega_c = D_b \qquad (5.47)$$

$$I_c \frac{d\omega_c}{dt} + (I_b - I_a)\,\omega_b\omega_a = D_c \,.$$

5.7.5 The Torque-free Symmetric Top

A symmetric top has two equal principal moments of inertia. If the symmetry axis of its inertial ellipsoid is the axis c we have $I_a = I_b \neq I_c$. For rotational symmetric bodies the symmetry axis is also called the *figure axis*. For a bicycle wheel as symmetric top this is the visible wheel axis (Fig. 5.32). Without any external torque ($D = 0$) the magnitude and the direction of the angular momentum L is constant. Such a top with $D = 0$ is called force-free top although it should be called more correctly torque-free top.

When the top rotates about its figure axis, L and ω coincide with this axis. The top rotates as if its axis would be hold by a stable mounting (see Sect. 5.6). If, however, ω points into an arbitrary direction which does not coincide with the figure axis the motion becomes complicated.

For the description of this motion one has to distinguish between three axes (Fig. 5.33a):

- The space-fixed angular momentum axis L
- The momentary (not space-fixed) rotation axis ω
- The figure axis of the symmetric top, which is only space-fixed, if L coincides with this axis.

We can win a qualitative picture for the motion of the figure axis by the following consideration: For $D = 0$ the angular momentum L and the rotational energy E_{rot} are both constant.

Figure 5.33 Figure axis c, angular momentum L and momentary rotation axis ω: **a** Decomposition of ω and L into the components parallel and perpendicular to the figure axis of the symmetric top. **b** Nutation cone of $\vec{L}$ and $\vec{\omega}$

Then we obtain from Eq. 5.43

$$L_x^2 + L_y^2 + L_z^2 = \text{const} = C_1 \,, \qquad (5.48a)$$

$$\frac{L_a^2}{I_a} + \frac{L_b^2}{I_b} + \frac{L_c^2}{I_c} = \text{const} = C_2 \,. \qquad (5.48b)$$

In a space-fixed coordinate system with the axes L_x, L_y, L_z (5.48a) represents the equation of a sphere. Equation 5.48b describes an ellipsoid in the principal axes system. Since the components of the space-fixed vector L must obey both equations, the endpoint of L can only move on the curve of intersection between sphere and ellipsoid (Fig. 5.34). Since the ellipsoid is determined by the principal axes system of the top, i.e. rotates with the top, while L is space-fixed, the top and therefore also its inertial ellipsoid move in such a way, that the endpoint of L always remains on the curve of intersection. This causes a nutation of the figure axis and the momentary rotation axis ω about the space-fixed axis L (Fig. 5.33b).

While the figure axis can be seen straight forward, the momentary rotation axis can be made visible by an experimental trick: A circular disc with red, black and white circular segments is
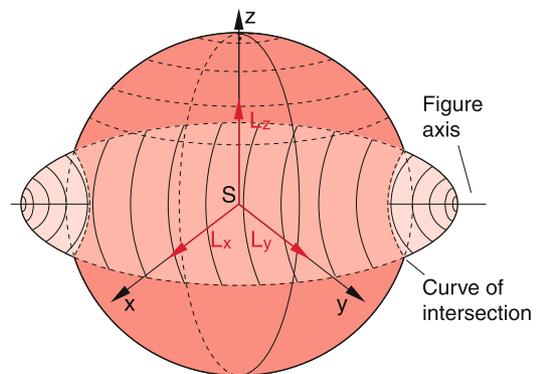

Figure 5.32 Bicycle wheel as symmetric top

Figure 5.34 The top of the angular momentum vector moves on the intersection curve of angular momentum sphere and energy ellipsoid
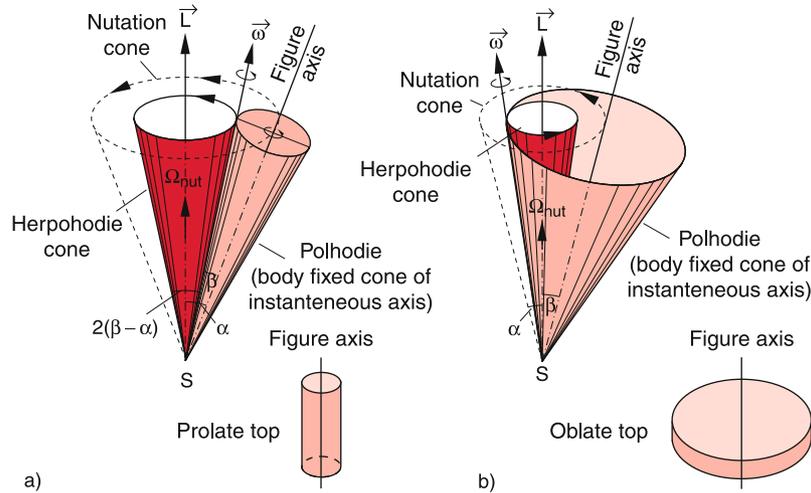
Figure 5.36 Nutation cone, herpolhode cone and polhode cone for **a** the prolate, **b** the oblate top
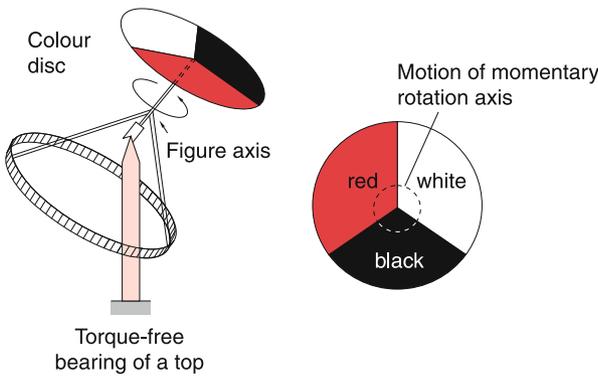

Figure 5.35 Visualization of the momentary rotational axis

centered on the peak of the figure axis (Fig. 5.35). When the top rotates the three colors blur to an olive-brown mixed color. Only at the intersection point with the momentary rotation axis one can see the color of the specific segment which wanders slowly from red over black to white which indicates the motion of the momentary rotation axis.

In order to calculate the motion of the figure axis and the momentary rotation axis more quantitatively we apply the Euler equations (5.47) to the special case $D = 0$ and $I_a = I_b$. They simplify to

$$\dot{\omega}_a + \Omega \omega_b = 0 \; ,$$
$$\dot{\omega}_b - \Omega \omega_a = 0 \; ,$$
$$\dot{\omega}_c = 0 \; ,$$

(5.49)

with the abbreviation $\Omega = ((I_c - I_a)/I_a)\omega_c$. The solutions of this system of equations are

$$\Omega_a = A \cos \Omega t \; , \quad \Omega_b = A \sin \Omega t \; ,$$
$$\Omega_c = C \quad \text{with} \quad A, C = \text{const} \; ,$$

(5.50)

as can be proved by inserting (5.50) into (5.49). While ω is the angular frequency of the spinning top, Ω is the frequency of the nutation. The solutions show, that the magnitude $\omega = |\boldsymbol{\omega}|$ is constant in the body-fixed system as well as in the lab-system, because $\omega^2 = \omega_a^2 + \omega_b^2 + \omega_c^2 = A^2 + C^2 = \text{const}$. However, the individual components ω_a and ω_b can change and therefore the direction of $\boldsymbol{\omega}$.

We separate ω into a component ω_c parallel to the figure axis and a component $\omega_\perp = \sqrt{(\omega_a^2 + \omega_b^2)} = A$ perpendicular to the figure axis (Fig. 5.33a). Squaring (5.42) yields then

$$\boldsymbol{L} = I_a \boldsymbol{\omega}_\perp + I_c \boldsymbol{\omega}_c \; .$$

(5.51)

The figure axis forms a constant angle α against the space-fixed angular momentum axis with

$$\tan \alpha = \frac{I_a \omega_\perp}{I_c \omega_c} = \frac{I_a}{I_c} \frac{\sqrt{\omega_a^2 + \omega_b^2}}{\omega_c} = \frac{I_a}{I_c} \cdot \frac{A}{\omega_c} \; .$$

This means that the figure axis migrates on a cone with the full aperture angle 2α around the space-fixed axis $\boldsymbol{L}$ (Fig. 5.33b and 5.36). This cone is called **nutation-cone**.

The vector $\boldsymbol{\omega}$ with its constant magnitude forms the constant angle β with the figure axis where $\sin \beta = \omega_\perp/\omega = A/\omega_c$. The momentary rotation axis $\boldsymbol{\omega}$ is also wandering on a cone with the opening angel $2(\beta - \alpha)$, called **herpolhode cone** around the space fixed axis of $\boldsymbol{L}$. This common motion of figure axis and momentary rotation axis without external torque is called **nutation**.

The common motion of figure axis and momentary rotation axis can be illustrated by two cones (nutation cone and herpolhode cone) with different opening angles centered around the space-fixed axis $\boldsymbol{L}$. A third cone (*polhode cone*) centered around the nutating figure axis touches the space fixed *herpolhode cone* along the momentary rotation axis and rolls on the outer surface (prolate top Fig. 5.36a) or the inner surface (oblate top Fig. 5.36b) of the herpolhode cone. The contact line shows the momentary rotation line $\boldsymbol{\omega}$. The apex of the three cones lies in the center of mass of the nutating body.

5.7.6   Precession of the Symmetric Top

If an external torque D acts on the body the angular momentum is no longer space-fixed, because of $D = \mathrm{d}L/\mathrm{d}t$. Depending on the direction of D relative to the figure axis the direction and also the magnitude of L changes with time. At first we will discuss the simplest case where the body rotates with the angular velocity ω around its figure axis c and all three axis L, ω and c coincide. In this case there is no nutation and for $D = 0$ the body would rotate with $\omega = \mathrm{const}$ about the space-fixed figure axis.

If the top is not supported in its CM, the gravitational force generates a torque

$$D = r \times m \cdot g \; ,$$

where r is the vector from the support point to the CM. If the symmetric top spins with the angular momentum L the torque is perpendicular to L and therefore changes only its direction but not its magnitude (Fig. 5.37). During the time interval $\mathrm{d}t$ the direction of L changes by the angle $\mathrm{d}\varphi$ and we can derive from Fig. 5.37

$$|\mathrm{d}L| = |L| \cdot \mathrm{d}\varphi \rightarrow D = \frac{\mathrm{d}L}{\mathrm{d}t} = |L|\frac{\mathrm{d}\varphi}{\mathrm{d}t} \; .$$

The angular momentum axis and with it the coincidental figure axis rotate with the angular velocity

$$\omega_{\mathrm{p}} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{D}{L} = \frac{D}{I\omega} \tag{5.52}$$

about an axis perpendicular to the plane of D and L where we have assumed that $\omega_{\mathrm{p}} \ll \omega$. This motion is called **precession**.

If the figure axis forms the angle α against the vertical direction the magnitude of the torque is $D = m \cdot g \cdot \sin\alpha$. The change $\mathrm{d}L$ of the angular momentum L is now for $\mathrm{d}L \ll L$ (Fig. 5.38)

$$\mathrm{d}L = |L|\sin\alpha \cdot \mathrm{d}\varphi \; .$$

For $\omega_{\mathrm{p}} \ll \omega$ therefore the equation for the precession frequency becomes

$$\omega_{\mathrm{p}} = \frac{mgr\sin\alpha}{I\omega\sin\alpha} = \frac{mgr}{I\omega} \; , \tag{5.53}$$
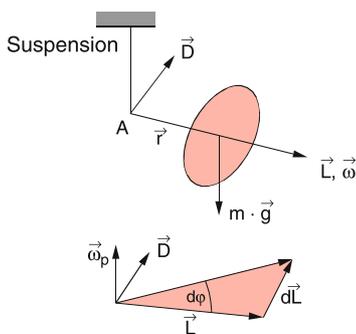

Figure 5.38   Precession of a spinning top

which shows that ω_{p} is independent of the space orientation of the figure axis and depends only on the angular momentum L and the torque D.

The general treatment of precession has to take into account the three vectors (Fig. 5.39)

1. The angular velocity ω_{F} about the figure axis
2. The angular velocity ω_{p} of precession around the vertical z-axis
3. The total angular velocity $\omega = \omega_{\mathrm{F}} + \omega_{\mathrm{p}}$

According to Fig. 5.39 we get the relations:

$$\begin{aligned} \omega_{\mathrm{F}} &= \omega \cdot e \quad \text{with } e = \{\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\} \\ \omega_{\mathrm{p}} &= \dot\varphi \cdot \{0, 0, 1\} \\ \omega &= \{\omega \cdot \sin\theta\cos\varphi, \omega \cdot \sin\theta\sin\varphi, \omega \cdot \cos\theta + \dot\varphi\} \; . \end{aligned} \tag{5.53a}$$

We separate ω into a component $\omega_{\parallel}$ parallel and $\omega_{\perp}$ perpendicular to the figure axis e.

$$\begin{aligned} \omega_{\parallel} &= e \cdot (\omega + \dot\varphi\cos\theta) \\ \omega_{\perp} &= e \times (\omega \times e) \\ &= \dot\varphi\sin\theta \cdot \{-\cos\theta\cos\varphi, -\cos\theta\sin\varphi, \sin\theta\} \; . \end{aligned} \tag{5.53b}$$

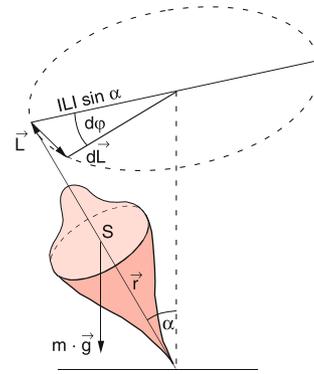**Figure 5.37**   The gravitational force causes a torque acting on a top, that is not supported in the Center of Mass

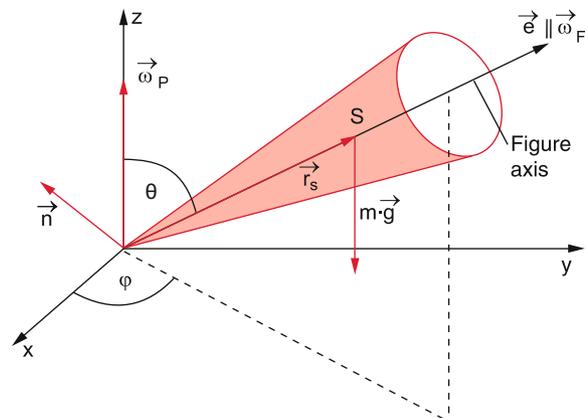**Figure 5.39**   Illustration of the equation of motion for a spinning top

The total angular momentum L is

$$
\begin{aligned}
L &= I_\parallel \boldsymbol{\omega}_\parallel + (mr_s^2 + I_\perp)\,\boldsymbol{\omega}_\perp \\
&= I_\parallel \cdot \boldsymbol{e}(\omega + \dot\varphi \cos\theta) \\
&\quad + (I_\perp + mr_s^2) \\
&\quad \cdot \dot\varphi \sin\theta\{-\cos\theta\cos\varphi, -\cos\theta\sin\varphi, \sin\theta\}\,,
\end{aligned}
\tag{5.53c}
$$

where $I_\parallel$ is the moment of inertia for a rotation about the figure axis and $I_\perp$ about an axis perpendicular to the figure axis.

Because ω, $\mathrm{d}\varphi/\mathrm{d}t$ and θ do not change with time the time derivative of (5.53c) is

$$
\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= I_\parallel(\omega + \dot\varphi\cos\theta)\cdot\dot{\boldsymbol{e}} \\
&\quad - (I_\perp + mr_s^2)\dot\varphi^2\cos\theta\sin\theta\{-\sin\varphi,\cos\varphi,0\} \\
&= \big[I_\parallel\cdot\sin\theta(\omega + \dot\varphi\cos\theta)\dot\varphi \\
&\quad - (I_\perp + mr_s^2)\dot\varphi^2\sin\theta\cos\theta\big]\cdot\hat{\boldsymbol{n}}\,.
\end{aligned}
\tag{5.53d}
$$

where $\boldsymbol{n} = \{-\sin\varphi,\cos\varphi,0\}$ is the unit vector in the direction of the torque D. With $\mathrm{d}L/\mathrm{d}t = D = m\cdot g\cdot r_s\cdot\sin\theta\cdot\boldsymbol{n}$ we obtain the equation

$$
\omega_p\cdot I_\parallel\cdot\omega + \omega_p^2\cos\theta(I_\parallel - I_\perp) = mgr_s\,.
\tag{5.53e}
$$

which has two solutions for the precession frequency ω_p. The difference between the two solutions depends on the difference $I_\parallel - I_\perp$ of the two moments of inertia [5.4].

5.7.7 Superposition of Nutation and Precession

In the general case the top does not rotate about its figure axis. Without external torque the top would perform a nutation around the space-fixed angular momentum axis L. With an external torque the angular momentum axis is no longer constant but precesses with the angular velocity ω_p around an axis through the underpinning point A parallel to the external force (Fig. 5.39) while the figure axis performs a nutation around the precessing axis L. With this combination of precession and nutation the end of the figure axis describes a complicated path (Fig. 5.40). The exact form of this trajectory depends on the ratio of nutation frequency Ω to the precession frequency ω_p.

For the demonstration of nutation and precession a special bearing of the top is useful called a *gimbal mounting* where the figure axis can be turned into arbitrary directions and the top is always "torque-free" (Fig. 5.41). This can be realized if the figure axis is mounted by ball bearings in a frame which can freely rotate around an axis perpendicular to the figure axis. The mounting of this axis can again rotate about a vertical axis. If the system is turned around the vertical axis, the figure axis of the top diverts from its horizontal direction. Reversal of the turning direction also reverses the direction of this diversion. If a short push is applied perpendicular to the top axis, the angular momentum
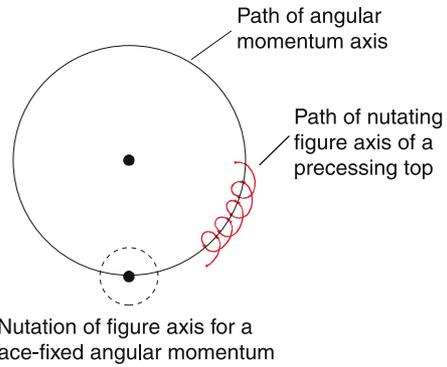

Figure 5.40 Path of figure axis when precession and mutation are superimposed

axis is forced into another direction and the top starts to nutate. When a mass m is attached to the first frame, a torque acts on the top which starts to precess around a vertical axis.

Before the invention of the GPS system the precession of the top was used for navigation purposes (gyro compass). Its function is explained in Fig. 5.42, where a rotating disc is suspended in a mounting B which can turn around a vertical axis a through the suspension point A. The top axis KA can freely turn only in a horizontal plane. The center of mass lies below the point A. Different from a torque-free top rotating around its figure axis where the figure axis and the angular momentum axis coincide, for the gyro compass the suspension axis through the point A is rigidly connected with the earth and participates in the earth rotation with the angular velocity ω. Therefore a torque is acting on the gyro perpendicular to the drawing plane. The gyro turns around the axis a until the figure axis is parallel to the rotation axis of the earth and points into the south-north direction. Now angular momentum axis L and the forced rotation axis ω_E are parallel (Fig. 5.42b), and the torque forcing the turn of the figure axis becomes zero.

This property can be experimentally studied with the gimbal mount by simulating the earth rotation by the rotation of the outer mounting in Fig. 5.41. The figure axis then turns into the vertical position.
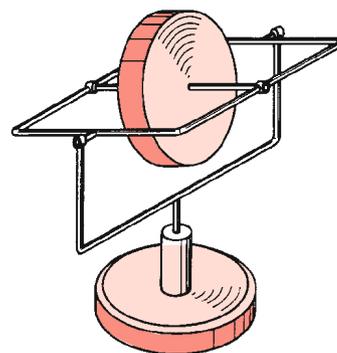

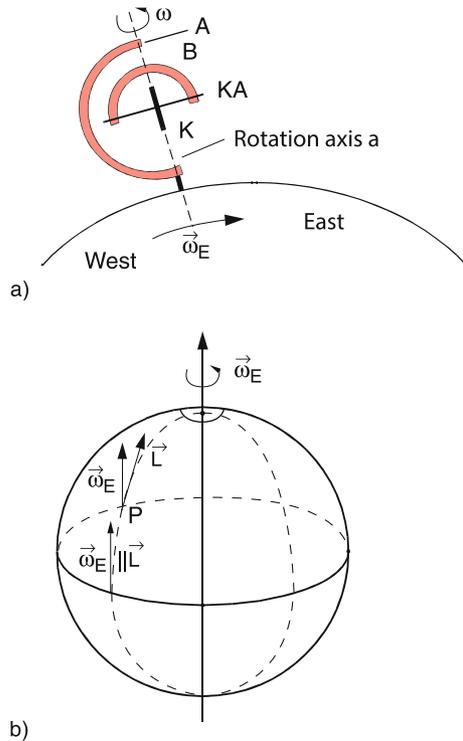**Figure 5.41** Gimbal mount of a symmetric top

Figure 5.42 Gyro-compass: **a** mount, **b** direction of L and ω_E at the equator and for higher lattitudes

Figure 5.43 The earth as symmetric top. The *arrows* indicate the difference forces $F_1 - F_2$ ath the Centers of Mass S_1 and S_2 of the two opposite sides of the bulge

At the earth equator ω_E and L are parallel. This is not the case for a point P on another circle of latitude because L has to lie in a horizontal plane. However, also in this case the gyro adjusts itself in such a way that the component L parallel to ω_E becomes maximum. The vector L becomes the tangent to the circle of longitude, points therefore again to the north. Only at the two poles of the earth the gyro fails, because here L is always perpendicular to ω_E.

5.8 The Earth as Symmetric Top

In a good approximation the earth can be described by a clinched rotational ellipsoid, i.e. an oblate symmetric top with $I_a = I_b < I_c$. The equator diameter is with $12\,756\,\text{km}$ by about $43\,\text{km}$ larger than the pole diameter with $12\,713\,\text{km}$. This clinch is caused by the centrifugal force due to the rotation of the earth (see Sect. 6.6). For the following considerations we will compose this oblate ellipsoid by a sphere plus additional bulges which have their maximum thickness at the equator (red area in Fig. 5.43).

Because of the inclination of the earth axis ($\varphi = 90° - 23.5° = 66.5°$) against the ecliptic (orbital plane of the earth's motion around the sun) the two centers of mass S_1 of the bulge towards the sun and S_2 of the bulge opposite to the sun are located above and below the ecliptic (Fig. 5.43) in contrast to the center of mass S of the sphere which lies in the ecliptic. While for the
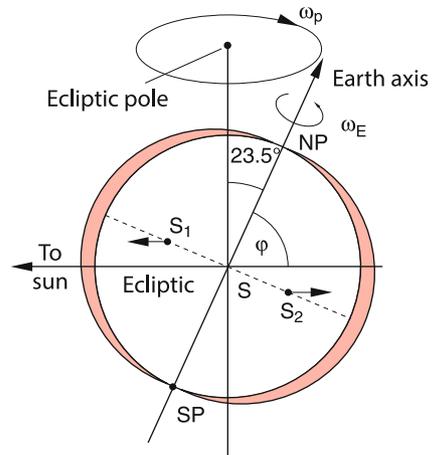
mass m_E of the sphere concentrated in S the centripetal force $F_1 = Gm_E M_\odot/r^2$ due to the gravitational attraction between earth and sun is just compensated by the equal but opposite centrifugal force $F_2 = m_E v_E^2/r$ this is no longer true for the centers of mass S_1 and S_2 of the bulges. Since S_1 is closer to the sun the centripetal force predominates while for S_2 the centrifugal force prevails. Since the net forces for S_1 and S_2 are antiparallel they form a couple of forces which act as a torque on the earth and cause the earth axis to precess (solar precession).

Besides the gravitational force between earth and sun the attraction between moon and earth must be taken into account. The calculation is here more complicated because the orbital plane of the moon around the earth is inclined by an angle of $5.1°$ against the ecliptic. The calculation shows that the influence on the earth is of the same order of magnitude than that of the sun.

Altogether both torques cause the lunar-solar precession where the earth axis propagates on a cone with an opening angle of $2 \times 23.5°$ by an angle $\varphi \approx 50''$ per year which gives a precession period of about $25\,750$ years for $\varphi = 2\pi$ (*Platonic year*). Within a Platonic year the cone is once circulated. The elongation of the earth axis describes a circle on the celestial sphere around the ecliptic pole (Fig. 5.44).

Remark. This precession causes a turn of the intersecting line between ecliptic and equatorial plane by $360°$ within $25\,850$ years. This shifts the vernal equinox (where day and night both last 12 hours) by about $50''$ per year. It causes furthermore a shift of the signs of the zodiac between their naming 2000 years ago and today by about one month. For example the real constellation of the *Gemini* (twins) coincides in our times with the sign of the zodiac *Cancer*. This is unknown to many astrological oriented people who come into trouble if they should explain whether the real stars or the signs of the zodiac are responsible for the fate of a person.

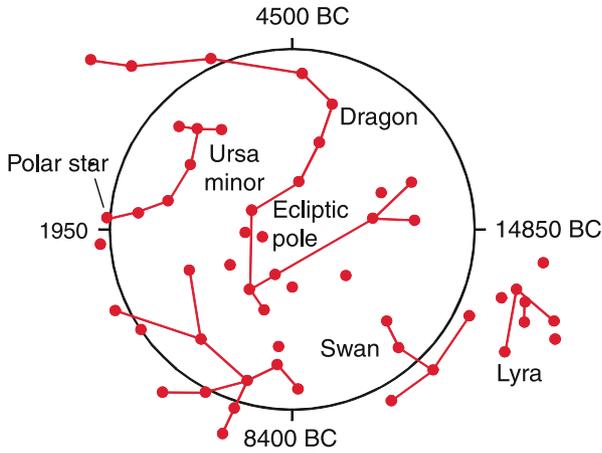The precession of the earth axis described above is not uniform because of the following reasons:

Chapter 5

Figure 5.44 Due to the precession of the earth its axis traverses a circle on the celestial sphere around the ecliptic pole. In 1950 it pointed towards the pole star

Figure 5.46 Nutation of the earth axis

- Because of the inclination of the earth axis the torque exerted by the sun changes periodically during the year (Fig. 5.45). It is maximal on December 22nd and June 21st and minimal at March 21st and September 23rd
- The torque exerted by the moon changes because the inclination of the moon's orbital plane changes with a period of 9.3 years.
- Also the other planets cause a small torque acting on the earth. Because the relative distances to the earth change in time, this causes a tiny variation of the precession.
- The motion of the earth around the sun proceeds on an elliptical path and therefore the distance r between earth and sun changes periodically. It is minimum in December and maximum in June (Fig. 5.45). Therefore the gravitational force acting on the earth changes correspondingly.

Astronomers call these short-period fluctuations of the precession *nutations* although they are strictly speaking no nutations but perturbations of the precession.

There are real torque-free nutations superimposed on the complicated precession. They are caused by the fact, that the figure axis of the earth and the rotation axis do not exactly coincide (Fig. 5.46). The figure axes (south-north pole intersection)
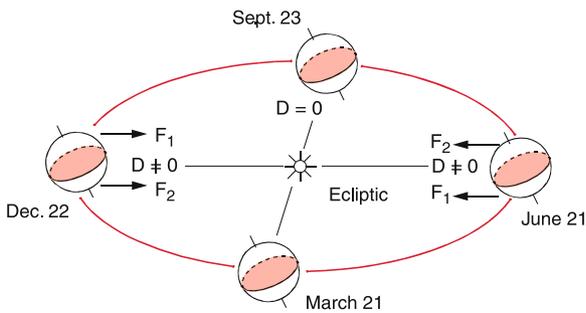
therefore nutates around the precessing angular momentum axis with a measure period of about 303 days. On the other hand the nutation period is

$$T_\text{nut} = \frac{2\pi}{\omega} \frac{I_a}{I_c - I_a} \ . \qquad (5.54)$$

From the measured nutation period one can therefore determine the difference $I_c - I_a$ of the inertial moments [5.5a, 5.5b].

Since the earth is no rigid body the mass distribution and therefore the inertial moments can change, for instance by volcanic eruptions or by convective currents in the liquid interior of the earth [5.6a, 5.6b]. This causes small fluctuations of the nutation. In Fig. 5.47 the wandering of the north-pole of the rotational axis during the year 1957 is shown.

The above discussion has shown, that the more precise measurements have been made, the more different influences on the motion of the earth axis have to be taken into account. Even today there are discussions about the best model for the earth motion [5.7, 5.8].

Figure 5.45 Position of the earth axis during the revolution of the earth about the sun. Note, that the direction of the angular momentum does not change during the year
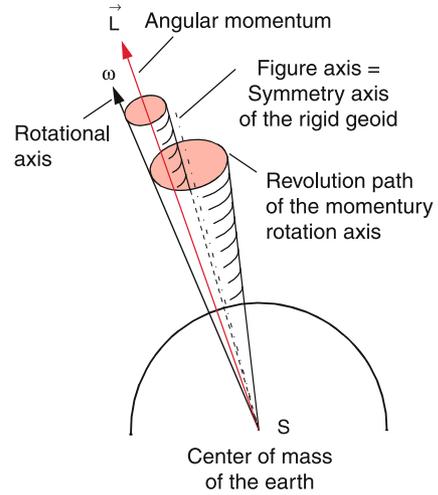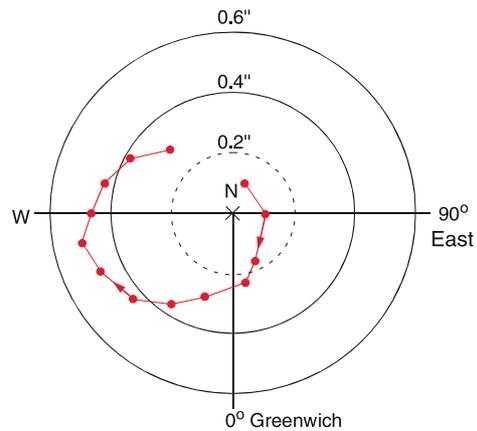

Figure 5.47 Migration of the north pole of the earth's rotation axis during the year 1957 about the average position of the period 1900–1905. One second of arc corresponds to about 30 m

Summary

- The model of the extended rigid body neglects all internal motions (Deformations and vibrations). The center of mass S has the coordinates

$$\boldsymbol{r}_S = \frac{1}{M} \int_V \boldsymbol{r}\varrho(\boldsymbol{r})\mathrm{d}V = \frac{1}{V} \int \boldsymbol{r}\mathrm{d}V \quad \text{for} \quad \varrho = \text{const} .$$

- The motion of a free rigid body can be always composed of a translation of the center of mass S with the velocity v_s and a rotation of the body around S with the angular velocity ω. The motion of the extended body has therefore 6 degrees of freedom.

- For the motion of an extended body not only magnitude and direction of the force acting on the body are important but also the point of action on the body.

- An arbitrary force acting on an extended body can always be composed of a force acting on the center of mass S (translational acceleration) and a couple of forces causing an accelerated rotation.

- The moment of inertia (rotational inertia) for a rotation about an axis through the center of mass S is $I_s = \int r_\perp^2 \varrho \mathrm{d}V$ where $r_\perp$ is the distance of the volume element $\mathrm{d}V$ from the rotation axis. The moment of inertia for a rotation around an arbitrary axis with a distance a from the parallel axis through S is $I = I_s + Ma^2$ (parallel axis theorem or Steiner's theorem).

- The kinetic energy of the rotational motion is $E_{\mathrm{rot}} = \frac{1}{2}I\omega^2$.

- The equation of motion for a body rotating about a space-fixed axis is $D_\| = I \cdot \mathrm{d}\omega/\mathrm{d}t$, where $D_\|$ is the component of the torque parallel to the rotation axis.

- The moment of inertia I_s depends on the direction of the rotation axis relative to a selected axis of the body. It can be described by a tensor. The directions of the axes with the maximum and the minimum inertial moment determine the principal axes system. In this system the tensor is diagonal. The diagonal elements are the principal moments of inertia.

- If two of the principal moments are equal, the body is a symmetric top. If all three are equal the body is a spherical top.

- Angular momentum L and angular velocity ω are related by $\boldsymbol{L} = I \cdot \boldsymbol{\omega}$, where I is the inertial tensor, which is diagonal in the principal axes system. In the general case L and ω are not parallel.

- If the body rotates about a principal axis, L and ω are parallel and without external torque their directions are space-fixed.

- For an arbitrary direction of ω the momentary rotation axis ω nutates around the angular momentum axis which is space-fixed without external torque.

- Under the action of an external torque the angular momentum axis L precedes around the external force and in addition the momentary rotation axis nutates around L. The relation between L and D is $\boldsymbol{D} = \mathrm{d}\boldsymbol{L}/\mathrm{d}t$.

- The general motion of a top is completely described by the Euler-equations.

- The earth can be approximately described by a symmetric top, which rotates about the axis of its maximum moment of inertia. The vector sum of the gravity forces exerted by the sun, the moon and the planets results in a torque which causes a periodic precession of the earth axis with a period of $25\,850$ years. In addition changes of the mass distribution in the earth cause a small difference between symmetry axis and momentary rotation axis. Therefor the earth axis performs an irregular nutation around the symmetry axis.

Problems

5.1 Determine the center of mass of a homogeneous sector of a sphere with radius R and opening angle α.

5.2 What are moment of inertia, angular momentum and rotational energy of our earth
a) If the density ϱ is constant for the whole earth
b) If for $r \le R/2$ the homogeneous density ϱ_1 is twice the density ϱ_2 for $r > R/2$?
c) By how much would the angular velocity of the earth change, if all people on earth ($n = 5 \cdot 10^9$ with $m = 70\,\mathrm{kg}$ each) would gather at the equator and would start at the same time to run into the east direction with an acceleration $a = 2\,\mathrm{m/s^2}$?

5.3 A cylindrical disc with radius R and mass M rotates with $\omega = 2\pi \cdot 10\,\mathrm{s}^{-1}$ about the symmetry axis ($R = 10\,\mathrm{cm}$, $M = 0.1\,\mathrm{kg}$).
a) Calculate the angular momentum L and the rotational energy E_{rot}.
b) a bug with $m = 10\,\mathrm{g}$ falls vertical down onto the edge of the disc and holds itself tight. What is the change of L and E_{rot}?
c) The bug now creeps slowly in radial direction to the center of the disc. How large are now $\omega(r)$, $I(r)$ and $E_{\mathrm{rot}}(r)$ as a function of the distance r from the center $r = 0$?

Chapter 5

5.4 The mass density ϱ of a circular cylinder (radius R, height H) increases in the radial direction as $\varrho(r) = \varrho_0(1 + (r/R)^2)$.

a) How large is its inertial moment for the rotation about the symmetry axis for $R = 10\,\text{cm}$ and $\varrho_0 = 2\,\text{kg/dm}^3$?
b) How long does it take for the cylinder to roll down an inclined plane with $\alpha = 10°$ from $h = 1\,\text{m}$ to $h = 0$?

5.5 Calculate the rotational energy of the Na_3-molecule composed of 3 Na atoms ($m = 23\,\text{AMU}$) which form an isosceles triangle with the apex angle $\alpha = 79°$ and a side length of $d = 0.32\,\text{nm}$ when it rotates around the three principal axes with the angular momentum $L = \sqrt{l(l+1)} \cdot \hbar$. Determine at first the three axis and the center of mass.

5.6 A wooden rod with mass $M = 1\,\text{kg}$ and a length $l = 0.4\,\text{m}$, which is initially at rest, can freely rotate about a vertical axis through the center of mass. The end of the rod is hit by a bullet ($m = 0.01\,\text{kg}$) with the velocity $v = 200\,\text{m/s}$, which moves in the horizontal plane perpendicular to the rod and to the rotation axis and which gets stuck in the wood.
What are the angular velocity ω and the rotational energy E_{rot} of the rod after the collision? Which fraction of the kinetic energy of the bullet has been converted to heat?

5.7 A homogeneous circular disc with mass m and radius R rotates with constant velocity ω around a fixed axis through center of mass S perpendicular to the disc plane. At the time $t = 0$ a torque $D = D_0 \cdot e^{-at}$ starts to act on the disc. What is the time dependence $\omega(t)$ of the angular velocity? Numerical example: $\omega_0 = 10\,\text{s}^{-1}$, $m = 2\,\text{kg}$, $R = 10\,\text{cm}$, $a = 0.1\,\text{s}^{-1}$, $D_0 = 0.2\,\text{Nm}$.

5.8 A full cylinder and a hollow cylinder with a thin wall and equal outer diameters roll with equal angular velocity ω_0 on a horizontal plane and then role up an inclined plane. At which height h do they return? (Friction should be neglected), numerical example: $R = 0.1\,\text{m}$, $\omega_0 = 15\,\text{s}^{-1}$.

References

5.1. Ph. Dennery, A. Krzwicki, *Mathematics for Physicists.* (Dover Publications, 1996)

5.2. J.C. Kolecki, *An Introduction to Tensors for Students of Physics and Engineering.*

5.3. https://en.wikipedia.org/wiki/Moment_of_inertia#Principal_moments_of_inertia

5.4. W. Winn, *Introduction to Understandable Physics.* (Authort House, 2010)

5.5a. K. Lambrecht, *The earth's variable Rotation. Geophysical Causes and Consequences.* (Cambridge University Press, Cambridge, 1980)

5.5b. W.H. Munk, G.J. MacDonald, *The Rotation of the Earth. A Geophysical Discussion.* (Cambridge Monographs, 2010)

5.6a. St. Marshak, *Earth: Portrait of a Planet,* 4th ed. (W.W. Norton & Company, 2011)

5.6b. I. Jackson, *The Earth's Mantle.* (Cambridge University Press, 2000)

5.7. V. Dehant, P.M. Mathews, *Precession, Nutation and Wobble of the Earth.* (Cambridge University Press, 2015)

5.8. A. Fothergill, D. Attenborough, *Planet Earth: As You've Never Seen it Before.* (BBC Books, 2006)