Mechanics of a Point Mass

2

W. Demtröder, *Mechanics and Thermodynamics*, Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-27877-3_2

As has been discussed in the previous chapter, the theoretical description of the physical reality often proceeds by successively refined models which approach the reality more and more with progressive refinement. In this chapter the motion of bodies under the influence of external forces will be depicted by the model of point masses, which neglects the spatial form and extension of bodies, which might influence the motion of these bodies.

2.1 The Model of the Point Mass; Trajectories

For many situations in Physics the spatial extension of bodies is of no importance and can be neglected because only their masses play the essential role. Examples are the motion of the planets around the sun where their size is very small compared with the distance to the sun. They can be described as point masses.

The position $P(t)$ of a point mass in the three-dimensional space can be described by its coordinates, which are defined if a suitable coordinate-system is chosen. These coordinates are $\{x, y, z\}$ in a Cartesian system, $\{r, \vartheta, \varphi\}$ in a spherical coordinate system and $\{\varrho, \vartheta, z\}$ in cylindrical coordinates (see Sect. 13.2).

The motion of a point mass is described as the change of its coordinates with time, for example in Cartesian coordinates

$$\left.\begin{aligned} x &= x(t) \\ y &= y(t) \\ z &= z(t) \end{aligned}\right\} \equiv \boldsymbol{r} = \boldsymbol{r}(t) \;,$$

where the position vector $\boldsymbol{r} = \{x, y, z\}$ combines the three coordinates x, y and z (Sect. 13.1).

Note: Vectors are always marked as bold letters.

The function $\boldsymbol{r}(t)$ represents a trajectory in a three-dimensional space, which is passed by the point mass in course of time (Fig. 2.1). The representation $\boldsymbol{r} = \boldsymbol{r}(t)$ is called parameter representation because the coordinates of the point $P(t)$ depend on the parameter t.

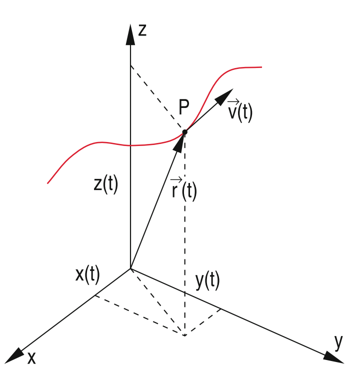

Figure 2.1 Illustration of a trajectory

The motion performed by $P(t)$ on its trajectory is called **translation**. Contrary to the point mass bodies with extended size can also perform **rotations** (Chap. 5) and **vibrations** (Chap. 6).

Note: The model of a point mass moving on a well-defined trajectory fails in micro-physics for the motion of atoms or elementary particles described correctly by quantum mechanics (Vol. 3), where position and velocity cannot be precisely given simultaneously. Instead of a precisely defined trajectory where the point mass can be find at a specific time with certainty at a well-defined position, only probabilities $\mathfrak{P}(x, y, z, t)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ can be given for finding the point mass in a volume $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ around the position (x, y, z). Strictly speaking a geometrical exact trajectory does not exist in the framework of quantum mechanics.

Examples

1. **Motion on a straight line**

$$x = a \cdot t \;, \quad y = b \cdot t \;, \quad z = 0 \;.$$

Elimination of t gives the usual representation $y = (b/a)x$ of a straight line in the (x, y)-plane.
The point mass moves in the x, y-diagram on the straight line with the slope (b/a) (Fig. 2.2).

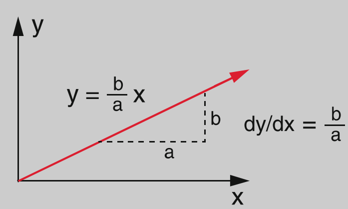

Figure 2.2 Motion on a straight line in the x-y plan

Motions where one of the coordinates are time-independent constants are named **planar motions**, because they are restricted to a plane (in our example the x, y-plane)

2. **Planar circular motion**
We can describe this motion by the coordinates R and φ (Fig. 2.3), where R is the radius of the circle and $\varphi(t)$ the angle between the x-axis and the momentary radius vector $\boldsymbol{R}(t)$. From Fig. 2.3 the relations

$$\begin{aligned} x &= R \cdot \cos \omega t \;, \quad & y &= R \cdot \sin \omega t \;, \\ R &= \text{const} \;, & \omega &= \mathrm{d}\varphi/\mathrm{d}t \;. \end{aligned}$$

can be derived. Squaring of x and y yields

$$x^2 + y^2 = R^2(\cos^2 \omega t + \sin^2 \omega t) = R^2 \;,$$

which is the equation of a circle with radius R. The point mass m with the coordinates $\{x, y, 0\}$ moves with the angular velocity $\omega = \mathrm{d}\varphi/\mathrm{d}t$ and the velocity $v = R \cdot \omega$ on a circle in the x, y-plane.

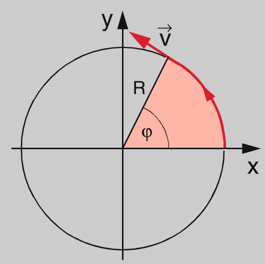

Figure 2.3 Circular motion ◀

Note: The point mass moves relative to a chosen coordinate system (in our case a plane system with the origin at $x = y = 0$). The description of this motion depends on the choice of the reference frame (coordinate system) (see Chap. 3).

Example

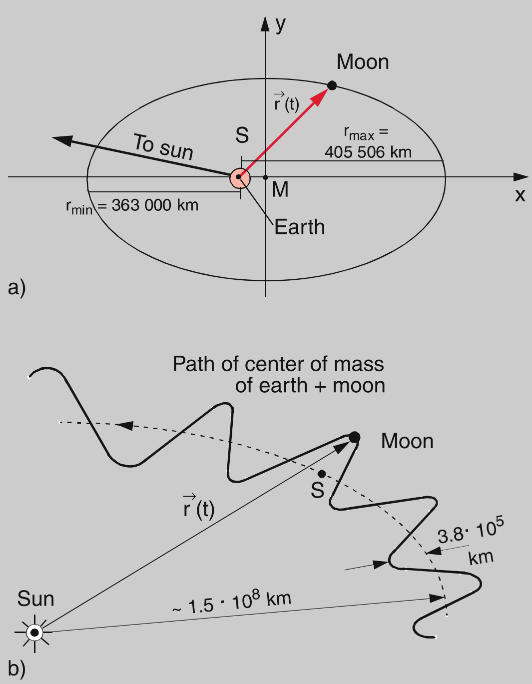

Figure 2.4 Part of the moon trajectory described in two different coordinate systems. **a** Origin in the mass centre of the moon-earth system, located in the focal point of the ellipse; **b** origin in the centre of the sun. The deviations from the elliptical path of the mass centre earth-moon are here exaggerated in order to illustrate these deviations. In reality the orbit of the moon around the sun is always concave, i. e. the curvature radius always points towards the sun. The orbital plane of the moon is inclined against that of the earth

The orbital motion of the moon around the earth is approximately an ellipse if $\boldsymbol{r}(t)$ is measured in a coordinate system with the origin in the centre of mass of the earth-moon system.(Fig. 2.4a). If one chooses, however, the centre of the sun as origin, the trajectory is much more complex (Fig. 2.4b), because now two motions are superimposed: the orbit around the centre of mass and the motion of the centre of mass around the sun. ◀

2.2 Velocity and Acceleration

For a uniformly moving point mass the position vector

$$\boldsymbol{r} = \boldsymbol{v} \cdot t \quad \text{with} \quad \boldsymbol{v} = \{v_x, v_y, v_z\} = \textbf{const} \,, \tag{2.1}$$

increases linearly with time. This means that in equal time intervals Δt equal distances $\boldsymbol{\Delta r}$ are covered.

The ratio $\boldsymbol{v} = \boldsymbol{\Delta r}/\Delta t$ is the velocity of the point mass. The unit of the velocity is $[v] = 1\,\text{m/s}$.

A motion where the magnitude and the direction of the velocity vector $\boldsymbol{v}$ is constant, i. e. does not change with time, is called **uniform rectilinear motion** (Fig. 2.5). In Cartesian coordinates with the unit vectors $\hat{e}_x, \hat{e}_y, \hat{e}_z$, the velocity vector $\boldsymbol{v}$ can be written as

$$\boldsymbol{v} = v_x\hat{\boldsymbol{e}}_x + v_y\hat{\boldsymbol{e}}_y + v_z\hat{\boldsymbol{e}}_z \quad \text{or} \quad \boldsymbol{v} = \left\{v_x, v_y, v_z\right\} \,.$$

Equation 2.1 reads for the components of v as

$$x = v_x t \,; \quad y = v_y t \,; \quad z = v_z t \,. \tag{2.1a}$$

Example

Uniform motion along the x-axis:

$$v_x = v_0 = \text{const}\,; \quad v_y = v_z = 0 \rightarrow \boldsymbol{v} = \{v_0, 0, 0\} \,.$$

The trajectory is the x-axis and the motion is $x = v_0 t$. ◀

In general the velocity will not be constant but can change with time its magnitude as well as its direction. Let us regard a point mass m, which is at time t in the position P_1 (Fig. 2.6). Slightly

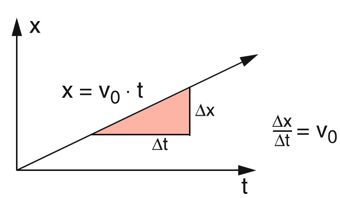

Figure 2.5 Uniform motion on a straight line

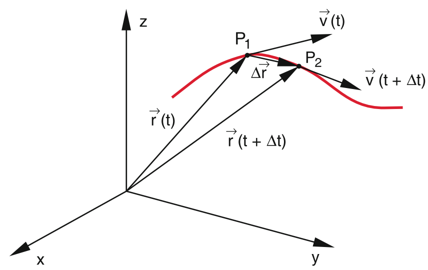

Figure 2.6 Non-uniform motion on an arbitrary trajectory in space

later at the time $t + \Delta t$ it has proceeded to the point P_2. The ratio

$$\frac{\overrightarrow{P_1P_2}}{(t+\Delta t)-t} = \frac{\boldsymbol{r}(t+\Delta t)-\boldsymbol{r}(t)}{\Delta t} = \frac{\Delta \boldsymbol{r}}{\Delta t} = \overline{\boldsymbol{v}}$$

is the average velocity $\overline{\boldsymbol{v}}$ over the distance $\overline{P_1P_2}$.

For $\Delta t \to 0$ the two points P_1 and P_2 merge together and we define as the momentary velocity $\boldsymbol{v}(t)$ the limiting value

$$\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t)-\boldsymbol{r}(t)}{\Delta t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \dot{\boldsymbol{r}} ,$$

which equals the time derivative of the function $\boldsymbol{r}(t)$. In order to distinguish this time derivative $\mathrm{d}r/\mathrm{d}t = \dot{\boldsymbol{r}}(t)$ from the spatial derivative $y'(x) = \mathrm{d}y/\mathrm{d}x$ the time derivative is marked by a point instead of an apostrophe.

Since the derivative $\mathrm{d}f/\mathrm{d}x$ of a function $f(x)$ gives the slope of the curve $f(x)$ at the point $P(x, y)$ the velocity v has at any point the direction of the tangent (Fig. 2.6). Its magnitude is in Cartesian coordinates:

$$v = |\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} . \tag{2.2}$$

Examples

1. Linear accelerated motion

$$z = a \cdot t^2 \to v_z = \dot{z} = 2a \cdot t .$$

For $a = $ const the velocity increases linearly with time. For $a = -g/2$ this describes the free fall with the initial velocity $v_z(t = 0) = 0$ (see Sect. 2.3.1). Here only the magnitude, not the direction of the velocity changes with time.

2. Uniform circular motion

$$\left.\begin{array}{lll} x = R \cdot \cos\omega t & \Rightarrow & \dot{x} = -R \cdot \omega \cdot \sin\omega t \\ y = R \cdot \sin\omega t & \Rightarrow & \dot{y} = \quad R \cdot \omega \cdot \cos\omega t \\ z = 0 & \Rightarrow & \dot{z} = \quad 0 \end{array}\right\}$$

$$\to |\boldsymbol{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = R \cdot \omega .$$

For $\omega = $ const the magnitude of $\boldsymbol{v}$ does not change, only its direction. ◀

We will now discuss the time dependence of the velocity $\boldsymbol{v}$ in more detail: Let us regard a point mass with the velocity $\boldsymbol{v}(t)$ at the point P_1 of the curve $v(t)$. At a slightly later time $t + \Delta t$ the point mass has arrived at P_2 and has there generally a different velocity $\boldsymbol{v}(t + \Delta t)$ (Fig. 2.7). We define the mean acceleration $\overline{\boldsymbol{a}}$ as

$$\overline{\boldsymbol{a}} = \frac{\boldsymbol{v}(t+\Delta t)-\boldsymbol{v}(t)}{\Delta t} .$$

Analogous to the definition of the momentary velocity the momentary acceleration is the limit

$$\boldsymbol{a}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{v}(t+\Delta t)-\boldsymbol{v}(t)}{\Delta t} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t)$$

$$\boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t) \tag{2.3}$$

The acceleration $\boldsymbol{a}(t)$ is the first time derivative $\mathrm{d}\boldsymbol{v}/\mathrm{d}t$ of the velocity $\boldsymbol{v}(t)$ and the second derivative $\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2$ of the position vector $\boldsymbol{r}(t)$. $\boldsymbol{a}(t) = \{a_x, a_y, a_z\}$ is a vector and has the dimensional unit $[a] = [1\,\mathrm{m/s^2}]$.

2.3 Uniformly Accelerated Motion

A motion with $\boldsymbol{a} = \mathbf{const}$ where the magnitude and the direction of a do not change with time is called uniformly accelerated motion. It is described by the equation

$$\ddot{\boldsymbol{r}}(t) = \boldsymbol{a} = \mathbf{const} . \tag{2.4}$$

Equation 2.4 is named *differential equation* because it is an equation between the derivative of a function and other quantities (here the constant vector $\boldsymbol{a}$).

The vector equation (2.4) can be written as the corresponding three equations for the components

$$\ddot{x}(t) = a_x$$
$$\ddot{y}(t) = a_y$$
$$\ddot{z}(t) = a_z .$$

The equation of motion (2.4) is readily solvable. The velocity if obtained by integrating (2.4) which yields:

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = \int \boldsymbol{a}\,\mathrm{d}t = \boldsymbol{a} \cdot t + \boldsymbol{b} . \tag{2.5}$$

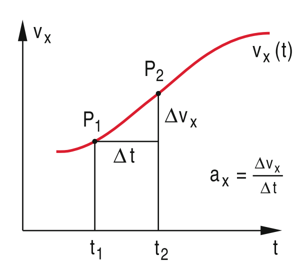

Figure 2.7 Definition of acceleration

The integration constant $\boldsymbol{b}$ ($\boldsymbol{b}$ is a vector with constant components) can be defined by choosing the initial conditions for the motion. For $t = 0$ is $\dot{\boldsymbol{r}}(0) = \boldsymbol{v}(0) = \boldsymbol{b}$. I. e. the constant $\boldsymbol{b}$ gives the initial velocity $\boldsymbol{v}(0) = \boldsymbol{v}_0$.

Further integration of (2.5) gives the trajectory $\boldsymbol{r}(t)$

$$\boldsymbol{r}(t) = \tfrac{1}{2}\boldsymbol{a}t^2 + \boldsymbol{v}_0 t + \boldsymbol{c} \quad \text{with} \quad \boldsymbol{c} = \boldsymbol{r}(0) = \boldsymbol{r}_0 \,. \tag{2.6}$$

This vector-equation can be written for the 3 components

$$\begin{aligned} x(t) &= \tfrac{1}{2}a_x \cdot t^2 + v_{0x}t + x_0 \,, \\ y(t) &= \tfrac{1}{2}a_y \cdot t^2 + v_{0y}t + y_0 \,, \\ z(t) &= \tfrac{1}{2}a_z \cdot t^2 + v_{0z}t + z_0 \,. \end{aligned} \tag{2.6a}$$

One should realize the following statement:

All functions $f(x) + c$ with arbitrary constants c have the same derivative $y' = f'(x)$ because the derivative of a constant is zero. This implies:

All functions $f(x) + c$, which represent an infinite parametric curve family, are solutions of the differential equation $y' = f'(x)$. Therefore infinitely many position vectors $r(t)$ are found for the same velocity $v(t)$. Only the initial conditions select one specific position vector.

We will illustrate this by several examples in the next sections.

2.3.1 The Free Fall

We choose the vertical direction as the z-axis. A body experiences in the gravitational field of the earth the acceleration

$$\begin{aligned} a_x &= a_y = 0 \,, \\ a_z &= -g = -9.81\,\text{m/s}^2 \,, \end{aligned}$$

where the numerical value is obtained from experiments.

When a body at rest falls at time $t = 0$ from the height h, the initial conditions are $x(0) = y(0) = 0$: $z(0) = h$; $v_x(0) = v_y(0) = v_z(0) = 0$.

With these initial conditions the system of equations (2.6a) reduces to

$$z(t) = -\tfrac{1}{2}gt^2 + h \,. \tag{2.7}$$

The derivative gives $v_z(t) = -g \cdot t$. The motion $z(t)$ plotted in the z-t-plane represents a parabola (Fig. 2.8). For $t = \sqrt{2h/g}$ the body has reached the ground at $z = 0$. The falling time for the distance h is

$$t_{\text{fall}} = \sqrt{2h/g} \,, \tag{2.8}$$

and the final velocity at $z = 0$ is $v_{\max} = \sqrt{2hg}$.

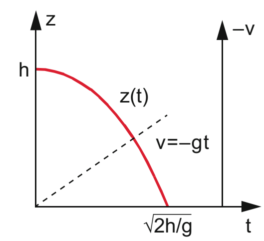

Figure 2.8 Path-time function $z(t)$ (*red curve*) and velocity-time function (*dotted line*)

2.3.2 Projectile Motion

As starting point we choose $x(0) = y(0) = 0$; $z(0) = h$; and the z-axis is again the vertical direction, while the x-axis marks the horizontal direction, so that the trajectory for the projectile is in the x-z-plane (Fig. 2.9). The initial velocity should be $\boldsymbol{v}_0 = \{v_{0x}, 0, v_{0z}\}$. The acceleration is $\boldsymbol{a} = \{9, 0, -g\}$. Equation 2.6 becomes then

$$\begin{aligned} x(t) &= v_{0x}t \,, \\ y(t) &= 0 \,, \\ z(t) &= -\tfrac{1}{2}gt^2 + v_{0z}t + h \,. \end{aligned}$$

The motion is therefore a superposition of a uniform straight motion into the x-direction and a uniformly accelerated motion into the z-direction. For $v_{0z} = 0$ we obtain the special case of the horizontal throw and for $v_{0x} = 0$ the vertical throw.

Elimination of $t = x/v_{0x}$ yields the projectile parabola

$$z(x) = -\frac{1}{2}\frac{g}{v_{0x}^2}x^2 + \frac{v_{0z}}{v_{0x}}x + h \,. \tag{2.9}$$

The value $x = x_s$ where the maximum occurs is found for $\mathrm{d}z/\mathrm{d}x = 0$.

$$x_S = \frac{v_{0x} \cdot v_{0z}}{g} = \frac{v_0^2 \cdot \sin\varphi \cdot \cos\varphi}{g} \,. \tag{2.10}$$

For a given value of the initial velocity v_0 the maximum of x_s is achieved for $\varphi = 45°$. In order to calculate the projectile range

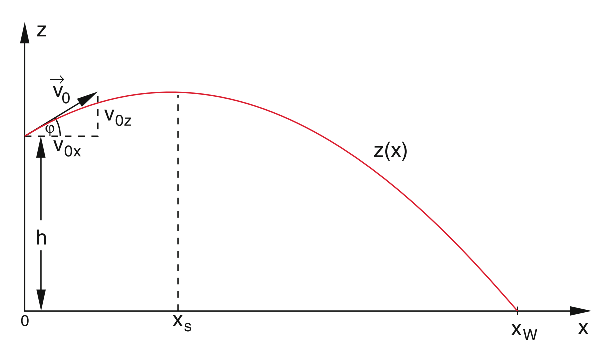

Figure 2.9 Projectile motion

Chapter 2

x_w we solve (2.9) for $z(x_w) = 0$. This gives

$$x_W = \frac{v_{0x} \cdot v_{0z}}{g} \pm \left[\left(\frac{v_{0x} \cdot v_{0z}}{g}\right)^2 + \frac{2v_{0x}^2}{g} \cdot h\right]^{1/2} . \qquad (2.11)$$

Since $x_w > 0$ only the positive sign is possible. With the relation $v_{z_0} \cdot v_{x_0} = \frac{1}{2} v_0^2 \cdot \sin 2\varphi$ we can transform (2.11) into

$$x_W = \frac{v_0}{2g} \sin 2\varphi \left[v_0 + \left(v_0^2 + \frac{2gh}{\sin^2\varphi}\right)^{1/2}\right] . \qquad (2.12)$$

The optimum angle φ_{opt} for achieving the largest throwing range for a given initial velocity v_0 is achieved when $dx_w/d\varphi = 0$. This gives

$$\varphi_{opt} = \arcsin\left(\frac{1}{\sqrt{2 + 2gh/v_0^2}}\right) . \qquad (2.13)$$

For the special case $h = 0$ (2.13) simplifies because of $\arcsin(\sqrt{2}/2) = \pi/4$ to $\varphi_{opt} = 45°$ (see the detailed derivation of (2.13) in the solution of Problem 2.5c).

2.4 Motions with Non-Constant Acceleration

While the differential equation for motions with constant acceleration is elementary integrable this might not be true for arbitrary time dependent accelerations. We will at first treat the simple example of the uniform circular motion, where the magnitude of the acceleration is constant but not the direction.

2.4.1 Uniform Circular Motion

For the uniform circular motion equal distances are gone for equal time intervals. This means that the magnitude of the velocity $\boldsymbol{v}$ is constant and the component a_φ of the acceleration $\boldsymbol{a} = \{a_r, a_\varphi\}$ in the direction of v must be therefore zero.

The path length Δs on the circle arc for the angle $\Delta\varphi$ is $\Delta s = R \cdot \Delta\varphi$ (Fig. 2.10a). The magnitude of the velocity is then

$$v = \frac{ds}{dt} = R \cdot \frac{d\varphi}{dt} = R \cdot \omega .$$

The quantity $\omega = d\varphi/dt$ is the **angular velocity** with the dimension $[\omega] = [\text{rad/s}]$.

The acceleration is now

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d}{dt}(v\hat{\boldsymbol{e}}_t) = \frac{dv}{dt}\hat{\boldsymbol{e}}_t + v\frac{d\hat{\boldsymbol{e}}_t}{dt}$$
$$= v\frac{d\hat{\boldsymbol{e}}_t}{dt} \quad \text{because} \quad v = \text{const} .$$

Because $\hat{\boldsymbol{e}}_t^2 = 1 \rightarrow 2\hat{\boldsymbol{e}}_t \cdot d\hat{\boldsymbol{e}}_t/dt = 0$.

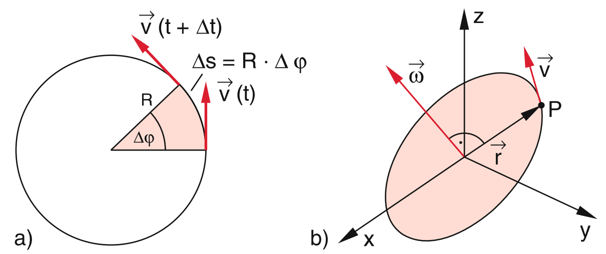

Figure 2.10 **a** uniform circular motion, **b** Illustration of the angular velocity

The scalar product of two vectors becomes zero, if either at least one of the vectors is zero or if the two vectors are orthogonal. Since $\hat{e}_t \neq \boldsymbol{0}$ and $d\hat{\boldsymbol{e}}_t/dt \neq 0$ it follows

$$\frac{d\hat{\boldsymbol{e}}_t}{dt} \perp \hat{\boldsymbol{e}}_t .$$

This means that the acceleration $\boldsymbol{a}$ is orthogonal to the velocity $\boldsymbol{v}$ which is collinear with $\hat{\boldsymbol{e}}_t$. The vector $d\hat{\boldsymbol{e}}_t/dt$ gives the angular velocity of the tangent to the circle. Since the radius vector R is orthogonal to the vector $\boldsymbol{v}$ both vectors turn with the angular velocity $\omega = d\varphi/dt$. This means that the magnitude is $|d\hat{\boldsymbol{e}}_t/dt| = \omega$. This gives for the acceleration

$$\boldsymbol{a} = v \cdot \frac{d\hat{\boldsymbol{e}}_t}{dt} = R \cdot \omega^2 \hat{\boldsymbol{e}}_a = -R\omega^2 \hat{\boldsymbol{r}} , \qquad (2.14)$$

where the unit vector $\boldsymbol{e}_a = -\boldsymbol{R}/R$ always points into the direction towards the centre of the circle, and $\hat{r} = \boldsymbol{r}/|\boldsymbol{r}|$ points into the opposite direction.

Proof

$$\boldsymbol{r} = \begin{Bmatrix} R \cdot \cos\omega t \\ R \cdot \sin\omega t \end{Bmatrix}$$
$$\boldsymbol{v} = \begin{Bmatrix} -R \cdot \omega \cdot \sin\omega t \\ R \cdot \omega \cdot \cos\omega t \end{Bmatrix}$$
$$\boldsymbol{a} = \begin{Bmatrix} -R\omega^2 \cos\omega t \\ -R\omega^2 \sin\omega t \end{Bmatrix} = -\omega^2 \cdot \boldsymbol{r} = -R\omega^2 \cdot \hat{\boldsymbol{r}} . \quad ◄$$

The vector of the acceleration for the uniform circular motion

$$\boldsymbol{a} = -R\omega^2 \hat{\boldsymbol{r}} \quad \text{with} \quad |\boldsymbol{a}| = R \cdot \omega^2$$

is called *centripetal-acceleration* because it points towards the centre of the circle (Fig. 2.11).

If also the orientation of the plane in the three-dimensional space should be defined, it is useful to define a vector $\boldsymbol{\omega}$ of the angular velocity which is vertical to the plane of motion (Fig. 2.10b) and has the magnitude $\omega = |\boldsymbol{\omega}| = d\varphi/dt = v/R$.

Figure 2.11 Rollercoaster, where the superposition of centripetal acceleration and gravity changes along the path and inluences the feelings of the passenger (with kind permission of Foto dpa)

2.4.2 Motions on Trajectories with Arbitrary Curvature

In the general case the velocity $\boldsymbol{v}$ will change its magnitude as well as its direction with time. However, the momentary velocity $\boldsymbol{v}(t)$ at time t is always the tangent to the trajectory in the point $P(t)$, while the acceleration $\boldsymbol{a}(t)$ can have any arbitrary direction (Fig. 2.12). The acceleration can be always composed of two components $a_t = dv/dt \cdot \hat{e}_t$ along the tangent to the curve (*tangential acceleration*) and a_n in the direction of the normal to the tangent, i. e. perpendicular to a_t (*normal acceleration*).

For $\boldsymbol{v} = v \cdot \hat{\boldsymbol{e}}_t$ where $\hat{\boldsymbol{e}}_t$ is the unit vector tangential to the trajectory, the acceleration $\boldsymbol{a}$ is

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{dv}{dt} \cdot \hat{\boldsymbol{e}}_t + v\frac{d\hat{\boldsymbol{e}}_t}{dt} = \boldsymbol{a}_t + \boldsymbol{a}_n \,. \qquad (2.15)$$

The change of the magnitude of the velocity is described by a_t while the change of the direction of v is described by a_n.

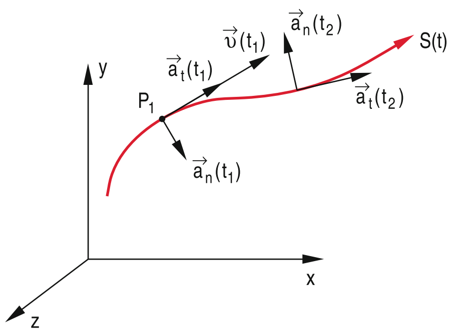

Figure 2.12 Tangential and normal acceleration

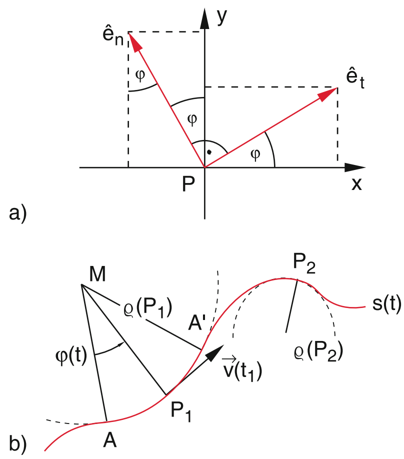

Figure 2.13 **a** Derivation of the normal acceleration. **b** Local radius of curvature of a trajectory with arbitrary curvature ϱ

For $a_n = 0$ the trajectory is a straight line, where the body moves with changing velocity if $a_t \neq 0$. For $a_t = 0$ the point mass moves with a constant velocity $|\boldsymbol{v}|$ on a curve which is determined by $\boldsymbol{a}_n(t)$. For the free fall of Sect. 2.3.1 is $a_n = 0$ and $a_t =$ const., while for the uniform circular motion $a_t = 0$ and $a_n =$ const.

For the motion on trajectories with arbitrary curvature the acceleration can be obtained as follows: We choose the x-y-plane as the plane of the two vectors $\boldsymbol{v}(t)$ and $\boldsymbol{a}(t)$, which implies that all vectors have zero z-components.

According to Fig. 2.13a the two mutually vertical unit vectors $\hat{e}_t$ and $\hat{e}_n$ can be composed as

$$\begin{aligned}\hat{e}_t &= \cos\varphi\hat{e}_x + \sin\varphi\hat{e}_y\\ \hat{e}_n &= \cos(\varphi + \frac{\pi}{2})\hat{e}_x + \sin(\varphi + \frac{\pi}{2})\hat{e}_y\\ &= -\sin\varphi\hat{e}_x + \cos\varphi\hat{e}_y\end{aligned}$$

There we get

$$\begin{aligned}\frac{d\hat{e}_t}{dt} &= -\sin\varphi\frac{d\varphi}{dt}\hat{e}_x + \cos\varphi\frac{d\varphi}{dt}\hat{e}_y\\ &= \frac{d\varphi}{dt}\hat{e}_n\end{aligned}$$

The normal acceleration is therefore

$$\boldsymbol{a}_n = v\frac{d\varphi}{dt}\hat{e}_n .$$

We regard in Fig. 2.13b an infinitesimal section between the points A and A' of an arbitrary curve and approximate this section by a circular arc $\overline{AA'}$ with the center of curvature M. Shortening the section $\overline{AA'}$ more and more, i.e. the points A and A' converge towards the point P_1 the curve section $\overline{AA'}$ approaches more and more the circular arc with radius $\overline{MP_1}$. The radius $\varrho = \overline{MP_1}$ is the radius of curvature of the curve in the point P_1.

For the small section of the curve we get

$$ds = \varrho d\varphi \tag{2.16a}$$

$$\frac{d\varphi}{dt} = \frac{d\varphi}{ds}\frac{ds}{dt} = \frac{d\varphi}{ds}v = \frac{1}{\varrho}v . \tag{2.16b}$$

The acceleration vector becomes

$$\boldsymbol{a} = \frac{dv}{dt}\hat{e}_t + \frac{v^2}{\varrho}\hat{e}_n \tag{2.16c}$$

Examples

1. Assume a motion on a straight line experiences the acceleration $a(x) = b \cdot x^4$.
   Calculate the velocity $v(x)$ for the initial condition $v(0) = v_0$.

Solution

$$a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot v , \quad \int_{x_0}^{x} a dx = \int_{v_0}^{v} v dv .$$

Inserting a and integration yields

$$\tfrac{1}{5}b\left(x^5 - x_0^5\right) = \tfrac{1}{2}\left(v^2(x) - v_0^2\right) .$$

Resolving this equation for $v(x)$ gives

$$v(x) = \sqrt{\tfrac{2}{5}b\left(x^5 - x_0^5\right) + v_0^2} .$$

2. The open parachute of a parachutist experiences, due to air friction, a negative acceleration besides the acceleration by gravity.

$$a = -b\cdot v^2 \quad \text{with} \quad b = 0.3\,\mathrm{m}^{-1} .$$

   a) What is his constant final velocity v_e?
   b) What is the time-dependent velocity $v(t)$, if the parachutist opens his parachute only after $t_0 = 10$ s free fall for which friction can be neglected?

Solution

a) A constant final velocity is reached, when the total acceleration becomes zero. This is the case when

$$g - b\cdot v_e^2 = 0 \rightarrow v_e = \sqrt{g/b} = 5.7\,\mathrm{m/s} .$$

b) The equation of motion after the parachute is opened is with the z-axis in the vertical direction

$$\ddot{z} = g - b\cdot\dot{z}^2 .$$

With $v = \mathrm{d}z/\mathrm{d}t$ and $\mathrm{d}v/\mathrm{d}t = \mathrm{d}^2z/\mathrm{d}t^2$ we obtain

$$\mathrm{d}v/\mathrm{d}t = b - b \cdot v^2 \,,$$

which leads to the equation

$$\int\limits_{v_0}^{v} \frac{\mathrm{d}v}{g - bv^2} = \frac{1}{g}\int\limits_{v_0}^{v} \frac{\mathrm{d}v}{1 - v^2/v_\mathrm{e}^2} = \int\limits_{t_0}^{t} \mathrm{d}t' = t - t_0 \,.$$

We substitute $v/v_\mathrm{e} = x$, for $x > 1$ i. e. for $v > v_\mathrm{e}$ we get

$$\int \frac{\mathrm{d}x}{1 - x^2} = \frac{1}{2}\ln\frac{x+1}{x-1}$$
$$\rightarrow t - t_0 = \frac{1}{2}\frac{v_\mathrm{e}}{g}\ln\frac{v + v_\mathrm{e}}{v - v_\mathrm{e}} + C \,.$$

For $t = t_0 \rightarrow v = v_0 = g \cdot t_0 = 98.1\,\mathrm{m/s}$. This gives for the integration constant C the value

$$C = -\frac{1}{2}\frac{v_\mathrm{e}}{g}\ln\frac{v_0 + v_\mathrm{e}}{v_0 - v_\mathrm{e}}$$
$$\rightarrow t - t_0 = \frac{1}{2}\frac{v_\mathrm{e}}{g}\ln\left[\frac{v + v_\mathrm{e}}{v - v_\mathrm{e}}\,\frac{v_0 - v_\mathrm{e}}{v_0 + v_\mathrm{e}}\right] \,.$$

Eliminating v from this equation for v yields

$$v(t) = v_\mathrm{e}\frac{d \cdot \mathrm{e}^{c(t-t_0)} + 1}{d \cdot \mathrm{e}^{c(t-t_0)} - 1} \quad \text{with}$$
$$d = \frac{v_0 + v_\mathrm{e}}{v_0 - v_\mathrm{e}} \quad \text{and} \quad c = 2g/v_\mathrm{e} \,.$$

The velocity decreases from the initial value $v(t_0) = v_0$ at t_0 exponentially to the final value v_e for $t = \infty$. However, already after $t - t_0 = 2v_\mathrm{e}/g = 1.16\,\mathrm{s}$ the velocity has reached 96.7% of its final value. ◀

2.5 Forces

We will now discuss the question, **why** a body performs that motion that we observe, why for instance the earth moves around the sun on an elliptical trajectory, or why a stone in a free fall moves on a vertical straight line to the ground.

Newton recognized that the cause for changes of a body's velocity must be interactions of the body with its surroundings. These can be long range interactions such as the gravitational interaction between the sun and the earth, or short range interaction which work for example in collisions between colliding billiard balls, or even ultrashort range strong interactions between neutrons in an atomic nucleus. All such interactions are described by the concept of **forces**. When a body changes its state of motion we say that a force acts upon the body.

If, for instance, two bodies collide we say: Each of the two bodies has exerted during the collision a force onto the other body, which causes a change of the state of motion for both bodies.

A body without any interaction with its surroundings (or for which the vector sum of all forces is zero), is called a *free body*. A free body does not change its state of motion. Strictly speaking there are in reality no free bodies without any interaction (because we would not see them). However, in many cases the interaction is so small, that we can neglect it. Examples are atoms in a tank where a very good vacuum has been established, or a sliding carriage on a nearly frictionless horizontal air track. Such free bodies move uniformly on a straight trajectory. For such cases the model of a free body is justified.

2.5.1 Forces as Vectors; Addition of Forces

Since velocity changes which are caused by forces are vectors, also forces must be described by vectors, i. e. they are defined by their magnitude and their direction.

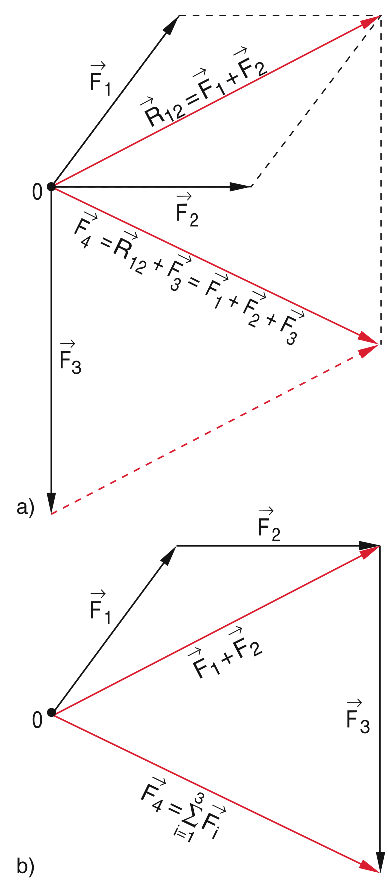

Figure 2.14 Vector sum of forces. **a** all forces act on the same point, **b** equivalent representation of the vector sum

Note: When forces act on extended bodies, also the point of origin is important (see Sect. 5.4).

A force, as any vector, can be reduced to the sum of its components. This reduction depends on the chosen coordinate system. For example in Cartesian coordinates the vector and its components are $\boldsymbol{F} = \{F_x, F_y, F_z\}$. If we choose the coordinate system in such a way that the z-direction points into the direction of $\boldsymbol{F}$, the component representation becomes $\boldsymbol{F} = \{0, 0, F_z = F\}$ with $F = |\boldsymbol{F}|$. Often the solution of a problem can be essentially simplified by choosing the optimum coordinate system (see Sect. 2.3.2). If several forces act on a body the total force is the vector sum of the individual forces (superposition principle)

$$\boldsymbol{F} = \sum_i \boldsymbol{F}_i \ .$$

This vector equation is equivalent to the three equations for the components

$$F_x = \sum_i F_{ix} \quad F_y = \sum_i F_{iy} \quad F_z = \sum_i F_{iz} \ .$$

The addition of several vectors is illustrated in Fig. 2.14a and b. Both ways to add vectors are equivalent, because the origin of the vectors can be shifted. If $\sum \boldsymbol{F}_i = \boldsymbol{0}$ the total force is zero and the body remains in its constant state of motion (either at rest or in a uniform motion on a straight line.

Examples

1. A body with mass m rests on a friction-free sloped plane (Fig. 2.15). The gravitational force can be regarded as the vecor sum of the two forces $\boldsymbol{F}_\perp$ perpendicular to the sloped plain and $\boldsymbol{F}_\parallel$ parallel to this plane. $\boldsymbol{F}_\perp$ exerts a force onto the surface of the plane and causes an opposite force $\boldsymbol{N}$ of equal magnitude by the elastic response of the surface. Only the force $\boldsymbol{F}_\parallel$ can cause an acceleration of the body. It can be compensated by an opposite force $\boldsymbol{Z}$ in order to reach a zero total force and keep the body at rest on the sloped plane. This situation can be described by the equation

$$m \cdot \boldsymbol{g} = \boldsymbol{F}_\parallel + \boldsymbol{F}_\perp = -(\boldsymbol{Z} + \boldsymbol{N}) \ .$$

   Attractive force $\boldsymbol{Z}$ and elastic force $\boldsymbol{N}$ compensate the gravitational force and the body remains at rest.

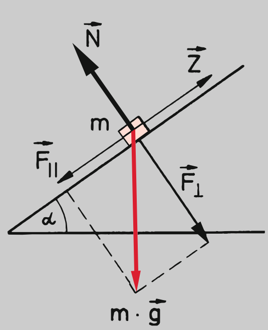

Figure 2.15 Equilibrium of forces for a body on an inclined plane

2. A circular pendulum is a mass m hold by a string which is fixed at a point P. The mass can move on a circle in the x-y-plane while the string movement forms the surface of a cone (Fig. 2.16). The total force $\boldsymbol{F} = m \cdot \boldsymbol{g} + \boldsymbol{F}_{\rm el}$ as the sum of gravitational force and elastic force of the stressed string acting on the mass m always points towards the centre of the circle in the x-y-plane and acts as centripetal force which causes the circular motion of m.

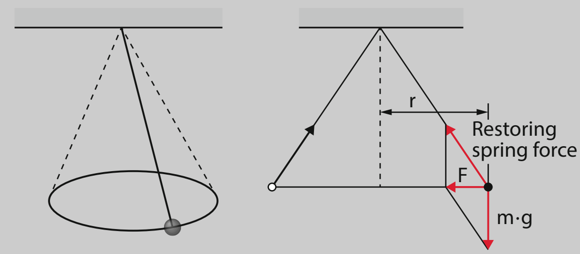

Figure 2.16 Circular pendulum with the vector diagram ◀

2.5.2 Force-Fields

Often the force acting on a body depends on the location. If it is possible to unambiguously assign to each point (x, y, z) a force with defined magnitude and direction the spatial force function $\boldsymbol{F}(x, y, z)$ is called a **force-field**. Its components depend on the chosen coordinate system:

$$\begin{aligned} &\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{F}(x, y, z) && \text{in Cartesian coordinates, or} \\ &\boldsymbol{F}(r, \vartheta, \varphi) && \text{in spherical coordinates, or} \\ &\boldsymbol{F}(r, \varphi, z) && \text{in cylinder coordinates.} \end{aligned}$$

In a graphical representation the direction of the force is illustrated by "force-lines" where the force at any point (x, y, z) is the tangent to the force-line (Fig. 2.17).

If the force has for any point in space only a radial component with a magnitude which depends on the distance r to the centre $r = 0$ the force field is centro-symmetric and is called a *central force field*. It can be written as

$$\boldsymbol{F} = f(r) \cdot \hat{\boldsymbol{r}} \ ,$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$ is the unit vector in radial direction. The sign of the scalar function $f(r)$ is: $f(r) < 0$ if the forces point to the centre and $f(r) > 0$ is it points from the centre away.

Surfaces where the force field has the same magnitude are called *equipotential surfaces*. (see Sect. 2.7.5)

Central force fields are spherical symmetric.

Examples

1. **Central force fields**

   a) ***Gravitational force field of the earth*** (Fig. 2.17a)
   $\boldsymbol{F}$ depends on the distance for the earth's centre. For the idealized case that the earth can be described by a homogeneous sphere with spherical symmetric mass distribution (see Fig. 2.9) the gravitational force is for $r > R$ (R = radius of the earth)

$$\boldsymbol{F} = -G\frac{m \cdot M}{r^2}\hat{\boldsymbol{r}}$$

   (M = mass of earth, m = mass of body, G = gravitation constant, unit vector $\hat{\boldsymbol{r}} = \boldsymbol{r}/|r|$)

   b) ***Force field of a positive electric charge*** Q (Fig. 2.17b).
   In the electric force field of an electric charge Q the force on a small test charge q is

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{q \cdot Q}{r^2}\hat{\boldsymbol{r}}\,;$$

   (ε_0 = dielectric constant see Vol. 2). The spherical symmetric force field has the same form as the gravitational force field.

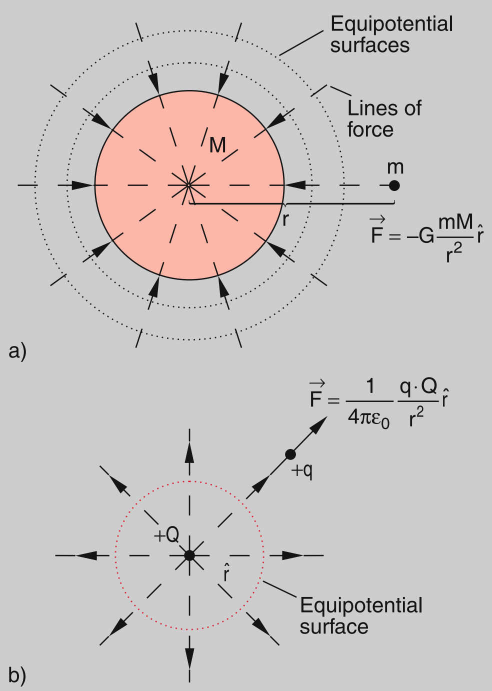

Figure 2.17 Spherical symmetric force fields **a** gravitational force field of a mass M (attractive force) and **b** electric force field of a positive charge Q and repulsive force on a positive test charge

2. **Non-central force fields**

   a) ***Dipole force field***
   The force field in the surrounding of two charges $+Q$ and $-Q$ with equal magnitude but opposite sign is no longer spherical symmetric. The force on a test charge not only depends on the distance from the centre of the two charges but also on the angle ϑ of the position vector against the connecting line of the two charges (Fig. 2.18). The calculation of the force field gives (see Sect. 1.5 of Vol. 2)

$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \frac{q \cdot Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1^2}\hat{\boldsymbol{r}}_1 - \frac{1}{r_2^2}\hat{\boldsymbol{r}}_2\right].$$

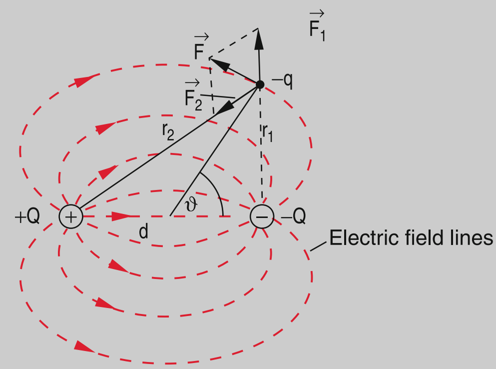

Figure 2.18 Force field of an electric dipole and the force on a negative test charge

   b) ***Force field of a planetary system***
   At each position $\boldsymbol{r}$ the gravitational forces on a test mass exerted by the sun, the planets and the moons superimpose. The force field $\boldsymbol{F}(\boldsymbol{r}) = \sum \boldsymbol{F}_i$ is very complex. It even can be zero at certain points in space, for example at a point N between earth and moon (neutral point) where the opposite gravitational forces from earth and moon just compensate (Fig. 2.19).

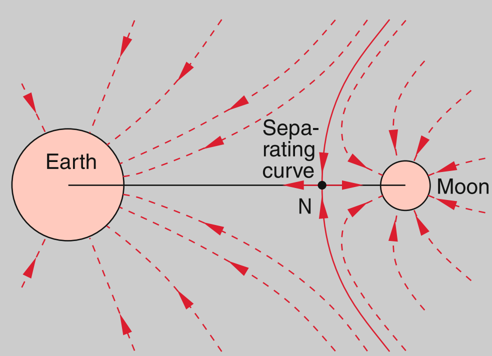

Figure 2.19 Gravitational field between earth and moon

c) ***Homogeneous force field of a parallel plate capacitor***
For a voltage V between the plates with a distance d the force on an electric charge $+q$ is $\boldsymbol{F} = +q \cdot (V/d) \cdot \hat{\boldsymbol{e}}_z$ vertical to the plates and pointing from the positively charged plate to the negative one (Fig. 2.20). The force $\boldsymbol{F}$ has at any point inside the capacitor the same magnitude and direction. Such a force field is called *homogeneous*.
Within a small volume also the gravitational field of the earth can be treated as a homogeneous force field as long as the vertical extension Δz of this volume is very small compared to the radius R of the earth. The force on a mass is then $\boldsymbol{F} = m \cdot \boldsymbol{g}$, where $|\boldsymbol{g}| = 9.81\,\mathrm{m/s^2}$ is the earth gravitational acceleration which remains constant in a small volume.

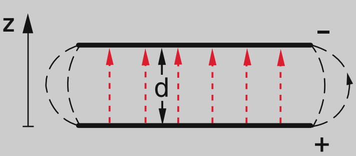

Figure 2.20 Homogeneous force field for electric charges inside a parallel plate capacitor

2.5.3 Measurements of Forces; Discussion of the Force Concept

Forces can be measured due to their effect on the deformation of elastic bodies (see Chap. 6). One example is the spring balance (Fig. 2.21). Here the elongation of a spring under the influence of a force is measured. Its displacement $x - x_0$ from the equilibrium position x_0 is proportional to the acting force

$$F_x = -D(x - x_0)\ . \tag{2.17}$$

If the spring constant $D = F/\Delta x$ is known, the determination of the force $\boldsymbol{F}$ is reduced to a length measurement $\Delta x = x - x_0$. The spring constant D can be obtained from measurements of the oscillation period of the spring balance. After a mass m has been

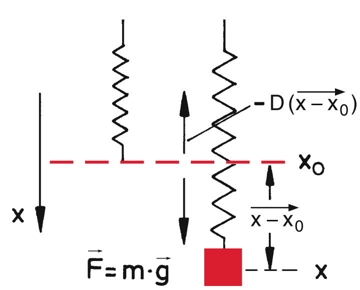

Figure 2.21 Spring balance for the measurement of forces

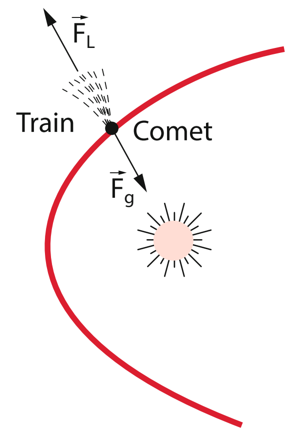

Figure 2.22 Interaction between sun and comet as an example for the far distance effect of forces

displaced from its equilibrium position x_0 and then released, it performs oscillations around x_0 (see Sect. 2.9.7).

Often forces can act on bodies without physical contact between them. Examples are the gravitational force between sun and earth or between sun and a comet (Fig. 2.22). In the latter case the comet is attracted by the sun due to the gravitational force and vice versa. Its tail is repelled because of the radiation pressure and the sun wind which is exerted by particles (protons and electrons) emitted from the sun.

Even if there is no direct contact between two bodies we say that a force acts on each body which causes the change of its motional state i. e. its velocity with time. Also for the investigation of atomic collision processes the information on the forces between the colliding atoms is obtained from the observed change of the velocities of the two collision partners (see Sect. 4.3). Here the change of the momentum $\mathrm{d}\boldsymbol{p}/\mathrm{d}t$ is used to determine the force. This explanation goes beyond the ordinary meaning of forces as directly perceptible phenomena as for instance the physical strength.

In all cases the force is a synonym for the interaction between bodies. The range of distances between the interacting bodies can reach from 10^{-17} m to infinity.

The question, what the real cause for this interaction is and whether it is transferred between the interacting bodies infinitely fast or with a finite speed can be up to now only partly answered and is the subject of intense research but is not yet fully understood. Theoretical predictions claim a finite transfer time which equals the speed of light. The description of the interaction between very fast moving bodies has therefore to take into account this finite transfer time (retardation, see Sect. 3.5). For velocities which are small compared to the speed of light this effect can be neglected (realm of non-relativistic physics).

We will now discuss more quantitatively the relations between forces and the change of motional states of bodies.

2.6 The Basic Equations of Mechanics

The mathematical description of the motion of bodies under the influence of forces can be reduced to a few basic equations. These equations are based on assumptions (axioms) which are suggested by experiments. They were first postulated by Isaac Newton in his famous multi-volume opus "*Philosophiae naturalis principia mathematica*" which was published in the years 1687–1726 [2.1].

2.6.1 The Newtonian Axioms

For the introduction of the force model and its relation with the state of motion of bodies Newton started from three basic assumptions which were taken from daily experience. They are called the *three Newtonian axioms* (sometimes also Newton's three laws).

First Newtonian Axiom

Each body remains in the state of rest or of uniform motion on a straight line as long as no force is acting on it.

As the measure for the state of motion of a body with mass m we define the **momentum**

$$\boldsymbol{p} = m \cdot \boldsymbol{v} \; .$$

The momentum $\boldsymbol{p}$ is a vector parallel to the velocity $\boldsymbol{v}$ and has the dimension $[p] = [\mathrm{kg \cdot m \cdot s^{-1}}]$. A particle on which no force is acting is called a **free particle**.

With this definition Newton's first law can be formulated as

The momentum of a free particle is constant in time.

This means: always when a particle changes its state of motion a force is acting on it, i. e. it interacts with other particles or it is moving in a force field (Fig. 2.23).

Second Newtonian Axiom

Since we attribute a force to any change of momentum we define the force $\boldsymbol{F}$ as

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \; . \qquad (2.18)$$

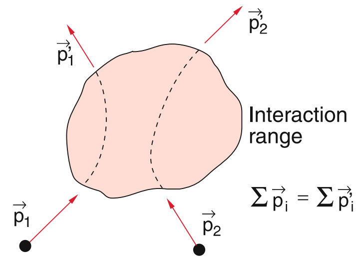

Figure 2.23 Forces as cause for a change of momentum

With $\boldsymbol{p} = m \cdot \boldsymbol{v}$ we can write this in the form

$$\boldsymbol{F} = m \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t} \cdot \boldsymbol{v} \; . \qquad (2.18a)$$

The second term describes a possible change of the mass m with the velocity of the particle. There are many situations where this second term becomes important, for instance when a rocket is accelerated by the expulsion of fuel (see Sect. 2.6.3) or when a particle is accelerated to very high velocities, comparable to the velocity c of light, where the relativistic mass $m(v)$ increase with velocity, cannot be neglected (see Sect. 4.4.1).

Example

A freight train moves with the velocity v in the horizontal x-direction (Fig. 2.24). It is loaded continuously with sand from a stationary reservoir above the train. The mass increase per time $\mathrm{d}m/\mathrm{d}t$ is assumed to be constant. When friction can be neglected the total force onto the train is zero. The equation of motion is then

$$0 = m \cdot \mathrm{d}v/\mathrm{d}t + A \cdot v \qquad (2.18b)$$

with $m = m_0 + A \cdot t$. Integration yields

$$\ln \frac{v}{v_0} = \ln \frac{m_0}{m_0 + A \cdot t}$$

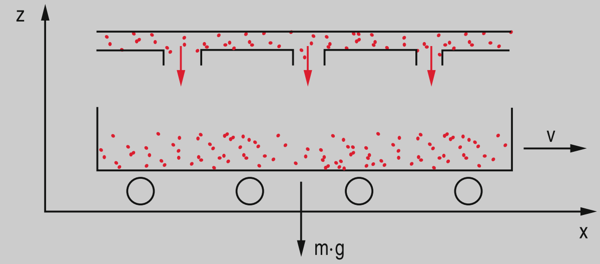

Figure 2.24 Example to Eq. 2.18a

with the solution

$$v(t) = v_0 \frac{1}{1 + (A/m_0) \cdot t} \; . \qquad (2.18c)$$

With $m_0 = 1000$ tons and $\mathrm{d}m/\mathrm{d}t = A = 1$ ton/s the train velocity $v(t) = v_0(1 + 1 + 10^{-3}\, t)^{-1}$ the velocity slows down to $v_0/2$ in 1000 s.

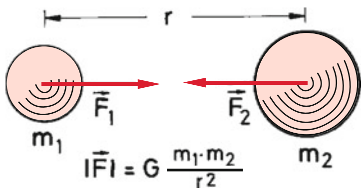

Figure 2.25 actio = reactio for the example of gravitational forces $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ between two masses

If the mass m is constant ($\mathrm{d}m/\mathrm{d}t = 0$) Eq. 2.18b takes the simple form

$$\boldsymbol{F} = m \cdot \boldsymbol{a} \quad \text{with} \quad \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} . \tag{2.18d}$$

The unit of the force is $[F] = 1\,\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-2} = 1\,\text{Newton} = 1\,\mathrm{N}$.

Third Newtonian Axiom

When two bodies interact with each other but not with a third partner the force acting on the first body has equal magnitude but opposite direction as the force on the second body (Fig. 2.25). Newton's formulation in Latin was

$$\textit{actio} = \textit{reactio}$$
$$\boldsymbol{F}_1 = -\boldsymbol{F}_2 .$$

We will apply Newton's axioms to a system of two masses m_1 and m_2 which interact with each other, i. e. they collide, but are otherwise completely isolated from their surroundings. Such a system is called a *closed system*.

Since there are no external forces on a closed system we can conclude in analogy to a free particle that the total momentum of the system remains constant:

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \mathbf{const} . \tag{2.19a}$$

Differentiating this equation yields

$$\frac{\mathrm{d}\boldsymbol{p}_1}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{p}_2}{\mathrm{d}t} = \mathbf{0} \Rightarrow \boldsymbol{F}_1 = -\boldsymbol{F}_2 . \tag{2.19b}$$

This axiom can be proved experimentally with two equal spring balances (Fig. 2.26a), which are connected to each other at one end. If one pulls at the two other ends into opposite directions they show that on each spring balance the same force is acting.

Another experimental verification is shown in Fig. 2.26b where a spring is compressed by two equal masses on an air track

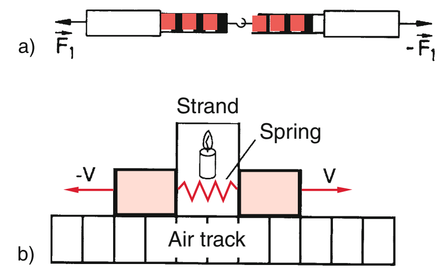

Figure 2.26 Experiment to prove the 3. Newtonian law **a** with two equal spring balances, **b** with two equal masses on an air track

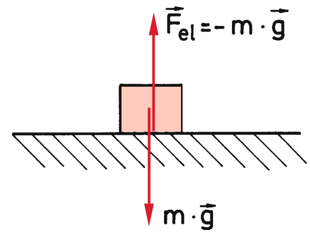

Figure 2.27 The gravitational force $\boldsymbol{F} = m \cdot \boldsymbol{g}$ of a mass m on a solid surface is compensated by the antiparallel deformation force of the solid surface

which are hold together by a string. If the string is burnt by a candle, the two masses are pushed by the expanding spring to opposite sides and slide on the air track with equal velocities, which means that they have equal but opposite momenta. The velocities can be accurately measured by photoelectric barriers.

Newton's third law can be also proved for resting bodies. A mass m resting on a solid surface acts with the gravitational force $\boldsymbol{F}_1 = m \cdot \boldsymbol{g}$ on the surface which is deformed and responds with an equal but opposite elastic force $\boldsymbol{F}_{\mathrm{el}} = -\boldsymbol{F}_1 = -m\boldsymbol{g}$ (Fig. 2.27).

2.6.2 Inertial and Gravitational Mass

The property of bodies to remain in their state of motion when left alone (i. e. when no force is acting on them) is called their **inertia**. Since the accelerating force is proportional to the mass of the body its mass can be regarded as the cause of the inertia and is therefore called the **inertial mass** m_{inertial}. Newton's second law means this inertial mass. There are many demonstration experiments which illustrate this inertia. Assume, for example, a glass of water standing on a sheet of paper. If the paper is pulled suddenly away, the glass remains a rest without moving, because of its inertia.

There is another property of masses which is the gravitational force ($\boldsymbol{F}_{\text{grav}} = m \cdot \boldsymbol{g}$ on the earth surface). This force is also called the weight of the mass. Experiments measure the weight of a mass of 1 kg as

$$\boldsymbol{F}_{\text{grav}} = 1\,\mathrm{kg} \cdot 9.81\,\mathrm{m/s^2} = 9.81\,\mathrm{N} .$$

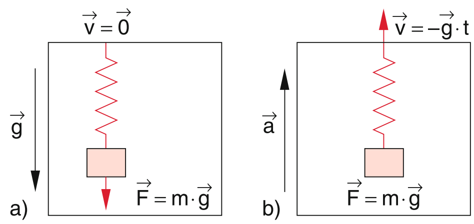

Figure 2.28 Einstein's Gedanken-experiment for the equivalence of gravitational and inertial mass **a** in the homogeneous gravitational field of the earth; **b** in a gravitation-free space inside an accelerated lift

Note: The gravitational force is always present when the mass m is attracted by another mass M and it is proportional to the product $m \cdot M$ (see Sect. 2.9.2).

The question is now: Are these two properties related to the same mass i. e. is $m_{\text{inertial}} = m_{\text{grav}}$?

Many detailed and accurate measurements for many different masses have proved that within the relative uncertainty of 10^{-10} there is no measurable difference between m_{inertial} and m_{grav}.

Starting from this experimental result Einstein has postulated the general *equivalence principle* **that inertial and gravitational masses are always equal**.

By the following "Gedanken-experiment" he has shown, that it doesn't make sense to distinguish between inertial and gravitational masses:

An observer in a closed lift measures a mass m hanging on a spring balance (Fig. 2.28). He cannot distinguish, whether the elevator is resting in a gravitational field with the gravitational force $\boldsymbol{F}_{\text{grav}} = m \cdot \boldsymbol{g}$ on the mass m (Fig. 2.28a) or whether the elevator moves upwards with the velocity $\boldsymbol{v} = -\boldsymbol{g}t$ and the acceleration $-\boldsymbol{g}$ in a force-free surrounding (Fig. 2.28b). Both situations lead to the same elongation of the spring balance. Any further experiment performed inside the closed elevator leads to the same results for the two situations (a) and (b).

For instance when the observer in the elevator throws a ball in the horizontal direction the trajectory of the ball is for both situations a parabola (see Fig. 2.9).

We will therefore no longer distinguish between inertial and gravitational mass and call it simply the mass m of a body which has the two characteristic features of inertia under acceleration and weight in gravitational fields.

Note: The question what the mass of a body really means is up to date not answered, although great efforts are undertaken to solve this problem.

2.6.3 The Equation of Motion of a Particle in Arbitrary Force Fields

Integration of Newton's equation of motion $\boldsymbol{F} = m \cdot \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ yields the equations

$$\boldsymbol{v}(t) = \frac{1}{m}\int \boldsymbol{F}\mathrm{d}t + \boldsymbol{C}_1 \,, \tag{2.20a}$$

$$\begin{aligned} \boldsymbol{r}(t) &= \int \boldsymbol{v}(t)\,\mathrm{d}t + \boldsymbol{C}_2 \\ &= \frac{1}{m}\int\left[\int \boldsymbol{F}\mathrm{d}t\right]\mathrm{d}t + \int \boldsymbol{C}_1\,\mathrm{d}t + \boldsymbol{C}_2 \end{aligned} \tag{2.20b}$$

For the velocity $\boldsymbol{v}(t)$ and the position vector $\boldsymbol{r}(t)$ with the integration constants $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ which are fixed by the initial conditions (e.g. $\boldsymbol{v}(t=0) = \boldsymbol{v}_0$ and $\boldsymbol{r}(t=0) = \boldsymbol{r}_0$).

Whether these equations are analytical solvable depends on the form of the force $\boldsymbol{F}$ which can be a function of position $\boldsymbol{r}$, velocity $\boldsymbol{v}$ or time $\boldsymbol{t}$. We will illustrate this by some examples.

Constant Forces

For the most simple case of constant forces $\boldsymbol{F} = \textbf{const}$, which do not depend on time nor on the position or velocity of the particle the integration of (2.20) immediately gives

$$\begin{aligned} \boldsymbol{F} &= m \cdot \boldsymbol{a} = \textbf{const} \,, \\ \boldsymbol{v}(t) &= \boldsymbol{a}t + \boldsymbol{C}_1 \quad \text{with} \quad \boldsymbol{C}_1 = \boldsymbol{v}_0 = \boldsymbol{v}(t=0) \,, \\ \boldsymbol{r}(t) &= \tfrac{1}{2}\boldsymbol{a}t^2 + \boldsymbol{v}_0 t + \boldsymbol{r}_0 \quad \text{with} \quad \boldsymbol{r}_0 = \boldsymbol{r}(t=0) \,. \end{aligned} \tag{2.21}$$

The trajectory of the particle can be directly determined, if the initial conditions are known. It is advisable to choose the coordinate system in such a way that the force coincides with one of the coordinate axes.

Example

The motion of a particle under the influence of the constant force $\boldsymbol{F} = \{0, 0, -mg\}$ pointing into the $-z$-direction gives the three equations for the 3 components of the force

$$\begin{aligned} \ddot{x} &= 0 \quad \Rightarrow \dot{x} = A_x \quad \Rightarrow x = A_x t + B_x \\ \ddot{y} &= 0 \quad \Rightarrow \dot{y} = A_y \quad \Rightarrow y = A_y t + B_y \\ \ddot{z} &= -g \Rightarrow \dot{z} = -gt + A_z \Rightarrow z = -\tfrac{1}{2}gt^2 + A_z t + B_z \,. \end{aligned} \tag{2.22}$$

These equations describe every possible motion of particles under the influence of the earth gravitation in a volume which is small compared with the dimensions of the earth where the gravitational force can be regarded as constant. From the many possible solutions of (2.22) the initial conditions with fixed values of A and B select special solutions (see examples in Sect. 2.3). ◀

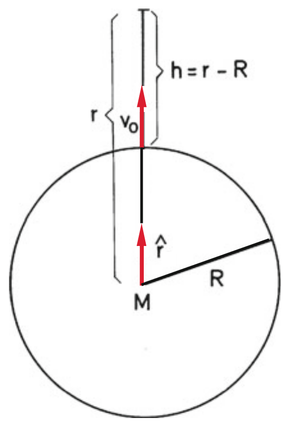

Figure 2.29 Launch of a body from the earth surface

Forces $F(r)$ that depend on the Position

As an example of position dependent forces we choose the gravitational force

$$\boldsymbol{F}(\boldsymbol{r}) = -G\frac{m \cdot M}{r^2}\hat{\boldsymbol{r}} \,.$$

The minus-sign indicates that the attractive force points into the direction of $-\boldsymbol{r}$.

The acceleration $\boldsymbol{a}$ has in this central force field only a radial component $a_r = -G \cdot M/r^2$. For vertical motions the velocity becomes in spherical coordinates $\boldsymbol{v} = \{v_r, 0, 0\}$ and its magnitude is $|\boldsymbol{v}| = v = v_r$. Our problem therefore becomes one-dimensional. From the relation

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}r} \cdot \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}r} \cdot v \,,$$

it follows: $v \cdot \mathrm{d}v = -(G \cdot M/r^2)\mathrm{d}r$.

Integration yields

$$\frac{1}{2}v^2 = \frac{GM}{r} + C_1 \,. \tag{2.23}$$

Let us discuss the case that a projectile is fired from the earth surface ($r = R$) upwards in vertical direction with the initial velocity v_0 (Fig. 2.29). The integration constant C_1 then becomes

$$C_1 = \frac{1}{2}v_0^2 - \frac{GM}{R} = \frac{1}{2}v_0^2 - g \cdot R \,,$$

because $a(R) = -g = -G(M/R^2)$. This gives

$$\frac{1}{2}v^2 = \frac{gR^2}{r} + \frac{1}{2}v_0^2 - g \cdot R \,. \tag{2.24}$$

At the maximum vertical height $r = r_{\max}$ the velocity becomes $v(r_{\max}) = 0$ and we obtain from (2.24)

$$r_{\max} = \frac{R}{1 - (v_0^2/2Rg)} \,. \tag{2.25}$$

For the initial velocity $v_0 \to \sqrt{2Rg}$ the maximum vertical height $r_{\max}$ becomes infinity and the projectile can leave the earth. This velocity is called *the escape velocity*. Inserting the numerical values for R and g gives

$$v_0 \geq v_2 = \sqrt{2Rg} = 11.2\,\mathrm{km/s} \,. \tag{2.26a}$$
(escape velocity)

The velocity v_2 is often named the 2nd cosmic velocity while the first cosmic velocity v_1 is the velocity of a projectile which is fired in horizontal direction and orbits around the earth on a circle closely above the earth surface. From the relation

$$\frac{v_1^2}{R} = \frac{GM}{R^2} \to v_1 = \sqrt{\frac{GM}{R}} = \sqrt{g \cdot R} \tag{2.26b}$$

the numerical value of v_1 becomes (when neglecting the earth rotation) $v_1 = v_2/\sqrt{2} \approx 7.9\,\mathrm{km/s}$.

Note: The general case of arbitrary motion in a central force field is treated in Sect. 2.9.

Time-dependent Forces

There are many situations where the force on a particle changes with time. One simple example is a mass hanging on a spring, which is induced to vertical oscillations, or a comet moving on a parabolic trajectory through the solar system. We will illustrate the solution of the equation of motion for time dependent forces by two numerical examples.

Examples

1. Assume the time dependent force $F = b \cdot t + c$ with $b = 120\,\mathrm{N/s}$ and $c = 40\,\mathrm{N}$, which points into the x-direction, is acting on the mass $m = 10\,\mathrm{kg}$. For $t = 0$ the mass should be at $x = 5\,\mathrm{m}$ with a velocity $v(0) = 6\,\mathrm{m/s}$. Calculate the position $x(t)$.

Solution

The straight motion proceeds along the x-axis. The acceleration is $a = F/m$ and the velocity

$$v(t) = \frac{1}{m}\int_0^t F(\tau)\mathrm{d}\tau = \frac{b}{2m}t^2 + \frac{c}{m}t + v_{0x} \,;$$

$$\begin{aligned} x(t) &= \int v_x \mathrm{d}\tau = \frac{b}{6m}t^3 + \frac{c}{2m}t^2 + v_{0x}t + x_0 \\ &= (2t^3 + 2t^2 + 6t + 5)m \quad \text{with } t \text{ in s} \,. \end{aligned}$$

2. What is the final velocity of a mass m initially at rest ($v(0) = 0$) which experiences a force $F(t) = A \cdot \exp[-a^2t^2]$?

Solution

$$m \cdot v(t=\infty) = \int F \,\mathrm{d}t = A \int \mathrm{e}^{-a^2t^2}\,\mathrm{d}t = \frac{A\sqrt{\pi}}{2a}\,;$$
$$v_\infty = \frac{A}{2}\frac{\sqrt{\pi}}{a \cdot m}\,.$$

Acceleration of a Rocket

In the example for position dependent forces we have assumed that the projectile starts with the initial velocity $v_0 > 0$. In reality it starts with $v_0 = 0$. However, the velocity $v > 0$ is reached within a short distance that is very small compared with the earth radius R. We will now study the acceleration during the start phase of the rocket in more detail. Within this small distance $d \ll R$, which the rocket passes during its acceleration, we can fairly assume the earth acceleration g to be constant.

During the burning phase the rocket is continuously accelerated by the recoil momentum of the propellant hot gases (Fig. 2.30).

With v' we denote the velocity of the propellant gases relative to the surface of the earth which represents our reference coordinate system, and with v the rocket velocity in this system. The escaping gas mass per second is $\Delta m/\Delta t$. The momentum of the rocket at time t is $p(t) = m \cdot v$. At time $t + \Delta t$ the mass of the rocked has been reduced by $-\Delta m$ (which equals the mass of the expanding gas during this time interval) and its velocity has increased by Δv while the gases have transported the momentum $\Delta m \cdot v'$. The total momentum of the system rocket + gas is then with $\Delta m > 0$

$$\boldsymbol{p}(t+\Delta t) = (m - \Delta m)(\boldsymbol{v} + \Delta \boldsymbol{v}) + \Delta m \cdot \boldsymbol{v}'\,. \tag{2.27a}$$

During the time interval Δt the momentum of the system has changed by

$$\begin{aligned}\Delta \boldsymbol{p} &= \boldsymbol{p}(t+\Delta t) - \boldsymbol{p}(t)\\ &= m \cdot \Delta \boldsymbol{v} + \Delta m(\boldsymbol{v}' - \boldsymbol{v}) - \Delta m \cdot \Delta v\,.\end{aligned} \tag{2.27b}$$

Figure 2.30 Acceleration of a rocket

For the limit $\Delta t \to 0$; $\Delta m/\Delta t \to \mathrm{d}m/\mathrm{d}t$ is $\lim_{\Delta t \to 0}(\Delta m \cdot \Delta v/\Delta t) = 0$.

Since the time derivative $\mathrm{d}\boldsymbol{p}/\mathrm{d}t$ of the momentum equals the force $F_g = m \cdot \boldsymbol{g}$ of gravity acting on the rocket we obtain

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}(\boldsymbol{v}' - \boldsymbol{v}) = m \cdot \boldsymbol{g}\,. \tag{2.27c}$$

The velocity $\boldsymbol{v}'$ of the propellant gases *relative to the earth* depends on the velocity $\boldsymbol{v}$ of the rocket. For $|\boldsymbol{v}| < |\boldsymbol{v}'|$ the direction of $\boldsymbol{v}'$ is downwards, for $|\boldsymbol{v}| > |\boldsymbol{v}'|$ it is upwards. It is therefore better to introduce the velocity $\boldsymbol{v}_\mathrm{e} = \boldsymbol{v}' - \boldsymbol{v}$ of the propellant gases *relative to the rocket*, which is independent of $\boldsymbol{v}$ and constant in time. This converts Eq. 2.27c into

$$m \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{\mathrm{d}m}{\mathrm{d}t}\boldsymbol{v}_\mathrm{e} = m \cdot \boldsymbol{g}\,. \tag{2.27d}$$

With $\boldsymbol{v} = \{0,0,v_z\}$, $\boldsymbol{v}_\mathrm{e} = \{0,0,v_\mathrm{e}\}$, $\boldsymbol{g} = \{0,0,-g\}$ this equation becomes after division by m and multiplication by $\mathrm{d}t$

$$\mathrm{d}v = -v_\mathrm{e}\frac{\mathrm{d}m}{\mathrm{d}t} - g \cdot \mathrm{d}t\,, \tag{2.27e}$$

Integration from $t = 0$ up to $t = T$ (propellant time of the rocket) yields

$$v(T) = v_0 + v_\mathrm{e} \ln \frac{m_0}{m} - gT\,, \tag{2.28}$$

where $v_0 = v(t=0)$.

Numerical Example

Launching of a *Saturn rocket* with $m_0 = 3 \cdot 10^6$ kg; $v_\mathrm{e} = 4000$ m/s, $T = 100$ s, $v_0 = 0$. Final mass at $t = T$ is $m(T) = 10^6$ kg, which means that the mass of the fuel is $2 \cdot 10^6$ kg. Equation 2.28 yields

$$\begin{aligned}v(T = 100\,\mathrm{s}) &= 0 + 4000\,\mathrm{m/s} \cdot \ln 3 - 9.81\,\mathrm{m/s^2}\,100\,\mathrm{s}\\ &= 3413.5\,\mathrm{m/s}\,.\end{aligned}$$

The heights $z(t)$ of the rocket during its burning time for constant loss of mass $q = \mathrm{d}m/\mathrm{d}t = \mathrm{const}$ is readily obtained. With $m(t) = m_0 - q \cdot t$, Eq. 2.28 becomes

$$v(t) = v_0 - v_\mathrm{e} \ln\left(1 - \frac{q}{m_0}t\right) - gt\,;$$
$$z(t) = v_0 t - v_\mathrm{e} \int \ln\left(1 - \frac{q}{m_0}t\right)\mathrm{d}t - \frac{1}{2}gt^2 + C_0\,,$$

and integration yields

$$z(t) = v_0 \cdot t - v_\mathrm{e} \int \ln\left(1 - \frac{q}{m_0}t\right)\mathrm{d}t - \frac{1}{2}gt^2 + C_0\,.$$

The integration constant is $C_0 = 0$ (because $z(0) = 0$).

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Since $\int \ln x dx = x \ln x - x$ the integration gives

$$z(t) = (v_0 + v_e)\,t + v_e \left[\frac{m_0}{q} - t\right] \ln\left(1 - \frac{q}{m_0} t\right) - \frac{1}{2} g t^2 \,. \tag{2.29}$$

Numerical Example

For our example above we obtain with $q = 2 \cdot 10^4$ kg/s, $v_0 = 0$; $v_e = 4000$ m/s $T = 100$ s

$$\begin{aligned} z(T) &= (4 \cdot 10^5 + 2 \cdot 10^3 \cdot \ln 0.33 - 4.9 \cdot 10^4 \text{ m} \\ &= (400 - 219.7 - 49)\,\text{km} = 131\,\text{km}\,, \\ v(T) &= \left[-4 \cdot 10^3 \cdot \ln(0.33) - 981\right] \text{m/s} = 3413\,\text{m/s}\,, \end{aligned}$$

This example illustrates that with $z(T) \ll R$ the earth acceleration does not change much and can be regarded as constant. It further demonstrates that with a single stage the escape velocity $v = 11200$ m/s of the rocket cannot be achieved with reasonable fuel masses. It is therefore necessary to use multistage rockets.

Numerical Example

After the end of the burning time T_1 of the first stage the velocity of the rocket in our example is $v(T_1) = 3400$ m/s. The second stage starts with a mass $m(T_1) = 9 \cdot 10^5$ kg (the fuel tank with $m = 10^5$ kg has been pushed off) including $m = 7 \cdot 10^5$ kg for the fuel. The burning time is again 100 s and the final mass $m(T_2) = 2.10^5$ kg. According to (2.28) the final velocity v is

$$\begin{aligned} v(T_1 + T_2) &= (3400 + 4000 \ln(9/2) - 9.81 \cdot 100) \\ &= 8435\,\text{m/s}\,. \end{aligned}$$

The third stage starts with a velocity $v = 8435$ m/s and a mass $m = 1.8 \cdot 10^5$ kg (the fuel tank of the 2nd stage with $m = 2 \cdot 10^4$ kg has been pushed off). With $T_3 = 100$ s we obtain the final velocity

$$\begin{aligned} v(T_{\text{final}}) &= (8400 + 4000 \ln 7.2 - 9.8 \cdot 100)\,\text{m/s} \\ &= 15{,}000\,\text{m/s} > v_{\text{escape}}\,. \end{aligned}$$

Note: For the second and third stage one should, strictly speaking, take into account the decrease of the earth acceleration g with increasing z. Instead of the constant g one should use the function $g(z) = G \cdot M/r^2$ with $r = z + R$ and M = mass of the earth. With the approximation $(1 + z/R)^{-2} \approx 1 - 2z/R$ one obtains instead of (2.27e) the equation

$$dv = -v_e \frac{dm}{m} - g(1 - 2z/R)dt\,. \tag{2.30}$$

This equation illustrates that even for $z = 100$ km the correction term $2z/R$ for g amounts only to 3%. This means for the calculation of the velocity v only a correction of 1%, because the term $g \cdot T$ in Eq. 2.28 represents only about $1/3v$.

The integration of (2.28) is now more tedious but an approximation is still possible, if the function (2.29) is inserted for $z(t)$.

2.7 Energy Conservation Law of Mechanics

In this section we will discuss the important terms "*work*", "*power*", "*kinetic and potential energy*" before we can formulate the **energy conservation law of mechanics**.

2.7.1 Work and Power

If a point mass m proceeds along the path element $\boldsymbol{\Delta r}$ in a force field $\boldsymbol{F}(\boldsymbol{r})$ (Fig. 2.31), the scalar product

$$\Delta W = \boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{\Delta r} \tag{2.31a}$$

is called **the *mechanical work***, due to the action of the force F on the point mass m.

The work is a scalar quantity!

Written in components of the vectors $\boldsymbol{F}$ and $\boldsymbol{r}$ Eq. 2.31a reads

$$\Delta W = F_x \Delta x + F_y \Delta y + F_z \Delta z\,. \tag{2.31b}$$

The unit of work is [work] = [force · length] = 1 N · m = 1 Joule = 1 J.

Remark. In the *cgs-system* the unit is [W] = 1 dyn · cm = 1 erg = 10^{-7} J.

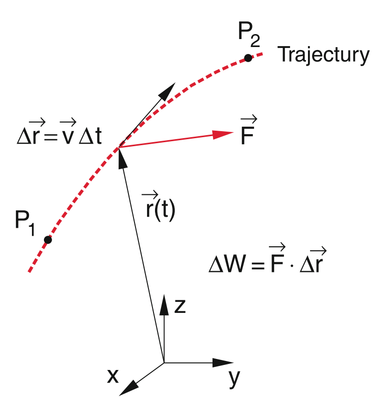

Figure 2.31 Definition of work

If the point mass moves under the action of the force F from point P_1 to point P_2 the total work on this path is the sum $W = \sum \Delta W_i$ of the different contributions $\Delta W_i = \boldsymbol{F}(\boldsymbol{r}_i) \cdot \Delta \boldsymbol{r}_i$ which converges in the limit $\Delta r_i \to 0$ to the integral

$$W = \int_{P_1}^{P_2} \boldsymbol{F} \cdot \mathbf{d}\boldsymbol{r} \,. \tag{2.32a}$$

The integral is called *line-integral* or *curvilinear integral*. Because of the relation $\boldsymbol{F} \cdot \mathbf{d}\boldsymbol{r} = F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z$ it can be reduced to a sum of simple Rieman integrals:

$$\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{x_1}^{x_2} F_x \mathrm{d}x + \int_{y_1}^{y_2} F_y \mathrm{d}y + \int_{z_1}^{z_2} F_z \mathrm{d}z \,, \tag{2.32b}$$

which can be readily calculated if the force is known (see the following examples). In Equation 2.32 is $W > 0$ for $\boldsymbol{F} \cdot \mathbf{d}\boldsymbol{r} > 0$ i.e. if the force $\boldsymbol{F}$ has a component in the direction of the movement. In this case the mass m is accelerated. According to this definition the work is positive if the energy of the mass m is increased. Work which is performed by the mass on other systems decreases its energy and is therefore defined as negative (see Sect. 2.7.3).

If $\boldsymbol{F}$ is perpendicular to $\boldsymbol{r}$ (and therefore also to the velocity $\boldsymbol{v}$) the work is $W = 0$, because then the scalar product $\boldsymbol{F} \cdot \mathbf{d}\boldsymbol{r} = \mathbf{0}$.

The work per time unit

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{2.33a}$$

is called the **power *P***. Its unit is $[P] = 1\,\mathrm{J/s} = 1\,\mathrm{Watt} = 1\,\mathrm{W}$.

$$\begin{aligned} P &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_0}^{t} \boldsymbol{F}(\boldsymbol{r}(t'), t') \cdot \dot{\boldsymbol{r}}(t') \mathrm{d}t' \\ &= \boldsymbol{F}(\boldsymbol{r}(t), t) \cdot \boldsymbol{v}(t) = \boldsymbol{F} \cdot \boldsymbol{v} \,. \end{aligned} \tag{2.33b}$$

Remark. In daily life the electrical work is defined in kWh. With $1\,\mathrm{J} = 1\,\mathrm{Ws}$ the relation is $1\,\mathrm{kWh} = 3.6 \cdot 10^6\,\mathrm{Ws}$.

Examples

1. Uniform circular motion under the action of a radial constant force. Here $\boldsymbol{v}$ always points in the direction of the tangent to the circle, but the force is always radial, i. e. $\boldsymbol{F} \perp \boldsymbol{v}$. The scalar product $\boldsymbol{F} \cdot \boldsymbol{v} = 0$ and therefore the work is zero.
2. A mass is moved with constant velocity without friction on a horizontal plane. (motion on a straight line). The gravitational force is always perpendicular to the motion, $\to \boldsymbol{F} \cdot \mathbf{d}\boldsymbol{r} = 0$. The work is zero.
3. The work performed by a mountaineer against the gravitational force (man + pack = 100 kg), who climbs up the Matterhorn ($\Delta z = 1800\,\mathrm{m}$) is $W = \int F_g \mathrm{d}z = -m \cdot g \cdot \Delta z = 10^2 \cdot 9.81 \cdot 1.8 \cdot 10^3\,\mathrm{kg \cdot m^2/s^2} = -17.6 \cdot 10^5\,\mathrm{J} \approx 0.5\,\mathrm{kWh}$.
   The work is negative, because the force is antiparallel to the direction of the movement. The mountaineer produces energy by burning his food and converts it into potential energy thus decreasing its internal energy. The prize for the electrical equivalent of 0.5 kWh is about 10 Cents!
4. In order to expand a coil spring one has to apply a force $\boldsymbol{F} = -\boldsymbol{F}_\mathbf{r}$ opposite to the restoring spring force $\boldsymbol{F}_\mathbf{r} = -D(\boldsymbol{x} - \boldsymbol{x}_0)$ which is proportional to the elongation $(x - x_0)$ of the spring from its equilibrium position x_0. The work which has to be applied is

$$\begin{aligned} W &= \int F_x \mathrm{d}x = D \int (x - x_0)\mathrm{d}x \\ &= \tfrac{1}{2} D (x - x_0)^2 \,. \end{aligned}$$

   This is equal to the area A in Fig. 2.32a between the x-axis and the straight line $F = D(x - x_0)$.

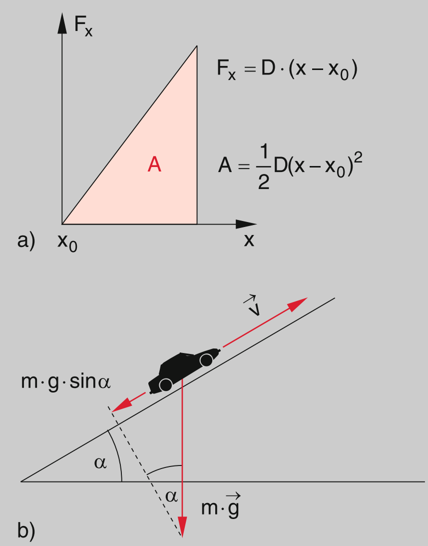

Figure 2.32 **a** Work for expanding a spring, **b** work of a car climbing up a slope

5. A car ($m = 1000\,\mathrm{kg}$) moves with constant velocity of 48 km/h on a straight line with a slope of 5° against the horizontal (Fig. 2.32b). What is the work the engine has to produce within 5 min, if friction effects can be neglected?
   The force in the direction of the motion is

$$F = -F_g \cdot \sin\alpha = m \cdot g \cdot \sin\alpha \,.$$

   The distance which the car moves within 5 min is

$$s = 48\,\mathrm{km} \cdot 5/60 = 4\,\mathrm{km} = 4000\,\mathrm{m} \,.$$

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The work is then with $1\,\text{kWh} = 10^3 \cdot 3.6 \cdot 10^3\,\text{Ws} = 3.6 \cdot 10^6\,\text{J}$

$$W = 4 \cdot 10^3 \cdot 9.81 \cdot \sin 5° \cdot 10^3\,\text{N} \cdot m$$
$$= 3.4 \cdot 10^6\,\text{J} \approx 1\,\text{kWh}\,.$$

The power is

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{3.4 \cdot 10^6\,\text{J}}{300\,\text{s}} \approx 1.13 \cdot 10^4\,\text{W}$$
$$= 11.3\,\text{kW}\,.$$

2.7.2 Path-Independent Work; Conservative Force-Fields

We regard a force field $\boldsymbol{F}(\boldsymbol{r})$ that depends only on the position $\boldsymbol{r}$ but not on time. When a mass m is moved from point P_1 to point P_2 on the path (a) (Fig. 2.33) the work necessary for this motion is

$$W_a = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{a}\,.$$

On the path (b) it is

$$W_b = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{b}\,.$$

If for arbitrary paths (a) and (b) always $W_\mathrm{a} = W_\mathrm{b}$ we name the integral *path-independent* and the force field $F(\boldsymbol{r})$ *conservative*.

With other words:

In conservative force fields the work necessary to move a mass m from a point $P(r_1)$ to a point $P(r_2)$ is independent of the path between the two points.

If we move the mass from P_1 to P_2 and back to P_1 the total work is then zero.

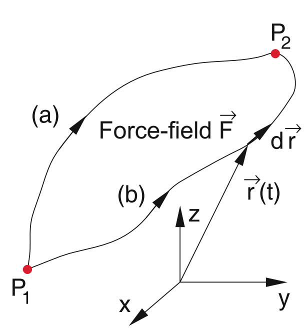

Figure 2.33 Path-independent work in a conservative force field

In conservative force fields the work for moving a mass m on a closed loop is zero.

$$\begin{aligned} W_\mathrm{a} - W_\mathrm{b} &= \int_{P_1}^{P_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{a} - \int_{P_1}^{P_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{b} \\ &= \int_{P_1}^{P_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{a} + \int_{P_2}^{P_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{b} \\ &= \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0\,. \end{aligned} \tag{2.34}$$

The work depends only on initial and final position of the motion, not on the chosen path between them.

In Vector-Analysis it is proved that the equivalent condition for a conservative force field $F(r)$ is $\textbf{curl}\,\boldsymbol{F} = \boldsymbol{0}$ (theorem of Stokes). For the definition of $\textbf{curl}\,\boldsymbol{F}$ see Sect. 13.1. It is

$$\begin{aligned} \textbf{curl}\,\boldsymbol{F} &= \textbf{rot}\,\boldsymbol{F} = \nabla \times \boldsymbol{F} \\ &= \left\{ \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\}. \end{aligned}$$

Conservative force fields are a special case of force fields $\boldsymbol{F}(\boldsymbol{r})$ that depend only on the position $\boldsymbol{r}$, not on time or velocity.

Note: Not every force field $\boldsymbol{F}(\boldsymbol{r})$ is conservative! (see Example below)

Examples

Conservative Force Fields

1. A homogeneous force field $\boldsymbol{F}(\boldsymbol{r}) = \{0, 0, F_z\}$ with $F_z = \text{const}$ (Fig. 2.34a) is conservative because

$$\begin{aligned} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &= F_z \mathrm{d}z \to W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \\ &= \int_{z_1}^{z_2} F_z \mathrm{d}z = -\int_{z_2}^{z_1} F_z \mathrm{d}z \to \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0\,. \end{aligned}$$

2. Every time-independent *central force field*, written in spherical coordinates (see Sect. 13.1) as $\boldsymbol{F} = \{F_\mathrm{r}, F_\vartheta = 0, F_\varphi = 0\}$, which depends only on the distance r from the centre $r = 0$ and not on the angles ϑ and φ is conservative.
   It can be written as $\boldsymbol{F}(\boldsymbol{r}) = f(r) \cdot \hat{\boldsymbol{r}}$, where $f(r)$ is a scalar function of r (Fig. 2.34b).

$$\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{r_1}^{r_2} F_\mathrm{r} \mathrm{d}r = -\int_{r_2}^{r_1} F_\mathrm{r} \mathrm{d}r \Rightarrow \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0\,.$$

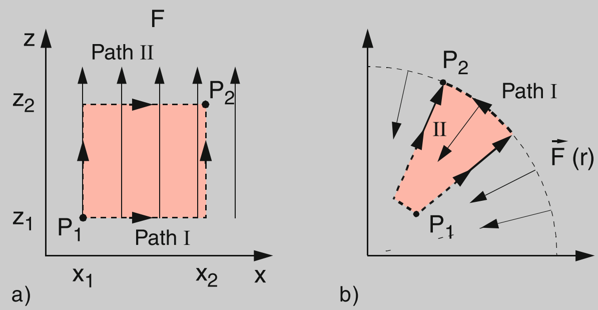

Figure 2.34 Examples for conservative force fields. **a** Homogeneous field, **b** central field

Non-conservative Force Fields

1. Position-dependent non-central force field

$$\boldsymbol{F}(\boldsymbol{r}) = y\boldsymbol{e}_x + x^2\boldsymbol{e}_y \; .$$

The work one has to expend for moving a body from point $P_1 = \{0, 0, 0\}$ to point $P_2 = \{2, 4, 0\}$ is

$$W = \int_0^P \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{x=0}^{2} F_x \mathrm{d}x + \int_{y=0}^{4} F_y \mathrm{d}y$$

$$= \int_{x=0}^{2} y\,\mathrm{d}x + \int_{y=0}^{4} x^2 \mathrm{d}y \; .$$

We choose two different paths (Fig. 2.35):
(a) along the straight line $y = 2x$
(b) along the parabola $y = x^2$.
On the path (a) is $y = 2x \Rightarrow x^2 = (y/2)^2$

$$\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{a} = \int_0^2 2x\mathrm{d}x + \int_0^4 \left(\frac{y}{2}\right)^2 \mathrm{d}y$$

$$= x^2\Big|_0^2 + \frac{y^3}{12}\Big|_0^4 = 4 + \frac{16}{3} = 28/3 \; ,$$

On the path (b) is $y = x^2$.

$$\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{b} = \int_0^2 x^2 \mathrm{d}x + \int_0^4 y\,\mathrm{d}y$$

$$= \frac{1}{3}x^3\Big|_0^2 + \frac{1}{2}y^2\Big|_0^4 = \frac{8}{3} + 8 = \frac{32}{3} \; .$$

$\Rightarrow \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \neq 0$. The force field is not conservative!

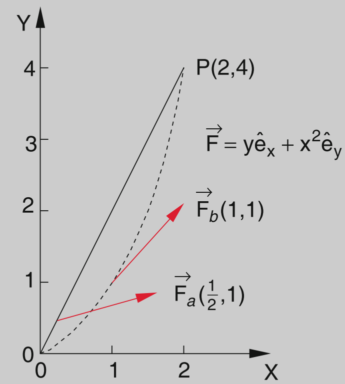

Figure 2.35 Movement in a non-conservative force field $\boldsymbol{F}(\boldsymbol{r}) = y \cdot \boldsymbol{e}_x + x^2 \cdot \boldsymbol{e}_y$

2. For time-dependent force fields the integral cannot be path-independent, because the force field varies during the travel of the body and therefore the work expended for the different paths is generally different.
3. If the force depends on the velocity of the body (for instance the friction for a body moving through a medium or on a surface, or the Lorentz-force $\boldsymbol{F} = q \cdot (\boldsymbol{v} \times \boldsymbol{B})$ on a charge q moving with the velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$) such fields are generally not conservative because the velocity differs generally on the different paths. For friction forces F_f the force is for small velocities v proportional to $v (F_f \sim v)$, when the body moves slowly through a liquid. For large velocities is $F_f \sim v^3$ for example when a body moves through turbulent air. For all friction forces heat is produced and therefore the mechanical energy cannot be preserved. In all these cases $\oint F \cdot \mathrm{d}r \neq 0$ (see also Sect. 6.5) ◀

Time-dependent or velocity-dependent forces are generally not conservative.

2.7.3 Potential Energy

When a body is moved in a conservative force field from a starting point $P_1(\boldsymbol{r}_1)$ to another point $P_2(\boldsymbol{r}_2)$ the work expended or gained during this movement does not depend on the path between the two points. If P_0 is a fixed point P_0 and $P(\boldsymbol{r})$ has an arbitrary position $\boldsymbol{r}$ the work solely depends on the initial point P_0 and the final point $P(r)$. It is therefore a function of $P(\boldsymbol{r})$ with respect to the fixed point P_0. This function is called the **potential energy $E_\mathrm{p}(P)$** of the body.

The work

$$\Delta W = \int_{P_1}^{P_2} \boldsymbol{F}\,\mathrm{d}\boldsymbol{r} \stackrel{\mathrm{Def}}{=} -\big(E_\mathrm{p}(P_2) - E_\mathrm{p}(P_1)\big) \; , \qquad (2.35\mathrm{a})$$

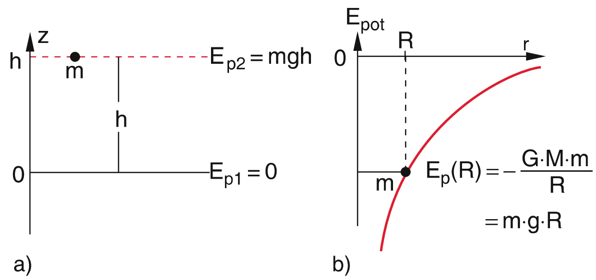

Figure 2.36 Different possibilities to choose the zero of the potential energy: **a** $E_p(z=0)=0$; **b** $E_p(r=\infty)=0$

which the force $\boldsymbol{F}(\boldsymbol{r})$ accomplishes on the body when it is moved between two points P_1 and P_2 is equal to the difference of the potential energies in these two points. For $\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}>0$ the force is directed into the direction of the motion. The potential energy difference $\Delta E_p = E_p(P_1)-E_p(P_2)$ is then negative. This means, that the mass m can deliver the work ΔW but looses potential energy.

One example is the free fall in the gravitational field of the earth, when a mass m falls from the height h with potential energy $m\cdot g\cdot h$ to the ground with $h=0$. When we lift the mass m from $h=0$ to $h>0$ against the gravitational force, the scalar product $\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ is negative and the potential energy increases (Fig. 2.36a). The work spend on the body to lift it against the force results in an increase of the potential energy. A body with a positive potential energy can convert this potential energy again into work. An example is water falling down through pipes and drives a turbine which drives maschines and produces electricity.

Note:

1. The sign of work and potential energy difference in (2.35a) has been chosen in such a way, that for $\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}<0 \to \Delta W<0$ but $\Delta E_p>0$, i. e. one has to spend work in order to move the body against the force which increases its potential energy. Work which the body can deliver to its surrounding for $\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}>0$ decreases its potential energy.
2. The defined zero $E_p=0$ for the potential energy is not fixed by the definition (2.35a). If we choose the fixed reference point P_0 as the zero point of the potential energy and define $E_p(P_0)=0$, then the absolute value of the potential energy in point P is given by
$$W=\int_{P_0}^{P}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}=-E_p(P)\,. \tag{2.35b}$$
For our example of the free fall we can choose $h=0$ as the reference point with $E_p(0)=0$. In many cases where a body can be moved to very large distances from the earth (for instance space crafts) it is more convenient to choose $r=\infty$ as the reference point for $E_p(\infty)=0$. We then have the definition
$$\int_{P}^{\infty}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}=E_p(P)-E_p(\infty)=E_p(P)\,, \tag{2.35c}$$
the potential energy $E_p(P)$ is then negative for $F\cdot\mathrm{d}r<0$. It is equal to the work one has to spend in order to bring the body from the point P to infinity. For instance the potential energy of a mass m in the gravitational field of the earth $F_g=-GMm/r^2$ at a distance $r=R$ from the centre of the earth is then
$$E_p(R)=-GMm/R\,, \tag{2.35d}$$
where G is Newton's constant of gravity and M is the mass of the earth (Fig. 2.36b).
3. The work which one has to spend on the body (for $F\cdot\mathrm{d}r<0$ or which can be gained from the body (for $F\cdot\mathrm{d}r>0$) when it is moved from point P_1 to point P_2 is of course independent of the choice of the zero point because it depends only on the difference $\Delta E_p = E(P_1)-E_p(P_2)$ of the potential energies.

Examples

1. A body with mass m is lifted in the constant gravitational force field $F=\{0,0,-mg\}$ from $z=0$ to $z=h$, where $h \ll R$ (earth radius). The necessary work to achieve this lift is
$$W=\int \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}=-\int_0^h m\cdot g\,\mathrm{d}z$$
$$=-m\cdot g\cdot h=E_p(0)-E_p(h)\,.$$
If we choose $E_p(z=0)=0$ the potential energy for $z=h$ is $E_p(h)=+mgh$ (Fig. 2.37a). The work applied to the mass m appears as potential energy.

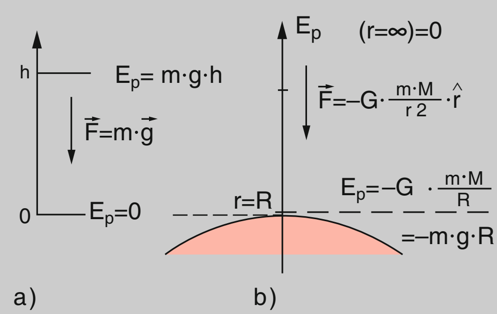

Figure 2.37 **a** Approximately homogeneous gravitational force field as small section of the spherical field of the earth in **b**. The selection of the definition $E_p=0$ is $E_p(z=0)=0$ in case **a** and $E_p(r=\infty)=0$ for case **b**

2. In an attractive force field, such as the gravitational field of the earth $\boldsymbol{F}=-(GMm/r^2)\boldsymbol{e}_r$ a mass m is moved from $r=R$ (earth surface) to $r=\infty$. In this case is $\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}<0$. The necessary work is negative:
$$W=-\int_r^\infty \frac{GMm}{r^2}\hat{\boldsymbol{r}}\mathrm{d}\boldsymbol{r}=-\int_r^\infty \frac{GMm}{r^2}\mathrm{d}r$$
$$=-\frac{GMm}{r}=E_p(r)\,. \tag{2.35e}$$

$E_p(r)$ is negative because $E(r=\infty)=0$. To raise the mass m work has to be applied, which is converted to the increase of potential energy (Fig. 2.37b).
For repulsive potentials (e.g. the Coulomb potential of two positive electrical charges q_1 and q_2)

$$\boldsymbol{F} = (q_1 \cdot q_2/r^2)\boldsymbol{e}_r$$

the potential energy is positive and one wins work when the charge separation increases, while the potential energy decreases.
When a body with mass m should be moved from the earth surface $r=R$ to $r=\infty$ one needs the work $W=-GMm/R$. With $g=GM/R^2$ this can be written as $W=-mgR$.
Numerical example: With $g = 9.81\,\text{m/s}^2$, $R = 6371\,\text{km}$, the work to launch a mass of 100 kg is $W = 6.25\cdot 10^9\,\text{J} = 1736\,\text{kWh}$. ◀

2.7.4 Energy Conservation Law in Mechanics

Multiplying the Newton equation

$$\boldsymbol{F} = m\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$$

scalar with the velocity $\boldsymbol{v}$ and integrating over time yields

$$\int \boldsymbol{F}\cdot\boldsymbol{v}\mathrm{d}t = m\int_{t_0}^{t_1}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\boldsymbol{v}\,\mathrm{d}t\,. \tag{2.36}$$

The integral on the left hand side gives with $v=\mathrm{d}\boldsymbol{r}/\mathrm{d}t$

$$\int \boldsymbol{F}\cdot\boldsymbol{v}\mathrm{d}t = \int_{P_0}^{P_1}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = E_p(P_0)-E_p(P_1)\,,$$

where the last equality is valid for conservative force fields.

The right hand side of (2.36) gives

$$m\cdot\int\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\boldsymbol{v}\,\mathrm{d}t = m\int_{v_0}^{v_1}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{v} = \frac{m}{2}v_1^2-\frac{m}{2}v_0^2\,.$$

The expression

$$E_{\text{kin}} = mv^2/2 \tag{2.37}$$

is called the **kinetic energy** of a body with mass m and velocity $v=|\boldsymbol{v}|$.

The integral $\int\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ represents the work W which is supplied to the body. The statement of Eq. 2.36 can therefore be formulated as:

$$\Delta E_{\text{kin}} = \Delta W\,. \tag{2.38a}$$

The increase of kinetic energy of a body is equal to the work supplied to this body.

In conservative force fields $\int\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ is equal to the change of potential energy. Then Eq. 2.36 states:

$$E_p(P_0)+E_{\text{kin}}(P_0) = E_p(P)+E_{\text{kin}}(P) = E\,. \tag{2.38b}$$

When a body is moved in a conservative force field from a point P_0 to a point P the total mechanical energy E (sum of potential and kinetic energy) is conserved, i. e. it has for all positions in the force field the same amount.

Examples

1. For the free fall starting from $z=h$ with the velocity $v(h)=0$ we choose $E_p(h=0)=0$. For arbitrary z the following equations hold:

$$E_p(z) = -\int_0^z -mg\mathrm{d}z = mgz\,.$$

With $v=g\cdot t$ and $s=h-z=\frac{1}{2}gt^2 \to \frac{1}{2}v^2 = \frac{1}{2}g^2t^2 = g(h-z)$ (see Sect. 2.3).
This gives

$$E_{\text{kin}}(z) = \tfrac{1}{2}mv^2 = m\cdot g\cdot(h-z)\,.$$

The sum $E_p(z)+E_{\text{kin}}(z) = mgh$ is independent of z and for all z equal to the total energy $E=mgh$.

2. A body with mass m oscillates in the x-direction, driven by the force $F=-D\cdot x$. For each point of its path the total energy is $E=E_p(x)+E_{\text{kin}}(x)=\text{const}$. For $x=0$ the potential energy is zero. In the upper turning points for $x=\pm x_m$ the velocity is zero and therefore $E_{\text{kin}}=0$. (Fig. 2.38).

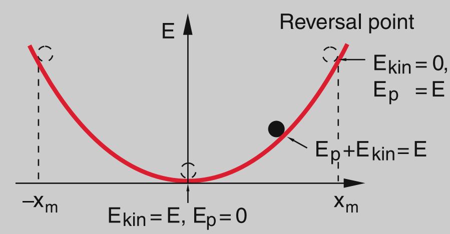

Figure 2.38 Example of energy conservation for a harmonic motion

The oscillation can be described by

$$x = x_m\sin\omega t \to v = \mathrm{d}x/\mathrm{d}t = x_m\omega\cos\omega t\,.$$

The potential energy is $E_p = \int Dx\mathrm{d}x = \frac{1}{2}Dx^2 = \frac{1}{2}Dx_m^2\sin^2\omega t$. The kinetic energy is $E_{\text{kin}} = \frac{1}{2}mv^2 =$

Chapter 2

$\frac{1}{2}m \cdot x_m^2\omega^2\cos^2\omega t$. From the Newton equation $F = ma = m \cdot \mathrm{d}^2x/\mathrm{d}t^2$ we obtain by comparison with $F = -Dx$ the relation $D = m \cdot \omega^2$. Inserting this into the expression for the potential energy we get

$$E = E_\mathrm{p} + E_\mathrm{kin} = \tfrac{1}{2}mx_m^2\omega^2(\sin^2\omega t + \cos^2\omega t) = \tfrac{1}{2}m \cdot x_m^2\omega^2 ,$$

which is independent of x. ◄

2.7.5 Relation Between Force Field and Potential

If a body in a conservative force field is moved from the point P by an infinitesimal small distance Δr to a neighbouring point P' (Fig. 2.39) the potential energy changes by the amount

$$\Delta E_\mathrm{p} = \frac{\partial E_\mathrm{p}}{\partial x}\Delta x + \frac{\partial E_\mathrm{p}}{\partial y}\Delta y + \frac{\partial E_\mathrm{p}}{\partial z}\Delta z , \tag{2.39}$$

where the partial derivative $\partial E/\partial x$ means that for the differentiation of the function $E(x, y, z)$ the two other variables are kept fixed (see Sect. 13.1.6).

The movement of the body from P to P' requires the work

$$\Delta W = \boldsymbol{F} \cdot \Delta \boldsymbol{r} = -\Delta E_p , \tag{2.40}$$

where $\boldsymbol{F}$ is an average of $\boldsymbol{F}(P)$ and $\boldsymbol{F}(P')$. The comparison between (2.39) and (2.40) yields

$$F\Delta r = F_x\Delta x + F_y\Delta y + F_z\Delta z = -\frac{\partial E_\mathrm{p}}{\partial x}\Delta x - \frac{\partial E_\mathrm{p}}{\partial y}\Delta y - \frac{\partial E_\mathrm{p}}{\partial z}\Delta z .$$

Since this equation holds for arbitrary paths, i. e. arbitrary values of Δx, Δy, Δz it follows that

$$F_x = -\frac{\partial E_\mathrm{p}}{\partial x}; \quad F_y = -\frac{\partial E_\mathrm{p}}{\partial y}; \quad F_z = -\frac{\partial E_\mathrm{p}}{\partial z} . \tag{2.41}$$

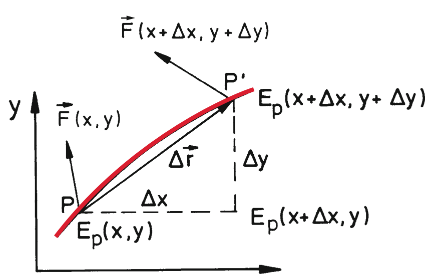

Figure 2.39 Relation between force and potential

Defining the gradient of the function $E_\mathrm{p}(x, y, z)$ as

$$\mathbf{grad}E_\mathrm{p} \stackrel{\mathrm{Def}}{=} \left\{\frac{\partial E_\mathrm{p}}{\partial x}, \frac{\partial E_\mathrm{p}}{\partial y}, \frac{\partial E_\mathrm{p}}{\partial z}\right\} , \tag{2.42}$$

the relations (2.41) for the components of F can be combined into the vector equation

$$\boldsymbol{F} = -\mathbf{grad}E_\mathrm{p} = -\nabla E_\mathrm{p} , \tag{2.41a}$$

where the symbol ∇ = *nabla* (∇ has the form of an old Egyptian string instrument called nabla) is an abbreviation to make the equation more simple to write.

The potential energy E_p of a body with mass m in the gravitational field of a mass M depends on both masses. However, for $m \ll M$ (for instance a mass m in the gravitational field of the earth with $M \gg m$) the small contribution of m to the gravitational field can be neglected. In such cases it is possible to define a function $V(P)$ for each point P, called the gravitational potential

$$V(P) \stackrel{\mathrm{Def}}{=} \lim_{m\to 0}\left(\frac{1}{m}E_\mathrm{p}(P)\right) ; \tag{2.42a}$$

which is the potential energy pro unit mass m in the limit of $m \to 0$ in the gravitational field of M. $V(P)$ is a scalar function which depends only on the position of P and on the mass M that generates the gravitational field.

The gravitational potential of the earth is for instance

$$V(r) = -G \cdot M_\mathrm{E}/r ,$$

where r is the distance from the centre of the earth.

The gravitational field strength is defined as

$$\mathcal{G} = -\mathbf{grad}V . \tag{2.43}$$

The force on a mass m is then

$$\boldsymbol{F}_\mathrm{G} = -m \cdot \mathcal{G} . \tag{2.44}$$

For the gravitational field of a spherical symmetric mass M one obtains

$$\mathcal{G} = G\frac{M}{r^2}\hat{\boldsymbol{r}} , \tag{2.43a}$$

and for the force on a body with mass m in this field Newton's gravitational law

$$\boldsymbol{F}_\mathrm{G} = -G\frac{m \cdot M}{r^2}\hat{\boldsymbol{r}} . \tag{2.44a}$$

These definitions are completely equivalent to their pendants in electrostatics: The electrical potential of an electric charge Q and the Coulomb law (see Vol. 2, Sect. 1.3).

2.8 Angular Momentum and Torque

Assume a point mass moving with the momentum $\boldsymbol{p} = m \cdot \boldsymbol{v}$ on an arbitrary path $\boldsymbol{r} = \boldsymbol{r}(t)$ (Fig. 2.40). We define its *angular momentum* L with respect to the coordinate origin $\boldsymbol{r} = \mathbf{0}$ as the vector product

$$\boldsymbol{L} = (\boldsymbol{r} \times \boldsymbol{p}) = m \cdot (\boldsymbol{r} \times \boldsymbol{v}) \,. \tag{2.45}$$

Note, that $\boldsymbol{L}$ is perpendicular to $\boldsymbol{r}$ and $\boldsymbol{v}$!

In Cartesian coordinates $\boldsymbol{L}$ has the components (see Sect. 13.4)

$$\begin{aligned} &L_x = yp_z - zp_y\,; \quad L_y = zp_x - xp_z\,; \\ &L_z = xp_y - yp_x\,. \end{aligned} \tag{2.46}$$

If the body moves in a plane but on an arbitrarily curved path we can compose the velocity in any point of the path of a radial component $\boldsymbol{v}_\mathrm{r} \parallel \boldsymbol{r}$ and a tangential component $\boldsymbol{v}_\varphi \perp \boldsymbol{r}$ using polar coordinates r and φ (Fig. 2.40). This gives the relations:

$$\begin{aligned} \boldsymbol{L} &= m \cdot [\boldsymbol{r} \times (\boldsymbol{v}_\mathrm{r} + \boldsymbol{v}_\varphi)] \\ &= m \cdot (\boldsymbol{r} \times \boldsymbol{v}_\varphi) \quad \text{because} \quad \boldsymbol{r} \times \boldsymbol{v}_\mathrm{r} = 0\,. \end{aligned}$$

The value of $\boldsymbol{L}$ is

$$|\boldsymbol{L}| = m \cdot r^2 \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}t} \quad \text{because} \quad |\boldsymbol{r} \times v_\varphi| = r^2 \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}t} = r^2 \cdot \omega\,.$$

These equations describe the following facts:

For planar motions the angular momentum L always points into the direction of the plane-normal perpendicular to the plane (Fig. 2.40). The vector product $(r \times v)$ forms a right-handed screw.

When the angular momentum is constant, the motion proceeds in a plane perpendicular to the angular momentum vector.

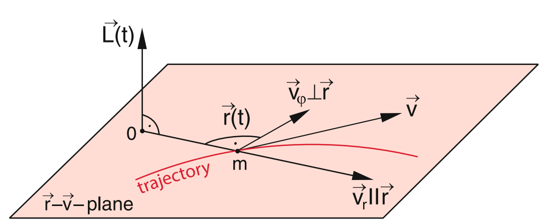

Figure 2.40 Angular momentum $\boldsymbol{L}$ referred to an arbitrarily chosen origin $\mathbf{0}$ for a plain motion of a point mass m

Example

For the uniform circular motion the constant angular momentum points into the direction of the axis through the circle centre perpendicular to the circular plane, i. e. into the direction of the angular velocity vector $\boldsymbol{\omega}$ (Fig. 2.41).

$$|\boldsymbol{L}| = L = m \cdot r \cdot v \cdot \sin(\boldsymbol{r}, \boldsymbol{v}) = m \cdot r \cdot v = m \cdot r^2 \cdot \omega\,.$$

$$\sin(\boldsymbol{r}, \boldsymbol{v}) = 1 \quad \text{because} \quad \boldsymbol{r} \perp \boldsymbol{v}\,. \tag{2.47}$$

For the uniform circular motion is $r = \text{constant}$ and $v = \text{constant} \to \boldsymbol{L} = \text{constant}$.

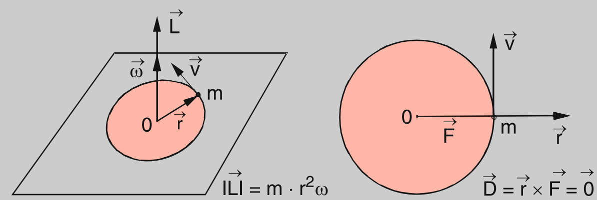

Figure 2.41 Constant angular momentum of the uniform circular motion

Differentiating (2.45) with respect to time we obtain

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} &= \left[\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p}\right] + \left[\boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right] \\ &= (\boldsymbol{v} \times \boldsymbol{p}) + (\boldsymbol{r} \times \dot{\boldsymbol{p}}) = (\boldsymbol{r} \times \dot{\boldsymbol{p}}), \quad \text{because} \quad \boldsymbol{v} \parallel \boldsymbol{p}\,, \\ \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} &= (\boldsymbol{r} \times \boldsymbol{F}), \quad \text{because} \quad \boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\,. \end{aligned} \tag{2.48}$$

The vector product

$$\boldsymbol{D} = (\boldsymbol{r} \times \boldsymbol{F}) \tag{2.49}$$

is the **torque of the force** around the origin $\boldsymbol{r} = \mathbf{0}$ acting on the mass m at the position $\boldsymbol{r}$. Equation 2.48 can then be written as

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{D}\,. \tag{2.49a}$$

The change of the angular momentum $\boldsymbol{L}$ with time is equal to the torque $\boldsymbol{D}$.

In other words: If the torque on a mass is zero, its angular momentum remains constant.

Note the equivalence between linear momentum $\boldsymbol{p}$ and angular momentum $\boldsymbol{L}$:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F}, \quad \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \boldsymbol{D}\,, \tag{2.50}$$

$\boldsymbol{p} =$ **constant** for $\boldsymbol{F} = \mathbf{0}$ and $\boldsymbol{L} =$ **constant** for $\boldsymbol{D} = \mathbf{0}$.

In central force fields $\boldsymbol{F}(r) = f(r) \cdot \hat{r}$ the torque $\boldsymbol{D} = r \times \boldsymbol{F} = \mathbf{0}$ because $\boldsymbol{F} \parallel \boldsymbol{r}$. Therefore the angular momentum is constant

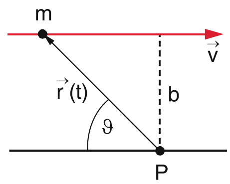

Figure 2.42 Illustration of angular momentum of a body moving on a straight line with respect to a point P which does not lie on the straight line

for all motions in a central force field. **This implies that all trajectories are in a plane, perpendicular to the angular momentum vector**.

Note: Angular momentum and torque are always defined **with respect to a selected point** (for instance the origin of the coordinate system). Even a body moving on a straight line can have an angular momentum with respect to a point, which is not on the straight line.

In Fig. 2.42 the amount L of the angular momentum $\boldsymbol{L}$ of the mass m moving with the constant velocity v on a straight line is with respect to the point P

$$L = m \cdot r \cdot v \cdot \sin\vartheta = m \cdot b \cdot v$$

where b (*called the impact parameter*) is the perpendicular distance of P from the straight line.

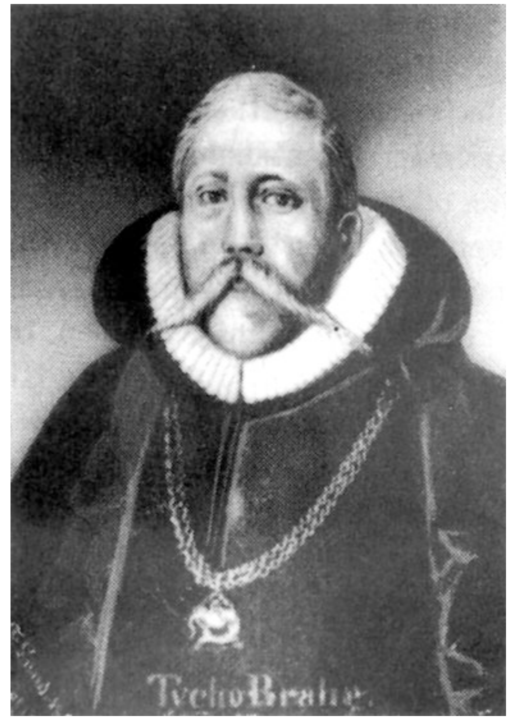

Figure 2.43 Tycho de Brahe (1546–1601) (with kind permission of "Deutsches Museum")

2.9 Gravitation and the Planetary Motions

In the previous section we have learned that in central force fields the angular momentum $\boldsymbol{L}$ is constant in time. The motion of a body therefore proceeds in a plane perpendicular to $\boldsymbol{L}$. The orientation of the plane is determined by the initial conditions (for instance by the initial velocity v_0) and is then fixed for all times. The most prominent example are the motions of the planets in the central gravitational field of the sun which we will now discuss.

2.9.1 Kepler's Laws

Based on accurate measurements of planetary motions (in particular the motion of Mars) by *Tycho de Brahe* (Fig. 2.43) *Johannes Kepler* (Fig. 2.44) could show, that the heliocentric model of *Copernicus* allowed a much simpler explanation of the observations than the old geocentric model of Ptolemy where the earth was the centre and the planets moved around the earth in complex trajectories (epicycles).

Kepler assumed at first circular trajectories because such motions seemed to him as perfect in harmony with God's creation. However, this assumption led to small inconsistencies between

Figure 2.44 Johannes Kepler (1571–1630) (with kind permission of "Deutsches Museum")

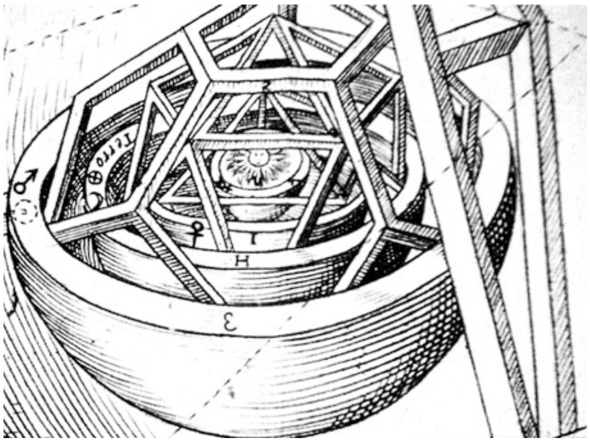

Figure 2.45 Initial model of Kepler illustrating the location of the planets at the corners of regular geometric figures (with kind permission of Prof. Dr. Ron Bienek)

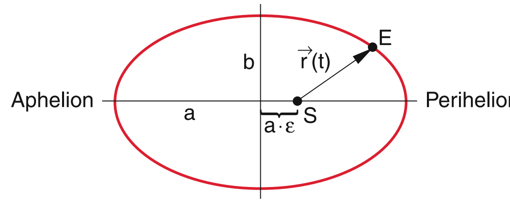

Figure 2.46 Kepler's first law

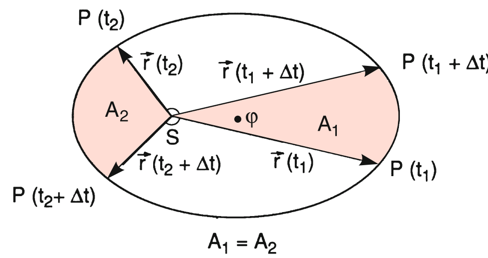

Figure 2.47 Kepler's second law. S: sun, φ: center of ellipse

calculated and observed motions of the planets which exceeded the error limits of the observations. After a long search with several unsuccessful models (for instance a model where the planets were located at the corners of symmetric figures which rotate around a centre (Fig. 2.45). Kepler finally arrived at his famous three laws which were published in his books: *Astronomia Nova* (1609) and *Harmonices Mundi Libri V* (1619).

Kepler's first law

The planets move on elliptical trajectories with the sun in one of the focal points (Fig. 2.46).

Kepler's second law

The radius vector from the sun to the planet sweeps out in equal time intervals equal areas (Fig. 2.47).

Kepler's third law

The squares of the full revolution times T_i of the different planets have the same ratio as the cubes of the large half axis a_i of the elliptical paths.

$$T_1^2/T_2^2 = a_1^3/a_2^3 \quad \text{or} \quad T_i^2/a_i^3 = \text{constant}\,,$$

where the constant is the same for all planets.

The 2. Kepler's law tells us that the areas A_i in Fig. 2.47 is for equal time intervals Δt always the same, i. e. the area $A_1 = SP(t_1)P(t_1 + \Delta t) = A_2 = SP(t_2)P(t_2 + \Delta t)$. For sufficiently small time intervals dt we can approximate the arc length $ds = P_1P_2 = v dt$ in Fig. 2.48b by the straight line P_1P_2. The area of the triangle SP_1P_2 is then

$$dA = \tfrac{1}{2} \cdot |r \times v| = \tfrac{1}{2}|r| \cdot |v| \cdot \sin\alpha = \tfrac{1}{2} \cdot \frac{|L|}{m}\,. \qquad (2.51)$$

Kepler's second law therefore states that the angular momentum of the planet is constant. Kepler's first law postulates that the motion of the planets proceeds in a plane. Since the angular momentum is perpendicular to this plane it follows that also the direction of $\boldsymbol{L}$ is constant.

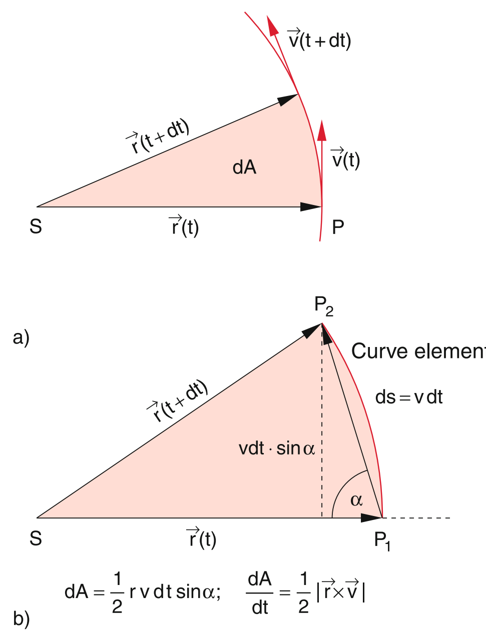

Figure 2.48 Kepler's second law as conservation of angular momentum. **a** schematic representation of the equal area law. **b** calculation of the area covered by the radius vector in the time interval dt

2.9.2 Newton's Law of Gravity

Newton came to the conclusion that the free fall of a body as well as the motion of the planets have a common cause: the gravitational attraction between two masses. In order to find a quantitative formulation of the gravitational force he started his considerations with Kepler's laws. Since the angular momentum of the planetary motion is constant the force field has to be a central force field

$$\boldsymbol{F}(\boldsymbol{r}) = f(r) \cdot \hat{r} .$$

The gravitational force which acts on a body with mass m at the surface of the earth with mass M (which is equal to its weight) is proportional to m. According to the principle *actio = reactio* and also because of symmetry principles the equal but opposite force acting on M should be also proportional to the mass M of the earth (Fig. 2.25). It is therefore reasonable to postulate that the gravitational force is proportional to the product $m \cdot M$ of the two masses. We therefore can write for the force between two masses m_1 and m_2

$$\boldsymbol{F}_{\mathrm{g}} = G \cdot m_1 \cdot m_2 \cdot f(r) \cdot \hat{\boldsymbol{r}} . \tag{2.52a}$$

The proportionality factor G is the *Newtonian gravitational constant*.

The function $f(r)$ can be determined from Kepler's third law. Since (2.52a) must be also valid for circular orbits we obtain for the motion of a planet with mass m around the sun with mass $M_\odot$ the equation

$$G \cdot m \cdot M_\odot \cdot f(r) = m \cdot \omega^2 \cdot r , \tag{2.52b}$$

because the gravitational force is the centripetal force which causes the circular motion of the planet with the angular velocity $\omega = v/r$. The revolution period of the planet is $T = 2\pi/\omega$. For the orbits of two different planets Kepler's third law postulates:

$$T^2/r^3 = \text{const} .$$

With $\omega = 2\pi/T$ this gives $\omega^2 \cdot r^3 = \text{const}$ or $\omega^2 \sim r^{-3}$.

Inserting this into (2.52b) yields $f(r) \sim r^{-2}$.

We then obtain Newton's law of gravity

$$F_{\mathrm{g}}(r) = -G \cdot \frac{m \cdot M_\odot}{r^2} \hat{\boldsymbol{r}} . \tag{2.52c}$$

The minus sign indicates that the force is attractive.

The gravitational force

$$F(r) = -G \cdot \frac{m_1 \cdot m_2}{r^2} \hat{\boldsymbol{r}}$$

acts not only between sun and planets but also between arbitrary masses m_1 and m_2 separated by the distance r. However, the force between masses realized in the laboratory is very small and it demands special very sensitive detection techniques in order to measure it. The gravitational constant G can be determined from such experiments in the lab. Among all physical constant it is that with the largest uncertainty. Therefore many efforts are undertaken to determine G with new laser techniques which should improve the accuracy [2.5a–2.5b]. The present accepted numerical value is

$$G = 6.67384(80) \cdot 10^{-11}\,\mathrm{N \cdot m^2/kg^2}$$

with a relative uncertainty of $1.2 \cdot 10^{-4}$.

Note: The gravitational force is always attractive, never repulsive! This differs from the static electric forces between two charges Q_1 and Q_2

$$F(r) \sim Q_1 \cdot Q_2 / r^{2+} ,$$

which can be attractive or repulsive, depending on the sign of the charges Q_i.

2.9.3 Planetary Orbits

Since the gravitational force field is conservative the sum of potential and kinetic energy of a planet is constant. Because it is a central field also the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is constant. This can be used to determine the orbit of a planet which proceeds in a plane with constant orientation perpendicular to $\boldsymbol{L}$. We use polar coordinates r and φ with the centre of the sun as coordinate origin (Fig. 2.49).

The kinetic energy is

$$\begin{aligned} E_{\mathrm{kin}} &= \frac{m}{2} v^2 = \frac{m}{2}\left(v_{\mathrm{r}}^2 + v_\varphi^2\right) \\ &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) . \end{aligned} \tag{2.53}$$

The amount $L = |\boldsymbol{L}|$ of the angular momentum $\boldsymbol{L}$ is

$$L = mr^2\dot{\varphi} = \text{const} . \tag{2.54}$$

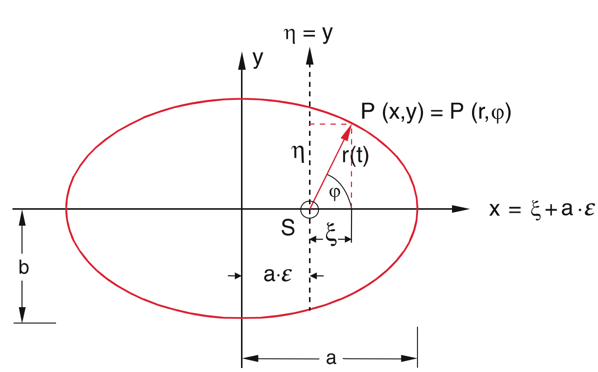

Figure 2.49 Elliptical orbit in Cartesian and in polar coordinates

Conservation of energy demands

$$E_p + \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} = E = \text{const} \ , \qquad (2.55)$$

where E and L^2 are temporally constant. Resolving (2.55) for dr/dt gives

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - E_p - \frac{L^2}{2mr^2}\right)} \ . \qquad (2.56)$$

For the angular variable $\varphi(t)$ one gets from (2.54)

$$\frac{d\varphi}{dt} = \frac{L}{mr^2} \ . \qquad (2.57)$$

Division of (2.57) by (2.56) yields

$$\frac{d\varphi}{dr} = \frac{L}{mr^2}\left[\frac{2}{m}\left(E - E_p - \frac{L^2}{2mr^2}\right)\right]^{-1/2} \ ,$$

integration gives

$$\begin{aligned}\int d\varphi &= \varphi - \varphi_0 \\ &= \frac{L}{m}\int \frac{dr}{r^2\sqrt{\frac{2}{m}\left(E - E_p - L^2/(2mr^2)\right)}} \ . \end{aligned} \qquad (2.58)$$

This allows to get the polar representation of the orbit in the following way:

With $E_p = -G \cdot M \cdot m/r$ the integral in (2.58) belongs to the type of elliptical integrals with the solution for the initial condition $\varphi(0) = \varphi_0 = 0$ (see integral compilation [2.6a–2.6b]):

$$\varphi = \arccos\left(\frac{L^2/r - Gm^2M}{\sqrt{(Gm^2M)^2 + 2mE \cdot L^2}}\right) \ . \qquad (2.59)$$

With the abbreviations

$$a = -\frac{GmM}{2E} \quad \text{and} \quad \varepsilon = \sqrt{1 + \frac{2EL^2}{G^2m^3M^2}} \ , \qquad (2.59a)$$

Eq. 2.59 can be written as

$$\varphi = \arccos\left(\frac{a(1-\varepsilon^2) - r}{\varepsilon \cdot r}\right) \ . \qquad (2.59b)$$

Solving for r gives

$$r = \frac{a\left(1-\varepsilon^2\right)}{1 + \varepsilon \cdot \cos\varphi} \ . \qquad (2.60)$$

This is the equation of a conic section (ellipse, hyperbola or parabola) in polar coordinates with the origin in the focal point S [2:6]. The minimum distance $r_{min} = a(1-\varepsilon)$ is obtained for $\cos\varphi = +1$, the maximum distance $r_{max} = a(1+\varepsilon)$ for $\cos\varphi = -1$. For the shortest distance (perihelion) and the largest distance (Aphelion) from the sun the derivative $dr/dt = 0$. Inserting this into (2.56) gives

$$E - \frac{GmM}{r} - \frac{L^2}{2m \cdot r^2} = 0 \ .$$

The solutions of this equation are

$$r_{min,max} = -\frac{GmM}{2E} \pm \left[\frac{G^2m^2M^2}{4E^2} + \frac{L^2}{2mE}\right]^{1/2} \ . \qquad (2.61)$$

We distinguish between three cases:

a) **$E < 0$.**
For $E < 0$ is the constant $a = -GmM/(2E) > 0$ and $\varepsilon < 1$. The orbit is an ellipse with the major axis a and the excentricity ε. This can be readily seen from (2.60), when the transformation $\xi = r \cdot \cos\varphi$ and $\eta = r \cdot \sin\varphi$ to Cartesian coordinates with the origin in the focal point S is applied. This gives

$$a\left(1-\varepsilon^2\right) - \varepsilon\xi = \sqrt{\xi^2 + \eta^2} \ . \qquad (2.61a)$$

When we shift the origin $\{0,0\}$ from S into the centre of the ellipse with the transformation $x = \xi + a\varepsilon$ and $y = \eta$ we obtain from (2.61a) the well-known equation for an ellipse in Cartesian coordinates

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{with} \quad b^2 = a^2\left(1-\varepsilon^2\right) \ . \qquad (2.61b)$$

For the special case $\varepsilon = 0 \Rightarrow a = b$ the orbit becomes a circle with $r = \text{const}$. From (2.54) it follows because of $L = \text{const}$ that $d\varphi/dt = const$ the planet proceeds with uniform velocity around the central mass M.

> For a negative total energy $E < 0$ the planet proceeds on an elliptical orbit (Kepler's first law).

b) **$E = 0$.**
For $E = 0$ one immediately obtains from (2.59)

$$r = \frac{L^2}{Gm^2M(1+\cos\varphi)} \ . \qquad (2.62)$$

This is the equation of a parabola [2.6a, 2.6b] with the minimum distance $r_{min} = L^2/(2Gm^2M)$ from the focal point for $\varphi = 0$.

c) **$E > 0$.**
Since in (2.61) the distance r has to be positive ($r > 0$) for $E > 0$ only the positive sign before the square root is possible. Therefore only one r_{min} exists and the orbit extends until infinity ($r = \infty$). For $E > 0 \Rightarrow \varepsilon > 0$ (see (2.59a)). The orbit is a hyperbola.

In Tab. 2.1 the relevant numerical data for all planets of our solar system are compiled, where the earth moon is included for comparison.

Table 2.1 Numerical values for the orbits of all planets in our solar system. The earth moon is included for comparison

Name	Symbol	Large semi axis a of orbit			Revolution period T	Mean velocity	Numerical excentricity	Inclination of orbit	Distance from earth	
		In AU	In 10^6 km	In light travel time t		In km s^{-1}			Minimum in AU	Maximum in AU
Mercury	☿	0.39	57.9	3.2 min	88 d	47.9	0.206	7.0°	0.53	1.47
Venus	♀	0.72	108.2	6.0 min	225 d	35.0	0.007	3.4°	0.27	1.73
Earth	♁	1.00	149.6	8.3 min	1.00 a	29.8	0.017	–	–	–
Mars	♂	1.52	227.9	12.7 min	1.9 a	24.1	0.093	1.8°	0.38	2.67
Jupiter	♃	5.20	778.3	43.2 min	11.9 a	13.1	0.048	1.3°	3.93	6.46
Saturn	♄	9.54	1427	1.3 h	29.46 a	9.6	0.056	2.5°	7.97	11.08
Uranus	⛢	19.18	2870	2.7 h	84 a	6.8	0.047	0.8°	17.31	21.12
Neptun	♆	30.06	4496	4.2 h	165 a	5.4	0.009	1.8°	28.80	31.33
Earth moon	☾	0.00257	0.384	1.3 s	27.32 d	1.02	0.055	5.1°	356410 km	406740 km

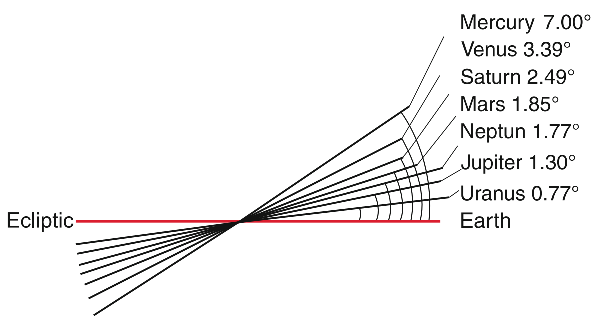

Figure 2.50 Inclination angles of the orbital planes for the different planets against the earth ecliptic

Remark.

1. Pluto is since 2006 no longer a planet but is now listed according to a decision of the International Astronomical Union in the group of *dwarf planets*. To this group also belong Ceres, Eris and about 200 additional dwarf planets in the Kuiper belt far beyond the orbit of Neptune.
2. The orientation of the orbital plane of a planet depends on the initial conditions when the solar system was created from a rotating gas cloud [2.7]. Since these initial conditions were different for the different planets the orbital planes are slightly inclined against each other (Fig. 2.50). Furthermore the gravitational interaction between the planets is small compared to the interaction with the sun, but not completely negligible. This disturbes the central force field and leads over longer time periods to a change of the orientation of the orbital planes.
3. For more accurate calculations (which are necessary for astronomical predictions) one has to take into account that the sun is not exactly located in a focal point of the ellipse. Because $M_\odot$ is not infinite, the sun and the planets move around the common centre of mass, which is, however, not far away from the focal point because $M_\odot \gg m$ [2.8]. For more accurate calculations one has to replace the mass m of a planet by the reduced mass $\mu = m \cdot M_\odot/(m + M_\odot)$ (see Sect. 4.1) where $M_\odot$ is 700 times larger than the mass of all planets ($M_\odot \approx 700 \cdot \sum m_i$). The constant a in Eq. 2.60 has to be replaced by

$$a = -\frac{G\mu M}{2E} = -\frac{GmM^2}{2E \cdot (m + M)} .$$

4. For the accurate calculation of the planetary orbits one has to take into account the interactions between the planets. Because of the small deviations from a central force field the angular momentum is no longer constant but shows slight changes with time.
5. Most of the comets have been formed within our solar system. They therefore have a negative total energy $E < 0$ and move on elongated elliptical orbits with $a \gg b$.

2.9.4 The Effective Potential

The radial motion of a body in a central force field, i. e. the solution of Eq. 2.56, can be illustrated by the introduction of the *effective potential.*

We decompose the kinetic energy in (2.53) into a radial part $(m/2)\dot{r}^2$ which represents the kinetic energy of the radial motion, and an angular part $\frac{1}{2}m \cdot r^2(\mathrm{d}\varphi/\mathrm{d}t)^2$ which stands for the kinetic energy of the tangential motion at a fixed distance r. The second part can be expressed by the angular momentum L

$$E_{\mathrm{kin}}^{\tan} = \frac{1}{2}mr^2\dot{\varphi}^2 = \frac{L^2}{2mr^2} \tag{2.63}$$

(see (2.55)). Since for a given constant L this part depends only on r but not on the angle φ or on the radial velocity $\dot{r}$, it is added to the potential energy E_p, which also depends only on r. The sum

$$E_\mathrm{p}^{\mathrm{eff}} = E_\mathrm{p}(r) + \frac{L^2}{2mr^2} \tag{2.64}$$

is the effective potential energy. Often the effective potential

$$V_\mathrm{p}^{\mathrm{eff}} = E_\mathrm{p}^{\mathrm{eff}}/m$$

is introduced which is the potential energy per mass unit. The part $L^2/(2m \cdot r^2)$ is called the centrifugal potential energy

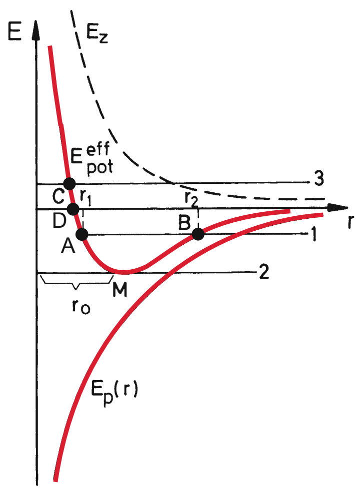

Figure 2.51 Effective potential energy $E_p^{eff}(r)$ as the sum of potential energy and centrifugal energy

and $L^2/(2m^2r^2)$ the *centrifugal potential*, while the radial part $E_p(r)/m$ is the *radial potential*.

The kinetic energy of the radial motion is then

$$E_{kin}^{rad} = \frac{1}{2}m\dot{r}^2 = E - E_p^{eff} , \tag{2.65}$$

where E is the constant total energy.

In the gravitational force field is

$$E_p^{eff} = -G \cdot \frac{mM}{r} + \frac{L^2}{2mr^2} . \tag{2.66}$$

Both parts are depicted in Fig. 2.51. The centrifugal-term E_z decreases with increasing r as $1/r^2$ and is for large r negligible while for small values of r it can overcompensate the negative radial part to make the total energy positive.

The minimum of E_p^{eff} is obtained from $dE_p^{eff}/dr = 0$. This gives

$$r_0 = \frac{L^2}{Gm^2M} . \tag{2.67}$$

The kinetic energy of the radial motion $E_{kin}(r) = E - E_p^{eff}(r)$ at the distance r from the centre is indicated in Fig. 2.51 as the vertical distance between the horizontal line E = constant and the effective potential energy. The body can only reach those intervals $\Delta r = r_{min} - r_{max}$ of r where $E - E_p^{eff} > 0$.

These intervals depend on the total energy E, as is illustrated in Fig. 2.51.

- $\boldsymbol{E < 0}$ but $\boldsymbol{E_{kin}^{rad} > 0}$. (horizontal line 1)
  The body moves between the points $A(r_{min})$ and $B(r_{max})$. They correspond to the radii $r = a(1 \pm \varepsilon)$ for the motion of planets on an ellipse around the sun.
- $\boldsymbol{E < 0}$ but $\boldsymbol{E_{kin}^{rad} = 0}$ (horizontal line 2)
  The orbital path has a constant radius r_0, which means it is a circle. In the diagram of Fig. 2.51 the body always remains at the point M in the minimum of E_p^{eff}.
- $\boldsymbol{E > 0}$ and $E_{kin}^{rad} < |E_p^{eff}(r = \infty)|$ (horizontal line 3)
  The body has the minimum value of r in the point C, where $E_{kin}^{rad} = 0$. It can reach $r = \infty$. Its orbit is a hyperbola.
- $\boldsymbol{E = 0}$
  From (2.65) it follows that $E_{kin}^{rad} = -E_p^{eff}$. The body reaches the minimum distance r_{min} in the point D on the curve $E(r)$. Here is $E_{kin}^{rad} = 0$ and $E_p^{eff} = 0$. It can reach $r = \infty$, where $E_{kin}^{rad} = 0$. The orbit is a parabola.

2.9.5 Gravitational Field of Extended Bodies

In the preceding sections we have discussed the gravitational field generated by point masses. We have neglected the spatial extension of the masses and have assumed that the total mass is concentrated in the centre of each body. This approximation is justified for astronomical situations because the distance between celestial objects is very large compared to their diameter.

Example

The radius of the sun is $R_\odot = 7 \cdot 10^8$ m, the mean distance sun–earth is $r = 1.5 \cdot 10^{11}$ m, i. e. larger by the factor 210! ◄

We will now calculate the influence of the spatial mass distribution on the gravitational field. We start with the field of a hollow sphere in a point P outside the sphere (Fig. 2.52). The hollow sphere should have the radius a and the wall-thickness $da \ll a$.

A disc with the thickness dx cuts a circular ring with the breadth $ds = dx/\sin\vartheta$ and the diameter $2y$. The mass of this ring (thickness da and breadth ds) is for a homogeneous mass density ϱ

$$\begin{aligned} dM &= 2\pi y \varrho \cdot ds \cdot da \\ &= 2\pi a \cdot \varrho \cdot dx \cdot da \quad \text{because} \quad y = a \cdot \sin\vartheta . \end{aligned}$$

All mass elements dM of this ring have the same distance to the point P. Therefore the potential energy of a small probe mass m in the gravitational field generated by dM is

$$dE_p = -G \cdot m \cdot dM/r = -G \cdot m \cdot 2\pi a \cdot \varrho \cdot da \cdot dx/r .$$

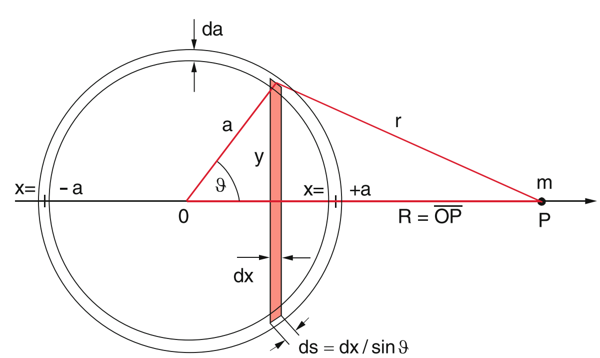

Figure 2.52 Potential and gravitational field-strength of a hollow sphere

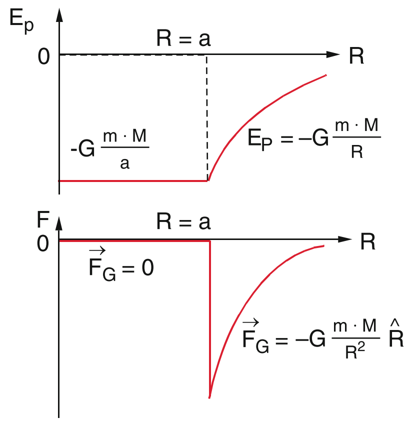

Figure 2.53 Potential energy of a sample mass *m* and gravitational field strength in the gravitational field of a hollow sphere with mass *M*

The gravitational field of the total mass M is obtained by integrating over x from $x = -a$ to $x = +a$.

$$E_p = -2\pi \varrho G m a \cdot da \int_{x=-a}^{+a} \frac{dx}{r} . \qquad (2.68)$$

From Fig. 2.52 the relations

$$\begin{aligned} r^2 &= y^2 + (R-x)^2 = y^2 + x^2 + R^2 - 2Rx \\ &= a^2 + R^2 - 2Rx; \quad r\,dr = -R\,dx \end{aligned}$$

can be verified. This yields

$$\begin{aligned} E_p &= \frac{2\pi \varrho a\,da \cdot m}{R} G \int_{r=R+a}^{R-a} dr \\ &= -G \cdot \frac{m \cdot M}{R} , \end{aligned} \qquad (2.69)$$

because $M = 4\pi a^2 \cdot \varrho da$ is the mass of the hollow sphere.

The gravitational force on the mass m is

$$\begin{aligned} \boldsymbol{F}_G &= -\mathbf{grad} E_p \\ &= -\frac{dE_p}{dR}\hat{\boldsymbol{R}} = -G \cdot \frac{m \cdot M}{R^2} \cdot \hat{\boldsymbol{R}} . \end{aligned} \qquad (2.70)$$

The gravitational field of a hollow sphere with mass M is outside the sphere exactly the same as if the mass M is concentrated in the centre of the sphere (Fig. 2.53).

For $R < a$ the calculation proceeds in the same way. Only the upper limit of the integration changes. For $x = +a$ the limit becomes $r = a - R$ as can be seen from Fig. 2.52. With

$$\int_{r=a+R}^{r=a-R} dr = -2R$$

the potential energy becomes

$$E_p = -G\frac{m \cdot M}{a} = \text{const} \quad \text{for} \quad R \le a . \qquad (2.71)$$

The gravitational force in the inner volume of the hollow sphere is then

$$\boldsymbol{F} = -\mathbf{grad} E_p = \mathbf{0} \quad \text{for} \quad R < a . \qquad (2.72)$$

In the inner volume of the hollow sphere there is no gravitational field. The force on a test mass m is zero. The contributions from the different parts of the hollow sphere cancel each other. In Fig. 2.53 the potential energy $E_p(R)$ and the force $F(R)$ are shown inside and outside of the hollow sphere.

A homogeneous full sphere can be composed of many concentric hollow spheres. Its mass is

$$M = \int_{a=0}^{R_0} \varrho \cdot 4\pi a^2 da .$$

For a test mass outside the sphere ($R > R_0$) we obtain from (2.69)

$$\begin{aligned} E_p &= -G\frac{4\pi}{R}\varrho m \int_0^{R_0} a^2\,da = -G\frac{4\pi}{3R}R_0^3 \varrho m \\ &= -G\frac{m \cdot M}{R} . \end{aligned} \qquad (2.71a)$$

For a point inside the sphere ($R < R_0$) we perform the integration in two steps over the ranges $0 \le a \le R$ and $R \le a \le R_0$. From the Eqs. 2.71 and 2.71a the potential energy can be derived as

$$\begin{aligned} E_p &= -4\pi \varrho G m \left[\int_{a=0}^{R} \frac{a^2\,da}{R} + \int_{a=R}^{R_0} a\,da \right] \\ &= -4\pi \varrho G m \left[\frac{R^2}{3} + \frac{1}{2}R_0^2 - \frac{1}{2}R^2 \right] ; \end{aligned} \qquad (2.73)$$

since $M = (4/3) \cdot \varrho \pi R_0^3$ this becomes

$$E_p = \frac{GMm}{2R_0^3}\left(R^2 - 3R_0^2\right) . \qquad (2.74)$$

The physical meaning of the two steps for the integration is the following: For a test mass in the point P(R) only the mass elements of the sphere with $r \le R$ contribute to the total gravitational force while the contributions of all mass elements with $r \ge R$ exactly cancel each other. The second term in (2.73) gives a constant part to the potential energy and therefore no contribution to the force. From (2.71) and (2.74) one obtains the force (Fig. 2.54 lower part)

$$\begin{aligned} \boldsymbol{F} &= -G\frac{Mm}{R^2}\hat{\boldsymbol{r}} \quad \text{for} \quad R \ge R_0 \\ \boldsymbol{F} &= -\frac{GMm}{R_0^3}R\hat{\boldsymbol{r}} \quad \text{for} \quad R \le R_0 . \end{aligned} \qquad (2.75)$$

Remark. The earth is not a sphere with homogeneous density

1. Because it is an oblate spheroid due to the rotation of the earth which deforms the plastic earth crust [2.10].

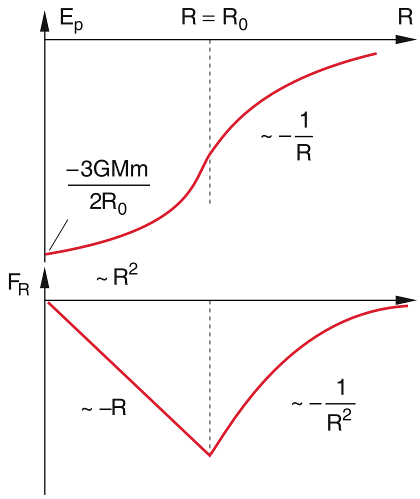

Figure 2.54 Potential energy E_p and gravitational force F of a sample mass m in the gravitational field of a full sphere with mass M

2. Because the density increases towards the centre. Therefore the mass $M(R)$ inside a sphere with radius $R < R_0$ increases with R only as R^n (with $n < 3$, Fig. 2.55). The earth acceleration g measured in a deep well therefore decreases with $r^q (q < 1)$ [2.11].

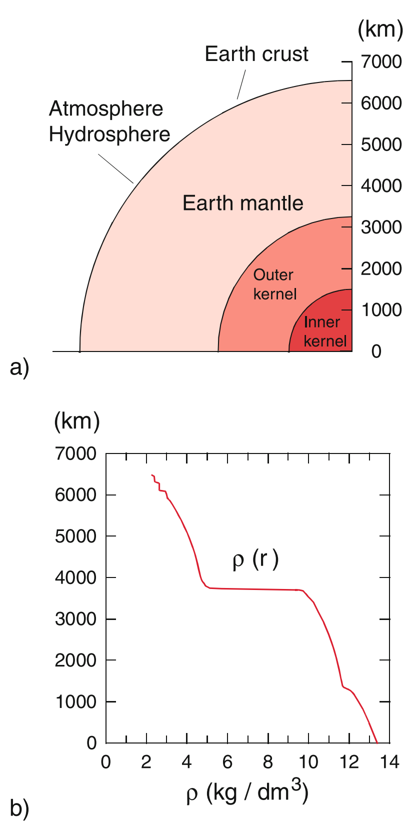

Figure 2.55 **a** Radial cut through the earth showing the different layers. **b** radial density function $\varrho(r)$

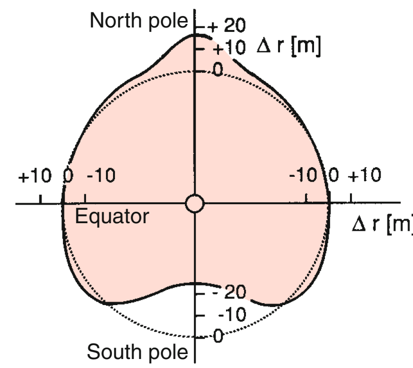

Figure 2.56 The shape of the earth as geoid. The deviation of the geoid from a spheroid with $(a - b)/a = 1/298.25$ (*dotted curve*) is shown 80 000 times exaggerated. Even the geoid gives only the approximate shape of the real earth

3. The mass distribution is not exactly spherical symmetric. The gravitational field of the earth is therefore not exactly a central force field. This implies that the angular momentum of a satellite, orbiting around the earth is not really constant. Measurements of the change of the orbital plane with time (the position $r(t)$ of a satellite can be determined with RADAR techniques with an uncertainty of a few cm!) allows the determination of the mass distribution $\varrho(\vartheta, \varphi)$ in the earth [2.9a, 2.9b].
4. The equipotential surfaces of the earth form a *geoid* (Fig. 2.56). One of these surfaces, which coincides with the average surface of the oceans is defined as the normal zero surface. All heights on earth are given with respect to this surface.

2.9.6 Measurements of the Gravitational Constant *G*

Measurements of the planetary motions allow only the determination of the product $G \cdot M_\odot$ of gravitational constant G and mass of the sun. The absolute value of G has to be measured by laboratory experiments. Such experiments were at first performed 1797 by *Henry Cavendish* and later on repeated by several scientists with increased accuracy [2.12a–2.14], where *Lorand Eötvös* (1848–1919) was especially of high repute because of his very careful and extensive precision experiments [2.2].

Most of these experiments use a **torsion balance** (Fig. 2.57). A light rod (1) with length $2L$ and two small lead balls with equal masses m hangs on a thin wire. Two large masses $M_1 = M_2 = M$ are placed on a rotatable rod (2), which can be turned to the two positions (a) or (b). Due to the gravitational force between m and M the light rod (1) is clockwise turned for the position (a) and counter-clockwise for the position (b) by an angle φ where the retro-driving torque

$$D_r = \frac{\pi}{2} G^* \frac{d^4}{16l} \cdot \varphi \qquad (2.76)$$

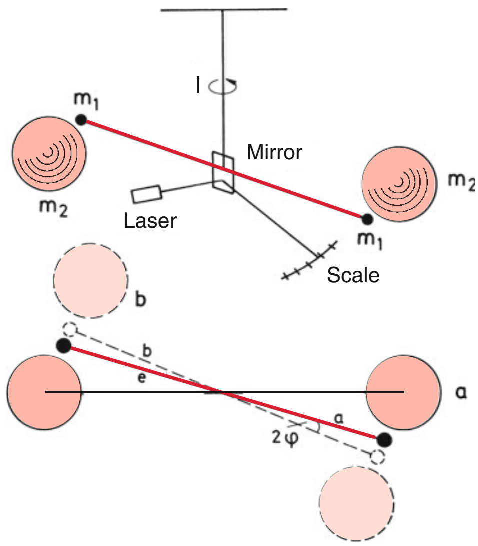

Figure 2.57 Eötvös' torsion balance for measuring Newton's gravitational constant G

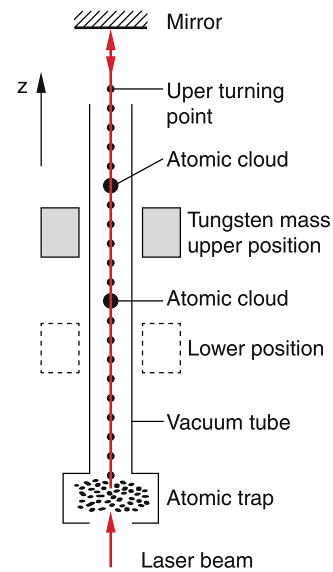

Figure 2.58 Atom interferometer for the measurement of the Newtonian gravitational constant G [2.13b]

of the twisted wire just compensates the torque with the amount $2L \cdot F_g$ generated by the gravitational forces

$$F_g = G \cdot \frac{m \cdot M}{r^2} = G \cdot \frac{16\pi^2}{9r^2} \varrho^2 R_1^3 R_2^3 \,. \qquad (2.77)$$

Here G^* is the torsion module of the wire, d its diameter and l its length, ϱ the mass density of the spheres, R_1 and R_2 their radii and r the distance between their centres. In the equilibrium position, where the two torques cancel, we have the condition $D_r = 2L \cdot F_g$. This gives for the gravitational constant

$$G = \frac{9G^*}{64\pi} \frac{r^2 (d/2)^4}{l \cdot L \cdot \varrho^2 R_1^3 R_2^3} \cdot \Delta\varphi \,. \qquad (2.78)$$

In order to maximize the force F_g, the density ϱ should be as high as possible, because the distance r between the masses m and M cannot be smaller than $r_{min} = R_1 + R_2$. The measurement of φ is performed by placing a mirror at the turning point of the rod with the masses m, which reflects a laser beam by an angle 2φ. On a far distant scale the deflection of the laser spot is a measure for the angle φ.

The most accurate measurement proceeds as follows: The masses M are turned into the position (a). The system now performs oscillations around the new equilibrium position φ_1 which can be determined as the mean of the turning points of the oscillations. Now the masses M are turned into the new position (b). Again oscillations start around the new equilibrium position φ_2, which is determined in the same way. The difference $\Delta\varphi = \varphi_1 - \varphi_2$ than gives according to (2.78) the gravitational constant G.

Equation 2.78 tells us, that the diameter d of the wire should be as small as possible. New materials, such as graphite composites, have a large tear strength. They can carry the masses m even for small values of d. This increases the sensitivity.

In recent years new methods for measuring G have been developed. Most of them are based on optical techniques. We will just discuss one of them: A collimated beam of very cold atoms (laser-cooled to $T < 1\,\mu$K) is sent upwards through an evacuated tube (Fig. 2.58). At the heights $z = h$ where $\frac{1}{2}mv^2 = mgh$ they reach their turning point where they fall down again. A large tungsten mass surrounds the tube and can be shifted upwards or downwards. Above the mass the atoms experience during their upwards motion an acceleration $-(g + \Delta g)$ due to the gravitational attraction by the earth (g) and the mass (Δg). Below the mass their acceleration is $-(g - \Delta g)$. These accelerations are measured via atom interferometry [2.13b].

Figure 2.59 gives the results of many experiments in the course of time, using different measuring techniques. This illustrates, that the error bars are still large but the differences between the results of many experiments are even larger, indicating the underestimation of systematic errors. The value accepted today

$$G = 6.67384(80)\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$$

is the weighted average of the different measurements where the number in the brackets give the standard deviation σ (see Sect. 1.8.2). The relative error is $1.2 \cdot 10^{-4}$ which illustrates that among all universal constants G is the one with the largest uncertainty.

2.9.7 Testing Newton's Law of Gravity

In order to test the validity of the $1/r^2$ dependence of the gravitational force (2.52) several precision experiments have been performed [2.13d]. An interesting proposal by Stacey [2.17] is based on the following principle: In the vertical tunnel within

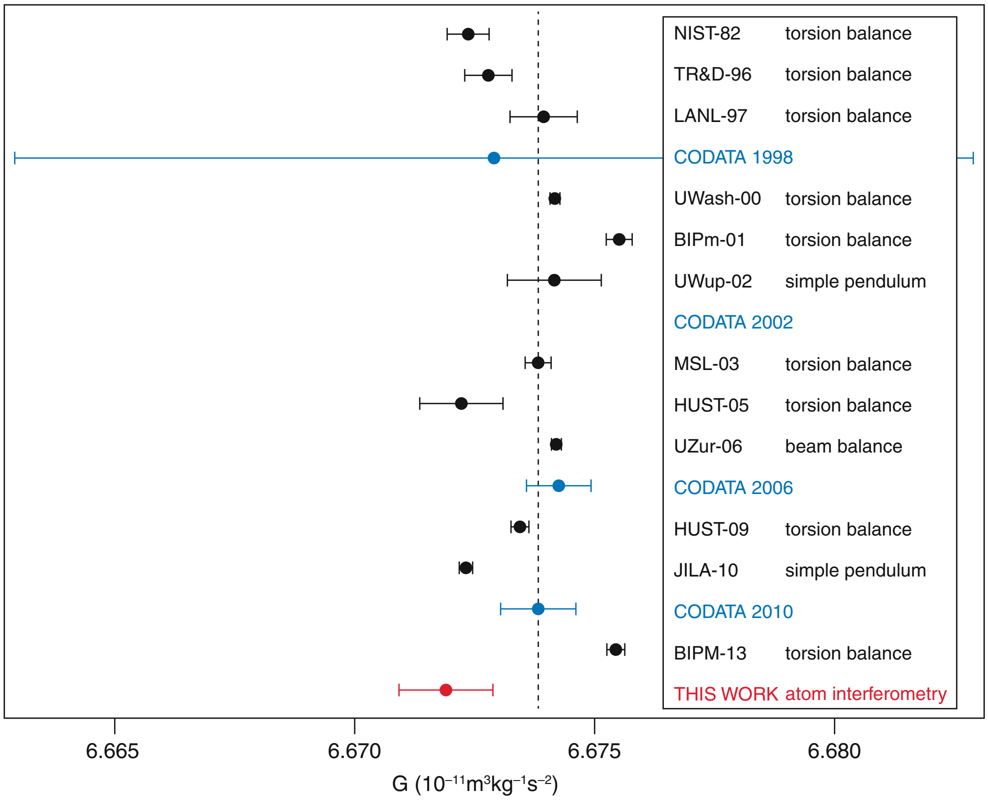

Figure 2.59 Results of different measurements of the Newtonian gravitational constant G [2.13b]

a large water reservoir a sensitive gravitation-balance is placed, where two masses m are hold at different heights, one above the water level and one below (Fig. 2.60). When the water level is lowered by Δh, the change of the gravitational force differs for the two masses. For the lower mass it increases by

$$\delta F_G = G \cdot m \cdot 2\pi\varrho \cdot \Delta h \qquad (2.78a)$$

because the water above the mass decreases, while the water below the mass stays constant. For the upper mass the force decreases because the distance between the mass and the water surface increases (see problem 2.34).

There is still an open question concerning the exact validity of the r^{-2} dependence in Newton's gravitation law over astronomical distances. Astronomical observations of the rotation of galaxies showed, that the visible mass distribution in the galaxy could not explain the differential rotation $\omega(R)$ as the function of the distance R from the galaxy centre, if Newton's law is assumed to be valid. There are two different explanations of this discrepancy: Either the $1/r^2$ dependence of F_G is not correct over large distances, or there exists invisible matter (*dark matter*) which interacts with the visible matter only by gravitation and therefore changes the gravitational force of the visible matter.

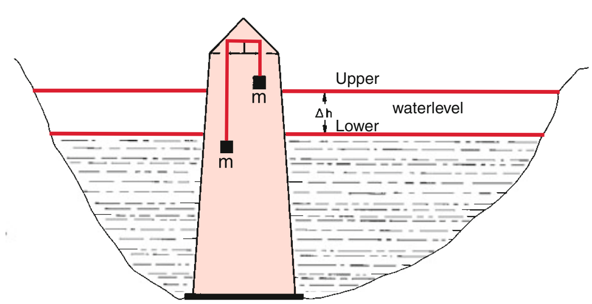

Figure 2.60 Possible method for measuring the $1/r^2$ dependence of the gravitational force

Such very difficult precision experiments have a great importance for testing fundamental physical laws. There are many efforts to develop theories which reduce the four fundamental forces (see Tab. 1.2) to a common origin and to understand more deeply the difference between energy and matter. One example of such precision experiments are tests of possible differences between gravitational and inertial mass as has been performed by *Eötvös* 1922 and *Dicke* 1960 and many other scientists.

Here the inertial mass is measured for different materials by the oscillation period of a gravitational torsion balance [2.18a]. The results obtained up to now show that the ratio m_{in}/m_g of inertial mass to gravitational mass does not differ from 1 within the error limits. For two different materials A and B a possible difference

$$\eta(A,B) = [m_{in}/m_g]_A - [m_{in}/m_g]_B < 10^{-12}$$

must be very small and lies below the detection limit of 10^{-12} with the presently achievable accuracy.

Table 2.2 Mass and mean density of sun, planets and the earth-moon

Planet	Symbol	Mass/earth mass	Mean density $\overline{\varrho}$ in 10^3 kg/m^3
Sun	☉	$3.33 \cdot 10^5$	1.41
Mercury	☿	0.0558	5.42
Venus	♀	0.8150	5.25
Earth	♁	1.0	5.52
Mars	♂	0.1074	3.94
Jupiter	♃	317.826	1.314
Saturn	♄	95.147	0.69
Uranus	⛢	14.54	1.19
Neptun	♆	17.23	1.66
Moon	☾	0.0123	3.34

From the revolution period $T = 2\pi/\omega$ of a satellite around the earth (e.g. the moon or an artificial satellite) the mass M of the earth can be determined. For a circular motion the gravitational force is equal to the centripetal force

$$m \cdot \omega^2 \cdot r = G \cdot mM/r^2 \, .$$

With the known gravitational constant G and the measured distance r of the satellite from the earth centre the mass of the earth is obtained from

$$M = \omega^2 \cdot r^3/G \, .$$

The experimental value is

$$M = 5.974 \cdot 10^{24}\,\text{kg} \, .$$

From measurements of the gravity acceleration g on the earth surface the equation

$$m \cdot g = G \cdot m \cdot M/R^2$$

yields the earth radius R. From M and R the mean density $\varrho = 3M/(4\pi R^3)$ can be derived.

A comparison of the densities of the different planets (Tab. 2.2) illustrates that the inner planets (Mercure, Venus, Earth and Mars) formed of rocks have comparable densities around $\varrho = 5\,\text{g/cm}^3$, while the outer gas planets and the sun have much lower densities. These differences give hints to the formation process of our solar system [2.7] (see Volume 4).

2.9.8 Experimental Determination of the Earth Acceleration *g*

The most accurate determination of g can be performed by measuring the oscillation period of a pendulum. This pendulum consist of a sphere with the mass m suspended by a string with length L (measured between suspension point A and the centre C of the sphere). If the mas of the string is negligibly small compared to m and the radius R of the sphere small compared

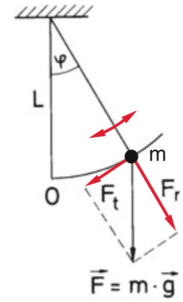

Figure 2.61 Measuring the free fall acceleration g with a pendulum

with L this device is called a *mathematical pendulum*. The motion of the pendulum under the influence of gravity can be best calculated when the force $\boldsymbol{F_g} = m \cdot \boldsymbol{g}$ is decomposed into the two components (Fig. 2.61):

- a radial component F_r in the direction of the string, which generates in the string an equal but opposite restoring force. Since the total force component in this direction is zero, it does not contribute to the acceleration.
- a tangential component $F_t = -m \cdot g \cdot \sin\varphi$ which causes a tangential acceleration $a_t = -g \cdot \sin\varphi$.

The pendulum represents an example of a position dependent force which is **not** a central force. The angular momentum is therefore not preserved. However, if the initial velocity for $\varphi \neq 0$ lies in the plane of the components F_r and F_t the motion remains in this plane. It can be therefore described by planar polar coordinates z and φ. The equation of motion reads

$$m \cdot g \cdot \sin\varphi = -m \cdot L \cdot \varphi \, . \tag{2.79a}$$

Expanding $\sin\varphi$ into a Taylor-series

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^7}{7!} - \cdots \, .$$

The higher order terms can be neglected for small elongations φ. For example is for $\varphi = 10° = 0.17\,\text{rad}$ the term $\varphi^3/3! = 8.2 \cdot 10^{-4}$ which means that the second term is already smaller by the factor 208 than the first term. The error in the approximation $\sin\varphi \approx \varphi$ is for $\varphi = 10°$ only $< 0.5\%$.

The equation of motion (2.79a) is then in the approximation $\sin\varphi \approx \varphi$

$$\ddot{\varphi} = -(g/L)\varphi \, . \tag{2.79b}$$

With the initial condition $\varphi(0) = 0$ the solution is

$$\varphi(t) = A \cdot \sin(\sqrt{g/L} \cdot t) \, . \tag{2.80}$$

The pendulum performs a periodic oscillation with the oscillation period

$$T = 2\pi \cdot \sqrt{L/g} \, . \tag{2.81}$$

Measuring the time for 100 periods with an uncertainty of 0.1 s allows the determination of T with an error of 10^{-3} s. The largest uncertainty comes from the measurement of the length L. The errors for L and T in the determination of

$$g = \frac{4\pi^2 \cdot L}{T^2}$$

give a total error of g according to

$$\left|\frac{\Delta g}{g}\right| \le 2\left|\frac{\Delta T}{T}\right| + \frac{\Delta L}{L} \,.$$

Example

$\Delta T/T = 5 \cdot 10^{-5}$, $\Delta L/L = 10^{-3}$ for $L = 1\,\mathrm{m}$. $\Rightarrow$ $\Delta g/g = 1.1 \cdot 10^{-3}$.

For a more accurate solution of (2.79a) we use the energy conservation law (see Sect. 2.7), which saves one integration. From Fig. 2.62 we see that

$$E_p = m \cdot g \cdot L \cdot (1 - \cos\varphi)$$
$$E_{kin} = \tfrac{1}{2} m \cdot v^2 = \tfrac{1}{2} m L^2 \cdot \dot\varphi^2 \,.$$

The constant total energy is

$$E = E_p + E_{kin} = \frac{m}{2} L^2 \dot\varphi^2 + mgL(1 - \cos\varphi) = mgL(1 - \cos\varphi_0) \,.$$

Where φ_0 is the angle at the turning point where $E_{kin} = 0$. Solving for φ gives

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sqrt{\frac{2g(\cos\varphi - \cos\varphi_0)}{L}} \,.$$

Integration yields

$$\sqrt{\frac{L}{2g}} \int\limits_{\varphi=0}^{\varphi_0} \frac{\mathrm{d}\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = \int\limits_{t=0}^{T/4} \mathrm{d}t = T/4 \,. \tag{2.82}$$

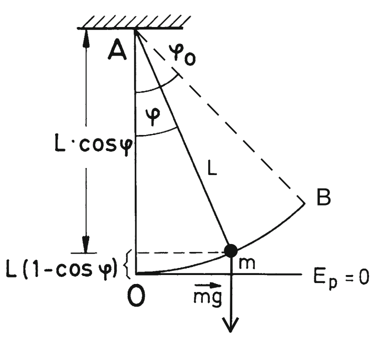

Figure 2.62 Illustration of the integration of the pendulum equation based on the energy conservation

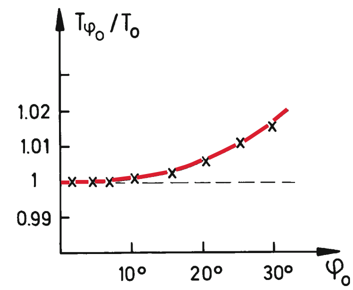

Figure 2.63 Dependence of the oscillation period on the deflection of the pendulum

With the substitution $\sin\xi = \sin(\varphi/2)/\sin(\varphi_0/2)$ the integral can be reduced to an elliptical integral

$$T = 4\sqrt{L/g} \int\limits_0^{\pi/2} \frac{\mathrm{d}\xi}{\sqrt{1 - k^2 \sin^2\xi}} \quad \text{with} \quad k = \sin(\varphi_0/2) \,, \tag{2.83}$$

which can be solved by a Taylor expansion of the integrand [2.18b]. The result is

$$T(\varphi_0) = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{16}\varphi_0^2 + \cdots\right) \,. \tag{2.84}$$

For the accurate determination of T the oscillation period is measured as a function of the elongation φ_0 and the measured values are extrapolated towards $\varphi_0 = 0$ (Fig. 2.63).

If the shape of the earth is approximated by a spheroid the dependence of g on the latitude $\beta = 90° - \vartheta$ can be approximated by the formula

$$g(\beta) \approx g_e\left(1 + 0.0053024 \sin^2\beta - 5.8 \cdot 10^6 \sin^2 2\beta\right) \tag{2.85}$$

where $g_e = g(\beta = 0) = 9.780327\,\mathrm{m/s^2}$ is the earth acceleration at the equator. This formula takes into account, that g is diminished by the centrifugal acceleration of the rotating earth which depends on β (see Sect. 3.2). Because of the inhomogeneous mass distribution of the earth additional local changes of g appear which are not considered in (2.85).

Instead of the pendulum nowadays modern gravimeters are used for the determination of g. These are sensitive spring balances which had been calibrated with a precision pendulum. The restoring force $\boldsymbol{F} = -D(x - x_0)$ is determined by measuring the displacement from the equilibrium position by a calibrated mass m and gets the local variation of the earth acceleration g according to [2.19]

$$m \cdot g = -D(x - x_0) \,.$$

Recently two identical satellites were launched which orbit around the earth on identical paths with an angle distance $\Delta\varphi$.

This distance can be measured very accurately (within a few millimetres) by the time laser pulses need to travel from one satellite to the other and back. Local variations of the gravity cause a different local acceleration which changes the distance $d = R \cdot \Delta\varphi$ between the satellites. This allows the determination of even tiny changes of the gravity force [2.20a, 2.20b, 2.20c].

Summary

- A body with mass m can be described by the model of a point mass as long as its spatial extensions are small compared to its distance to other bodies.
- The motion of a body is described by a trajectory $\boldsymbol{r}(t)$, which the body traverses in the course of time. Its momentary velocity is $\boldsymbol{v}(t) = \dot{\boldsymbol{r}} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ and its acceleration is $a(t) = \mathrm{d}v/\mathrm{d}t = \mathrm{d}^2r/\mathrm{d}t^2$.
- Motions with $\boldsymbol{a}(t) = \boldsymbol{0}$ are called uniform straight-line motions. Magnitude and direction of the velocity are constant.
- For the uniform circular motion the magnitude $|\boldsymbol{a}(t)|$ is constant, but the direction of $\boldsymbol{a}(t)$ changes uniformly with the angular velocity $\boldsymbol{\omega}$.
- A force acting on a freely movable body causes an acceleration and therefore a change of its state of motion.
- A body is in an equilibrium state if the vector sum of all acting forces is zero. In this case it does not change its state of motion.
- The state of motion of a body with mass m and velocity $\boldsymbol{v}$ is defined by the momentum $\boldsymbol{p} = m \cdot \boldsymbol{v}$.
- The force $\boldsymbol{F}$ acting on a body is defined as $F = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ (2. Newton's law).
- For two bodies with masses m_1 and m_2 which interact with each other but not with other bodies the 3. Newtonian law is valid: $F_1 = -F_2$ (F_1 is the force acting on m_1, F_2 acting on m_2).
- The work executed by the force $\boldsymbol{F}(\boldsymbol{r})$ on a body moving along the trajectory $\boldsymbol{r}(t)$ is the scalar quantity $W = \int \boldsymbol{F}(\boldsymbol{r})\mathrm{d}\boldsymbol{r}$.
- Force fields where the work depends only on the initial point P_1 and the final point P_2 but **not** on the choice of the path between P_1 and P_2 are called **conservative**. For such fields is $\mathbf{rot}\,\boldsymbol{F} = \boldsymbol{0}$. All central force fields are conservative.
- To each point P in a conservative force field a potential energy $E_\mathrm{p}(P)$ can be attributed. The work $\int F(r)\mathrm{d}r = E(P_1) - E(P_2)$ executed on a body to move it from P_1 to P_2 is equal to the difference of the potential energies in P_1 and P_2. The choice of the point of zero energy is arbitrarily. Often one chooses $E(r = 0) = 0$ or $E(r = \infty) = 0$.
- The potential energy $E(P)$ and the force $F(r)$ in a conservative force field are related by $\boldsymbol{F}(\boldsymbol{r}) = -\mathbf{grad}E_\mathrm{p}$.
- The kinetic energy of a mass m moving with the velocity v is $E_\mathrm{kin} = \frac{1}{2}mv^2$.
- In a conservative force field the total energy $E = E_p + E_\mathrm{kin}$ is constant (law of energy conservation).
- The angular momentum of a mass m with momentum $\boldsymbol{p}$, referred to the origin of the coordinate system is $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m \cdot (\boldsymbol{r} \times \boldsymbol{v})$. The torque acting on a body in a force field $\boldsymbol{F}(\boldsymbol{r})$ is $\boldsymbol{D} = \boldsymbol{r} \times \boldsymbol{F}$. It is $\boldsymbol{D} = \mathrm{d}\boldsymbol{L}/\mathrm{d}t$.
- All planets of our solar system move in the central force field $F(r) = -G \cdot (m \cdot M/r^2)\hat{\boldsymbol{r}}$ of the sun. Therefore their angular momentum is constant. Their motion is planar. Their trajectories are ellipses with the sun in one focal point.
- The gravitational field of extended bodies depends on the mass distribution. For spherical symmetric mass distributions with radius R the force field outside the body ($r > R$) is exactly that of a point mass, inside the body ($r < R$) the force $F(r)$ increases for homogeneous distributions linearly with r from $\boldsymbol{F} = 0$ at the centre $r = 0$ to the maximum value at $r = R$.
- The free fall acceleration $\boldsymbol{g}$ of a body with mass m equals the gravitational field strength $\boldsymbol{G} = \boldsymbol{F}/m$ at the surface $r = R$ of the earth with mass M. With Newton's law of gravity $\boldsymbol{g}$ can be expressed as $\boldsymbol{g} = G \cdot (M/R^2)\hat{\boldsymbol{r}}$ (G = gravitational constant). It can be determined from the measured oscillation period $T = 2\pi\sqrt{L/g}$ of a pendulum with length L, or with gravitational balances.

Problems

2.1 A car drives on a road behind a foregoing truck (length of 25 m) with a constant safety distance of 40 m and a constant velocity of 80 km/h. As soon as the driver can foresee a free distance of 300 m he starts to overtake. Therefore he accelerates with $a = 1.3\,\mathrm{m/s^2}$ until he reaches a velocity of $v = 100\,\mathrm{km/h}$. Can he safely overtake? How long are time and path length of the overtaking procedure if he considers the same safety distance after the overtaking? Draw for illustration a diagram for $s(t)$ and $v(t)$.

2.2 A car drives half of a distance x with the velocity $v_1 = 80\,\mathrm{km/h}$ and the second half with $v_2 = 40\,\mathrm{km/h}$. Estimate and calculate the mean velocity $\langle v \rangle$ as the function of v_1 and v_2. Make the same consideration if $x_1 = 1/3x$ and $x_2 = 2/3x$.

2.3 A body moves with constant acceleration along the x-axis. It passes the origin $x = 0$ with $v = 6\,\mathrm{cm/s}$. 2 s later it arrives at $x = 10\,\mathrm{cm}$. Calculate magnitude and direction of the acceleration.

2.4 An electron is emitted from the cathode with a velocity v_0 and experiences in an electric field over a distance of 4 cm a constant acceleration $a = 3 \cdot 10^{14}\,\mathrm{m/s^2}$, reaching a velocity of $7 \cdot 10^6\,\mathrm{m/s}$. How large was v_0?

2.5 A body is thrown from a height $h = 15\,\mathrm{m}$ with an initial velocity $v_0 = 5\,\mathrm{m/s}$
a) upwards,
b) downwards.
Calculate for both cases the time until it reaches the ground.
c) Derive Eq. 2.13.

2.6 Give examples where both the magnitude and the direction of the acceleration are constant but the body moves nevertheless not on a straight line. Which conditions must be fulfilled for a straight line?

2.7 A car crashes with a velocity of 100 km/h against a thick tree. From which heights must it fall down in order to experience the same velocity when reaching the ground? Compare this with two equal cars with velocities of 100 km/h crashing head on against each other.

2.8
a) A body moves with constant angular velocity $\omega = 3\,\mathrm{rad/s}$ on a vertical circle in the x-z-plane with radius $R = 1\,\mathrm{m}$ in the gravity field $F = \{0, 0, -g\}$ of the earth. How large are its velocities at the lowest and the highest point on the circle? How large is the difference between the two values? Could you relate this to the potential energy?
b) A body starts with $v_0 = 0$ from the point $A(z = h)$ in Fig. 2.64 on the frictionless looping path. How large are velocities and accelerations in the points B and C of the circular path with radius R? What is the maximum ratio R/h to prevent that the body falls down in B? How large is then the velocity $v(B)$?

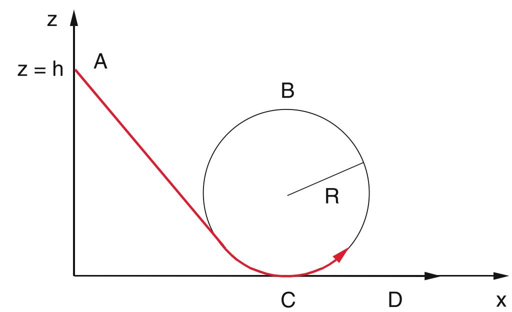

Figure 2.64 Looping path (Probl. 2.8 b)

2.9 How large is the escape velocity
a) of the moon ($d = 384\,000\,\mathrm{km}$) in the gravitational field of the earth?
b) of a body on the surface of the moon in the gravitational field of the moon?

2.10 What is the minimum fuel mass of a one stage rocket with a payload of 500 kg for a horizontal launch at the equator to bring the rocket to the first escape velocity of $v_1 = 7.9\,\mathrm{km/s}$ when the velocity of the propellant gas relative to the rocket is $v_e = 4.5\,\mathrm{km/h}$
a) in the east direction
b) in the west direction?

2.11 Check the energy conservation law for the examples given in the text. Show, that (2.26) follows directly from the condition $E_{kin} \geq E_p$, i. e. $\frac{1}{2}mv^2 \geq m \cdot g \cdot R$.

2.12 A rocket to the moon is launched from a point at the equator. How much energy is saved compared to a vertical launch, when it is shot in the eastern direction under 30° against the horizontal?

2.13 A wooden cylinder (radius $r = 0.1\,\mathrm{m}$, heights $h = 0.6\,\mathrm{m}$) is vertically immersed in water with 2/3 of its length which is its equilibrium position. Which work has to be performed when it is pulled out of the water? How is the situation if the cylinder lies horizontally in the water? How deep does it immerse?

2.14 A body with mass $m = 0.8\,\mathrm{kg}$ is vertically thrown upwards. In the heights $h = 10\,\mathrm{m}$ its kinetic energy is 200 J. What is the maximum heights it can reach?

2.15 A spiral spring of steel with length $L_0 = 0.8\,\mathrm{m}$ is expanded by the force $F = 20\,\mathrm{N}$ to a length $L = 0.85\,\mathrm{m}$. Which work is needed to expand the spring to twice its initial length, if the force is always proportional to the expansion $\Delta L = L - L_0$?

2.16 What is the minimum initial velocity of a body at a vertical launch from the earth when it should reach the moon?

2.17 What is the distance of a geo-stationary satellite from the centre of the earth? Which energy is needed to launch it? How accurate has its distance to the earth centre be stabilized in order to maintain its position relative to a point on earth within $0.1\,\mathrm{km/d}$?

2.18 What is the change of potential, kinetic and total energy of a satellite when its radius r on a stable circular orbit around the earth centre is changed? What is the ratio $E_{\mathrm{kin}}/E_{\mathrm{p}}$? Does it depend on r? Express the total energy E by m, g, r and the mass M_{E} of the earth. Are these quantities sufficient or are more needed?

2.19 Prove, that the force $\boldsymbol{F} = m \cdot g \cdot \sin\varphi \cdot \boldsymbol{e}_{\mathrm{t}}$ for the mathematical pendulum is conservative and that for arbitrary values of φ conservation of energy $E_{\mathrm{kin}} + E_{\mathrm{p}} = \mathrm{const}$ holds.

2.20 Assume one is able to measure the length $L = 10\,\mathrm{m}$ of a pendulum within 0.1 mm and the period T within 10 ms. How many oscillation periods have to be measured in order to equalize the contribution of ΔL and ΔT to the accuracy of g? How large is then the uncertainty of g?

2.21 How much accuracy is gained for the determination of G with the gravity balance if the large masses M are increased by a factor of 10? How accurate has the measurement of the angle φ to be in order to determine G with an accuracy of 10^{-4}? Give some physical reasons for the limits of the accuracy of φ.

2.22 The comet Halley has a period of 76 years. His smallest distance to the sun is 0.59 AU. How large is its maximum distance to the sun and what is the eccentricity of its elliptical orbit? Hint: Look for a relation between T and $r_{\min} = a(1-\varepsilon)$ and $r_{\max} = a(1+\varepsilon)$.

2.23 Assume that the gravity acceleration at the equator of a rotating planet is $11.6\,\mathrm{m/s^2}$, the centripetal acceleration $a = 0.3\,\mathrm{m/s^2}$ and the escape velocity for a vertical launch $23.6\,\mathrm{m/s}$. At the heights $h = 5000\,\mathrm{km}$ above the surface is $g = 8.0\,\mathrm{m/s^2}$. What are the radius R and the mass M of the planet. How fast is it rotating? Which planet meets these requirements?

2.24 The gravitational force exerted by the sun onto the moon is about twice as large as that exerted by the earth. Why is the moon still circling around the earth and has not escaped?

2.25 Which oscillation period would a pendulum have on the moon, if its period on the earth is 1 s?

2.26 A vertical straight tunnel is cut through the earth between opposite points A to B on the earth surface.
a) Show that without friction a body released in A performs a harmonic oscillation between A and B.
b) What is the oscillation period?
c) Compare this value with the period of a satellite, which circles around the earth closely above the surface.
d) A straight tunnel is cut between London and New York. What is the travel time of a train without friction and extra driving force (besides gravity) which starts in London with the velocity $v_0 = 0$? How much does the time change, if $v_0 = 40\,\mathrm{m/s}$?

2.27 Calculate the distance earth-moon from the period of revolution of the moon $T = 27\,\mathrm{d}$ (mass of the earth is $M = 6 \cdot 10^{24}\,\mathrm{kg}$).

2.28 Saturn has a mass $M = 5.7 \cdot 10^{26}\,\mathrm{kg}$ and a mean density of $0.71\,\mathrm{g/cm^3}$. How large is the gravitational acceleration on its surface?

2.29 How large is the relative change of the gravity acceleration g between a point on the earth surface and a point with $h = 160\,\mathrm{km}$ above the surface?

2.30 How large is the change Δg of the earth acceleration due to the attraction by
a) the moon and
b) the sun?
Compare the two changes and discuss them. How large is the relative change $\Delta g/g$?

2.31 Two spheres made of lead with masses $m_1 = m_2 = 20\,\mathrm{kg}$ are suspended by two thin wires with length $L = 100\,\mathrm{m}$ where the suspension points are 0.2 m apart. What is the distance between the centres of the spheres, when the gravitational field of the earth is assumed to be spherical symmetric?
a) without
b) with the gravitational force between the two masses.

2.32 Based on the energy conservation law determine the velocity of the earth in its closest distance from the sun (Perihelion) and for the largest distance (aphelion). How large is the difference Δv to the mean velocity? Discuss the relation between the eccentricity of the elliptical orbit and Δv.

2.33 A satellite orbiting around the earth has the velocity $v_A = 5\,\mathrm{km/s}$ in the aphelion and $v_P = 7\,\mathrm{km/s}$ in its perihelion. How large are minor and major half axes of its elliptical orbit?

2.34 Prove the equation (2.78a).

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