Moving Coordinate Systems and Special Relativity

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For the description of the location and the velocity of a body in a three-dimensional space one needs a coordinate system where the position vectors $\boldsymbol{r}(t)$ and its time derivative $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \boldsymbol{v}(t)$ are defined. Of course are all physical processes independent of the choice of the coordinate system. However, their mathematical formulation can be much simpler in a suitable coordinate system than in other systems. It is therefore essential to choose that system which allows the optimum description of a process and to find the transformation equations to change from one to another coordinate system.

For example is the coordinate system connected with the earth which moves around the sun, the best choice for the description of measurements on earth. For astronomical observations the results of such measurements must be transformed into a galactic coordinate system which has its origin in the galactic centre and moves with the rotating galaxy, in order to eliminate the complex motion of the earth relative to the galactic centre. For coordinate systems at rest these transformations impose no problems. The situation is different for systems which move against each other.

In this chapter we will discuss question which arise for transformations between moving coordinate systems when physical processes are described in different systems. It turns out that many concepts derived from daily life experience which were taken for granted, had to be revised. The mathematical framework for such revisions is the special relativity theory developed by *Albert Einstein*, which will be briefly treated in this chapter.

3.1 Relative Motion

An observer, sitting in the origin O of a coordinate system looks at two objects A and B with the coordinates $\boldsymbol{r}_\mathrm{A}$ and $\boldsymbol{r}_\mathrm{B}$ and the relative distance

$$\boldsymbol{r}_\mathrm{AB} = \boldsymbol{r}_\mathrm{A} - \boldsymbol{r}_\mathrm{B} \,, \tag{3.1}$$

which move with the velocities

$$\boldsymbol{v}_\mathrm{A} = \frac{\mathrm{d}\boldsymbol{r}_\mathrm{A}}{\mathrm{d}t} \quad \text{and} \quad \boldsymbol{v}_\mathrm{B} = \frac{\mathrm{d}\boldsymbol{r}_\mathrm{B}}{\mathrm{d}t}$$

relative to the coordinate system O (Fig. 3.1). The velocity of A relative to B is then

$$\boldsymbol{v}_\mathrm{AB} = \frac{\mathrm{d}\boldsymbol{r}_\mathrm{AB}}{\mathrm{d}t} = \boldsymbol{v}_\mathrm{A} - \boldsymbol{v}_\mathrm{B} \,, \tag{3.2a}$$

while the velocity of B relative to A

$$\boldsymbol{v}_\mathrm{BA} = \boldsymbol{v}_\mathrm{B} - \boldsymbol{v}_\mathrm{A} = -\boldsymbol{v}_\mathrm{AB} \,. \tag{3.2b}$$

This illustrates that position vector and velocity do depend on the reference system.

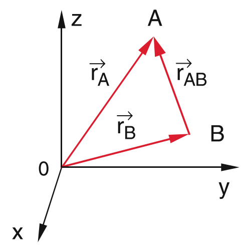


Figure 3.1 Definition of the relative distance

3.2 Inertial Systems and Galilei-Transformations

Two observers B and B′ sit in the origins O and O' of two coordinate systems $S(x, y, z)$ and $S'(x', y', z')$ which move against each other with the constant velocity $\boldsymbol{u}$ (Fig. 3.2). Both observers measure the motion of an object A. which has the position vector $\boldsymbol{r}(x, y, z)$ in the system S and $\boldsymbol{r}'(x', y', z')$ in the system S'.

As can be erived from Fig. 3.2 it is

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{u} \cdot t \,, \tag{3.3}$$

which can be written for the components as

$$\left\{ \begin{aligned} x'(t) &= x(t) - u_x \cdot t \\ y'(t) &= y(t) - u_y \cdot t \\ z'(t) &= z(t) - u_z \cdot t \\ t' &= t \end{aligned} \right\} \,, \tag{3.3a}$$

where $t = t'$ means that both observers use synchronized equal clocks for their time measurements. This is **not** obvious and is generally not true if the velocity $\boldsymbol{u}$ approaches the velocity of light (see Sect. 3.4). For the velocity of A the two observers find

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \quad \text{and} \quad \boldsymbol{v}' = \frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t} \,. \tag{3.4}$$

From (3.3) follows

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{u} \,. \tag{3.5}$$

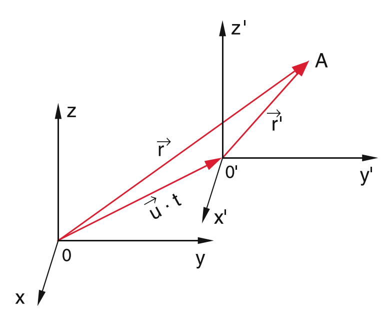


Figure 3.2 The coordinates of a point A, described in two different systems O and O' which move against each other with the constant velocity $\boldsymbol{u}$

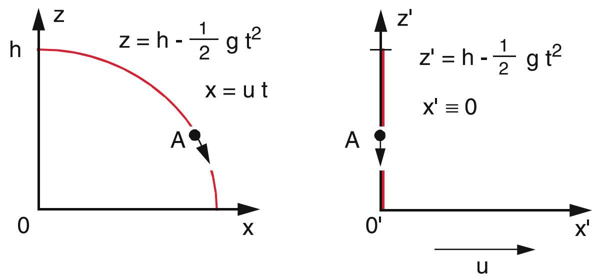


Figure 3.3 Description of the free fall in two different inertial systems

The acceleration a of A can be derived from (3.5) as

$$\boldsymbol{a}' = \frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{a} \,. \tag{3.6}$$

Both observers in the systems which move with constant velocity $\boldsymbol{u}$ against each other, measure the same value for the acceleration $\boldsymbol{a}$. Because the force on a body with mass m is $\boldsymbol{F} = m \cdot \boldsymbol{a}$ both observers come to the same conclusion about the force acting on A and find the same relations for dynamical processes in the two systems.

Such systems which move with a constant relative velocity $\boldsymbol{u}$ against each other are named **inertial systems**.

Between the quantities $\boldsymbol{r}$, $\boldsymbol{v}$ and t for the motion of an object A measured in two different inertial systems the Galilei transformations pertain

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}' + \boldsymbol{u}t \,, \\ \boldsymbol{v} &= \boldsymbol{v}' + \boldsymbol{u} \Rightarrow \boldsymbol{a} = \boldsymbol{a}' \quad \text{and} \quad \boldsymbol{F} = \boldsymbol{F}' \,, \\ t &= t' \,, \end{aligned} \tag{3.7}$$

where $u = |\boldsymbol{u}| \ll c$ is the constant velocity of S against S'.

Because of $\boldsymbol{F} = \boldsymbol{F'}$ both observers measure the same forces and derive identical physical laws. This can be illustrated by the example of the free fall observed in the two systems S and S' moving with the velocity $u = u_x$ in the x-direction against each other (Fig. 3.3):

A body A which is released at the heights $z = h$ falls down in the system S' along the z'-axis ($x' = y' = 0$), which moves with the velocity u against the z-axis in the system S. For the observer O' in S' the motion of A appears as vertical free fall. For the observer O in S the body A starts at $z = z' = h$ with the velocity $v(h) = u$ in the x-direction, which bends down into the $-z$-direction because of the gravitation. The trajectory of A is for O a parabola (horizontal throw see Sect. 2.3.2). However, both observes measure the same fall acceleration $\boldsymbol{g} = \{0, 0, -g\}$ and the same fall times. They derive the same law for the free fall.

All inertial systems are equivalent for the description of physical laws.

In other words: An observer siting in a train who does not look out through the window cannot decide by arbitrary many experiments whether he sits in a train at rest or in a train moving against another reference system with constant velocity.

3.3 Accelerated Systems; Inertial Forces

If the two observers sit in two systems which move against each other with a velocity $\boldsymbol{u}(t)$ changing with time resulting in an acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ they measure for the motion of a body A relative to their system different accelerations and therefore conclude that different forces act on A.

The observer in an accelerated system can, however, ascertain that his system moves accelerated against another system. If he takes into account this acceleration he comes to the same conclusions about physical laws for the observed motion of the body A as an observer in an inertial system.

We will discuss this for two different accelerated motions:

a) rectilinear motion of S against S' with constant acceleration
b) rotation of S against S' around the common origin $0 = 0'$.

Remark. In the following sections we will always assume that the observers O and O' sit in the origins 0 and 0′ of the systems S and S'.

The discussion of the description of physical processes in accelerated coordinate systems leads to the introduction of special forces (inertial forces), which are often confusing students. Therefore these forces will be discussed as vivid as possible in order to illustrate that these forces are no real forces but are only necessary, when the observer in the accelerated system does not take into account the acceleration of his system.

3.3.1 Rectilinear Accelerated Systems

If the origin 0′ of the system S' moves along the x-axis of S with the time dependent velocity $u(t) = u_0 + a \cdot t(\boldsymbol{a} = a_x \boldsymbol{e}_x$ with $a_x = \mathrm{d}u/\mathrm{d}t = \mathrm{d}^2x/\mathrm{d}t^2)$ against S, only the magnitude of the velocity changes not its direction (Fig. 3.4). An example is an observer in a train accelerating on a straight track.

For a body A with the coordinates (x', y', z') in the system S' the observer in S measures the coordinates $x = u_0 t + \frac{1}{2}at^2 + x'$, $y = y'$, $z = z'$, if for $t = 0$ the two origins of S and S' coincide and the relative velocity $u(t)$ between S and S' at time $t = 0$ is u_0. The velocity of A is then $v' = \{v'_x, v'_y, v'_z\}$ for O' and $v = \{v_x = u_0 + a \cdot t + v'_x,\ v_y = v'_y,\ v_z = v'_z\}$ for O.

The description of different situations by O (sitting in a system S at rest) and O' (sitting in the accelerated System S') shall be illustrated by three examples. **Note** that S' is no inertial system!

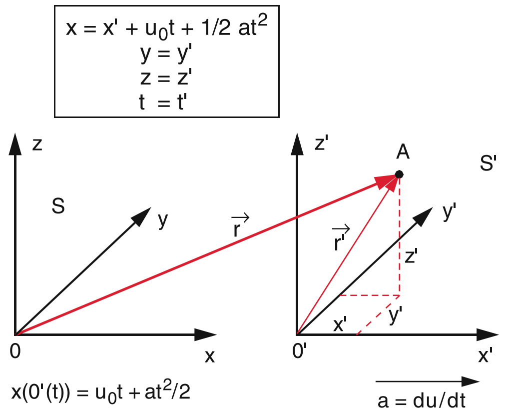


Figure 3.4 Coordinates of point A in a system S with origin O and a system S' with origin O', that moves against O with the acceleration a in x-direction

Examples

1. The observer O' is sitting on a carriage at a fixed table with plain tabletop. On the tabletop rests a ball A without friction (Fig. 3.5a). If the system S' is accelerated to the left (i. e. in $-x$-direction), both observers O (in the system S at rest) and O' see that the ball moves accelerated towards O'. Both O and O' make the same observation but interpret this in a different way:
   O' says: The ball moves accelerated towards me. Therefore a force $\boldsymbol{F} = m \cdot \boldsymbol{a}$ must act on the ball.
   O says: The system S' moves with the acceleration $-\boldsymbol{a}$ to the left, while the ball does not participate in the acceleration and stays at rest. This means: Not the ball is accelerated towards O', but O' is accelerated towards the resting ball. Therefore no force is acting on A.

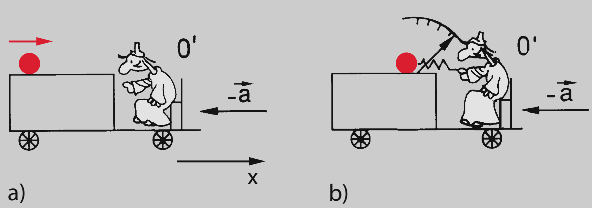


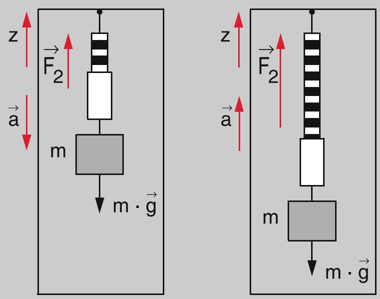
Figure 3.5 **a** A freely movable ball; **b** a ball fixed to a spring balance, both on a table that is accelerated into the $-x$-direction with constant acceleration a

   Note: If O' knows that his system S' is accelerated, he also knows that the ball must stay at rest, because it is frictionless and therefore not linked with the table, which means that it will not participate in the motion of the table. In order to explain his observation of the acceleration $\boldsymbol{a}$ of the ball he introduces a force $\boldsymbol{F} = m \cdot \boldsymbol{a}$ which he calls *fictitious force* (often named *pseudo-force*), because he knows that this is not a real force but merely the description of a virtual acceleration $\boldsymbol{a}$ of the ball when its motion is described in a reference system which itself is accelerated with the acceleration $-\boldsymbol{a}$. Often the notation "*inertial force*" is used in order to point to the *inertial mass* of the ball which prevents it to follow the acceleration of the table.
2. The observer O' connects the ball with an elastic spring scale and holds the other end with his hand (Fig. 3.5b). If the system S' is now accelerated with the acceleration $-\boldsymbol{a}$ to the left O' observes that the spring is compressed. The spring balance measures the force $\boldsymbol{F}_1 = -m \cdot \boldsymbol{a}$. He must apply an equal but opposite force $\boldsymbol{F}_2 = +m \cdot \boldsymbol{a}$ in order to keep the ball at rest.
   O' says: The total force $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2$ acting on the ball is zero in accordance with my observation that the ball rests.
   The observer O in the rest system S says: Since the ball is now connected with the table in S' it participates in the acceleration $-\boldsymbol{a}$ of S'. The observer O' has to apply the force $\boldsymbol{F} = -m \cdot \boldsymbol{a}$ in order to transfer the same acceleration $-\boldsymbol{a}$ to the ball as the system S' and to keep the ball at rest relative to the system S'.
3. A mass m in an elevator is suspended by a spring balance (Fig. 3.6). If the elevator moves with the acceleration $\boldsymbol{a} = \{0, 0, -a\}$ downwards (Fig. 3.6a) the spring balance measures the force $\boldsymbol{F} = m(\boldsymbol{g} - \boldsymbol{a})$, if the elevator moves upwards with the acceleration $+a$ the balance measures $\boldsymbol{F} = m(\boldsymbol{g} + \boldsymbol{a})$ where $\boldsymbol{g} = \{0, 0, -g\}$ is the earth acceleration. The observer O', sitting in the elevator, says: The body is at rest. Therefore the total force acting on it must be zero. The total force $F = F_1 + F_2 + F_3$ (Fig. 3.6c) is the sum of

$$\begin{aligned} \boldsymbol{F}_1 &= m \cdot \boldsymbol{g} && = \text{the weight of the mass } m \\ \boldsymbol{F}_2 &= -\boldsymbol{m}(\boldsymbol{g} - \boldsymbol{a}) && = \begin{array}{l}\text{opposite force of} \\ \text{the spring balance}\end{array} \\ \boldsymbol{F}_3 &= -\boldsymbol{m} \cdot \boldsymbol{a} && = \text{inertial force} \end{aligned} \quad .$$

   O' must introduce the inertial force F_3 in order to explain his observation.
   The observer O outside the elevator at rest says: The body with mass m is connected with the elevator. It therefore participates in the acceleration of the elevator. This demands the force $\boldsymbol{F} = m \cdot \boldsymbol{a}$. The total force acting on the body is the sum of its weight $\boldsymbol{F}_1 = m \cdot \boldsymbol{g}$ and the restoring force $\boldsymbol{F}_2 = -m \cdot (\boldsymbol{g} - \boldsymbol{a})$ of the spring balance. Which gives, as expected the total force $\boldsymbol{F} = m \cdot \boldsymbol{g} - m \cdot (\boldsymbol{g} - \boldsymbol{a}) = m \cdot \boldsymbol{a}$.
   If the suspension cable of the elevator is ruptured and the elevator goes down in a free fall its acceleration is $a = g$. For O' the total force remains $\sum F_i = 0$ while for O the total force becomes $F = m \cdot g$.
   These examples illustrate, that the inertial forces are introduced only for measurements in accelerated coordinate systems if the acceleration of the system is not taken into account. They are therefore also called *fictitious forces* or *pseudo-forces*. A transformation to an inertial system lets all pseudo-forces vanish. This

means an observer O in an inertial system does not need any pseudo-force for the explanation of the observed physical processes.



a) b)

O'	O
$\vec{F}_1 = m \cdot \vec{g}$; $\vec{F}_2 = -m(\vec{g} - \vec{a})$; $\vec{F}_3 = -m \cdot \vec{a}$	$\vec{F}_1 = m \cdot \vec{g}$; $\vec{F}_2 = -m(\vec{g} - \vec{a})$
$\Sigma \vec{F}_i = 0$	$\Sigma \vec{F}_i = m \cdot \vec{a}$

c)

Figure 3.6 Elevator experiment. Description of the forces acting on a mass m, that hangs on a spring balance in an elevator accelerated downwards in **a** and upwards in **b**. In **c** the forces are listed as observed by O' in the elevator (*left hand side*) and by O at rest outside the elevator (*right hand side*)

3.3.2 Rotating Systems

We regard two coordinate systems $S(x, y, z)$ and $S'(x', y', z')$ with the unit vectors $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$, $\hat{\boldsymbol{e}}_z$ and $\hat{\boldsymbol{e}}_{x'}$, $\hat{\boldsymbol{e}}_{y'}$, $\hat{\boldsymbol{e}}_{z'}$ of the coordinate axes and a common origin $0 = 0'$. S' rotates against S with the constant angular velocity $\boldsymbol{\omega} = \{\omega_x, \omega_y, \omega_z\}$ around $0 = 0'$ (Fig. 3.7). S' is therefore no inertial system. We assume that for all times $0 = 0'$.

A point A should have at time t in the system S the position vector

$$\boldsymbol{r}(t) = x(t) \cdot \hat{\boldsymbol{e}}_x + y(t) \cdot \hat{\boldsymbol{e}}_y + z(t) \cdot \hat{\boldsymbol{e}}_z \tag{3.8}$$

and the velocity

$$\boldsymbol{v}(t) = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\boldsymbol{e}}_x + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\boldsymbol{e}}_y + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\boldsymbol{e}}_z \,. \tag{3.9}$$

The same point A has in the system S' the position vector

$$\boldsymbol{r}'(t) = \boldsymbol{r}(t) = x'\hat{\boldsymbol{e}}_{x'} + y'\hat{\boldsymbol{e}}_{y'} + z'\hat{\boldsymbol{e}}_{z'} \,. \tag{3.10}$$

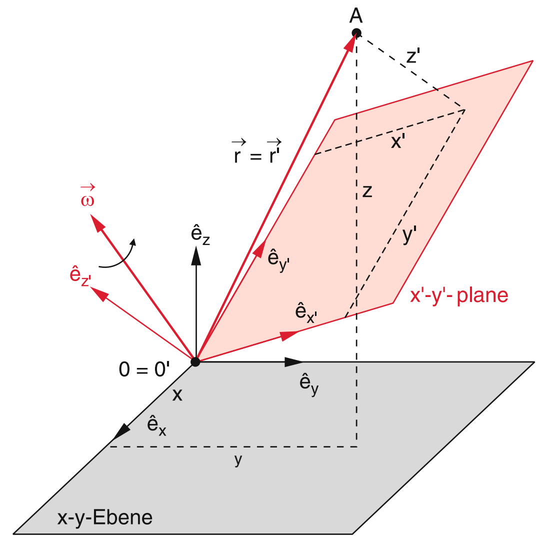


Figure 3.7 A system S', that rotates around the axis $\boldsymbol{\omega}$ against S. Both systems have the same origin $O = O'$

Note: $\boldsymbol{r} = \boldsymbol{r}'$ means that we regard the same vector in both systems with the same magnitude but different components.

If the observer O' does not take into account that his system rotates, he will define as the velocity of A in his system

$$\boldsymbol{v}'(t) = \frac{\mathrm{d}\boldsymbol{r}'}{\mathrm{d}t} = \frac{\mathrm{d}x'}{\mathrm{d}t}\hat{\boldsymbol{e}}_{x'} + \frac{\mathrm{d}y'}{\mathrm{d}t}\hat{\boldsymbol{e}}_{y'} + \frac{\mathrm{d}z'}{\mathrm{d}t}\hat{\boldsymbol{e}}_{z'} \,. \tag{3.11}$$

However, if the observer O in the inertial system S describes the velocity of A in the coordinates of S'. he knows that the axis of S' are rotating and therefore not constant in time. He therefore must write:

$$\begin{aligned} \boldsymbol{v}(x', y', z') &= \left(\frac{\mathrm{d}x'}{\mathrm{d}t}\hat{\boldsymbol{e}}_{x'} + \frac{\mathrm{d}y'}{\mathrm{d}t}\hat{\boldsymbol{e}}_{y'} + \frac{\mathrm{d}z'}{\mathrm{d}t}\hat{\boldsymbol{e}}_{z'}\right) \\ &\quad + \left(x'\frac{\mathrm{d}\hat{\boldsymbol{e}}_{x'}}{\mathrm{d}t} + y'\frac{\mathrm{d}\hat{\boldsymbol{e}}_{y'}}{\mathrm{d}t} + z'\frac{\mathrm{d}\hat{\boldsymbol{e}}_{z'}}{\mathrm{d}t}\right) \\ &= \boldsymbol{v}' + \boldsymbol{u} \,. \end{aligned} \tag{3.12}$$

The endpoints of the unit vectors $\hat{\boldsymbol{e}}_{x'}$, $\hat{\boldsymbol{e}}_{y'}$, $\hat{\boldsymbol{e}}_{z'}$ perform a circular motion with the angular velocity ω around $0 = 0'$. Their velocity is then

$$\frac{\mathrm{d}\hat{\boldsymbol{e}}_{x'}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}_{x'}; \quad \frac{\mathrm{d}\hat{\boldsymbol{e}}_{y'}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}_{y'}; \quad \frac{\mathrm{d}\hat{\boldsymbol{e}}_{z'}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}'_z \,. \tag{3.13}$$

Inserting this into (3.12) the second term in (3.12) becomes

$$\begin{aligned} \boldsymbol{u} &= (\boldsymbol{\omega} \times \hat{\boldsymbol{e}}_{x'})x' + (\boldsymbol{\omega} \times \hat{\boldsymbol{e}}_{y'})y' + (\boldsymbol{\omega} \times \hat{\boldsymbol{e}}_{z'})z' \\ &= \boldsymbol{\omega} \times (\hat{\boldsymbol{e}}_{x'}x' + \hat{\boldsymbol{e}}_{y'}y' + \hat{\boldsymbol{e}}_{z'}z') \\ &= \boldsymbol{\omega} \times \boldsymbol{r}' = \boldsymbol{\omega} \times \boldsymbol{r}, \quad \text{because } \boldsymbol{r} \equiv \boldsymbol{r}' \,. \end{aligned}$$

We therefore get the transformation between the velocity v of the point A measured by O in the system S and the velocity v' measured by O' in the system S'

$$\boldsymbol{v} = \boldsymbol{v}' + (\boldsymbol{\omega} \times \boldsymbol{r}) \,. \tag{3.14}$$

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Note: v' is the velocity measured by O', if he does not take into account, that his system S' rotates with the angular velocity $\boldsymbol{\omega}$, while $\boldsymbol{v}$ in (3.9) is the velocity in the resting system S and $\boldsymbol{v}$ in (3.14) the velocity of A measured by O but expressed in the coordinates of the rotating system S'.

The acceleration a can be obtained by differentiating (3.14). The result is

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} + \left(\boldsymbol{\omega} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) , \tag{3.15}$$

because we have assumed that $\omega = \text{const}$. The observer O' gets the result for a, expressed in the coordinates of his system S':

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} &= \left(\hat{\boldsymbol{e}}_{x'}\frac{\mathrm{d}v'_x}{\mathrm{d}t} + \hat{\boldsymbol{e}}_{y'}\frac{\mathrm{d}v'_y}{\mathrm{d}t} + \hat{\boldsymbol{e}}_{z'}\frac{\mathrm{d}v'_z}{\mathrm{d}t}\right) \\ &\quad + \left(\frac{\mathrm{d}\hat{\boldsymbol{e}}_{x'}}{\mathrm{d}t}v'_x + \frac{\mathrm{d}\hat{\boldsymbol{e}}_{y'}}{\mathrm{d}t}v'_y + \frac{\mathrm{d}\hat{\boldsymbol{e}}_{z'}}{\mathrm{d}t}v'_z\right) \\ &= \boldsymbol{a}' + (\boldsymbol{\omega} \times \boldsymbol{v}') ,\end{aligned} \tag{3.16}$$

where a' is again the acceleration of A measured by O' in the system S'. We therefore obtain with (3.15)

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{a}' + (\boldsymbol{\omega} \times \boldsymbol{v}') + (\boldsymbol{\omega} \times \boldsymbol{v}) .$$

Inserting for v the expression (3.14) we finally obtain from (3.15)

$$\boldsymbol{a} = \boldsymbol{a}' + 2(\boldsymbol{\omega} \times \boldsymbol{v}') + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) , \tag{3.17}$$

and for $\boldsymbol{a}'$

$$\begin{aligned}\boldsymbol{a}' &= \boldsymbol{a} + 2(\boldsymbol{v}' \times \boldsymbol{\omega}) + \boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) \\ &= \boldsymbol{a} + \boldsymbol{a}_\mathrm{C} + \boldsymbol{a}_\mathrm{cf} .\end{aligned} \tag{3.18}$$

While the observer in his resting system S measures the acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$, the observer O' in his rotating system S' has to add additional terms for the acceleration in order to describe **the same motion of A**. These are

the Coriolis-acceleration

$$\boldsymbol{a}_\mathrm{C} = 2(\boldsymbol{v}' \times \boldsymbol{\omega}) , \tag{3.19a}$$

the centrifugal acceleration

$$\boldsymbol{a}_\mathrm{cf} = \boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) . \tag{3.20a}$$

Special Cases: If the point A moves parallel to the rotation axis we have $\boldsymbol{v} \parallel \boldsymbol{\omega}$ and therefore the Coriolis acceleration becomes $\boldsymbol{a}_\mathrm{C} = \boldsymbol{0}$. The Coriolis acceleration appears only, if $\boldsymbol{v}'$ has a component perpendicular to $\boldsymbol{\omega}$. When we choose the z-axis as the direction of $\boldsymbol{\omega}$ (Fig. 3.8), both the Coriolis acceleration a_C and the centrifugal acceleration a_cf lie in the x-y-plane. The centrifugal acceleration points outwards in the radial direction. The direction of the Coriolis acceleration depends on the direction of the velocity v' in the coordinate system (x', y', z'). Since the v'_z-component does not contribute to a_C only the projection $v_\perp = \{v'_x, v'_y\}$ is responsible for the determination of the vector

$$\boldsymbol{a}_\mathrm{C} = \boldsymbol{\omega} \cdot \{v'_y, -v'_x, 0\} .$$

The vector $\boldsymbol{a}_\mathrm{C}$ is perpendicular to $\boldsymbol{v}_\perp$ as can be immediately seen when forming the scalar product $\boldsymbol{a}_\mathrm{C} \cdot \boldsymbol{v}'_\perp$.

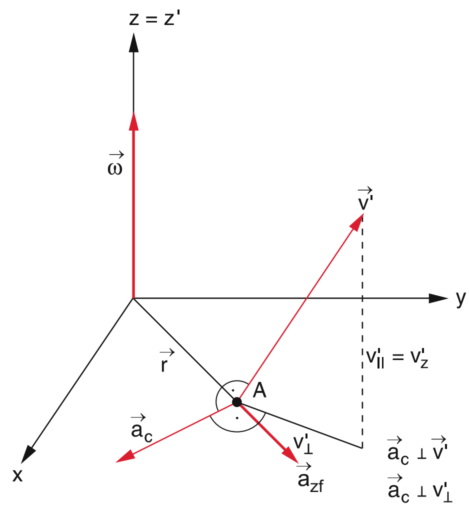

Figure 3.8 Centrifugal- and Coriolis-force acting on a mass m in A$(x, y, z = 0)$ described in a system S', that rotates with constant angular velocity $\boldsymbol{\omega}$ around the z-axis

3.3.3 Centrifugal- and Coriolis-Forces

According to Newton's laws accelerations are caused by forces. Therefore the observer O', who measures in his rotating system S' additional accelerations has to introduce additional forces based on the equation $\boldsymbol{F} = m \cdot \boldsymbol{a}$ These are the Coriolis force

$$\boldsymbol{F}_\mathrm{C} = 2\,\mathrm{m} \cdot (\boldsymbol{v}' \times \boldsymbol{\omega}) , \tag{3.19b}$$

and the centrifugal force

$$\boldsymbol{F}_\mathrm{cf} = m \cdot \boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) . \tag{3.20b}$$

Both forces are *inertial forces* or *virtual forces* because they are not real forces due to interactions between bodies. They have only to be introduced if the rotation of the coordinate axes of the rotating system S' are not taken into account. If the same motion of the body A are described in an inertial system S or in the rotating system S' where the rotation of the coordinate axes in considered, these forces do not appear.

We will illustrate these important facts by some examples.

Examples

1. A mass m is attached to one end of a string with length L while the other end is connected to the end of a bar with length d which rotates with the angular velocity ω around a vertical axis fixed to the centre of a rotating table (Fig. 3.9). In the equilibrium position the string forms an angle α against the vertical direction, where α depends on ω, d and L. The observer O in the resting frame S and the observer O' sitting on the rotating table describe their observations as follows:

O says: Since m moves with constant angular velocity ω on a circle with radius $r = d + L \cdot \sin\alpha$ a centripetal force $\boldsymbol{F}_{cp} = -m \cdot \omega^2 \cdot \boldsymbol{r}$ acts on m which is the vector sum of its weight $m \cdot g$ and the restoring force F_r of the string (Fig. 3.9a).
O' says: Since m is resting in my system S' the total force on m must be zero, i. e. $\sum F_i = 0$. The vector sum $m \cdot g + F_r$ has to be compensated by the centrifugal force $\boldsymbol{F}_{cf} = +m\omega^2 \boldsymbol{r}$ (Fig. 3.9b). He has to introduce the virtual force F_{cf} if he does not take into account the rotation of his system.

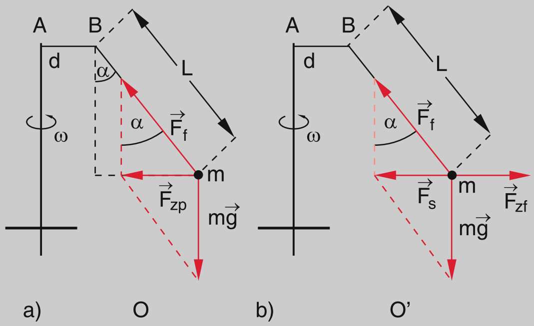

Figure 3.9 Forces on a rotating string pendulum, described by the observer O at rest and O' rotating with the pendulum

2. In a satellite, circling around the earth with constant angular velocity ω experiments are performed concerning the "weightlessness" (Fig. 3.10). For example an astronaut can freely float in his satellite without touching the walls.
   The observer O' in the satellite (i. e. the astronaut) says: I know that the gravity force
   $$\boldsymbol{F}_g = -(GmM/r^2)\hat{\boldsymbol{e}}_r$$
   acts on me, where r is the distance to the centre of the earth. It is compensated by the opposite centrifugal force
   $$\boldsymbol{F}_{cf} = +m\omega^2 \cdot r \cdot \hat{\boldsymbol{e}}_r .$$
   The total force acting on me is zero and therefore I can freely float.

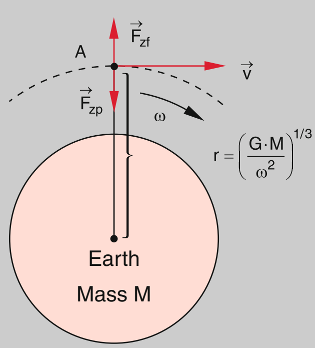

Figure 3.10 Force-free conditions in a satellite orbiting around the earth

Note: The state of the astronaut should be better called "force-free" instead of "weightlessness".
The observe O in a resting system S (for example the galactic coordinate system) says: The gravity force F_g acts as centripetal force on both the satellite and the astronaut. Both are therefore forced to move on a circle around the earth. The acceleration $\boldsymbol{a} = -(GM/r^2)\hat{\boldsymbol{e}}_r$ is the same for the astronaut and the satellite and the difference of the accelerations is zero. Therefore the astronaut can freely float in his satellite.
Note: Both observes can describe consistently the situation of the astronaut, however the observer S' has to introduce the inertial force F_{cf} if he does not take into account the accelerated motion of his space ship.

3. A sled moves with constant velocity v on a linear track and writes with a pen on a rotating disc (Fig. 3.11). The marked line on the rotating disc is curved where the curvature depends on the velocity v of the sled, the perpendicular distance d of the track from the centre of the disc and the angular velocity ω of the rotating disc. The two observers O and O' describe the observed curve as follows:
   O says: The sled moves with constant velocity on a straight line, as can be seen from the marked line outside the disc. Therefore no force is acting on the sled and its acceleration is zero. The curved path marked on the disc is due to the fact that the disc is rotating.
   O' says: I observe a curved path. Therefore a force has to act on the sled. By experiments with different values of v, ω and d he finds:
   For $d = 0$ is $|\boldsymbol{a}'| \propto v' \cdot \omega$; $\boldsymbol{a} \perp \boldsymbol{v}'$ and $\boldsymbol{a} \perp \omega$.
   For $d \neq 0$ is $a = c_1\omega + c_2\omega^2$ with $c_1 \propto v$ and $c_2 \propto r$, where r is the distance of the sled from the centre of the disc. The quantitative analysis of his measurements gives the result:
   $$\boldsymbol{a}' = 2(\boldsymbol{v}' \times \boldsymbol{\omega}) + \boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega}) ,$$
   which is consistent with (3.18) and shows that the acceleration of the sled measured by O' is the sum of centrifugal and Coriolis accelerations.

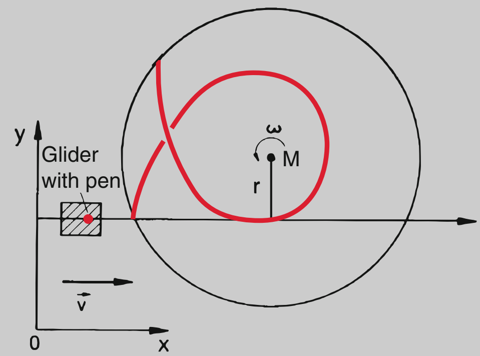

Figure 3.11 Experimental illustration of the inertial forces. A glider, moving on a straight line above a rotating disc writes with a pen its path on the rotating disc which appears as a curved trajectory

This example illustrates clearly that the two accelerations and the corresponding forces are only virtual, because the sled moves in fact with constant velocity on a straight line and therefore experiences no real forces.

4. A hollow sphere filled with sand hangs on a string which is connected to a fixed suspension point and swings in the fixed x-z-plane of a resting system S, driven by the gravity force $\boldsymbol{F}_g = m \cdot \boldsymbol{g}$ with $\boldsymbol{g} = \{0, 0, -g\}$.
Below the swinging pendulum is a rotating table in the x-y-plane which rotates with the angular velocity ω around a vertical axis through the minimum position of the pendulum.
If the sand flows through a small hole in the hollow sphere it draws for $\omega = 0$ a straight line on the table while for $\omega \neq 0$ a rosette-like figure is drawn (Fig. 3.12) with a curvature which depends on the ratio of oscillation period T_1 of the pendulum to rotation period T_2 of the rotating table.
The two observers give the following explanations:
O says: The x-z-oscillation plane remains constant because the driving force $F_g = m \cdot g \cdot \sin\alpha$ (see Sect. 2.9.7) lies always in the x-z-plane and therefore the motion must stay in this plane. The projection of the trajectory onto the x-y-plane should be a straight line. The curved trajectory drawn on the rotating table is caused by the rotation and not by an additional force.
O' says sitting on the rotating table: I see a curved path which must be caused by forces, which depend on ω, v' and r. Its form can be explained by the centrifugal and the Coriolis forces. My careful measurements prove that the paths is due to the action of the total acceleration $a' = a_{cf} + a_C$ in accordance with Eq. 3.18.

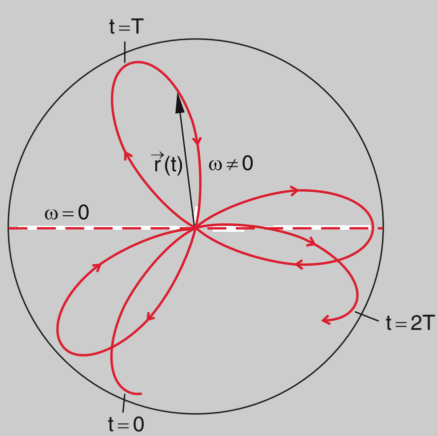

Figure 3.12 Apparent trajectory written on a rotating disc by a sand pendulum than oscillates in a constant plane

5. Foucault pendulum. Since our earth is a rotating system, the path drawn by a linearly swinging pendulum onto the ground must show curved lines as discussed in example 4). However, because of the slow earth rotation ($\omega = 7.3 \cdot 10^{-5}\,\text{s}^{-1}$) the curvature is very small. Using a pendulum with a large length L and a corresponding large oscillation period T the rotation of the earth under the linearly swinging pendulum could be first demonstrated 1850 by Leon Foucault (1819–1868) who used a copper ball ($m = 28\,\text{kg}$) suspended by a 67 m long string ($T = 16.4\,\text{s}$). The turn of the oscillation plane against the rotating ground occurs with the angular velocity $\omega_s = \omega \cdot \sin\varphi$ where φ is the geographic latitude of the pendulum (Fig. 3.13). in Kaiserslautern with $\varphi = 49°$ the pendulum plane turns in 1 h by $11°32'$, which can be readily measured. Using shadow projection of the pendulum string defining the oscillation plane this turn can be quantitatively measured within a physics lecture.

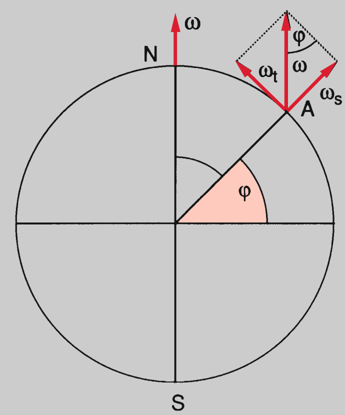

Figure 3.13 Explanation of the turning plane of oscillation of the Foucault-pendulums on the surface the rotating earth

6. An impressive demonstration of the Coriolis force is provided by the motion of cloud formations around a low pressure region as for instance realized by tornados or typhoons (Fig. 3.14). For an observer on the rotating earth looking from above onto the ground the wind does not blow radially into the low pressure region but rotates on the northern hemisphere anticlockwise around it, on the southern hemisphere clockwise. Around a high pressure region the rotation is clockwise on the northern hemisphere and anticlockwise on the southern.

Note: If a small balloon which floats in the air is used as indicator of the wind flow an observer on earth would see the balloon moving on one of the lines in Fig. 3.14. An observer O at a fixed position outside the earth, would however see, that the balloon moves on a straight line radially into the centre of the deep pressure region or out from the centre of a high pressure region. These centres are fixed at a point on earth and rotate with the earth.

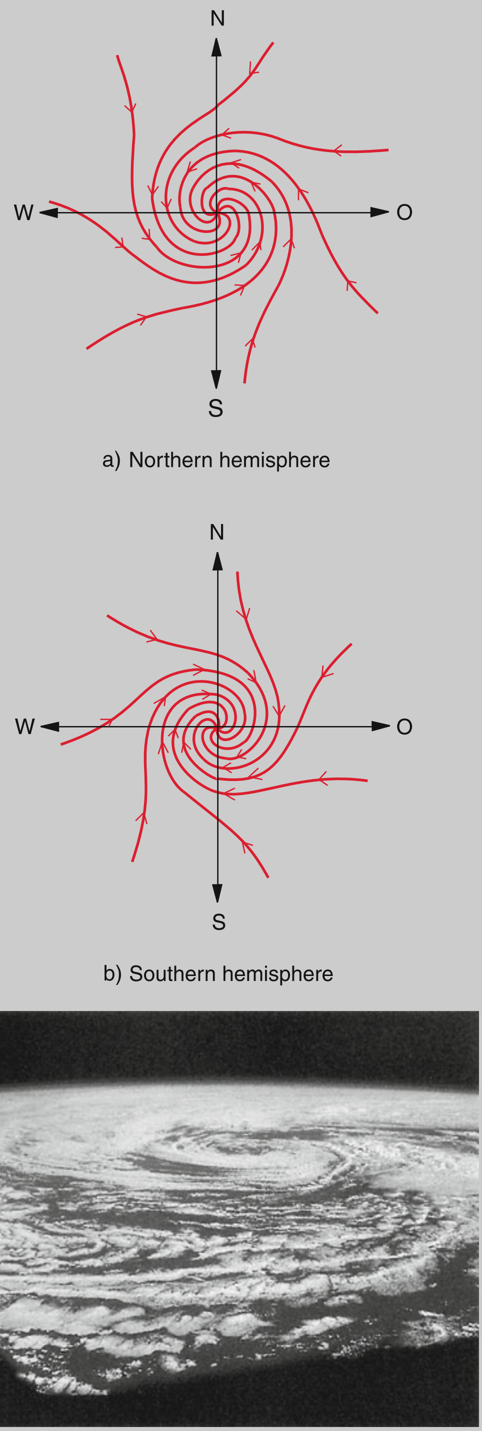

Figure 3.14 Stream-lines of the air around a deep-pressure region. **a** on the northern hemisphere; **b** on the southern hemisphere. On the northern hemisphere the Coriolis force acts (seen in the wind direction from above) in the right direction against the radial force of the pressure gradient, on the southern hemisphere in the left direction. **c** Satellite photo of the "death-hurricane" north of Hawaii (with kind permission of NASA photo HP 133)

3.3.4 Summary

Inertial forces (virtual forces) have to be introduced, if the motion of bodies are described in accelerated coordinate systems. These forces are not caused by real interactions between bodies but only reflect the acceleration of the coordinate system. They do not appear if the same motion is described in an inertial system.

In rotating systems with a fixed centre the inertial forces are centrifugal and Coriolis forces. In systems with arbitrarily changing velocities further inertial forces have to be introduced.

3.4 The Constancy of the Velocity of Light

We consider a body A which has the velocity $\boldsymbol{v}$ measured in the system S but the velocity $\boldsymbol{v}'$ in a system S', which moves itself with the velocity $\boldsymbol{u}$ against the resting system S. According to the Galilei transformations the different velocities are related through the vector sum (Fig. 3.15)

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{u} \,. \tag{3.21a}$$

Therefore one might suggest, that also the velocity of light, emitted from a light source which is fixed in a system S' moving with the velocity $\boldsymbol{u}$ against the system S, should be measured in the system S as the vector sum

$$\boldsymbol{c} = \boldsymbol{c}' + \boldsymbol{u} \,, \tag{3.21b}$$

where c' is the velocity measured by O' in his system S'. This means that the observer O should measure the velocity $\boldsymbol{c}_1 = c' + u$ if c' and u have the same direction, and $c_2 = c' - u$ if they have opposite directions.

Very careful measurements performed 1881 by Albert Abraham Michelson and Edward Morley [3.2a, 3.3] and later on by many other researchers [3.4a, 3.4b] produced evidence that the velocity of light is independent of the relative velocity u between source and observer. For example measurements of the velocity of light from a star at different times of the year always

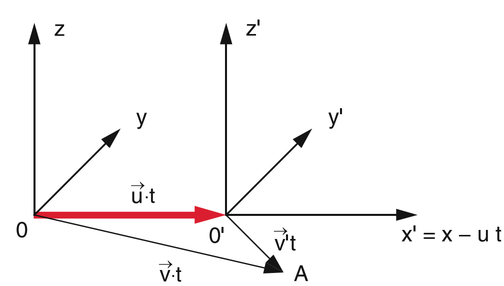

Figure 3.15 Galilei transformations of velocities in two inertial systems

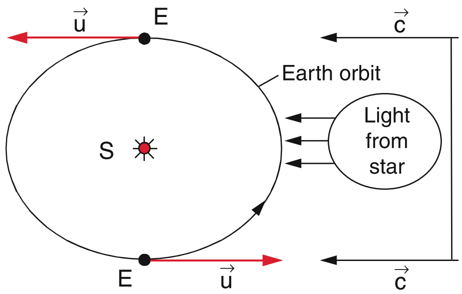

Figure 3.16 Experimental possibility to prove the constant velocity of light, by measuring the velocity of light from a distant star at two different days with a time interval of half a year, when the earth on its way around the sun moves towards the star and away from it

brought the same result although the earth moved with a velocity of 30 km/s at one time of the year against the star and half a year later away from the star (Fig. 3.16). This result was very surprising and brought about many discussions but induced the formulation of the theory of special relativity by Albert Einstein.

According to these unambiguous experimental results we must conclude:

> The velocity of light is the same in all inertial systems, independent of their velocity against the light source.

The Galilei-transformations (3.7) which appear very plausible apparently fail for very large velocities. It turns out, that in particular the assumption $t = t'$ in Eq. 3.3a needs a critical revision. It must be precisely defined what "simultaneity" means for two events at different locations. The question is: How does one measure the times of two events at different locations?

To illustrate this point we regard in Fig. 3.17 two systems S and S' where light pulses are emitted from the points A and B in the system S and from A′ and B′ in the system S'. If the two systems do not move against each other (Fig. 3.17a) the situation is clear: The observers O and O' measure the arrival time of the two light pulses in O resp. O' and can decide, whether the pulses had been sent from A and B resp. from A′ and B′ simultaneously or at different times. For the first case they arrive in O or O' simultaneously. Both observers come to the same result.

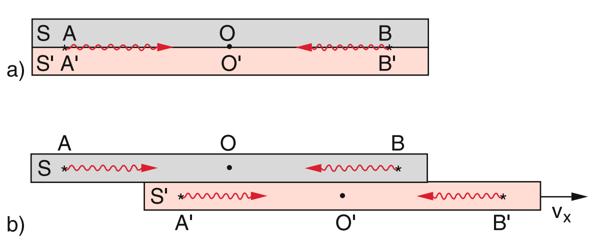

Figure 3.17 Illustration of the problem of measurements of simultaneous events in A and B resp. in A′ and B′ in two different systems: **a** which are at rest, **b** which move against each other

The situation is more difficult, if S' moves with the velocity v_x against S (Fig. 3.17b). We assume, that at time $t = 0$ the origins of both systems coincide, i. e. $O = O'$ and therefore also A and A′ as well as B and B′ coincide. If now at $t = 0$ two pulses are emitted form A and B in S and from A′ and B′ in S' the observer O in the rest frame measures their arrival times in O. During the light travel time Δt for the pulses from A or B the system S' has moved over the distance $\Delta x = v_x \cdot \Delta t$ to the right side in Fig. 3.17b. The pulses from B′ therefore arrive earlier in O' than those from A′. Therefore O' concludes that the pulses from B′ had been sent earlier than those from A′.

Now we will take the standpoint of O', who assumes that his system S' is at rest and that S moves with the velocity $-v_x$ to the left in Fig. 3.17b. He now defines the Simultaneity of the events in A′ and B′ if he receives the light pulses at O' simultaneously. Now the pulses from A arrive for O earlier that those from B. This illustrates that the definition of simultaneity depends on the system in which the pulses are measured. The reason for this ambiguity is the finite velocity of light. If this velocity would be infinite, the problem of simultaneity would not exist because then the travel time for the signals from the two points A and B would be always zero.

The question is now: what are the true equations for the transformation between different inertial systems?

3.5 Lorentz-Transformations

We regard two inertial systems $S(x, y, z)$ and $S'(x', y', z')$ with parallel axes and with $O(t = 0) = O'(t' = 0)$ which move with the constant velocity $v = \{v_x, 0, 0\}$ against each other in the x-direction (Fig. 3.18). Assume, that a short light pulse is emitted at $t = 0$ from $O = O'$. The observer O measures that the pulse has reached the point A after a time t. He describes his observation by the equation

$$\boldsymbol{r} = \boldsymbol{c} \cdot t \quad \text{or:} \quad x^2 + y^2 + z^2 = c^2 \cdot t^2 \,. \tag{3.22a}$$

The observer O' in S' measures that the pulse has arrived in A after the time t'. he therefore postulates:

$$\boldsymbol{r}' = \boldsymbol{c} \cdot t' \quad \text{or:} \quad x'^2 + y'^2 + z'^2 = c^2 \cdot t'^2 \,. \tag{3.22b}$$

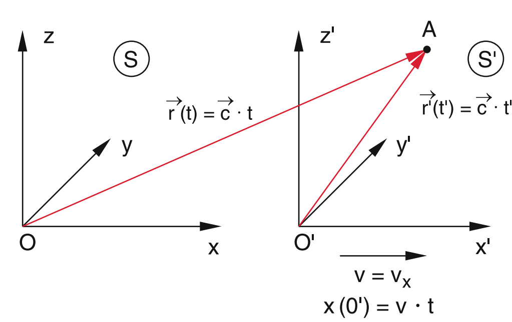

Figure 3.18 Schematic diagram for deriving the Lorentz-transformations

Both observers know about the result of the Michelson experiment. They therefore assume the same velocity of light c. The coordinate x of the origin O' measured in the system S is

$$x(O') = v \cdot t \quad \text{for} \quad x' = 0\,.$$

Since the coordinate x' refers to the system S' the transformation to the coordinates $x(\mathrm{A})$ of the point A, expressed in the system S must depend on the argument $(x - v \cdot t)$. We make the ansatz

$$x' = k(x - v \cdot t)\,, \tag{3.23}$$

where the constant k has to be determined. At time $t = 0$ the two observers were at the same place $x = x' = 0$ and have simultaneously started their clocks, i.e. $t(x = 0) = t'(x' = 0) = 0$. However, the time measurements for $t > 0$ are not necessarily the same for the two observers, because they are no longer at the same place but move against each other with the velocity v. The simplest transformation between t and t' is a linear transformation

$$t' = a(t - bx)\,, \tag{3.24}$$

where the constants a and b have again to be determined. Inserting (3.23) and (3.24) into (3.22b) yields with $y = y'$ and $z = z'$

$$\begin{aligned} k^2\left(x^2 - 2vxt + v^2t^2\right) + y^2 + z^2 \\ = c^2a^2\left(t^2 - 2bxt + b^2x^2\right)\,. \end{aligned}$$

Rearrangement gives

$$\begin{aligned} \left(k^2 - b^2a^2c^2\right)x^2 - 2\left(k^2v - ba^2c^2\right)xt + y^2 + z^2 \\ = \left(a^2 - k^2v^2/c^2\right)c^2t^2\,. \end{aligned}$$

This has to be identical with (3.22a) for all times t and all locations x. Therefore the coefficients of x and t have to be identical. This gives the equations

$$\left.\begin{aligned} k^2 - b^2a^2c^2 &= 1 \\ k^2v - ba^2c^2 &= 0 \\ a^2 - k^2v^2/c^2 &= 1 \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} a = k &= \frac{1}{\sqrt{1-v^2/c^2}} \\ b &= v/c^2\,. \end{aligned} \tag{3.25}$$

Inserting the expressions for a, b and k into (3.23) and (3.24) gives the special **Lorentz-Transformations**

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}}\,, \quad y' = y\,, \quad z' = z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}\,, \end{aligned} \tag{3.26}$$

between the coordinates (x, y, z) and (x', y', z') of two inertial systems which move against each other with the constant velocity $\boldsymbol{v} = \{v, 0, 0\}$. These equations were first formulated 1890 by Hendrik Lorentz [3.5]. They show, that for $v \ll c$ the Lorentz transformations converge towards the Galilei transformations (because for $v^2/c^2 \ll 1 \rightarrow \sqrt{1 - v^2/c^2} \approx 1$), which are therefore a special approximation for small velocities v:

Example

For $v = 10\,\mathrm{km/s}$ ($36\,000\,\mathrm{km/h}$) is $v/c \approx 3 \times 10^{-5}$ and $(1 - v^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}v^2/c^2 = 1 + 10^{-10}$. The difference between Galilei and Lorentz transformations is then only $5 \cdot 10^{-10}$ and therefore smaller than the experimental uncertainty. ◀

With the abbreviation $\gamma = (1 - v^2/c^2)^{-1/2}$ the Lorentz transformations can be written in the clear form

$$\begin{aligned} x' &= \gamma(x - vt) & x &= \gamma(x' + vt') \\ y' &= y & y &= y' \\ z' &= z & z &= z' \\ t' &= \gamma(t - vx/c^2) & t &= \gamma(t' + vx'/c^2) \end{aligned}\,. \tag{3.26a}$$

Note: The Lorentz transformations have, compared to the Galilei transformations, only one additional assumption: The constancy of the velocity of light and its independence of the special inertial system, which was used in (3.22a,b), where both observers anticipate the same value of the velocity of light.

We will now discuss, how the velocity u of a body A, measured in the system S transforms according to (3.26) into the velocity u' of A, measured by O' in S'.

For O pertains:

$$u_x = \frac{\mathrm{d}x}{\mathrm{d}t}; \quad u_y = \frac{\mathrm{d}y}{\mathrm{d}t}; \quad u_z = \frac{\mathrm{d}z}{\mathrm{d}t}\,, \tag{3.27}$$

while for O' applies

$$\boldsymbol{u}' = \{u'_x, u'_y, u'_z\} = \left\{\frac{\mathrm{d}x'}{\mathrm{d}t'}, \frac{\mathrm{d}y'}{\mathrm{d}t'}, \frac{\mathrm{d}z'}{\mathrm{d}t'}\right\}\,.$$

Using (3.26) and considering that $x = x(t)$ depends on t, we get:

$$\begin{aligned} u'_x &= \frac{\mathrm{d}x'}{\mathrm{d}t'} = \frac{\mathrm{d}x'}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}t'} = \frac{\mathrm{d}x'}{\mathrm{d}t} \Big/ \frac{\mathrm{d}t'}{\mathrm{d}t} \\ &= \frac{\gamma\left(\frac{\mathrm{d}x}{\mathrm{d}t} - v\right)}{\gamma\left(1 - \frac{v}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}\right)} = \frac{u_x - v}{1 - \frac{v \cdot u_x}{c^2}}\,. \end{aligned}$$

Solving for u_x gives the back-transformation

$$u_x = \frac{u'_x + v}{1 + u'_xv/c^2}\,. \tag{3.28a}$$

In the same way one obtains

$$u'_y = \frac{u_y}{\gamma\left(1 - vu_x/c^2\right)}; \quad u_y = \frac{u'_y}{\gamma\left(1 + vu'_x/c^2\right)}\,, \tag{3.28b}$$

$$u'_z = \frac{u_z}{\gamma\left(1 - vu_x/c^2\right)}; \quad u_z = \frac{u'_z}{\gamma\left(1 + vu'_x/c^2\right)}\,. \tag{3.28c}$$

These equations demonstrate that the velocity components u_y and u_z perpendicular to the velocity $v = v_x$ of S' against S

transforms differently from the component u_x parallel to v_x. For $v_x \cdot u \ll c^2$ one obtains again the Galilei transformations.

If the body A moves parallel to the velocity v i. e. parallel to the x-axis and therefore also to the x'-axis, we have $u_y = u_z = 0 \Rightarrow u = u_x$, the Lorentz transformations simplify to

$$u' = \frac{u - v}{1 - vu/c^2} \; . \tag{3.28d}$$

For $u = c$ we get

$$u' = \frac{c - v}{1 - v/c} \equiv c \; , \tag{3.28e}$$

which means that O and O' measure the same value for the light velocity in accordance with the results of the Michelson experiment.

3.6 Theory of Special Relativity

Starting with the results of the Michelson experiment and the Lorentz transformations Einstein developed 1905 his theory of special relativity [3.5–3.8], which is based on the following postulates:

- All inertial systems are equivalent for all physical laws
- The velocity of light in vacuum has the same value in all inertial systems independent of the motion of the observer or the light source.

For the comparison of measurements of the same event by two observers in two different systems S and S', which move against each other the time definition plays an important role. The Lorentz-transformations (3.26) show that also the time has to be transformed when changing from S to S'. We will therefore at first discuss the relativity of simultaneity. The presentation in Sect. 3.6 follows in parts the recommendable book by French [3.8].

3.6.1 The Problem of Simultaneity

We will now treat the problem of simultaneity in different inertial systems in some more detail. We regard three points A, B and C which rest in the system S and have equal distances, i. e. $AB = BC = \Delta x$. In an x-t-coordinate-system with rectangular axis. For $t = 0$ the three points are located on the x-axis (Fig. 3.19a). In the x-t-diagram the points A, B and C proceed in the course of time on vertical straight lines since they are fixed in the system S and have therefore constant positions x. At time $t = 0$ a light pulse is emitted from point B. The light pulse, however proceeds on an inclined straight line with an inclination angle α_1 with $\tan\alpha_1 = t_1/\Delta x$. This line intersects the vertical position lines of A and C at the points A_1 and C_1. The connecting line through A_1 and C_1 is the horizontal line $t = t_1$. Since the light travels with the same speed c in all directions the pulses reach the points A and C at the same time $t_1 = \Delta x/c$.

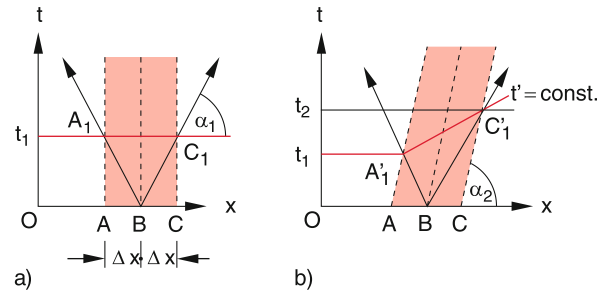

Figure 3.19 Illustration of the different results in a space-time diagram, when measuring simultaneous events in two different systems that move against each other

Now we regard the same situation in the system S' which moves with the velocity $v = v_x$ against S (Fig. 3.19b). The points A, B and C should rest in the system S', they therefore move with the velocity v_x against the system S and pass in the x-t-diagram of S inclined straight lines with the inclination angle α_2 and the slope $\tan\alpha_2 = \mathrm{d}t/\mathrm{d}x = 1/v$. The light pulses travel with the same velocity c as in S. At time $t = 0$ both systems S and S' should coincide. The light pulse, emitted from B at $t = 0$ now reaches the two points A and C for the observer O **not simultaneously** but in A at $t = t_1$ and in C at $t = t_2$, which correspond with the intersection points A'_1 and C'_1 in Fig. 3.19b. The reason is that A propagates towards the light pulse but C from it away.

Since for the observer O' in S' the points A, B and C are resting in S' the events A'_1 and C'_1 (we define as event the arrival of the light pulse in the point A'_1 or C'_1) has to occur simultaneously, equivalent to the situation for S in Fig. 3.19a because all inertial systems at rest are completely equivalent. In the x'-t'-diagram the line through the points A'_1 and C'_1 has to be a line $t' = \text{const}$ i. e. it must be parallel to the x'-axis (Fig. 3.20). One has to choose for the moving system S' other x'- and t'-axes which are inclined against the x- and t-axes of the system S. The x'-axis ($t' = 0$) and the t'-axis ($x' = 0$) are generally not perpendicular to each other.

One obtains the t'-axis in the following way: If O' moves with the velocity $v = v_x$ against O he propagates in the system S' along the axis $x' = 0$ which is the t'-axis (because he is in his

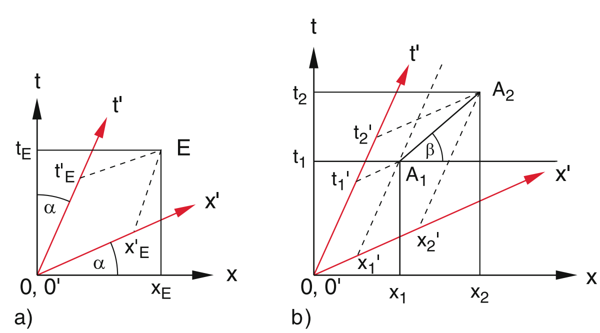

Figure 3.20 **a** Space axis and time axis in a moving inertial system S' are inclined by the angle α against the axes in a system S at rest. **b** Definition of the velocity u of a point A in the two systems S and S'

system S' always resting at the origin $x' = 0$). In the system S this axis is $x = v \cdot t$ which is inclined against the t-axis $x = 0$ by the angle α with $\tan\alpha = v/c$. The slope of the t'-axis against the x-axis in the system S is $\mathrm{d}t/\mathrm{d}x = 1/v$.

Any event E is completely defined by its coordinates (x, t) in S or (x', t') in S'.

Note, however, that for the same event E the spatial and time coordinates (x_E, t_E) for O in S are different from (x'_E, t'_E) for O' in S' (Fig. 3.20)

For each observer the simultaneity of two events at different spatial points depends on the coordinate system in which the events are described.

We regard a point mass A which moves with the velocity u_x against O and with u'_x against O'. Its velocity is determined by O and O' by measuring the coordinates $x_1(t_1)$ and $x_2(t_2)$ in S resp. $x'_1(t'_1)$ and $x'_2(t'_2)$ in S' (Fig. 3.20b).

$$O \text{ obtains:} \quad u_x = \frac{x_2 - x_1}{t_2 - t_1} ,$$

$$O' \text{ obtains:} \quad u'_x = \frac{x'_2 - x'_1}{t'_2 - t'_1} .$$

The velocity u_x is represented in S by the reciprocal slope $\Delta x/\Delta t = u_x$ of the straight line A_1A_2, In S' however by $u'_x = \Delta x'/\Delta t'$. One can see already from, Fig. 3.20b that $u_x \neq u'_x$, which is quantitatively described by Eq. 3.28.

3.6.2 Minkowski-Diagram

The relativity of observations and their dependence of the reference system can be illustrated by space-time-diagrams as shown in Fig. 3.20. Each physical event which occurs at the location $\boldsymbol{r} = \{x, y, z\}$ at time t can be represented by a point in the four-dimensional space-time $\{x, y, z, t\}$. For simplicity we will restrict the following to one spatial dimension x and the relative motion of S' against S should occur only in the x-direction. Then the four-dimensional representation reduces to a two-dimensional one. Furthermore the time axis t is changed to $c \cdot t$ in order to have the same physical dimension [m] for both axes. Such a depiction is called ***Minkowski-diagram*** (Fig. 3.21).

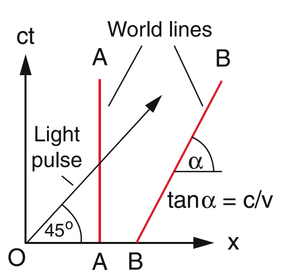

Figure 3.21 Minkowski diagram showing the world lines of a point A resting in the system, of a point B moving in the system with the velocity u and a light pulse emitted at time $t = o$ from the origin

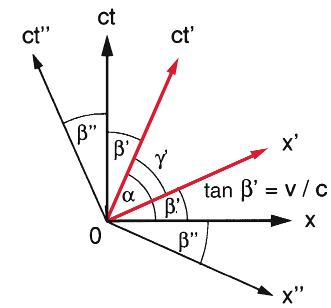

Figure 3.22 Minkowski diagram of the axes (x, t) of a system S at rest, of the axes (x', t') of a system S' that moves with the velocity v against S and of the axes (x'', t'') of a system S'', moving with $-v$ against S

A body A at rest propagates in an orthogonal (x, ct) diagram on a vertical line while a body B with the constant velocity v relative to O propagates on a sloped straight line with the slope $c \cdot \Delta t/\Delta x = c/v$. A light pulse which is emitted from $x = 0$ at $t = 0$ and propagates with the velocity c into the x-direction traverses on a straight line with the inclination of 45° against the x-axis because the slope is $\tan\alpha = c/c = 1$. It is represented by the diagonal in an orthogonal (x, ct)-diagram. Such lines for moving bodies or for light pulses are called *world lines* or *space-time-lines*, which can be also curved. Two events A and B occur in the system S simultaneously, if their points in the Minkowski-diagram lie on the line $t = t_1$ parallel to the x-axis (Fig. 3.19). The ct'-axis in S' is the world line of O'.

We had already discussed in the preceding section that the axis of two inertial systems S and S', which move against each other with the constant velocity v_x are inclined against each other. If the x- and the ct-axes in system S are orthogonal the ct'-axis has the slope $\tan\alpha = c/v_x$ against the x-axis. Also the x'-axis is inclined against the x-axis. According to the Lorentz-transformations the relation $t' = 0 \Rightarrow t = v \cdot x/c^2$ must be satisfied (Fig. 3.22). Its slope against the x-axis is therefore $\mathrm{d}t'/\mathrm{d}x = \tan\beta' = v/c$. The angle between the x' and the ct' axes is $\gamma = \alpha\text{-}\beta' = \arctan(c/v) - \arctan(v/c)$.

For illustration also a third system S'' is shown in Fig. 3.22, which moves with the velocity $v = -v_x$ against S. The slope of the x''-axis against the x-axis is now $\tan\beta'' = -v_x/c$. the angle between ct''-axis and ct-axis is also β''. The ct''-axis forms an angle $\delta = 2(\beta' + \beta'') + \gamma > 90°$ against the x''-axis.

3.6.3 Lenght Scales

Not only the inclination of the axis but also their scaling is different in the systems S, S' and S''. Since the velocity of light is the same in all inertial systems (which implies $c = \mathrm{d}x/\mathrm{d}t =$

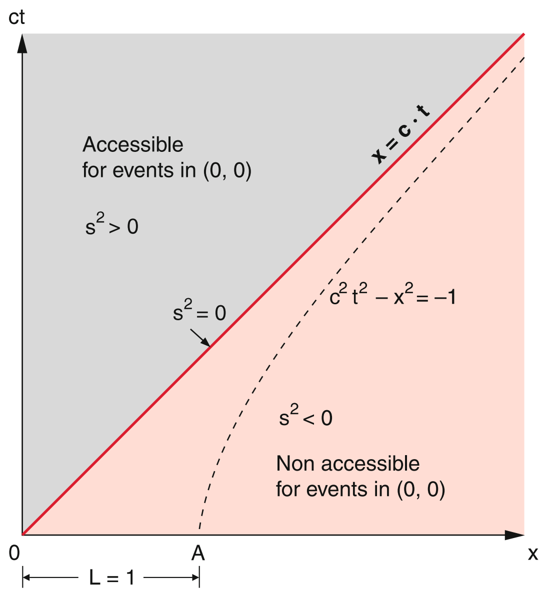

Figure 3.23 Illustration of the invariant s^2

$c' = \mathrm{d}x'/\mathrm{d}t' = \text{const}$) the quantity

$$s^2 = (ct)^2 - x^2 = (ct')^2 - x'^2 \tag{3.29}$$

must be equal in all inertial systems. This can be also seen from the Lorentz-equations (3.26). The quantity s^2 is therefore invariant under transformation between different inertial systems. For $s^2 = 0$ the world-line $x = \pm ct$ of a light pulse is obtained. For the motion of a body with velocity $v < c$ starting at $t = 0$ and $x = 0$ it follows $x^2(t) < (ct)^2 \Rightarrow s^2 > 0$.

In the (x, ct)-diagram no points with $x^2 > (ct)^2 (s^2 < 0)$ can be reached by signals emitted by O at $t = 0$. The area in Fig. 3.23 with $s^2 < 0$ is *non-accessible*, while all points with $s^2 > 0$ can be reached by such signals.

Such an invariant quantity like s^2 can be used to fix the scale length in Minkowski-diagrams. If we allow also imaginary values of s, the square s^2 can be also negative. For $s^2 = -1$ we obtain from (3.29) for all inertial systems (i. e. for S as well as for S') the hyperbola

$$x^2 - (ct)^2 = x'^2 - (ct')^2 = 1 \, ,$$

which is drawn in Fig. 3.23. It intersects in the system S the x-axis ($t = 0$) in the point A at $x = 1$. This defines the scale length $L = 1$ for the system S.

Also in the system S' is $x' = 1$ for $t' = 0$, which gives the scale length $L' = 1$ for the observer O'. However, for the observer O in S the length L' appears as $L \neq 1$ as can be seen from Fig. 3.24 where $L = OA$ but $L' = 0B$. Each observer measures for the length in his own system another value than for the length in a system moving against his system. This seems very strange but is a consequence of the problem of simultaneity, because in order to measure the lengths L and L', O has to measure the endpoints 0 and A or 0 and B simultaneously, i. e. at the same time t, while O' measures them at the same time t'.

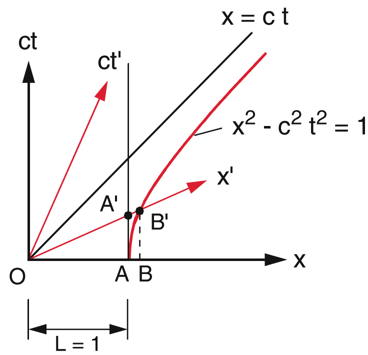

Figure 3.24 A yardstick with length OA that rests in the system S, appears shortened in the system S' moving against S

This shows that length standards in different inertial systems can be in fact different. If O in his system S measures distances in another system S', moving against S, he uses a larger scale, which means that the length of distances appears shorter.

3.6.4 Lorentz-Contraction of Lengths

One of the surprising results of the Lorentz-transformations is the contraction of the length of bodies in Systems S' moving against the observer in a rest frame S. In the foregoing section we have already indicated that this contraction is caused by the change of the length scale L' and that it can be ascribed to the problem of simultaneity.

Assume a rod with the endpoints P'_1 and P'_2 rests in the moving system S'. The coordinates x'_1 and x'_2 therefore move in the course of time on straight lines parallel to the t'-axis (Fig. 3.25). The observer O' measures at time t'_1 the length

$$L' = P'_1 P'_2 = x'_2(t'_1) - x'_1(t'_1) \, .$$

For the observer O in S the rod resting in S' moves with the system S' with the velocity v in the x-direction. In order to determine the length of the rod, O has to measure the endpoints x_1

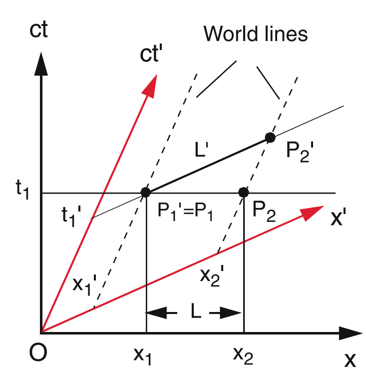

Figure 3.25 Graphical illustration of the Lorentz-contraction of a yard stick with length L resting in the moving system S', when O in S measures the length L', expressed in the Minkowski diagram of S'

and x_2 simultaneously, i. e. for $t = t_1$. These endpoints are for $t = t_1$ at the intersection points $P_1(t_1) = P_1'$ and $P_2(t_1) \neq P_2'$ of the world-lines $x_1'(t)$ and $x_2'(t)$ with the horizontal line $t = t_1$ in Fig. 3.25. For O is therefore the length of the rod

$$L = P_1 P_2 = x_2 - x_1 \ ,$$

where x_1 and x_2 are the vertical projections of P_1 and P_2 onto the x-axis $t = 0$ (Fig. 3.25).

Since $\Delta x'$ differs from Δx the two lengths L and L' are different. Because the scale lengths s and s' are different one cannot directly geometrically compare the length of the rod measured in S and S' from Fig. 3.25, but has to use the Lorentz-transformations.

$$\begin{aligned} & x_1' = \gamma(x_1 - vt_1); \qquad x_2' = \gamma(x_2 - vt_2) \\ & \Rightarrow x_2' - x_1' = \gamma(x_2 - x_1) \quad \text{for} \quad t_1 = t_2 \\ & \Rightarrow L' = \gamma \cdot L \Rightarrow L < L'; \quad \text{because} \quad \gamma > 1 \ . \end{aligned} \tag{3.30}$$

The lengths of a moving rod seems for an observer to be shorter than that of the same rod at rest.

- The contraction does not depend on the sign of the velocity $v = \pm v_x$.
- The contraction is really relative as can be seen from the following example: Two rods should have the same length $L_1 = L_2$ if both are resting in the same system S. Now L_2 is brought into a moving system S' where it rests relative to the origin O' of S'. For the observer O the length L_2 seems to be shorter then L_1 but for O' L_1 seems to be shorter than L_2. This implies that the Lorentz contraction is symmetric. This is no contradiction, because the different length measurements are due to the different observations of simultaneity as has been discussed before.
  Each observer can only make statements of events and times with respect to his own system S. If he transfers measurements of events in moving systems S' to his own system S, he has to take into account the relative velocity of S' against S and must use the Lorentz transformations. Then O and O' come to the same results.

Note that both observers O and O' come to consistent results for measurements in their own system and in the other system which moves against their own system, if they use consequently the Lorentz-transformations.

The answer to the often discussed question whether there is a "real contraction" depends on the definition of "real". The only information we can get about the length of the rod is based on measurements of the distance between its endpoints. For rods moving against the observer the locations of the two endpoints have to be measured simultaneously, which gives the results discussed above.

The relativity of the contraction can be visualized in the Minkowski diagram of Fig. 3.26. We regard again two iden-

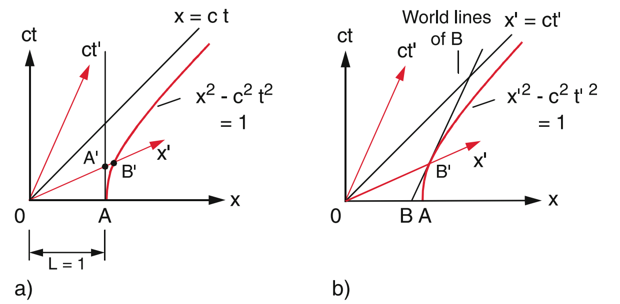

Figure 3.26 Relativity of the Lorentz contraction: **a** The yardstick $OA = 1$ rests in S, **b** the yardstick $OB = 1$ rests in the moving system S'

tical yardsticks with the scale $L = 1$, which rest in the system S resp. S'. The yardstick in S has for the observer O the endpoints O and A with the distance $OA = 1$. The world line for O is the ct-axis $x = 0$ and for A the parallel vertical line $x = 1$. In Fig. 3.26a also the world line $x = c \cdot t$ of a light pulse and the hyperbola $x^2 - c^2t^2 = 1$ are drawn. The intersection of the hyperbola with the x-axis $t = 0$ defines the scale $L = 1$, in the system S.

How is the situation in the system S'? The world line of A intersects the x'-axis $ct' = 0$ in the point A$'$. Therefore the distance OA$'$ is for O the length $L' = 1$ of the yardstick. However, for O' in his system S' the length of the yardstick is $x' = 1$ given by the distance OB$'$ where B$'$ is the intersection point of the parabola $x'^2 - (ct')^2 = 1$. For O is the length of the moving yardstick therefore smaller than for O', who regards the stick resting in his system.

Note that the parabola is the same in both systems S and S' (see Sect. 3.6.3).

For O' is the scale of O which he measures as OA$'$ shorter than his own scale OB$'$ this means that it appears for O' shorter.

Now we take a scale OB$'$ which rests in the system S' and has there the length $x' = 1$ because B$'$ is the intersection point of the parabola $x'^2 - (ct)^2 = 1$ with the x'-axis $ct' = 0$ (Fig. 3.26b). The world line of O' is the ct'-axis $x' = 0$ and that of the point B the line through B$'$ parallel to the ct' axis. This line intersects the x-axis in the point B. The observer O measures the length of the scale $x' = 1$ as the distance OB which is shorter than the distance OA with $x = 1$. Now the scale $x' = 1$ of the observer O' is shorter for the observer O.

This illustrates that the length contraction is due to the different prolongation of the scale which is caused by the different simultaneity for measurements of the endpoints by *O* and *O'*.

Note that both observers O and O' come to contradiction-free statements concerning measurements in their own system and in the other system if they use the Lorentz-transformations.

3.6.5 Time Dilatation

We regard a clock, which rests in the origin O of system S. We assume that this clock sends two light pulses at times t and $t+\Delta t$ with a time delay Δt of the second pulse. An observers O at the location x_0 in the system S receives the light pulses at times t_1 and t_2, at the event points A and B of his world line $x = x_0 =$ const (Fig. 3.27). For O the time interval between the two pulses is

$$\Delta t = \mathrm{AB} = t_2 - t_1 \ .$$

An observers O' sitting at $x' = x_0'$ in the system S' which moves with the velocity v against S receives the light pulses at the intersection points A′ and B′ of his world line $x' = x_0'$ with the two axes $x' = ct'$ and $x' = c(t' + \Delta t')$ which are observed at times t_1' and t_2' measured with his clock in S'.

The observer O in S knows, that these times t_1' and t_2' are transformed into his measured times t_1 and t_2 by the Lorentz transformations

$$t_1' = \gamma \frac{t_1 - v \cdot x_0}{c^2} t_2' = \gamma \frac{t_2 - v \cdot x_0}{c^2} \ .$$

According to these equations he determines the time difference in the moving system S' as

$$\Delta t' = t_2' - t_1' = \gamma \cdot \Delta t \ . \tag{3.31}$$

Since $\gamma = (1 - v^2/c^2)^{-1/2} > 1$ the observer O at rest measures for the moving system S' a longer time interval $\Delta t'$ between the two pulses than the moving observer O'. Because the clock resting in S moves for the observer S' he measures, that this clock runs slower than his own clock. This can be expressed by: **Moving clocks run slower**. Equivalent to the length contraction also the time dilatation is caused by the different observations of simultaneity in the systems S and S'. This effect increases with increasing velocity v and reaches essential values only for velocities v close to the velocity c of light. However this time dilatation can be measured with very precise clocks already for smaller velocities. For example, if two clocks are synchronized in Paris and transported by a fast plane (such as the concorde with $v = 2400\,\mathrm{km/h} = 667\,\mathrm{m/s} \Rightarrow \gamma = 1 + 8.9 \cdot 10^{-12}$) to New York the difference between Δt and $\Delta t'$ during a flight time of 3 hours is $8.9 \cdot 10^{-9}\,\mathrm{s} = 8.9\,\mathrm{ns}$.

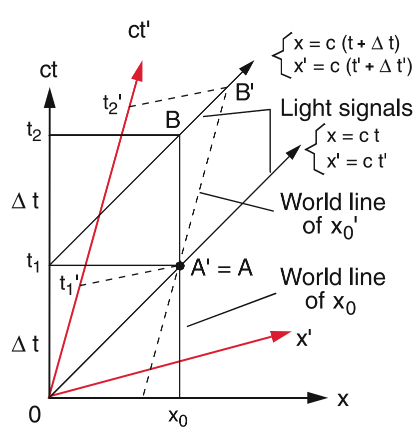

Figure 3.27 Minkowski diagram for illustration of the time dilatation. Two signals with the time difference $\Delta t = t_2 - t_1$ in the resting system S reach the moving observer O' in S' with the time difference $\Delta t' = \gamma \cdot \Delta t$

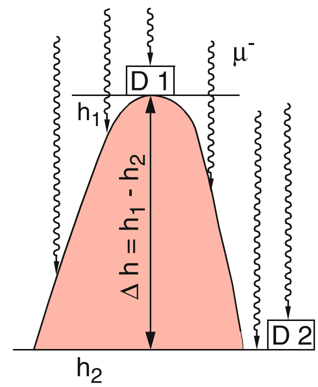

Figure 3.28 Measurement of the lifetime of relativistic muons with two detectors at different heights h_1 and h_2 above sea level

A much more precise measurement of the time dilatation can be obtained with faster moving clocks. Examples of such fast moving clocks are fast elementary particle such as electrons, protons or muons which move with velocities $v \approx c$.

The cosmic radiation (electrons and protons with very high energy) produce in the upper earth atmosphere at collisions with the atomic nuclei of the atmospheric molecules muons μ^- with velocities $v \approx c$ which reach nearly the velocity of light. Part of these muons reach the earth surface, while part of them decay during their flight through the atmosphere according to the scheme

$$\mu^- \xrightarrow{\tau} e^- + \nu_\mu + \overline{\nu_e} \tag{3.32}$$

into an electron and two neutrinos (see Vol. 4). The lifetime of decelerated resting muons can be precisely measured as $\tau = 5 \cdot 10^{-6}\,\mathrm{s}$.

In order to measure the lifetime τ' of fast flying muons the rate of muons, incident onto a detector is measured at different altitudes above sea level, for instance at the altitude $h = h_1$ on the top of a mountain and at $h = h_2$ at the bottom of the mountain (Fig. 3.28). For a mean decay time τ' of the muons moving with the velocity v the relative fraction $\mathrm{d}N/N$ decays during the time interval $\mathrm{d}t/\tau'$

$$\mathrm{d}N = -a \cdot (N/\tau')\mathrm{d}t \ .$$

Integration yields

$$N(h_2) = N(h_1) \cdot \mathrm{e}^{-\Delta t/\tau'} \quad \text{with} \quad \Delta t = \frac{h_1 - h_2}{v} \ ,$$

where the factor $a < 1$ takes into account the scattering of muons by the atmospheric molecules. This factor can be calculated from known scattering data. The often repeated measurements clearly gave essentially higher lifetimes $\tau' = 45 \cdot 10^{-6}\,\mathrm{s}$

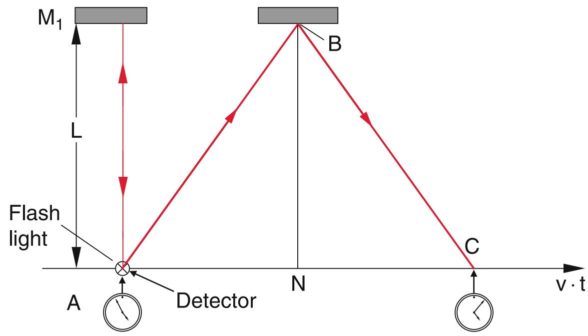

Figure 3.29 Einstein's "light-clock" for illustration of the time dilatation

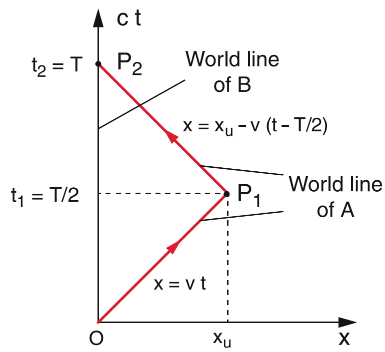

Figure 3.30 Minkowski diagram for the twin paradox

of the moving muons than $\tau = 5 \cdot 10^{-6}$ s for muons at rest. From $\tau' = \gamma \cdot \tau$ follows $\gamma = 9 \rightarrow v = 0.994c$. The muons have a velocity v which is smaller than c by only 6 ‰.

Meanwhile many short-lived elementary particles can be produced and accelerated to high velocities. Comparing the lifetimes of these particles at rest and while moving, unambiguously confirms the time dilatation postulated by the Lorentz transformations.

The relativistic time dilatation can be illustrated by an "gedanken-experiment" proposed by Einstein using a light pulse clock (Fig. 3.29). The system consists of a box with length L. On one side a flash lamp is mounted at the point A and on the opposite end a mirror M_1. The flash lamp emits a short light pulse and starts a clock. The light pulse reflected by M_1 is received by a detector which stops the clock. The time interval $\Delta t_0 = 2L/c$ is used as time scale in the system S in which the light clock rests.

Now we let the system S move with the velocity v relative to a system S' in a direction perpendicular to the length L. For the observer O' in S' the light pulse now travels from A to B and is reflected to C. With AN = $NC = v \cdot \Delta t/2$ it follows from Fig. 3.29

$$\begin{aligned} \mathrm{AB} + \mathrm{BC} &= 2 \cdot \left[L^2 + \left(v \frac{\Delta t'}{2} \right)^2 \right]^{1/2} = c \cdot \Delta t' \\ \Rightarrow \Delta t' &= \frac{2L}{(c^2 - v^2)^{1/2}} \, . \end{aligned} \tag{3.33}$$

The observer O measures $\Delta t = 2L/c$. The comparison between Δt and $\Delta t'$ gives:

$$\Delta t' = \frac{\Delta t}{(1 - v^2/c^2)^{1/2}} = \gamma \cdot \Delta t \, , \tag{3.34}$$

which turns out to be identical with (3.31).

3.6.6 The Twin-Paradox

No other problem of special relativity has aroused so many controversial discussions as the twin paradox (often called the clock-paradox), discussed by Einstein in his first paper 1905 about relativity. It deals with the following situation:

Two clocks which are synchronized show equal time intervals when sitting in the same system S at rest. One of the clocks is taken by O' on a fast moving spacecraft and returns after the travel time T (measured by O in S) back to the other clock which always had stayed in the system S. A comparison of the two clocks shows that the moved clock is delayed, that means that it shows a smaller value T' than T [3.8–3.11].

This "gedanken-experiment" has meanwhile be realized and the time dilatation has been fully verified (see previous section). For manned space missions this means that an astronaut A after his return to earth after a longer journey through space is younger than his twin brother B who has stayed at home. The "gained" time span is, however, for velocities of spacecrafts which can be realized up to now, very small and therefore insignificant. Nevertheless an understanding of the twin paradox is of principal significance because it illustrates the meaning of relativity, which is often used in a popular but wrong way.

We have discussed in the previous section that the time dilatation is relative, i. e. for each of the two observers O and O' the time scale of the other seems to be prolonged. Why is it then possible to decide unambiguously that A and not B is younger after his return?

The essential point is that A is not strictly in an inertial system, even if he moves with constant velocity, because at his returning point he changes the system from one that moves with the velocity $+v$ into one that moves with $-v$ against B. This shows that the measurements of A and B are not equivalent.

In order to simplify the discussion we will categorize the journey of A into three sections, which are illustrated in the Minkowski diagram of the resting system S of B in Fig. 3.30.

- A starts his journey from $x = 0$ at time $t = t' = 0$, reaches in a negligibly small time interval his final speed v until he arrives at his point of return $P_1(x_r, T/2)$ after the time $t_1 = T/2$.
- At time $t_1 = T/2$ he decelerates to $v = 0$ and accelerates again to $-v$. This should all happen within a time interval which is negligibly small compared with the travel time T.
- Astronaut A flies with $v_2 = -v$ back to B and reaches B in $x = 0$ at $t_2 = T$.

While the world line of B in Fig. 3.30 is the vertical line $x = 0$, A follows the line $x = v \cdot t \rightarrow ct = (c/v)x$ until the point of return P_1 from where he travels on the line $x = x_r - v(t - T/2) \Rightarrow ct = (c/v)(x_r - x) + cT/2$, until the point $P_2(0, T)$ where he meets with B.

From (3.29) we obtain for the invariant

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 = c^2\mathrm{d}t'^2 - \mathrm{d}x'^2 \,.$$

This yields the different travel times for A and B: For B is always $\mathrm{d}x = 0$. We therefore get for the total distance s in the Minkowski diagram:

$$s = \int \mathrm{d}s = c \cdot \int \mathrm{d}t = c \cdot T \,.$$

For the moving astronaut A the resting observer B measures on the way OP_1: $\mathrm{d}x = v \cdot \mathrm{d}t \rightarrow \mathrm{d}s^2 = c^2\mathrm{d}t^2 - v^2\mathrm{d}t^2$, which gives for the total path

$$\int \mathrm{d}s = \sqrt{c_2 - v^2} \int \mathrm{d}t = \frac{c \cdot T}{2\gamma} = \frac{cT'}{2} \,,$$

and on the way P_1P_2 back: $\mathrm{d}x = -v\mathrm{d}t$:

$$\int \mathrm{d}s = \sqrt{c^2 - v^2} \int \mathrm{d}t = \frac{c \cdot T}{2\gamma} = \frac{cT'}{2} \,.$$

The total travel time measured by B in S for the system S' of his twin A is then $T' = T/\gamma < T$. This result can be also explained by the Lorentz contraction: For A is the path L shortened by the factor γ. Therefore the travel time T for A is shorter by the factor γ since A as well as B measure the same velocity v of A relative to B.

The asymmetry of the problem can be well illustrated by regarding light pulses sent at constant intervals by A to B and by B to A. Both the observer B and the astronaut A send these light signals at the frequency f_0 measured with their clocks. The sum of the sent pulses at a frequency $f_0 = 1/s$ gives the total travel time in seconds (Fig. 3.31).

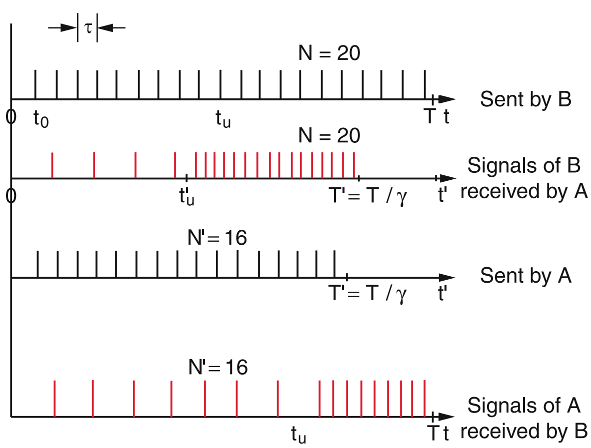

Figure 3.31 Illustration of the twin paradox, using the signals sent and received by A and B

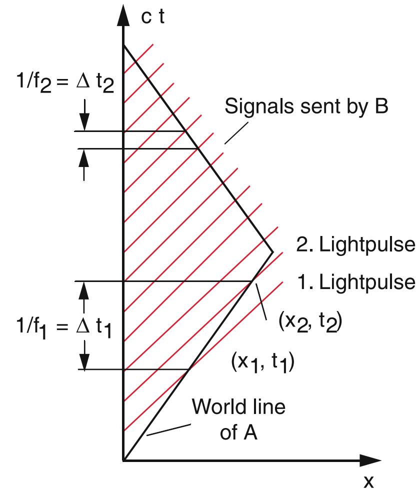

Figure 3.32 Doppler-effect of the signal frequency illustrated in the Minkowski diagram

While A moves away from B both observers receive the pulses at a lower frequncy f_1 because each successive pulse has to travel a longer way than the preceding pulse. The asymmetry occurs at the turning point P_1. While A on his way back now receives the pulses with a higher frequency f_2 directly after he turns around, B receives the pulses from A with the frequency f_2 only at times $\Delta t \le c \cdot x_0$ after the return time. He receives the same total number N' of pulses as has been sent by A but he receives for a longer time signals with lower frequency than A. Therefore he measures a longer travel time for A than A himself.

This is illustrated in Fig. 3.31 for $v = 0.6c$. B sends during the travel time of A altogether 20 pulses, which are all received by A. While B sends his signals at constant time intervals τ_0, A receives them on the outbound trip with larger time intervals τ_1 i.e. lower frequency and on the way back with shorter time intervals τ_2, i.e. higher frequency. Measured in the system S' of A the travel time T' is shorter by the factor γ. The astronaut A sends during this time only $N' = N/\gamma = 16$ pulses which are all received by B. Since B receives the signal sent by A at the return time only delayed, he receives signals with the larger interval τ_1 (lower frequency) for a longer time and only after the time $t'_r + L/2c$ the signals with the shorter intervals τ_2.

This is further illustrated by the Minkowski diagram of Fig. 3.32 which explains the relativistic Doppler-effect. The astronaut A is at time $t = 0$ at the point $(x = 0, t = 0)$ in the Minkowski diagram. He moves, measured by B on the line

$$x = v \cdot t \rightarrow c \cdot t = \frac{c}{v}x \,.$$

The observer B sitting always at $x = 0$ sends light pulses at a repetition frequency f_0. A pulse sent by B at time t_0 travels in the (x, ct)-diagram on lines with a 45° slope which intersects the world line of A in the point (x_1, t_1), where it is received by A. The next pulse is sent by B at time $t = t_0 + \tau = t_0 + 1/f_0$ and reaches A at (x_2, t_2). According to Fig. 3.31 the following

Table 3.1 Measurement of multiple physical quantities of resident and traveler (according to [3.7])

Physical quantity	Measurement of B (resident)	Measurement of A (traveler)
Total travel time	$T = \frac{2L}{v}$	$T' = \frac{2L}{\gamma v}$
Total number of sent signals	$f \cdot T = \frac{2fL}{v}$	$f \cdot T' = \frac{2fL}{\gamma v}$
Reversal time of A	$t_u = \frac{L}{v} + \frac{L}{c} = \frac{L}{v}(1+\beta)$	$t'_u = \frac{L}{\gamma v}$
Number of received signals with frequency $f'\left(f' = f \cdot \left(\frac{1-\beta}{1+\beta}\right)^{1/2}\right)$	$f' t_u = f \cdot \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \cdot \frac{L}{v}(1+\beta)$ $= \frac{fL}{v}(1-\beta^2)^{1/2}$	$f' t'_u = f \cdot \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \cdot \frac{L}{v}(1-\beta^2)^{1/2}$ $= \frac{fL}{v}(1-\beta)$
Travel time after reversal	$t_2 = \frac{L}{v} - \frac{L}{c} = \frac{L}{v}(1-\beta)$	$t'_2 = \frac{L}{\gamma v} = \frac{L}{v}\frac{1}{(1-\beta^2)^{1/2}}$
Number of received signals with frequency $f'' = f \cdot \left(\frac{1+\beta}{1-\beta}\right)^{1/2}$	$f'' t_2 = f \cdot \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \cdot \frac{L}{v}(1-\beta)$ $= \frac{fL}{v}(1-\beta^2)^{1/2}$	$f'' t'_2 = f \cdot \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \cdot (1-\beta^2)^{1/2}$ $= \frac{fL}{v}(1+\beta)$
Total number of received signals $N = f' t_u + f'' t_2, \quad N' = f' t'_u + f'' t'_2$	$N = f' t_u + f'' t_2 = \frac{2fL}{v}(1-\beta^2)^{1/2}$ $= \frac{2fL}{\gamma \cdot v}$	$N' = f' t'_u + f'' t'_2 = \frac{2fL}{v}$
Conclusion regarding the time measured by the other	$T' = \frac{2L}{\gamma v}$ $\beta = v/c,\ \gamma = (1-\beta^2)^{-1/2}$	$T = \frac{2L}{v}$

relations apply:

$$x_1 = c \cdot (t_1 - t_0) = x_0 + v \cdot t_1$$
$$x_2 = c \cdot (t_2 - t_0 - \tau) = x_0 + v \cdot t_2 \,.$$

Subtraction of the first from the second equation yields

$$t_2 - t_1 = \frac{c \cdot \tau}{c - v}; \qquad x_2 - x_1 = \frac{v \cdot c \cdot \tau}{c - v} \,.$$

Figure 3.32 illustrates that for A the time intervals τ' are longer on the outward flight than on the return flight. Astronaut A measures in his system S' according to the Lorentz transformations

$$\tau' = t'_2 - t'_1 = \gamma \cdot \left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]$$
$$= \gamma \cdot (1+\beta) \cdot \tau \,, \quad \text{with } \beta = v/c \,.$$

With $\gamma = (1-\beta^2)^{-1/2}$ this becomes

$$\tau' = \tau \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \Rightarrow f' = \frac{1}{\tau'} = f_0 \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \,.$$

Astronaut A measures therefore on the outward flight the smaller repetition frequency f_1 which is smaller than f_0 by the factor $[(1-\beta)/(1+\beta)]^{1/2}$ and on the return flight with the velocity $-v$ he measures the higher repetition rate $f_2 = [(1+\beta)/(1-\beta)]^{1/2} f_0$.

In Tab. 3.1 the different measurements of A and B are summarized. The table shows again, that the total number of pulses sent by B is equal to the number received by A but different from the number sent by A. The last line in Tab. 3.1 makes clear, that B can conclude the travel time measured by A from the number of pulses received from A and vice versa can A conclude the time measured by B. **Both observers are therefore in complete agreement in spite of the different times measured in their systems. This shows that there are no contradictions in the description of the twin paradox.** Observer B knows, that the travel time T' measured by A is shorter than the time measured by himself because A is sitting in a moving system, and A knows that B measures in his resting system a longer time.

3.6.7 Space-time Events and Causality

Since the speed of light is the upper limit for all velocities with which signals can be transmitted from one space-time point (x_1, t_1) to another point (x_2, t_2) all space-time events can be classified into those which can be connected by signals and those which cannot. In the first case an event in (x_2, t_2) can be caused by an event in (x_1, t_1).

In the Minkowski diagram of Fig. 3.33 the two diagonal lines $x = \pm c \cdot t$ are the worldliness of light signals passing through

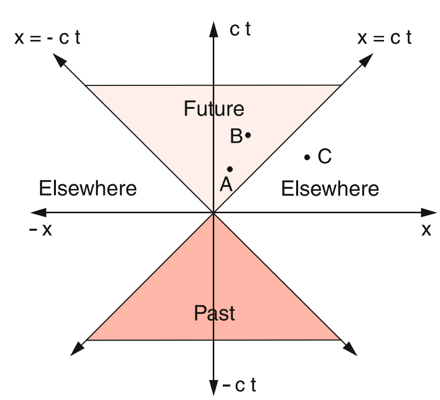

Figure 3.33 Two-dimensional Minkowski diagram with the shaded areas for past and future and the white areas for non-accessible space-time point $(\boldsymbol{r}, t)$

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the point ($x = 0, t = 0$). These world lines divided the space-time into different regions: All regions with ($x, t > 0$) with $|x|| \leq ct$ represent *the future* seen from ($x = 0, t = 0$). They can be reached by signals sent from $(0, 0)$, while the region with ($x, t < 0$) form the *past*.

This can be also expressed in the following way: All events in space-time point (x, t) can be causally connected with each other, i. e. an event in (x_2, t_2) can be caused by an event in (x_1, t_1) if both points lie in the red shaded regions in Fig. 3.33. this means signals can be transferred between these points and interactions between bodies in these points are possible. For instance the event A can influence the event B in Fig. 3.33 but not the event C.

An observer in the space-time point (x, t) with $|x| \leq |ct|$ can never receive a signal from points in the white regions with $|x| > |ct|$. We call these regions therefore "elsewhere".

In a three-dimensional space-time diagram (x, y, ct) the surfaces $x^2 + y^2 = c^2t^2$ form a cone called the light-cone. Past and future are inside the cone. "elsewhere" is outside. In a four-dimensional space-time diagram (x, y, z, ct) this light cone becomes a hyper-surface.

Very well written introductions to the special relativity and its consequences without excessive Mathematics, which are also understandable to undergraduate students can be found in [3.7–3.11].

Summary

- For the description of motions one needs a coordinate system. Coordinate systems in which the Newtonian Laws can be formulated in the form, discussed in Sect. 2.6 arte called inertial systems. Each coordinate system which moves with constant velocity v against another inertial system is also an inertial system.
- The transformation of coordinates (x, y, z), of time t and of velocity v and therefore also of the equation of motion from one to another inertial system is described by the Lorentz transformations. They are based on the constancy of the speed of light c, confirmed by experiments, which is independent of the chosen inertial system and has the same value in all inertial systems. For small velocities $v \ll c$ the Lorentz transformations approach the classical Galilei transformations.
- The description of motions in accelerated systems demand additional accelerations, which are caused by "inertial or virtual" forces. In a rotating system with constant angular velocity these are the Coriolis force $\boldsymbol{F}_C = 2\,\mathrm{m}(\boldsymbol{v}' \times \boldsymbol{\omega})$ which depends on the velocity v' of a body relative to the rotating system, and the centrifugal force $\boldsymbol{F}_{cf} = m \cdot \boldsymbol{\omega} \times (\boldsymbol{r} \times \boldsymbol{\omega})$ which is independent of v'.
- The theory of special relativity is based on the Lorentz transformations and discusses the physical effects following from these equations when the motion of a body is described in two different inertial systems which move against each other with constant velocity v. An essential point is the correct definition of simultaneity of two events. Many statements of special relativity can be illustrated by space-time diagrams (x, ct) (Minkowski diagrams), as for instance the length-contraction or the time-dilatation. Such diagrams show that these effects are relative and symmetric, which means that each observers measures the lengths in a system moving against his system contracted and the time prolonged. The description of the two observers O and O' are different but consistent. There is no contradiction.
- For the twin-paradox an asymmetry occurs, because the astronaut A changes its inertial system at the point of return. It is therefore possible to attribute the time dilation unambiguously to one of the observers.
- The statements of special relativity have been fully confirmed by numerous experiments.

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Problems

3.1 An elevator with a cabin heights of 2.50 m is accelerated with constant acceleration $a = -1\,\mathrm{m/s^2}$ starting with $v = 0$ at $t = 0$. After 3 s a ball is released from the ceiling.
a) At which time reaches it the bottom of the cabin?
b) Which distance in the resting system of the elevator well has the ball passed?
c) Which velocity has the ball at the time of the bounce with the bottom in the system of the cabin and in the system of the elevator well?

3.2 From a point A on the earth equator a bullet is shot in horizontal direction with the velocity $v = 200\,\mathrm{m/s}$.
a) in the north direction
b) in the north-east direction 45° against the equator
c) In the north-west direction 135° against the equator
What are the trajectories in the three cases described in the system of the rotating earth?

3.3 A ball hanging on a 10 m long string is deflected from its vertical position and rotates around the vertical axis with $\omega = 2\pi \cdot 0.2\,\mathrm{s}^{-1}$. What is the angle of the string against the vertical and what is the velocity v of the ball?

3.4 In the edge region of a typhoon over Japan (geographical latitude $\varphi = 40°$) the horizontally circulating air has a velocity of 120 km/h. What is the radius of curvature r of the path of the air in this region?

3.5 A fast train ($m = 3 \cdot 10^6\,\mathrm{kg}$) drives from Cologne to Basel with a velocity of $v = 200\,\mathrm{km/h}$ exactly in north-south direction passing 48° latitude. How large is the Coriolis force acting on the rail? Into which direction is it acting?

3.6 A body with mass $m = 5\,\mathrm{kg}$ is connected to a string with $L = 1\,\mathrm{m}$ and rotates
a) in a horizontal plane around a vertical axis
b) in a vertical plane around a horizontal axis
At which angular velocity breaks the string in the cases a) and b) when the maximum tension force of the string is 1000 N?

3.7 A plane disc rotates with a constant angular velocity $\omega = 2\pi \cdot 10\,\mathrm{s}^{-1}$ around an axis through the centre of the disc perpendicular to the disc plane. At time $t = 0$ a ball is launched with the velocity $v = \{v_r, v_\varphi\}$ with $v_r = 10\,\mathrm{m/s}$, $v_\varphi = 5\,\mathrm{m/s}$ (measured in the resting system) starting from the point A ($r = 0.1\,\mathrm{m}, \varphi = 0°$). At which point (r, φ) does the ball reach the edge of the disc?

3.8 A bullet with mass $m = 1\,\mathrm{kg}$ is shot with the velocity $v = 7\,\mathrm{km/s}$ from a point A on the earth surface with the geographical latitude $\varphi = 45°$ into the east direction. How large are centrifugal and Coriolis force directly after the launch? At which latitude is its impact?

3.9 Two inertial systems S and S' move against each other with the velocity $v = v_x = c/3$. A body A moves in the system S with the velocity $\boldsymbol{u} = \{u_x = 0.5c, u_y = 0.1c, u_z = 0\}$. What is the velocity vector $\boldsymbol{u}'$ in the system S' when using
a) the Galilei transformations and
b) the Lorentz transformations?
How large is the error of a) compared to b)?

3.10 A meter scale moves with the velocity $v = 2.8 \cdot 10^8\,\mathrm{m/s}$ passing an observer B at rest. Which length is B measuring?

3.11 A space ship flies with constant velocity v to the planet Neptune and reaches Neptune at its closest approach to earth. How large must be the velocity v if the travel time, measured by the astronaut is 1 day? How long is then the travel time measured by an observer on earth?

3.12 Light pulses are sent simultaneously from the two endpoints A and B of a rod at rest. Where should an observer O sit in order to receive the pulses simultaneously? Is the answer different when A, B and O moves with the constant velocity v? At which point in the system S an observer O' moving with a velocity v_x against S receives the pulses simultaneously if he knows that the pulses has been sent in the system S simultaneously from A and B?

3.13 At January 1st 2010 the astronaut A starts with the constant velocity $v = 0.8c$ to our next star α-Centauri, with a distance of 4 light years from earth. After arriving at the star, A immediately returns and flies back with $v = 0.8c$ and reaches the earth according to the measurement of B on earth at the 1st of January 2020. A and B had agreed to send a signal on each New Year's Day. Show that B sends 10 signals, but A only 6. How many signals does A receive on his outbound trip and how many on his return trip?

3.14 Astronaut A starts at $t = 0$ his trip to the star Sirius (distance 8.61 light years) with the velocity $v_1 = 0.8c$. One year later B starts with the velocity $v_2 = 0.9c$ to the same star. At which time does B overtake A, measured
a) in the system of A,
b) of B and
c) of an observer C who stayed at home?
At which distance from C measured in the system of C does this occur?

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