Solutions of the Problems

14

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14.1 Chapter 1

1.1 180 km/h.

1.2 The length measurement can be performed in different ways:
For example the period of the earth's rotation depends on the moment of inertia and is therefore proportional to R^{-2}. Independent measurements of length and time can decide the question.

1.3 This is one, but not the only requirement for a length standard. Similarly important is the accuracy of the comparison between the length to be measured and the standard.

1.4 The length T of a day increases per year by 10^{-4} s. The relative prolongation of a day per year is then

$$dT/T = a = \frac{10^{-4}}{24 \cdot 3600} = 1.1 \cdot 10^{-9} .$$

a) Since the length of a day increases per year by 10^{-4} s, it is after 10^4 years longer by 1 s.
b) How often must a leap second be inserted?
The length of a day increases per day by $\delta t = 10^{-4}\,\text{s}/365\,\text{d} = 2.74 \cdot 10^{-8}$ s/d. The total time delay after x days is

$$\Delta t = \delta t \int_0^x n\,\mathrm{d}n = \frac{1}{2}\delta t \cdot n^2\Big|_0^x$$
$$= \tfrac{1}{2}x^2\delta t = 1\,\text{s} \to x^2 = \frac{2}{\delta t}$$
$$\to x = 8600\,\text{d} = 23.5\,\text{years} .$$

The distance $x_{n+1} - x_n = (2/aT_0)^{1/2} \cdot (\sqrt{n+1} - \sqrt{n})$ between two leap times becomes shorter and shorter and reaches for $n \gg 1$ the value $1/(aT_0)^{1/2}/\sqrt{2n}$.

1.5 1 light year $= 9.46 \cdot 10^{15}\,\text{m} \Rightarrow T = 4.5$ years. The distance is 1.39 parsec, the angle $1/1.39 = 0.7''$.

1.6 $L = 2 \cdot 10^3 \cdot \tan(\alpha/2) = 17.45\,\text{m}$ for $\alpha = 1°$. For $\alpha = 1° \pm 1' \Rightarrow L = 17.45 \pm 0.29\,\text{m}$.

1.7 Since the orbital speed of the earth varies during one revolution, the time between two culminations of the sun also varies (see Fig. 1.22 and 1.23). Further reasons are variations of the mass distribution inside the earth, which changes the moment of inertia, caused by magma flow, earth quakes and melting of glaciers.

1.8 The mass of a hydrogen atom is $m_H = 1.673 \cdot 10^{-27}\,\text{kg} \Rightarrow N = 5.98 \cdot 10^{26}$ /kg.

1.9 The mass of a H_2O-molecules is $m_{H_2O} = 3.0 \cdot 10^{-26}$ kg; $\varrho_{H_2O} = 1$ kg/litre; $\Rightarrow N = 3.0 \cdot 10^{25}$ /litre.

1.10 The mass of the uranium nucleus is $m(^{238}\text{U}) = 1.661 \cdot 238 \cdot 10^{-27}$ kg. Its density is then $\varrho = m/(4 \cdot \frac{1}{3}\pi r^3) = 1.4 \cdot 10^{17}\,\text{kg/m}^3$.

1.11 From $s = 1/2gt^2$ it follows for the falling time $t = \sqrt{2s/g} = 0.45$ s.

$$\sigma_m = \left[\frac{\sum(\bar{x} - x_i)^2}{n(n-1)}\right]^{1/2} = \left[\frac{40 \cdot 0.01}{40 \cdot 39}\right]^{1/2} \text{s}$$
$$= 1.6 \cdot 10^{-2}\,\text{s}$$
$$\Rightarrow \sigma_m/\bar{x} = \frac{1.6 \cdot 10^{-2}}{0.45} = 3.5\% .$$

1.12 a) $e^{-x^2/2} = 0.5 \Rightarrow x^2 = 2\ln 2 \Rightarrow x = \sqrt{2\ln 2} \approx 1.177$;
b) $e^{-x^2/2} = 0.1 \Rightarrow x^2 = 2\ln 10 \Rightarrow x = \sqrt{2\ln 10} \approx 2.156$.

1.13 $A = x - y^2 \Rightarrow \partial A/\partial x = 1$ and $\partial A/\partial y = -2y$

$$\sigma_A = \left[(1000 \cdot 10^{-3} \cdot 1)^2 + (30 \cdot 3 \cdot 10^{-3} \cdot 60)^2\right]^{1/2}$$
$$= [1 + 29]^{1/2} \approx 5.5 .$$

1.14 Quartz clock $\Delta T_{max} = 10^{-9} \cdot 3.16 \cdot 10^7\,\text{s} \approx 0.03\,\text{s} = 30$ ms.
Atomic Clock: $\Delta T_{max} = 0.3\,\mu\text{s}$.

1.15 For the five points we obtain from (1.35) with $n = 5$:

$$a = \frac{5 \cdot \sum x_i y_i - \sum x_i \sum y_i}{5 \cdot (\sum x_i^2) - (\sum x_i)^2}$$
$$= \frac{5 \cdot (3 + 6 + 20 + 25) - 12 \cdot 18}{5 \cdot (1 + 4 + 16 + 25) - 12^2} = 0.628 ,$$

$$b = \frac{(\sum x_i^2) \cdot (\sum y_i) - (\sum x_i) \cdot (\sum x_i y_i)}{5 \cdot (\sum x_i^2) - (\sum x_i)^2}$$
$$= \frac{828 - 684}{230 - 144} = 2.093$$
$$\Rightarrow y = 0.628x + 2.093 ;$$
$$\sigma_y = \sqrt{\frac{0.430}{n-2}} = 0.38 .$$

Note, that here $(n-2)$ instead of $(n-1)$ has to be used, because already two values are determined by the equation $y = ax + b$.

$$\sigma_a^2 = \frac{5 \cdot \sigma_y^2}{86} = 0.006 \Rightarrow \sigma_a = 0.077 ,$$
$$\sigma_b^2 = \frac{\sigma_y^2 \cdot \sum x_i^2}{86} = \frac{0.102 \cdot (1 + 4 + 16 + 25)}{86} = 0.055$$
$$\Rightarrow \sigma_b = 0.23 .$$

14.2 Chapter 2

2.1 a) The acceleration time t_1 can be obtained from

$$v_1 = v_0 + at_1$$
$$\Rightarrow t_1 = \frac{v_1 - v_0}{a} = \frac{(100 - 80)\,\text{m/s}}{3.6 \cdot a\,\text{m/s}^2} = 4.27\,\text{s} .$$

b) The time t_2 from the end of the acceleration period until the end of overtaking is obtained from the equation for the distance

$$s = v_0 t_1 + \tfrac{1}{2} a t_1^2 + v_1 t_2 \; .$$

This distance s is

$$s = [v_0(t_1 + t_2) + (40 + 25 + 40)]\,\mathrm{m} \; .$$

where the first term gives the distance, the truck has passed during the overtaking time. The comparison yields

$$v_0 t_1 + \tfrac{1}{2} a t_1^2 + v_1 t_2 = v_0(t_1 + t_2) + 105 \; .$$

With $t_1 = 4.27\,\mathrm{s} \Rightarrow t_2 = 16.77\,\mathrm{s}$.
c) The total overtaking time is then $t = t_1 + t_2 \approx 21\,\mathrm{s}$ and the total overtaking distance 570.6 m.
The overtaking would have been therefore not successful but lethal and should not have been tried!

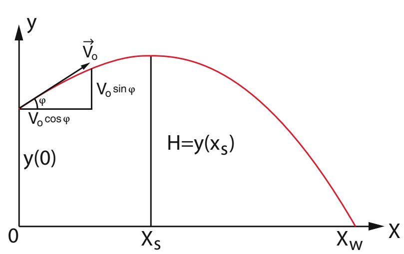
2.2 The driving times are: $t_1 = \frac{x}{2v_1}$; $t_2 = \frac{x}{2v_2}$.
The total driving time is $t = t_1 + t_2$.
The average velocity is

$$\langle v \rangle = \frac{x}{t_1 + t_2} = \frac{2 v_1 v_2}{v_1 + v_2} = \frac{2 \cdot 40 \cdot 80}{120} = 53.33\,\mathrm{km/h} \; .$$

2.3 From $s = v_0 t + \frac{1}{2} a t^2$ one obtains: $a = -1\,\mathrm{cm/s^2}$.

2.4 From $v = v_0 + at \Rightarrow t = (v - v_0)/a$. Inserting into $s = v_0 t + \frac{1}{2} a t^2 = 0.04\,\mathrm{m}$ gives $v_0 = 5 \cdot 10^6\,\mathrm{m/s}$.

2.5 From $s = h + v_0 t - \frac{1}{2} g t^2 = 0$ it follows:
a) $t_1(v_0 = 5\,\mathrm{m/s}) = 2.3\,\mathrm{s}$,
b) $t_2(v_0 = -5\,\mathrm{m/s}) = 1.3\,\mathrm{s}$,
c) derivation of Eq. 2.13 (see figure)



$v_x = v_0 \cdot \cos\varphi$, $v_y = v_0 \cdot \sin\varphi$, t_s = rise time, t_f = fall time, H = rise height

$$y(t) = -\frac{g}{2} t^2 + v_0 \sin\varphi + h \; ,$$

$$g \cdot t_s = v_0 \sin\varphi \Rightarrow t_s = \frac{v_0 \sin\varphi}{g} \; ,$$

$$H = y(t_s) \; .$$

For t_f the equation holds (free fall)

$$H = \frac{1}{2} g t_f^2 \Rightarrow t_f = \sqrt{\frac{2H}{g}} \; ,$$

$$x_w = v_x \cdot (t_s + t_f) = v_0 \cdot \cos\varphi \left[\frac{v_0 \cdot \sin\varphi}{g} + \sqrt{\frac{2H}{g}} \right] \; ,$$

$$H = -\frac{g}{2} t_s^2 + v_0 \cdot \sin\varphi + h \; .$$

Inserting H into x_w yields

$$x_w = \frac{v_0^2}{g} \cdot \cos\varphi \left[\sin\varphi + \sqrt{\frac{2gh}{v_0^2} + \sin^2\varphi} \right] \; .$$

2.6 If the vector of acceleration is not parallel to the tangent on the trajectory the result is a curved trajectory and no straight line.

2.7 $v = gt$;

$$s = \frac{1}{2} g t^2 = \frac{1}{2} v^2 / g = \frac{1}{2} \left(\frac{100}{3.6} \right)^2 \frac{1}{9.81}\,\mathrm{m} = 39.3\,\mathrm{m} \; .$$

2.8 a) If ω is constant the acceleration $a = \omega^2 \cdot R$ is also constant. This demands an additional tangential acceleration, that compensates in each point of the trajectory the tangential component $g \cdot \cos\alpha$ of the gravity acceleration, where α is the angle between the vertical direction and the tangent to the trajectory.
b) Velocity in point C: $v = \sqrt{2gh}$; in the point B: $v = \sqrt{2g(h - 2R)}$.
From the condition $(v^2/R) > g \Rightarrow v(R) > \sqrt{R \cdot g} \Rightarrow h > \frac{5}{2} R$; $v_{\min}(B) = \sqrt{g \cdot R}$.

2.9 a) Potential energy of the moon

$$E_p = -G \frac{M_{Mo} M_E}{r} \; .$$

Kinetic energy:

$$E_{kin} = \tfrac{1}{2} M_{Mo} r^2 \omega^2 \; .$$

From $M_{Mo} \omega^2 \cdot r = G \cdot M_{Mo} \cdot M_E / r^2 \Rightarrow \omega_{Mo} = (G \cdot M_E / r^3)^{1/2}$.
The condition $E_p + E_{kin} = E > 0$ yields

$$\omega^2 > \frac{2G \cdot M_E}{r^3} = 2\omega_{Mo}^2 \; .$$

The velocity of the moon at the same radius r must be enlarged by the factor $\sqrt{2}$.
b) $v > (2G \cdot M_{Mo}/r_{Mo})^{1/2}$ gives with $M_{Mo} = 7.36 \cdot 10^{22}\,\mathrm{kg} \Rightarrow v = 2.38\,\mathrm{km/s}$.

2.10 Velocity of the starting point at the equator:

$$v_0 = \pm \frac{4 \cdot 10^7}{24 \cdot 3600}\,\mathrm{m/s} = \pm 4.6 \cdot 10^2\,\mathrm{m/s} \; .$$

Rocket equation (neglecting the vertical acceleration if $g \cdot T \ll v_0$):

$$v = v_0 + v_e \ln(m_0/m) .$$

With $m_0 = m + m_x$ (m_x = fuel mass) it follows:

$$\ln\left(1 + \frac{m_x}{m}\right) = (7.9 \cdot 10^3 \mp 4.6 \cdot 10^2)/(4.5 \cdot 10^3)$$

$\Rightarrow m_x = 2103\,\text{kg}$ for a launch into the east direction and $m_x = 2705\,\text{kg}$ into the west direction, if the mass of the fuel container is neglected.

2.11 $(mv_0^2/2) > \frac{G \cdot m \cdot M_E}{R} = mgR \Rightarrow v_0 > \sqrt{2gR}$.

2.12 Because the velocity of the earth at the launch point is $v_E = 463\,\text{m/s}$. The initial velocity of the rocket is not zero but $v_0 = v_E \cdot \cos 30° = 400\,\text{m/s}$. One wins the initial kinetic energy $\frac{1}{2}mv_0^2$. For a vertical launch one would need the escape velocity $v_E = \sqrt{2gR} = 11{,}200\,\text{m/s}$ (see (2.30)). Therefore one only needs to accelerate from 400 m/s until 11,200 m/s. This requires the kinetic energy $\Delta E_{kin} = \frac{1}{2}m \cdot (v_E^2 - v_0^2)$. The relative energy saving is $(1 - \Delta E/E) = 0.004$. One saves only 0.4%.

2.13 The total force is $F = F_A - m \cdot g = g \cdot \pi r^2 (z \cdot \varrho_w - h \cdot \varrho_H)$. For $z = \frac{2}{3}h \to F = 0$,

$$\Rightarrow \varrho_H = \tfrac{2}{3}\varrho_w .$$

The work is

$$W = \int_{z=2h/3}^{0} F\,dz = \int_{2h/3}^{0} g\pi r^2 (z\varrho_w - h\varrho_H)\,dz$$

$$= \tfrac{2}{9} g\pi r^2 h^2 \varrho_w = 24.7\,\text{J} .$$

Without water twice the work would be necessary.

2.14 $E_{kin}(h_1) = \frac{1}{2}mv^2(h_1) = 200\,\text{N} \cdot \text{m}$,

$$mg(h_2 - h_1) = E_{kin}(h) ,$$

$$\Rightarrow h_2 = \frac{E_{kin}(h_1)}{mg} + h_1 = 35.5\,\text{m} .$$

2.15 $F = -Dx_1 \Rightarrow D = \frac{F_1}{x_1} = 400\,\text{N/m}$.

$$W = \int_0^{l_0} Dx\,dx = \tfrac{1}{2}Dl_0^2 = 128\,\text{N} \cdot \text{m} .$$

2.16 The neutral point between earth and moon, where the opposite gravitational forces just compensate, has the distance r_2 from the earth and r_1 from the moon. From $F = 0$ it follows:

$$\frac{G \cdot M_{Mo}}{r_1^2} = \frac{G \cdot M_E}{r_2^2} \quad \text{and} \quad r = r_1 + r_2$$

$$\Rightarrow r_2 = \frac{r}{1 + (M_{Mo}/M_E)^{1/2}} = \frac{3.84 \cdot 10^8\,\text{m}}{1 + 0.11}$$

$$\approx 3.46 \cdot 10^8\,\text{m} .$$

In order to reach the distance r_1 for the earth the initial kinetic energy must be

$$\frac{1}{2}mv_0^2 \geq G \cdot M_E m \int_R^{r_2} \frac{dr}{r^2} .$$

Since $M_{Mo} = 0.012 M_E$ we can neglect the attraction by the moon at the start of the rocket. It follows:

$$v_0^2 \geq 2G \cdot M_E \left(\frac{1}{R} - \frac{1}{r_2}\right) \approx 0.98\, v_0^2(\infty) .$$

The energy saving is 2% compared with the case where the second escape velocity is required to reach $r = \infty$.

2.17 a) $m\omega^2 r = mG \cdot M_E/r^2 \Rightarrow r^3 = G \cdot M_E/\omega^2$,

$$T = \frac{2\pi}{\omega} = 1\,\text{day} = 24 \cdot 3600\,\text{s}$$

$$\Rightarrow \omega = 7.2 \cdot 10^{-5}\,\text{s}^{-1}$$

$$\Rightarrow r = 4.25 \cdot 10^7\,\text{m} = 42{,}500\,\text{km} .$$

b) The energy of the body in the geostationary orbit compared to a body resting on the earth surface is

$$E = E_{kin} + E_p = \frac{mv^2}{2} + \int_{r=R}^{r_S} \frac{GmM_E}{r^2}\,dr$$

$$= \frac{m\omega^2 r_S^2}{2} + GmM_E\left(\frac{1}{R} - \frac{1}{r_S}\right)$$

with $r_S = 42{,}500\,\text{km}$. It needs the energy supply

$$E = m\left[\tfrac{1}{2} \cdot \omega^2 r_S + gR(1 - R/r_S)\right] .$$

c) In order to get the accuracy 0.1 km/day of its position the upper limit for the accuracy of the angular velocity is

$$\frac{\Delta\omega}{\text{day}} = \frac{0.1}{42{,}500} = 2.4 \cdot 10^{-6}\ \text{per day} .$$

The minimum relative stability has to be $\Delta\omega/\omega \leq 2.4 \cdot 10^{-6}/2\pi = 3.8 \cdot 10^{-7}$

Since $\omega^2 = G \cdot M_E/r^3 \Rightarrow \Delta r/r = -2/3\Delta\omega/\omega \Rightarrow \Delta r \leq 10.6\,\text{m}$.

2.18 $E_p = -G \cdot m \cdot \dfrac{M_E}{r} = -m \cdot g \cdot \dfrac{R^2}{r}$ with R = radius of earth.

$$E_{kin} = +G \cdot m \cdot \frac{M_E}{2r} = -\frac{1}{2}E_p ,$$

$$E = E_p + E_{kin} = -G \cdot m \cdot \frac{M_E}{2r} = -E_{kin} .$$

2.19 $E_p = mgL(1 - \cos\varphi)$; $E_{kin} = \dfrac{mv^2}{2} = \dfrac{m}{2}L^2\dot{\varphi}^2$;

$$E = E_{kin} + E_p .$$

The equation of motion is $m \cdot L \cdot d^2\varphi/\mathrm{d}t^2 = -m \cdot g \cdot \sin\varphi$. Multiplication with $L \cdot d\varphi/\mathrm{d}t$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{2}L^2\dot{\varphi}^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}(m \cdot g \cdot L \cdot \cos\varphi)\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{kin}} = \frac{\mathrm{d}}{\mathrm{d}t}(E - E_{\mathrm{p}})$$
$$\Rightarrow E_{\mathrm{kin}} + E_{\mathrm{p}} = E = \mathrm{const}\,.$$

2.20 $T = 2\pi\sqrt{L/g} \Rightarrow g = 2\pi L/T^2$

$$\Delta g = \left[\left(\frac{\mathrm{d}g}{\mathrm{d}L}\Delta L\right)^2 + \left(\frac{\mathrm{d}g}{\mathrm{d}T}\Delta T\right)^2\right]^{1/2}$$
$$= g\left[\left(\frac{\Delta L}{L}\right)^2 + \left(\frac{2\Delta T}{T}\right)^2\right]^{1/2},$$
$$\frac{\Delta L}{L} = 10^{-5}\,.$$

The uncertainty of the length measurement results in a relative error of time determination

$$\frac{\Delta T_1}{T} = \frac{1}{2}\frac{\Delta L}{L} = 5 \cdot 10^{-6}\,.$$

The uncertainty of time measurement $\Delta T = 10^{-2}$ s gives with $T = 6.34$ s a relative error of

$$\frac{\Delta T_2}{T} = 1.5 \cdot 10^{-3}\,.$$

The uncertainty $2\Delta T_2/nT = 10^{-5}$ which corresponds to the error in the length measurement can be only achieved for $n \cdot T = 2000\,\mathrm{s} \to n \geq 316$.
With this uncertainty the relative error $\Delta g/g = \{2 \cdot 10^{-10}\}^{1/2} \approx 1.4 \cdot 10^{-5} \Rightarrow \Delta g = 1.37 \cdot 10^{-4}\,\mathrm{m/s^2}$.

2.21 From (2.84) one obtains $\Delta G/G = \Delta\varphi/\varphi = 10^{-4} \Rightarrow \Delta\varphi = 10^{-4} \cdot \varphi \propto \varrho \cdot R_2^3/r^2$. Since $r > R_1 + R_2 \approx R_2$ the maximum elongation angle is $\varphi_{\max} \propto R_2$. For a tenfold mass the elongation angle increases only by $10^{1/3} \approx 2.1$. For the angle φ the limitation is $\varphi \leq R_2/L$ (Fig. 2.60). If the measuring uncertainty $\Delta\varphi$, which is due to air turbulence and vibrations of the ground is reduced by a factor of 10, the uncertainty of the value of G is only reduced by a factor $10^{1/3}$.

2.22 According to Kepler's 3rd law the major axis a of the comet trajectory is

$$a = \left[\frac{T^2}{4\pi^2}GM_\odot\right]^{1/3} = 2.68 \cdot 10^{12}\,\mathrm{m}\,.$$

With $r_{\min} = a(1-\varepsilon) = 0.59\,\mathrm{AU} = 0.88 \cdot 10^{11}\,\mathrm{m} \Rightarrow \varepsilon = 1 - r/a = 0.967$.

2.23 The escape velocity is $v_0 = 23.6$ km/s, $g = 11.6\,\mathrm{m/s^2}$.

$$v_0 = \sqrt{2Rg} \Rightarrow R = v_0^2/2g = 2.4 \cdot 10^7\,\mathrm{m}\,.$$

The centripetal acceleration is $a = \omega^2 \cdot R$

$$\Rightarrow \omega = \sqrt{a/R} = 1.12 \cdot 10^{-4}\,\mathrm{s^{-1}}$$
$$\Rightarrow T = \frac{2\pi}{\omega} = 5.71 \cdot 10^4\,\mathrm{s} = 15.8\,\mathrm{h}$$
$$g = G \cdot M/R^2 \Rightarrow M = g \cdot R^2/G$$
$$= 11.6 \cdot 2.4^2 \times 10^{14}/(6.67 \cdot 10^{-11})\,\mathrm{kg}$$
$$= 1 \cdot 10^{26}\,\mathrm{kg}\,.$$

The wanted planet is Neptune.

2.24 The gravitational force between the sun and the earth-moon system causes the accelerated motion of the system around the sun. In order to remove the moon from its orbit around the earth, the difference-acceleration $\Delta a = a_1 - a_2$ between a_1 (sun–moon) and a_2 (sun–earth) must be larger than the acceleration a_3 (earth–moon). A fast estimation shows that this is not the case.

2.25 The pendulum period is $T = 2\pi \cdot \sqrt{L/g_{\mathrm{Mo}}} = (g_{\mathrm{E}}/g_{\mathrm{Mo}})^{1/2} \cdot T_{\mathrm{E}}$. Because $g = G \cdot M/R^2 \Rightarrow (g_{\mathrm{E}}/g_{\mathrm{Mo}})^{1/2}$. $T_{\mathrm{E}} = (R_{\mathrm{Mo}}/R_{\mathrm{E}}) \cdot (M_{\mathrm{E}}/M_{\mathrm{Mo}})^{1/2} = 2.47 \Rightarrow T = 2.47$ s.

2.26 From (2.81) one obtains for the force, that causes the acceleration $F = -a \cdot R$ with $a = G \cdot M_{\mathrm{E}} \cdot m/R_0^3$.
The force is proportional to R and the motion of the body therefore a harmonic oscillation $R = R_0 \cdot \cos(\omega t)$ with $\omega^2 = a/m = G M_{\mathrm{E}}/R_0^3$.
The travel time T_{t} (half the oscillation period)

$$T_{\mathrm{t}} = \frac{\pi}{\omega} = \pi R_0\sqrt{R_0/G \cdot M_{\mathrm{E}}}$$

is exactly as long as that of the satellite flying around half of the earth at a low distance above the surface.

2.27 From $\omega^2 \cdot r = G \cdot M_{\mathrm{E}}/r^2 = g \cdot R^2/r^2$

$$\Rightarrow r = \left[\frac{g \cdot R^2T^2}{4\pi^2}\right]^{1/3} = 3.8 \cdot 10^8\,\mathrm{m}\,.$$

2.28 $M = \frac{4}{3}\pi R^3\varrho \Rightarrow R = [3M/4\pi\varrho]^{1/3} = 5.8 \cdot 10^7\,\mathrm{m}$

$$\Rightarrow g = G \cdot M/R^2 = 11.3\,\mathrm{m/s^2}\,.$$

2.29
$$\frac{\Delta g}{g} = \frac{1/R^2 - 1/(R+h)^2}{1/R^2} = 1 - \frac{R^2}{(R+h)^2} \approx \frac{2h}{R}$$
$$= \frac{320}{6380} = 0.05 = 5\%\,.$$

2.30 The acceleration which the moon causes on the earth due to the gravitational force is $g_{\mathrm{Mo}} = G \cdot M_{\mathrm{Mo}}/r^2 \approx 3.3 \cdot 10^{-6} g_{\mathrm{E}}$. It causes the accelerated motion of the earth around the common centre of mass of the earth-moon-system. For the sun we get: $g_\odot = G \cdot M_\odot/r^2 \approx 5.4 \cdot 10^{-4} g$. However, measurements on earth detect only the difference between the gravitational attraction by the sun acting on the centre of the earth (which is compensated by the centrifugal force of the earth motion around the sun) and the effect on the earth surface. This difference causes the contribution of the sun to the tides on earth (see Chap. 6).

2.31 The distance between the centres of the balls is
a) $d = 0.2(1 - L/R) = 0.2(1 - 1.57 \cdot 10^{-5})$.
b) Due to the gravitational attraction between the balls the balls do not hang exactly vertical but form an angle $\Delta\varphi = G \cdot m/(d^2 \cdot g) \approx 3.4 \cdot 10^{-9}$ against the vertical direction. The distance between the balls changes therefore by $\Delta d = L \cdot \Delta\varphi = 3.4 \cdot 10^{-7}\,\mathrm{m} = 0.34\,\mu\mathrm{m}$.

2.32 $E_{kin} = E - E_p = E + G \cdot M_E M_\odot / r$ with $E =$ const. In the perihelion is $r = r_{min} = a(1 - \varepsilon)$ with the eccentricity $\varepsilon = 0.0167$, in the aphelion is $r = r_{max} = a(1 + \varepsilon)$. The potential energy changes between perihelion and aphelion by

$$\frac{\Delta E_p}{E_p} = \frac{2\varepsilon}{1 - \varepsilon^2} = 0.033 = 3.3\% \ .$$

Since $E_{kin} \approx -\frac{1}{2} E_p$ is $\Delta E_{kin}/E_{kin} = -3.3\%$. Because $\Delta v/v = \frac{1}{2}\Delta E_{kin}/E_{kin} \Rightarrow \Delta v/v \approx 1.65\%$. With $\overline{v} = 2\pi a/T \approx 30\,\mathrm{km/s} \Rightarrow v_{max} = 30.25\,\mathrm{km/s}$ and $v_{min} = 29.75\,\mathrm{km/s}$.

2.33 Conservation of energy demands:

$$\frac{1}{2} m v^2 = E + G\frac{mM_E}{r} \ .$$

With $GM_E = gR^2$ we get

$$v_{max}^2 \ \frac{2E}{m} + \frac{gR^2}{a(1 - \varepsilon)}, \qquad v_{min}^2 \ \frac{2E}{m} + \frac{gR^2}{a(1 + \varepsilon)} \ ,$$

subtraction yields

$$\overline{v}\Delta v = gR^2 \frac{\varepsilon}{a(1 - \varepsilon^2)} \ .$$

The semi-major axis can be obtained from

$$\overline{v}^2/a = gR^2/a^2 \ .$$

The result is $a = 1.1 \cdot 10^7$ m. The solution of the quadratic equation for ε gives

$$\varepsilon = 0.268 \ \Rightarrow \ r_{max} = 13{,}950\,\mathrm{km} \ ; \quad r_{min} = 8050\,\mathrm{km} \ .$$

14.3 Chapter 3

3.1 For the motion of the ball relative to the elevator we get
a) $s = 2.50\,\mathrm{m} = \frac{1}{2}(g - a)t^2 = \frac{1}{2} \cdot 8.81 t^2 \Rightarrow t = \sqrt{5/8.81}\,\mathrm{s} \approx 0.75\,\mathrm{s}$ after the release at $t = 0$.
b) The fall distance in the lift shaft is

$$s_2 = \left[\tfrac{1}{2}a(t + t_0)^2 + s\right] = 9.35\,\mathrm{m} \quad \text{with } t_0 = 3\,\mathrm{s} \ . \tag{14.1a}$$

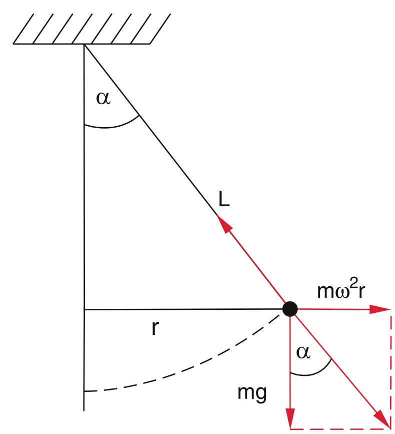
In the coordinate system at rest one obtains:

$$s = \tfrac{1}{2}gt^2 + v_0 t = 2.5\,\mathrm{m} + \tfrac{1}{2}at^2 + v_0 t \ , \tag{14.1b}$$

where $v_0 = 3\,\mathrm{m/s}$ is the velocity of the lift at the time of the release $t = 0$. The result is, of course, identical with that obtained in the moving system.
c) In the rest system the ball has the velocity $v = v_0 + g \cdot t = (3a + 9.81 \cdot 0.7)\,\mathrm{m/s} = 9.87\,\mathrm{m/s}$. In the system of the moving lift, the ball has the impact velocity $v_2 = (g – a)t = 6.17\,\mathrm{m/s}$.

3.2 a) When launching into the north direction is $\boldsymbol{v} \parallel \boldsymbol{\omega} \Rightarrow a_c = 0$. The trajectory of the rocket is along a circle of longitude (meridian).
b) When launching into the north-east direction (45° against the equator) the magnitude of the Coriolis acceleration is $|\boldsymbol{a}_c| = 2v' \cdot \omega \cdot \sin 45° = v'\omega \cdot \sqrt{2} = 4.2 \cdot 10^{-2}\,\mathrm{m/s^2}$. The acceleration $\boldsymbol{a}_c$ points into the radial direction away from the centre of the earth. The effective acceleration is the difference between $\boldsymbol{g}$ and $\boldsymbol{a}_c$. The bullet flies on a slightly upwards curved trajectory in the north-east direction.
c) When $\boldsymbol{v}$ points into the north-west direction, a_c is pointing radially downwards towards the centre of the earth. The two accelerations $\boldsymbol{g}$ and $\boldsymbol{a}_c$ are parallel and must be added. The trajectory if curved downwards.

3.3 $\tan\alpha = \omega^2 r/g$; $r/L = \sin\alpha = \omega^2 r/\sqrt{\omega^4 r^2 + g^2} \Rightarrow r = \sqrt{L^2 - g^2/\omega^4} = 7.836\,\mathrm{m} \Rightarrow \sin\alpha = 0.7836 \Rightarrow \alpha = 51.6°$; $v = \omega r = 9.85\,\mathrm{m/s}$.



3.4 The vertical component of the angular velocity of the rotating earth is $\omega_s = \omega \cdot \sin\varphi \approx 4.7 \cdot 10^{-5}\,\mathrm{s^{-1}}$. The Coriolis acceleration $a_c = 2\omega \cdot v \cdot \sin\varphi = 9.4 \cdot 10^{-5} \cdot 33.3\,\mathrm{m/s^2}$ points into the horizontal direction. It causes the curvature of the air flow which would stream radially into the centre of the deep pressure region for a non-rotating earth. For the radius r of curvature one obtains from $a_c = v^2/r \Rightarrow r = v^2/a_c = 3.5 \cdot 10^5\,\mathrm{m} = 350\,\mathrm{km}$.

3.5 $F_c = m \cdot a_c = 2m \cdot \omega \cdot v \cdot \sin\varphi = 1.8 \cdot 10^4\,\mathrm{N}$. The Coriolis force is directed toward west.

3.6 The centrifugal force is a) for a horizontal motion

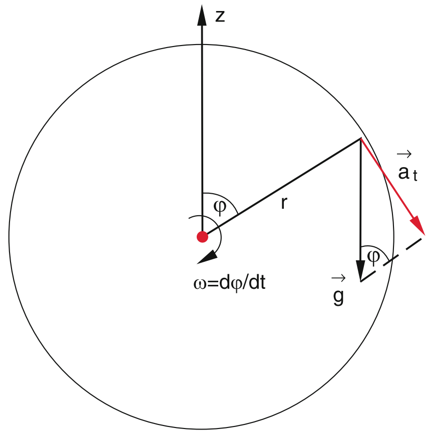
$$F_{cf} = m\omega^2 r \Rightarrow \omega = (F/mr)^{1/2}$$

$$\omega_{max} = (1000/5)^{1/2}\,\mathrm{s^{-1}} = 14.14\,\mathrm{s^{-1}}$$

$$\Rightarrow \nu_{max} = 2.25\,\mathrm{s^{-1}} \ .$$

For constant ω the force on the string is constant.
b) If the body rotates around a horizontal axis in the gravity field of the earth ω is not constant. The tangential acceleration is

$$a_t = g \cdot \sin\varphi(t) \ ,$$



where φ is the angle between the radius vector $\boldsymbol{r}$ and the vertical z-axis. The angular velocity is then

$$\omega = \omega_0 + (g/r) \int_0^{\varphi} \sin\varphi(t)\,dt\,,$$

where $\omega_0 = \omega(\varphi = 0)$ is the velocity at the upper point of the circle. The maximum value of ω is reached for $\varphi = \pi$ at the lowest point of the circle. The following relations hold:

$$\begin{aligned}\omega_{max} &= \omega_0 + \left(\frac{g}{r}\right) \int_0^{\pi} \sin\varphi(t)dt \\ &= \omega_0 + \left(\frac{g}{r}\right) \frac{(\cos\varphi)}{\dot\varphi}\bigg|_{\pi}^{0}\,.\end{aligned} \tag{14.2}$$

With $d\varphi/dt = \omega$ we obtain

$$\begin{aligned}&\to \omega_{max} = \omega_0 + \frac{g}{(r\cdot\omega_0)} + \frac{g}{(r\cdot\omega_{max})}\,, \\ &\to \omega_{max} = \frac{1}{2r}\left(r\cdot\omega_0 + \frac{g}{\omega_0}\right) \\ &\quad + \frac{1}{2}\sqrt{\omega_0^2 + \frac{6g}{N} + \frac{g^2}{N^2\omega_0^2}}\,.\end{aligned} \tag{14.3}$$

For ω_{min} the plus sign before the square root must be replaced by a minus sign.
The maximum force onto the string occurs at the lower point $\varrho\varphi = \pi$.
The condition that the string does not break gives the relation

$$\begin{aligned}&F = m\omega_{max}^2 \cdot r + m\cdot g \le 1000\,\mathrm{N} \\ &\to \omega_{max} \le \left[\frac{(1000/5 - g)}{r}\right]^{1/2} = 13.8\,\mathrm{s}^{-1}\,.\end{aligned}$$

The maximum allowed angular velocity ω_0 at the lower point $\varphi = 0$ can be calculated either from (3) or from the law of energy conservation

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv_{max}^2 - 2mg\cdot r \quad \text{with} \quad v = r\cdot\omega\,.$$

Inserting the numerical values gives

$$\omega_0 = 12.3\,\mathrm{s}^{-1}\,.$$

The difference between the velocities in the upper and lower point is

$$\tfrac{1}{2}v_{min}^2 + 2rg = \tfrac{1}{2}v_{max}^2 \Rightarrow \omega_{max}^2 - \omega_{min}^2 = \frac{4g}{r}\,.$$

3.7 At a radial velocity $v_r = 10\,\mathrm{m/s}$ the ball needs $10^{-2}\,\mathrm{s}$ to reach the outer edge of the disc. For a non-rotating disc the ball would fly on a straight line with the velocity $v = \{v_r, v_\varphi\}$ in the lab system as well as in the system of the disc (which are identical for the non-rotating disc). It would reach the edge of the disc at the displacement $R\cdot\varphi$ from the position $\varphi = 0$ which can be obtained from $R\cdot\varphi = \Delta t\cdot v_\varphi = 0.05\,\mathrm{m}$ at the angle $\varphi = 0.05\,\mathrm{m}/0.2\,\mathrm{m} = 0.25\,\mathrm{rad} = 14.8°$. When the disc rotates with $\omega = 2\pi\cdot 10\,\mathrm{s}^{-1}$, its edge turns during the time $T = 0.01\,\mathrm{s}$ by $R\varphi = R\omega T = 0.126\,\mathrm{m} \Rightarrow \varphi = 39°$. For a radial velocity ($d\varphi/dt = 0$) the bullet would reach the edge at $\varphi = -39°$. With $v_\varphi = 5\,\mathrm{m/s}$ the bullet reaches the edge at $\varphi = 14.5° - 39° = -24.5°$. From the point of view of the observer at rest the bullet flies on a straight line, but viewed by an observer on the disc it flies along a curved line with the tangential acceleration

$$a_\varphi = 2v_r\cdot\omega\,.$$

The trajectory on the disc is a parabola. The velocity is $v = \{v_r, v_\varphi - 2v_r\omega t\}$ where v_r and v_φ are the velocity components in the rest frame while v_r and $v_\varphi - 2v_r\omega t$ are the components in the system of the rotating disc.

3.8 The centrifugal acceleration is

$$\begin{aligned}a_{cf} &= \omega^2 r = \omega^2 R\cos\varphi \quad (R = \text{earth radius}) \\ &= 3.7\cdot 10^{-9}\cdot 6.37\cdot 10^6\,\mathrm{m/s^2} \\ &= 2.36\cdot 10^{-2}\,\mathrm{m/s^2}\,.\end{aligned}$$

It is acting radially outwards, perpendicular to the angular velocity $\boldsymbol{\omega}$.
The Coriolis acceleration is parallel to $\boldsymbol{a}_{cf}$. Numerical values: $a_{cf} = 0.023\,\mathrm{m/s^2}$, $a_C = 1.02\,\mathrm{m/s^2}$. The Coriolis acceleration is for this case 44 times as large as the centrifugal force.

3.9 a) According to the Galilei-transformations is:

$$\begin{aligned}u'_x &= u_x - v_x = 0.5c - \tfrac{1}{3}c = \tfrac{1}{6}c \\ u'_y &= u_y = 0.1\,\mathrm{c} \\ u'_z &= u_z = 0\,.\end{aligned}$$

b) The Lorentz-transformations give:

$$\begin{aligned}&u'_x = \tfrac{1}{5}c\,; \quad u'_y = 0.113c\,; \quad u'_z = 0 \\ &\Rightarrow \boldsymbol{u}' = \{0.2,\,0.113,\,0\}c\,.\end{aligned}$$

The relative errors of the Galilei transformations are with $\Delta u = u_{\text{Lo}} - u_{\text{Ga}}$

$$\frac{\Delta u'_x}{u'_x} = \frac{1/5 - 1/6}{1/5} = \frac{1}{6} \approx 16.7\% ,$$

$$\frac{\Delta u'_y}{u'_u} = \frac{0.013}{0.113} \approx 11.5\% ,$$

$$\Delta u'_z = 0 .$$

3.10 $\gamma = (1 - v^2/c^2)^{-1/2} = 2.785 \Rightarrow L' = L/\gamma = 0.36L$.

3.11 The nearest distance Earth–Neptune is (Tab. 2.1)

$$L = 28.8\,\text{AU} = 4.3 \cdot 10^{12}\,\text{m} .$$

The travel time according to the measurement of the pilot:

$$T' = \frac{2L}{\gamma v} = \frac{2L}{v}\sqrt{1 - v^2/c^2} = 1\,\text{d} \mathrel{\hat{=}} 8.64 \cdot 10^4\,\text{s} .$$

Resolving for v yields

$$v = \frac{2L}{(T^2 + 4L^2/c^2)^{1/2}} = 0.94 \cdot 10^8\,\text{m/s} = 0.3c$$
$$\Rightarrow \gamma = 1.048 .$$

Travel time according to the observers at rest on earth:

$$T = T'\gamma = 9.05 \cdot 10^4\text{s}$$
$$= 1\,\text{d} + 1.41 \cdot 10^4\,\text{s} = 1\,\text{d} + 1.15\,\text{h} .$$

3.12 a) In the rest frame the observer O sits in the middle between A and B. This is also true, when A, B and O move with the same constant velocity v.
b) When O' moves with the velocity v_x against the length $\overline{AB}$, he measures the simultaneous arrival of the light pulses from A and B at the point C, if C is away from A by $(L/2)(1 - v/c)$. C is therefore closer to A than to B.

3.13 $v = 0.8c \Rightarrow \gamma = 5/3$.
The travelling time is according to B: $T = 2L/v = 10$ years. According to A is $T' = 2L/(v\gamma) = 6$ years.
The number of pulses sent by B is

$$N = f\,T = 1 \cdot 10 = 10 .$$

Number of pulses sent by A:

$$N' = f\,T' = 6 .$$

Number of pulses received by A during the journey out:

$$N'_1 = (L/v)(1 - \beta) = 5 \cdot 0.2 = 1 .$$

On the journey back:

$$N'_2 = (L/v)(1 + \beta) = 5 \cdot 1.8 = 9 .$$

3.14 For C the two astronauts A and B meet after a distance x (in ly) with $x = 0.8ct + 0.1ct = 0.9ct$, when t is measured in years.
$\Rightarrow t = 8$ years after the departure of B, $\Rightarrow x = 7.2$ ly.
For A the travelling time is $t'_{\text{A}} = 1/\gamma_{\text{A}}$ with $\gamma_{\text{A}} = (1 - 0.8^2)^{-1/2} = 1.67 \Rightarrow t'_{\text{A}} = 4.8$ years.
For B is $t'_{\text{B}} = 1/\gamma_{\text{B}}$ with $\gamma_{\text{B}} = (1 - 0.9^2)^{-1/2} = 2.3 \Rightarrow t'_{\text{B}} = 3.49$ years.

14.4 Chapter 4

4.1 All particles move into the $\pm x$-direction, $\Rightarrow \boldsymbol{v} = \{v_x, 0, 0\}$; $|\boldsymbol{v}| = v$.
The centre of mass velocity is

$$v_{\text{CM}} = \frac{mv - 3mv}{4m} = -\frac{1}{2}v .$$

The particle velocity in the CM-system is

$$v_{1\text{CM}} = v_1 - v_{\text{CM}} = \tfrac{3}{2}v ,$$
$$v_{2\text{CM}} = v_2 - v_{\text{CM}} = \tfrac{1}{2}v .$$

a) Elastic collision:

$$\left.\begin{aligned} v'_{1\text{CM}} &= -v_{1\text{CM}} = -\tfrac{3}{2}v , \\ v'_{2\text{CM}} &= -v_{2\text{CM}} = +\tfrac{1}{2}v \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} v'_1 &= v'_{1\text{CM}} + v_{\text{CM}} = -2v , \\ v'_2 &= v'_{2\text{CM}} + v_{\text{CM}} = 0 \end{aligned}\right\}$$

$$\Rightarrow \begin{aligned} E'_{\text{kin}}(m_1) &= \tfrac{m}{2}{v'_1}^2 = 2mv^2 , \\ E'_{\text{kin}}(m_2) &= 0 . \end{aligned}$$

Before the collisions was

$$E_{\text{kin}}(m_1) = \frac{m}{2}v^2 ; \quad E_{\text{kin}}(m_2) = \frac{3}{2}mv^2$$
$$\Rightarrow \sum E_{\text{kin}} = \sum E'_{\text{kin}} .$$

b) Completely inelastic collision: The two particles stay together after the collision. The total mass $M = 4\,\text{m}$ moves with the velocity $v_{\text{CM}} = v'_{\text{CM}} = -\frac{1}{2}v$

$$\Rightarrow E'_{\text{kin}} = \frac{4m}{2}v^2_{\text{CM}} = \frac{1}{2}mv^2 .$$

The rest $(3/2)mv^2$ of the initial energy $2mv^2$ is transferred into heat energy, $\Rightarrow$ 75% are converted into heat, only 25% remain as kinetic energy.

4.2 The momentum of the bullet: $m_2 v$
$\Rightarrow$ the velocity of wooden block + bullet is

$$v' = \frac{m_2 v}{M} \quad \text{with} \quad M = m_1 + m_2 .$$

$$\Rightarrow \quad E_{\text{kin}} = \tfrac{1}{2}Mv'^2 = \tfrac{1}{2}\tfrac{m_2^2}{M}v^2$$
$$= E_{\text{p}} = MgL(1 - \cos\varphi_0)$$

$$\Rightarrow \quad \cos\varphi_0 = 1 - \frac{1}{2}\frac{m_2^2}{M^2 gL}v^2 = 1 - 0.196 = 0.804$$
$$\Rightarrow \quad \varphi_0 = 36.5° \,.$$

4.3 We assume that the incident proton moves into the $+x$-direction.
a) The momentum conservation for the x- and the y-direction demands:

$$x: \quad mv_1' \cos\theta_1 + 2mv_2' \cos 45° = mv_1 \qquad (14.4a)$$
$$y: \quad mv_1' \sin\theta_1 = 2mv_2' \sin 45° \,. \qquad (14.4b)$$

Division by m gives for (14.4b):

$$v_1' = 2v_2' \frac{\sin 45°}{\sin\theta_1} \,.$$

Inserting in (14.4a) gives:

$$v_2' = \frac{1}{2}\frac{v_1}{\cos 45° + \sin 45°/\tan\theta_1} = \frac{v_1}{\sqrt{2}\,(1+\cot\theta_1)} \,,$$
$$v_1' = \frac{v_1}{\sin\theta_1 + \cos\theta_1} \,.$$

Energy conservation demands

$$v_1^2 = v_1'^2 + 2v_2'^2$$
$$\Rightarrow 1 = \frac{1}{(\sin\theta_1+\cos\theta_1)^2} + \frac{1}{(1+\cot\theta_1)^2}$$
$$\Rightarrow \tan\theta_1 = 2 \Rightarrow \theta_1 = 63.435° \,.$$

b) $$v_{CM} = \frac{m_1 v_1}{m_1+m_2} \quad \text{with} \quad 2m_1 = m_2$$
$$\Rightarrow v_{CM} = \tfrac{1}{3}v_1 = v_{CM}' \,.$$

c) $$v_1'^2 = \frac{v_1^2}{\left(1+\frac{m_2}{m_1}\right)^2}\left[\left(\frac{m_2}{m_1}\right)^2 + 2\frac{m_2}{m_1}\cos\vartheta_1 + 1\right]$$
$$= v_1^2 \frac{4 + 4\cos 63.435° + 1}{9} = 0.75\,v_1^2$$
$$\Rightarrow v_1' = 0.866\,v_1 \,.$$

$$v_2'^2 = \frac{1}{2}(v_1^2 - v_1'^2) = \frac{1}{2}0.25\,v_1^2 = 0.125\,v_1^2$$
$$\Rightarrow v_2' = 0.35\,v_1 \,.$$

4.4 a) Energies in the Lab-system:

$$E_{kin}(m_1) = \frac{m}{2}\left(v_x^2 + v_y^2 + v_z^2\right) = 1\cdot(9+4+1) = 14\,\text{N m} \,,$$
$$E_{kin}(m_2) = 36\,\text{N m} \,.$$

Velocity of the centre of mass:

$$\boldsymbol{v}_{CM} = \frac{1}{M}\sum m_i \cdot \boldsymbol{v}_i = \left\{v_{xCM}, v_{yCM}, v_{zCM}\right\} = \{0,2,2\}\ \text{m/s} \,.$$

Relative velocities:

$$\left.\begin{aligned}\boldsymbol{v}_{1CM} &= \boldsymbol{v}_1 - \boldsymbol{v}_{CM} = \{3,0,-3\}\,\text{m/s}\\ \boldsymbol{v}_{2CM} &= \boldsymbol{v}_2 - \boldsymbol{v}_{CM} = \{-2,0,2\}\,\text{m/s}\end{aligned}\right\} \Rightarrow$$
$$E_{kin}^{(CM)}(m_1) = \frac{m_1}{2}v_{1CM}^2 = 18\,\text{N m} \,,$$
$$E_{kin}^{(CM)}(m_2) = \frac{m_2}{2}v_{2CM}^2 = 12\,\text{N m} \,.$$

b) The centre of mass momentum equals the momentum of the compound particles after the collision.

$$M\boldsymbol{v}_{CM} = M\{0,2,2\}\,\text{kg m/s} \,,$$
$$E_{kin}'(M) = \frac{M}{2}v_{CM}^2 = 20\,\text{N m} \,.$$

c) The fraction of the converted kinetic energy is

$$\eta = 1 - \frac{E_{kin}'(M)}{E_{kin}(m_1)+E_{kin}(m_2)} = \frac{50-20}{50} = 0.6 \,.$$

In the centre of mass system is $E_{kin}^{CM} = 0$. The total kinetic energy is converted into heat.

4.5 We choose the x-axis as the direction of v_1.
a) Conservation of momentum for the x- and y-components yields:

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v_{1x}' + m_2 v_{2x}' \,,$$
$$\boldsymbol{v}_1 = \{4,0\}\,\text{m/s} \,; \quad \boldsymbol{v}_1' = \{2,2\}\,\text{m/s} \,;$$
$$\boldsymbol{v}_2' = \{1,-1\}\,\text{m/s} \;\Rightarrow\; v_{2x} = 0 \,,$$
$$m_1 v_{1y} + m_2 v_{2y} = m_1 v_{1y}' + m_2 v_{2y}'$$
$$0 + 2v_{2y} = 2 - 2\cdot 1\,\text{m/s} \,,$$
$$\Rightarrow v_{2y} = 0\,\text{m/s}$$
$$\Rightarrow \boldsymbol{v}_2 = \{0,0\}\,\text{m/s} \,,$$

i. e. m_2 was at rest before the collision.
b) Energy conservation (4.17) gives:

$$Q = E_{kin}' - E_{kin} = \tfrac{1}{2}\left(m_1 v_1'^2 + m_2 v_2'^2 - m_1 v_1^2 - m_2 v_2^2\right) = -2\,\text{N m} \,.$$

$E_{kin} = 8\,\text{N m} \Rightarrow 25\%$ of the initial energy is converted into heat.
The centre of mass velocity is

$$v_{CM} = \frac{1}{M}\left\{m_1 v_{1x} + m_2 v_{2x}; m_1 v_{1y} + m_2 v_{2y}\right\} = \tfrac{1}{3}\{4,0\}\,\text{m/s} \,.$$

The energy of the centre of mass is

$$E_{\text{kin}}^{(\text{CM})} = \tfrac{1}{2} M v_{\text{CM}}^2 = 2.66\,\text{N m} .$$

For a completely inelastic collision the fraction $Q = E_{\text{kin}} - E_{\text{kin}}^{(\text{CM})}$ is converted into heat. Since the collision of our example is not a central collision, $|Q|$ is smaller. In the C-system 37.5% are converted.
c) Velocities in the CM-system:

$$\boldsymbol{v}_{1\text{CM}} = \boldsymbol{v}_1 - \boldsymbol{v}_{\text{CM}} = \left\{\tfrac{8}{3}, 0\right\}\ \text{m/s} ,$$

$$\boldsymbol{v}'_{1\text{CM}} = \boldsymbol{v}'_1 - \boldsymbol{v}_{\text{CM}} = \left\{\tfrac{2}{3}, 2\right\}\ \text{m/s} ;$$

$$\cos\vartheta_1 = \frac{\boldsymbol{v}_{1\text{CM}} \cdot \boldsymbol{v}'_{1\text{CM}}}{|v_{1\text{CM}}||v'_{1\text{CM}}|} = \frac{16/9}{\sqrt{\frac{64}{9} \cdot \frac{40}{9}}} = 0.316$$

$$\Rightarrow \vartheta_1 = 71.578^\circ ;$$

$$\boldsymbol{v}_{2\text{CM}} = \left\{-\tfrac{4}{3}, \tfrac{2}{3}\right\}\ \text{m/s} ;\ \boldsymbol{v}'_{2\text{CM}} = \left\{-\tfrac{1}{3}, -\tfrac{7}{3}\right\}\ \text{m/s}$$

$$\Rightarrow \vartheta_2 = 121.6^\circ .$$

4.6 Conservation of momentum gives

$$m_1 \boldsymbol{v}'_1 + m_2 \boldsymbol{v}'_2 = m_2 \boldsymbol{v}_2 .$$

Conservation of energy gives:

$$m_1 v_1'^2 + m_2 v_2'^2 = m_2 v_2^2 .$$

a) After the collision is $v'_2 = -v'_1$

$$\Rightarrow \boldsymbol{v}'_2 \left(1 - \frac{m_1}{m_2}\right) = \boldsymbol{v}_2 , \quad v_2'^2 \left(1 + \frac{m_1}{m_2}\right) = v_2^2$$

$$\Rightarrow m_1/m_2 = 3 .$$

b) The travel time for m_1 resp. m_2 until the left barrier are

$$t_1 = \frac{1.6\,\text{m}}{v'_1} > t_2 = \frac{2.4\,\text{m}}{v'_2} \Rightarrow \frac{v'_2}{v'_1} > 1.5 .$$

Energy conservation demands with $x = m_1/m_2$

$$\frac{v_2'^2}{v_1'^2} = \frac{v_2^2}{v_1'^2} - x .$$

Using momentum conservation gives:

$$\frac{v'_2}{v'_1} = \frac{1}{2}(x-1) \Rightarrow x > 4 .$$

c) The velocity of the CM is

$$v_{\text{CM}} = v'_{\text{CM}} = \frac{1}{3} v_2 .$$

The velocities in the lab-system are after the collision:

$$v'_1 = \frac{2}{3} v_2 ; \quad v'_2 = -\frac{1}{3} v_2 .$$

The two masses meet for the first time at $x_0 = 1.6\,\text{m}$ at the time $t_1 = 0$, for the second time t_2 at the location x ($x = 0$ is at the left wall). According to the calculation in a) the masses meet for the second time only after the reflection of m_1 at the left wall. It is:

$$t_2 = \frac{x_0 + x}{v'_1} = \frac{0.8 + x_0 - x}{v'_2}$$

$$\Rightarrow x = 1.07\,\text{m} .$$

The two masses meet at $x = 1.07\,\text{m}$ from the left wall after m_1 has suffered a reflection at the left wall and m_2 at the right wall.

4.7 The velocity of the steel ball at the impact is

$$m_1 L g = \tfrac{1}{2} m_1 v_1^2 \Rightarrow v_1 = \sqrt{2gL} = 4.43\,\text{m/s} .$$

The energy transferred to m_2

$$\Delta E = 4 \frac{m_1 m_2}{M^2} E_1 .$$

The steel ball has therefore the energy after the collision

$$E'_{\text{kin}} = \left(1 - 4\frac{m_1 m_2}{M^2}\right) E_1 = \frac{4}{9} E_1 .$$

It rises up to the height $H = L(1 - \cos\varphi) = \frac{4}{9} L \Rightarrow \cos\varphi = \frac{5}{9} \Rightarrow \varphi = 56.15^\circ$.

4.8 The distance between ball and lift is $\Delta s = 20\,\text{m}$. The time until the impact onto the ceiling of the lift is obtained from

$$\tfrac{1}{2} g t_1^2 + v t_1 = \Delta s = 20\,\text{m} \Rightarrow t_1 = 1.8\,\text{s} .$$

During this time the lift has moved over the distance $v t_1 = 3.6\,\text{m}$. The impact point is therefore 26.4 m below A.
b) In the lab system the impact velocity of the ball is $v_1 = g t_1 = 17.66\,\text{m/s}$. The centre of mass moves because $M \gg m$ with the velocity $v_{\text{CM}} = v = 2\,\text{m/s}$ upwards. In the centre of mass system (which is nearly identical with the system of the lift) the ball has the velocity $v_{1\text{CM}} = v_1 + v_{\text{CM}} = 19.66\,\text{m/s}$ downwards. After the completely elastic reflection at the lift ceiling the ball has the upward velocity $v'_{1\text{CM}} = 19.66\,\text{m/s}$.
In the Lab system is $v'_1 = v'_{1\text{CM}} + v_{\text{CM}} = 21.66\,\text{m/s}$. The ball has won twice the velocity of the lift by the reflection at the moving lift. It rises now by the distance $\Delta h_1 = v_1'^2/2g$ above the impact point. Inserting the numerical values gives: $\Delta h_1 = 23.9\,\text{m}$. Its upper return point is then 2.5 m below A.
c) It hits the ceiling of the lift for a second time at the time t_2. During the time $\Delta t = t_2 - t_1$ the lift has moved upwards by $\Delta h_2 = v \Delta t$.
The ball needs a rise time Δt_1 obtained from $v(\Delta h_1) = 0 = v'_1 0 g \Delta t_1 \Rightarrow \Delta t_1 = v'_1/g = 2.2\,\text{s}$. Its drop time is $\Delta t_2 = 1.9\,\text{s}$ which is obtained from

$$\tfrac{1}{2} g \Delta t_2^2 = \Delta h_1 - v(2.2\,\text{s} + \Delta t_2) .$$

This gives the time $\Delta t = \Delta t_1 + \Delta t_2 = 4.1\,\text{s} \Rightarrow \Delta h_2 = 8.2\,\text{m}$. The second impact occurs 8.2 m above the first impact point, i. e. 18.2 m below A.

4.9 a) The α-particle should fly into the $+x$-direction. For the y-components of the momenta we get

$$0 = m_1 v_1' \sin 64^\circ - m_2 v_2' \sin 51^\circ$$

$$\Rightarrow \frac{v_1'}{v_2'} = 4 \cdot \frac{\sin 51^\circ}{\sin 64^\circ} = 3.46 \quad \text{since} \quad m_2 = 4m_1 \, .$$

b) $$\frac{E'_{\text{kin}}(m_1)}{E'_{\text{kin}}(m_2)} = \frac{m_1 v_1'^2}{m_2 v_2'^2} = \frac{1}{4} \cdot 3.46^2 \approx 3.0 \, .$$

4.10 $$E = c\sqrt{m_0^2c^2 + p^2} \, ,$$

with $E = 6\,\text{GeV}$ and $pc = 4\,\text{GeV}$

$$\Rightarrow m_0c^2 = \sqrt{20}\,\text{GeV} \, ,$$

$$\Rightarrow E' = \sqrt{20 + 25}\,\text{GeV} = 6.71\,\text{GeV} \, .$$

With

$$\left.\begin{array}{l} E = mc^2 \\ E' = m'c^2 \end{array}\right\} \Rightarrow \frac{m'}{m} = \frac{6.7}{6} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \frac{v}{c} = 0.445 \, .$$

The two systems move with $v = 0.445c$ against each other.

14.5 Chapter 5

5.1 When we cut a cone with full aperture angle 2α out of a sphere we choose the origin of our coordinate system at the peak of the cone. The z-axis is the symmetry axis. Then the coordinates of the centre of mass are

$$x_{\text{CM}} = y_{\text{CM}} = 0 \, ,$$

$$z_{\text{CM}} = \frac{1}{V} \int\limits_{r=0}^{R} \int\limits_{\vartheta=\pi/2-\alpha}^{\pi/2} \int\limits_{\varphi=0}^{2\pi} r^3 \cos\vartheta \sin\vartheta \, \mathrm{d}r \, \mathrm{d}\vartheta \, \mathrm{d}\varphi$$

$$= \frac{1}{V} \frac{\pi}{4} R^4 \left[1 - \sin^2\left(\frac{\pi}{2} - \alpha\right)\right]$$

$$= \frac{1}{V} \frac{\pi}{4} R^4 \sin^2\alpha \, .$$

The volume of the cone is $V = \frac{2}{3}\pi R^3(1 - \cos\alpha)$. Then we get

$$z_{\text{CM}} = \frac{3}{8} R \left(\frac{\sin^2\alpha}{1 - \cos\alpha}\right) = \frac{3}{8} R(1 + \cos\alpha) \, .$$

5.2 a) $$I_{\text{CM}} = \tfrac{2}{5} MR^2 = 9.7 \cdot 10^{37}\,\text{kg}\,\text{m}^2,$$

$$L = I_{\text{CM}}\omega = 7.07 \cdot 10^{33}\,\text{kg}\,\text{m}^2\,\text{s}^{-1}$$

$$\Rightarrow E_{\text{curl}} = \frac{1}{2} I_{\text{CM}}\omega^2 = \frac{1}{5} MR^2\omega^2 = 2.57 \cdot 10^{29}\,\text{J} \, .$$

b) The mass of the earth is for this case

$$M_{\text{E}} = \frac{4}{3}\pi\varrho_1 \frac{R^3}{8} + \frac{4}{3}\pi\varrho_2 \left(R^3 - \frac{1}{8}R^3\right) = \frac{4}{3}\pi\varrho R^3 \, .$$

With $\varrho = M/V =$ mean density

$$\Rightarrow \varrho_1 + 7\varrho_2 = 8\varrho \, .$$

With $\varrho_1 = 2\varrho_2$

$$\Rightarrow \varrho_2 = \frac{8}{9}\varrho \, , \quad \varrho_1 = \frac{16}{9}\varrho \, .$$

The moment of inertia is therefore

$$I_{\text{CM}} = \frac{2}{5} \cdot \frac{4}{3}\pi \left(\varrho_1 \left(\frac{R}{2}\right)^3 \left(\frac{R}{2}\right)^2 + \varrho_2 \left[R^3R^2 - \left(\frac{R}{2}\right)^3 \left(\frac{R}{2}\right)^2\right]\right)$$

$$= \frac{8}{15}\pi \left(\varrho_1 \frac{R^5}{32} + \varrho_2 \frac{31}{32} R^5\right)$$

$$= \frac{1}{60}\pi R^5 \left(\frac{16}{9}\varrho + \frac{31 \cdot 8}{9}\varrho\right)$$

$$= \frac{22}{45}\pi R^5 \varrho = \frac{11}{30} MR^2 = 0.367\,MR^2 \, .$$

This should be compared with the moment of inertia $I_{\text{CM}} = (2/5)MR^2 = 0.4MR^2 = 9.72 \cdot 10^{37}\,\text{kg}\,\text{m}^2$ of the homogeneous earth.

c) If all $N = 5 \cdot 10^9$ adults on earth would run simultaneously on the equator eastwards their torque exerted on the earth would be $D = N \cdot m \cdot a \cdot R = 5 \cdot 10^9 \cdot 70 \cdot 2 \cdot 6.37 \cdot 10^6\,\text{N} \cdot \text{m} = 4.46 \cdot 10^{18}\,\text{N}\,\text{m}$. This would lead to a relative decrease $\Delta\omega/\omega = \Delta L/L$ of the earth rotation. Inserting the numerical value for the angular momentum L

$$L = I_{\text{CM}}\omega = 0.71 \cdot 10^{34}\,\text{kg}\,\text{m}^2/\text{s}$$

we get

$$\frac{\Delta\omega/\Delta t}{\omega} = \frac{1}{L}\frac{\Delta L}{\Delta t} = \frac{D}{L} = 6.3 \cdot 10^{-16}\,\text{s}^{-1} \, ,$$

which is so small, that it falls below the detection limit.

5.3 a) $$I_0 = \tfrac{1}{2} MR^2 = 5 \cdot 10^{-4}\,\text{kg}\,\text{m}^2$$

$$L = I_0\omega_0 = \tfrac{1}{2} MR^2\omega_0 = 3.14 \cdot 10^{-2}\,\text{N}\,\text{m}\,\text{s} \, ,$$

$$E^0_{\text{curl}} = \tfrac{1}{2} I_0\omega_0^2 = 0.987\,\text{N}\,\text{m} \, .$$

b) $I = I_0 + mR^2 = (5 + 1) \cdot 10^{-4}\,\text{kg}\,\text{m}^2 = 6 \cdot 10^{-4}\,\text{kg}\,\text{m}^2$. The angular momentum does not change, because the bug falls onto the disc parallel to the rotation axis.

$$\Rightarrow \omega = \frac{L_0}{I} = \frac{5}{6}\omega_0$$

$$E_{\text{curl}} = \frac{1}{2} I\omega^2 = \frac{5}{6} E^0_{\text{curl}} = 0.823\,\text{N}\,\text{m} \, .$$

The energy difference $\Delta E = 0.164\,\mathrm{N \cdot m}$ is converted by friction into heat energy, which is lost during the equalization of the tangential velocities of bug and disc (which are here assumed to occur instantaneously).

c)
$$\omega(r) = \frac{1}{1 + mr^2/I_0}\omega_0$$

$$L(r) = L_0, \text{ independent of } r\ ,$$

$$E_{\mathrm{curl}} = \frac{E^0_{\mathrm{curl}}}{1 + mr^2/I_0}\ .$$

5.4 a)
$$I = \int_V r^2 \varrho \,\mathrm{d}V = 2\pi H \varrho_0 \int_{r=0}^{R} \left[1 + \left(\frac{r}{R}\right)^2\right] r^3 \,\mathrm{d}r$$
$$= 2\pi\varrho_0 H\left[\frac{1}{4}R^4 + \frac{1}{6}R^4\right] = \frac{10\pi}{12}\varrho_0 H R^4$$
$$= \frac{5}{6}\varrho_0 R^2 V\ .$$

The mass is $M = \int_V \varrho \,\mathrm{d}V = \frac{3}{2}\pi\varrho_0 H R^2$

$$\Rightarrow I = \frac{5}{9}MR^2\ .$$

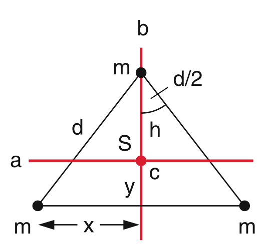
Numerical values: $M = 18.85\,\mathrm{kg}$, $I = 0.105\,\mathrm{kg\,m^2}$.

b) $a = \dfrac{g\sin\alpha}{1 + I_{\mathrm{CM}}/(MR^2)} = \dfrac{g\sin 10°}{14/9}$

$$h = \tfrac{1}{2}at^2 \Rightarrow t = (2h/a)^{1/2} = 1.35\,\mathrm{s}\ .$$

5.5 For the isosceles triangle with height h and side length d the centre of mass $S = (x_{\mathrm{CM}}, y_{\mathrm{CM}})$ has the coordinates $x_{\mathrm{CM}} = 0$, $y_{\mathrm{CM}}(\alpha)$.
The moments of inertia around the principal axes are

$$I_a = 2my^2_{\mathrm{CM}} + m(h - y_{\mathrm{CM}})^2\ ,$$
$$I_b = 2mx^2\ ,$$
$$I_c = m(h - y_{\mathrm{CM}})^2 + 2m(x^2 + y^2_{\mathrm{CM}}) = I_a + I_b\ ,$$
$$x = d\sin(\alpha/2) = 0.204\,\mathrm{nm}\ ,$$
$$h = d\cos(\alpha/2) = 0.247\,\mathrm{nm}\ ,$$
$$y_{\mathrm{CM}} = \tfrac{1}{3}h = 0.082\,\mathrm{nm}\ .$$



For the moments of inertia we get:

$$I_a = 0.93\,\mathrm{AMU\,nm^2}\ ,$$
$$I_b = 1.91\,\mathrm{AMU\,nm^2}\ ,$$
$$I_c = 2.85\,\mathrm{AMU\,nm^2}\ .$$

$1\,\mathrm{AMU} = 1.67 \cdot 10^{-27}\,\mathrm{kg}$.
The rotational energy is then

$$E_{\mathrm{rot}} = \frac{L^2}{2I} \quad \text{with} \quad L^2 = l\cdot(l+1)\hbar^2\ .$$

Where $l = 1, 2, 3, \ldots$ and $\hbar = h/2\pi = 1.06 \cdot 10^{-34}\,\mathrm{J\cdot s}$ is the reduced Planck constant and it is the smallest unit of the rotational angular momentum.
This gives with $1\,\mathrm{eV} = 1.6 \cdot 10^{-19}\,\mathrm{J}$

$$E_a = 2.2 \cdot 10^{-5}\,\mathrm{eV}\ ,$$
$$E_b = 1.1 \cdot 10^{-5}\,\mathrm{eV}\ ,$$
$$E_c = 0.73 \cdot 10^{-5}\,\mathrm{eV}\ .$$

5.6 The inertial moment of a rod with length L is

$$I_{\mathrm{CM}} = \tfrac{1}{12}ML^2 = 1.33 \cdot 10^{-2}\,\mathrm{kg\,m^2}\ .$$

The angular momentum of the bullet referred to the CM of the rod is

$$L_{\mathrm{B}} = |\boldsymbol{r} \times \boldsymbol{p}| = \tfrac{1}{2}Lmv = 0.4\,\mathrm{N\,m\,s}\ .$$

The rotational velocity of the rod is

$$\omega = \frac{L_{\mathrm{B}}}{I} = \frac{L_{\mathrm{B}}}{I_{\mathrm{CM}} + m(L/2)^2}$$
$$= 29.2\,\mathrm{s^{-1}} \Rightarrow \nu = 4.65\,\mathrm{s^{-1}}\ .$$
$$\Rightarrow E_{\mathrm{rot}} = \tfrac{1}{2}I\omega^2 = 5.67\,\mathrm{N\,m}\ ,$$
$$E_{\mathrm{kin}} = \tfrac{1}{2}mv^2 = 200\,\mathrm{N\,m}\ ,$$
$$\Rightarrow E_{\mathrm{rot}}/E_{\mathrm{kin}} = 2.8 \cdot 10^{-2} = 2.8\%\ .$$

97.1% of the kinetic energy of the bullet is lot as heat energy. Compare this with the case of a completely inelastic central collision of a bullet with mass m hitting a free mass M. Here the ratio

$$E'_{\mathrm{kin}}/E_{\mathrm{kin}} = \frac{(m+M)v^2_{\mathrm{CM}}}{mv^2} \quad \text{(see Sect. 4.2.4)}\ .$$

With $v_{\mathrm{CM}} = \dfrac{m}{M+m}v \Rightarrow E'_{\mathrm{kin}}/E_{\mathrm{kin}} \approx \dfrac{m}{M} = 0.01.$

Question: Why is the transfer of kinetic energy of the bullet into rotational energy more efficient?

5.7 $I_{CM} = \frac{1}{2}MR^2$; $D = I_{CM} \cdot \frac{d\omega}{dt}$

$$\Rightarrow \omega = \omega_0 + \frac{1}{I_{CM}} \int_0^t D\, dt'$$

$$= \omega_0 + \frac{D_0}{I_{CM}} \int_0^t e^{-at'}\, dt'$$

$$= \omega_0 + \frac{2D_0}{aMR^2}\left[1 - e^{-at}\right] .$$

For $t \to \infty \Rightarrow \omega(\infty) = \omega_0 + (2D_0/aMR^2)$.
Numerical example: $\omega(t = 10\,s) = 136.4\,s^{-1}$ (because $\omega_0 = 10\,s^{-1}$).

5.8 $E_{kin} = E_{rot} + E_{trans} = \frac{1}{2}I_{CM}\omega_0^2 + \frac{1}{2}MR^2\omega_0^2$

$$E_p = Mgh = E_{kin}$$

$$\Rightarrow h = \frac{\omega_0^2}{2Mg}(I_{CM} + MR^2) .$$

a) Full cylinder: $I_{CM} = \frac{1}{2}MR^2 \Rightarrow h_1 = \frac{3}{4}\frac{\omega_0^2 R^2}{g}$.

b) Hollow cylinder: $I_{CM} = MR^2 \Rightarrow h_2 = \frac{\omega_0^2 R^2}{g}$.

Numerical example: $h_1 = 17.2\,cm$; $h_2 = 22.9\,cm$.

14.6 Chapter 6

6.1 Tensile strength in the height z above the end of the rope:

$$\sigma = \varrho \cdot g \cdot z .$$

Relative elongation:

$$\varepsilon(z) = \frac{1}{E}\sigma(z) .$$

Total elongation

$$\Delta L = \int_0^L \varepsilon(z)\, dz = \frac{1}{E}\int_0^L \sigma(z)\, dz$$

$$= \frac{\varrho g}{E}\int_0^L z\, dz = \frac{\varrho g}{2E}L^2 .$$

a) $\varrho_{St} = 7.7 \cdot 10^3\,kg/m^3$, $E = 2 \cdot 10^{11}\,N/m^2 \Rightarrow \Delta L = 15.3\,m$.
b) $\Delta\varrho = \varrho_{St} - \varrho_W = 6.67 \cdot 10^3\,kg/m^3 \Rightarrow \Delta L = 13.3\,m$.
c) The maximum tensile stress $\sigma_{max} = \varrho \cdot g \cdot L$ appears for $z = L$ at the upper end of the rope. It should be smaller than $\sigma_{tear} = 8 \cdot 10^8\,N/m^2$

$$\Rightarrow L < \frac{\sigma_{tear}}{\varrho g} = 10^4\,m .$$

6.2 The maximum deflection is according to (6.23)

$$s = \frac{L^3}{3EI}F \quad \text{with} \quad I = \text{cross sectional moment of inertia} .$$

a) $I = \frac{1}{12}d^3 b = 4.2 \cdot 10^{-6}\,m^4 \Rightarrow s = 0.4\,m$.
b) $I = \frac{1}{12}(b_1 d_1^3 - b_2 d_2^3) = 7.8 \cdot 10^{-6}\,m^4$

$$\Rightarrow s = 0.22\,m .$$

The two cross sectional areas are
a) $5 \cdot 10^{-3}\,m^2$, b) $7.5 \cdot 10^{-3}\,m^2$.
Although the area in b) is only 1.5 times larger than in a) the double-T-profile has twice the stability in the z-direction and 10 times higher stability when bending into the y-direction.

6.3 $p(h = 10{,}000\,m) \approx 10^8\,Pa \approx 10^3\,atm$.
$F = 4\pi r^2 \cdot p = 2.8 \cdot 10^9\,N$. This force equals the weight of $2.8 \cdot 10^5$ tons.
According to (6.9) is $\Delta V = -p \cdot V/K$. After Tab. 6.1 is $K = 1/\kappa = 1.56 \cdot 10^{11}\,N/m^2$.
a)
$$\Rightarrow \frac{\Delta V}{V} = -\frac{10^{18}}{1.56 \cdot 10^{11}} = 6.4 \cdot 10^{-4}$$

$$\Rightarrow \frac{\Delta r}{r} = \frac{1}{3}\frac{\Delta V}{V} \approx 2.1 \cdot 10^{-4} .$$

The radius of the solid sphere decreases by 0.3 mm. This can be also obtained in the following way: $\Delta V/V = -p \cdot \kappa$ and $\kappa = (3/E)(1 - 2\mu) \Rightarrow \Delta r/r = -p/E(1 - 2\mu)$. Inserting the numerical values for E and μ from Tab. 6.1 one obtains the same results for $\Delta r/r$.
b) Compression of a hollow sphere with radius r and wall thickness d: Now the elastic back pressure during the compression is missing since the inner sphere with radius $(r - d)$ is a gas volume, where the compression modulus is smaller by 3 orders of magnitude. We therefore get for $d \ll r$ the pressure

$$p = -\frac{E}{1 - 2\mu}\left(\frac{\Delta r}{r} - \frac{\Delta r}{r - d}\right) \approx \frac{E}{1 - 2\mu}\frac{d\,\Delta r}{r^2}$$

$$\Rightarrow \frac{\Delta r}{r} \approx -\frac{p}{E}\frac{r}{d}(1 - 2\mu) .$$

For $d = 0.2\,m$ and $r = 1.5.\,m \Rightarrow r/d = 7.5 \Rightarrow \Delta r/r 1.5 \cdot 10^{-3}$. The compression is larger by a factor 7.5 compared to the solid sphere.

6.4 The tangential force acting on the wave is

$$F = \frac{\text{power}}{\text{lenght/time}} = \frac{3 \cdot 10^5}{2\pi R \cdot 25}\,N = 3.8 \cdot 10^4\,N .$$

a) The torque acting on the axis is

$$D = FR = \frac{\pi}{2}G\frac{R^4}{L}\varphi \Rightarrow \varphi = \frac{2FL}{\pi G R^3}$$

$$= \frac{1}{G}3.87 \cdot 10^9\,rad .$$

With $G = 8 \cdot 10^{10}\,N/m^2 \Rightarrow \varphi = 5.2 \cdot 10^{-2}\,rad \approx 3°$.

6.5 From $\kappa = -(1/V)\,\mathrm{d}V/\mathrm{d}p$ (6.32) $\Rightarrow \mathrm{d}V/V = -\kappa \mathrm{d}p$. Integration yields

$$\ln V = -\kappa \cdot p + C \quad \text{with} \quad C = \ln V(p=0) = \ln V_0$$
$$\Rightarrow V = V_0 \mathrm{e}^{-\kappa p}$$
$$\Rightarrow \varrho = \varrho_0 \mathrm{e}^{+\kappa p}$$

with $\kappa = 4.8 \cdot 10^{-10}\,\mathrm{m^2/N}$ and a pressure $p = 10^8\,\mathrm{N/m^2}$ at 10^4 m water depth we get $\kappa \cdot p = 0.048$. This gives

$$\varrho = \varrho_0 \cdot \mathrm{e}^{0.048} \approx \varrho_0(1 + 0.048)\,.$$

The density rises by 4.8%.

6.6 $M = \varrho[1\,\mathrm{m}^3 - (1-2d)^2(1\,\mathrm{m} - d)] = \varrho \cdot 0.0968\,\mathrm{m}^3 = 755\,\mathrm{kg}$.
The cube immerses about 0.755 m. Its centre of gravity S_b is 0.4069 m above its lower edge, i. e. 0.348 m below the water surface. The centre of gravity of the displaced water is 0.3775 m below the water surface i. e. below S_b.
For a tilt angle $\varphi = 24°$ the deeper upper edge of the open cube comes below the water surface. The cube runs full with water. For this angle φ the meta-centre M is still above S_b, i. e. the position of a closed cube would be stable.

6.7
$$W = g\Bigg[(\varrho_\mathrm{b} - \varrho_\mathrm{l})a^3(h-a) + \int_0^a \left[(\varrho_\mathrm{b} - \varrho_\mathrm{l})a^2(a-z) + \varrho_\mathrm{b} a^2 z\right]\mathrm{d}z\Bigg]$$
$$= gha^3\left[\varrho_\mathrm{b} - \varrho_\mathrm{l}(1 - a/2h)\right]\,.$$

With $\varrho_\mathrm{b} = 7.8 \cdot 10^3\,\mathrm{kg/m^3} \Rightarrow W = 2.51 \cdot 6.85 \cdot 10^3\,\mathrm{N \cdot m} = 1.72 \cdot 10^4\,\mathrm{N \cdot m}$.
The lift in air would require the work $mgh = gh \cdot a^3 \varrho_\mathrm{b} = 1.96 \cdot 10^4\,\mathrm{N \cdot m}$.

6.8 $F = A\,\Delta p = \pi r^2\,\Delta p = \frac{1}{4}\pi d^2\,\Delta p = 2.5 \cdot 10^4\,\mathrm{N}$ for each of the two semi-spheres., i. e. each horse had to pull with 3.125 N $\widehat{=}$ 318 kp. If one side of the sphere had been tied to a tree, 8 horses with the pulling force 318 kp each would have been sufficient but less impressive.

6.9 a) The ratio of the two measured values is

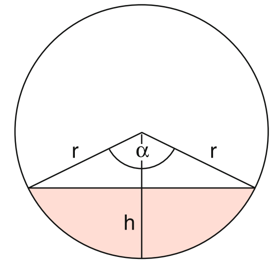
$$\frac{\varrho_\mathrm{gold} V}{(\varrho_\mathrm{gold} - \varrho_\mathrm{l})V} = \frac{19.3}{18.3} = 1.0546\,.$$

b)
$$\frac{0.8\varrho_\mathrm{gold} + 0.2\varrho_\mathrm{copper}}{0.8\varrho_\mathrm{gold} + 0.2\varrho_\mathrm{copper} - 1} = \frac{17.2}{16.2} = 1.062\,.$$

c)
$$\frac{1.0550 - 1.0546}{1.0546} \approx 3.8 \cdot 10^{-4}\,.$$

6.10 $M_\mathrm{wood} = L\pi r^2 \varrho_\mathrm{s} = \varrho_\mathrm{l} V_\mathrm{i}$; $V_\mathrm{e}/V_\mathrm{cyl} = 0.525/1 = \varrho_\mathrm{wood}/\varrho_\mathrm{l}$, where $r = d/2$ and V_i is the immersed volume.
$\Rightarrow M = 16.5\,\mathrm{kg} \Rightarrow V_\mathrm{i} = 1.65 \cdot 10^{-2}\,\mathrm{m}^3$.
a) The immersed segment of the cylinder has the volume (see the figure)

$$V_\mathrm{i} = \tfrac{1}{2}L\left[r^2\alpha - (r-h)\sin(\alpha/2)r\right]\,,$$



where h is the height below the water surface and α is the segment angle. With

$$h = r(1 - \cos\alpha/2) \Rightarrow V_\mathrm{i} = \tfrac{1}{2}Lr^2\left(\alpha - \tfrac{1}{2}\sin\alpha\right)\,.$$

Inserting $V_\mathrm{i} = 1.65 \cdot 10^{-2}\,\mathrm{m}^3$ and $r = 0.1\,\mathrm{m}$ we get $\alpha \approx 184.5° \Rightarrow h = 0.108\,\mathrm{m}$. The cylinder immerses slightly more than half of it height.
b) The sphere has the volume $V_\mathrm{k} = \frac{4}{3}\pi R^3$ and the mass $M_\mathrm{k} = V_\mathrm{k} \cdot \varrho_\mathrm{steel}$. It experiences a buoyancy $g \cdot V_\mathrm{k} \cdot \varrho_\mathrm{l} = g \cdot M_\mathrm{k} \cdot \varrho_\mathrm{l}/\varrho_\mathrm{steel}$. At equilibrium is reached by an immersion depth where the buoyancy just compensates the weight of the sphere.

$$(M_\mathrm{cyl} + M_\mathrm{K})g = (\pi r^2 \varrho_\mathrm{l} H + M_\mathrm{K}\varrho_\mathrm{l}/\varrho_\mathrm{steel})g$$
$$\Rightarrow h = 0.553\,\mathrm{m} \text{ at } \varrho_\mathrm{steel} = 7800\,\mathrm{kg/m^3}\,.$$

14.7 Chapter 7

7.1
$$g(h) = G\frac{M_\mathrm{E}}{(R+h)^2} = G\frac{M_\mathrm{E}}{R^2(1+h/R)^2} \approx g(h=0)(1 - 2h/R)\,.$$

Inserting into (7.5a) gives

$$\frac{\mathrm{d}p}{p} = -\frac{\varrho_0}{p_0} g \cdot (1 - 2h/R)\,\mathrm{d}h\,.$$

Integration yields

$$\ln p = -\frac{\varrho_0}{p_0} gh + \frac{\varrho_0}{p_0 R} gh^2 + C$$
$$\Rightarrow p = p_0 \exp\left[-\frac{\varrho_0 g \cdot (h - h^2/R)}{p_0}\right]\,.$$

7.2 $p_0 = 1$ bar. The altitude where p=1 mbar is obtained from

$$\mathrm{e}^{-h/8.33\,\mathrm{km}} = 10^{-3}$$
$$\Rightarrow h = 8.33 \ln(10^3)\,\mathrm{km} = 57.5\,\mathrm{km}\,.$$

7.3
$$p(h = 100\,\mathrm{km}) \approx 6 \cdot 10^{-6}\,\mathrm{bar} = 0.6\,\mathrm{Pa}\,.$$

For $T = 250\,\mathrm{K} \Rightarrow n = \frac{p}{kT} \Rightarrow n = 1.7 \cdot 10^{20}\,/\mathrm{m}^3 = 1.7 \cdot 10^{14}\,\mathrm{cm}^3$. $\varrho = nm = 1.7 \cdot 10^{20} \cdot 28\,\mathrm{AMU} = 8 \cdot 10^{-6}\,\mathrm{kg/m^3}$ (for N_2).

7.4 The buoyancy G_A = weight of the displaced air $\Rightarrow G_A = \varrho(h)gV = \varrho_0 g e^{-1/8.33} \cdot 3 \cdot 10^3\,\mathrm{m}^3 = 3.37 \cdot 10^4\,\mathrm{N}$.
The mass of balloon + load + gas fill can be at most $3.44 \cdot 10^3$ kg. The pressure of the fill gas is $p(h) = 0.887\,p_0 = 8.87 \cdot 10^4$ Pa.
The mass of the fill gas
a) helium: $\varrho_0 = 0.1785\,\mathrm{kg/m^3} \Rightarrow \varrho(h) = 0.1583\,\mathrm{kg/m^3} \Rightarrow m_{He} = 475\,\mathrm{kg}$, $\Rightarrow$ mass of balloon + load can be at most $= 2965$ kg.
b) H_2: $\varrho_0 = 0.09\,\mathrm{kg/m^3} \Rightarrow \varrho(h) = 0.08\,\mathrm{kg/m^3} \Rightarrow m_{H_2} = 240\,\mathrm{kg} \Rightarrow$ mass of balloon + load should be smaller than 3200 kg.

7.5 For $h = 0$ the pressure is $p = p_0 = 10^5$ Pa and $x = x_0 = 0.2$ m. According to the Boyle–Marriott law is

$$(p_0 + \varrho g h)Ax = p_0 A x_0$$
$$\Rightarrow h = a\frac{x_0 - x}{x} \quad \text{with} \quad a = \frac{p_0}{\varrho g} = 10.2\,\mathrm{m}$$
$$\Rightarrow \frac{\Delta h}{\Delta x} \approx -a\frac{x_0}{x^2} < 0$$
$$\Rightarrow x = \sqrt{a x_0 \left|\frac{\Delta x}{\Delta h}\right|}\,,$$

with $\Delta x = 10^{-3}$ m and $\Delta h = 1$ m. The device is usable down to a depth of 35 m with an accuracy of ± 1 m.

7.6 The number of particles that have passed at least the distance x without collisions is according to (7.33)

$$N(x) = N_0 e^{-x/\Lambda}\,.$$

a) $N(x \geq \Lambda) = N_0 e^{-1} \Rightarrow N(\Lambda)/N_0 = 0.368 = 36.8\%$.
b) $N(x \geq 2\Lambda) = N_0 e^{-2} \Rightarrow N(2\Lambda)/N_0 = 13.5\%$.

7.7 The probability W is

$$W = \int_{v_1}^{v_2} f(v)\,dv = \frac{4}{\sqrt{\pi}\,v_W^3}\int_{v_1}^{v_2} v^2 e^{-v^2/v_W^2}\,dv$$
$$\approx \frac{4\overline{v}^2}{\sqrt{\pi}\,v_W^3}\Delta v e^{-\overline{v}^2/v_W^2}\,,$$

with $\overline{v} = (v_1 + v_2)/2 = 950$ m/s and $\Delta v = (v_1 - v_2) = 100$ m/s. For N_2-molecules at $T = 300$ K is $v_W = 422$ m/s.

$$\Rightarrow W = \frac{4 \cdot 950^2 \cdot 100}{\sqrt{\pi} \cdot 422^3} \cdot e^{-5.06} = 1.7 \cdot 10^{-2}\,.$$

7.8 From (7.6) we obtain

$$\ln\frac{p_1}{p_2} = \frac{\varrho_1}{p_1} g\,\Delta h\,.$$

The density is obtained from $\varrho_1 = m \cdot p_1/(kT)$ with $m = 0.71 m_{N_2} + 0.29 m_{O_2}$ as $\varrho_1 = 1.24\,\mathrm{kg/m^3}$

$$\Rightarrow \Delta h = 866\,\mathrm{m}\,.$$

7.9 $\Delta\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2 \Rightarrow (\Delta\boldsymbol{v})^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos\alpha$.
a) Since the direction s of the velocity vectors are uniformly distributed the average values $\overline{v_1^2} = \overline{v_2^2} = \overline{v^2}$ and $\overline{\cos\alpha} = 0$:

$$\overline{(\Delta v)^2} = 2\overline{v^2} \Rightarrow \sqrt{\overline{(\Delta v)^2}} = \sqrt{2}\sqrt{\overline{v^2}}\,,$$
$$\Delta v = |\Delta\boldsymbol{v}|$$
$$= \sqrt{(v_{x1} - v_{x2})^2 + (v_{y1} - v_{y2})^2 + (v_{z1} + v_{z2})^2}$$
$$= \left(v_{x1}^2 + v_{x2}^2 + v_{y1}^2 + v_{y2}^2 + v_{z1}^2 + v_{z2}^2 - 2(v_{x1}v_{x2} + v_{y1}v_{y2} + v_{z1}v_{z2})\right)^{1/2};$$

with $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \frac{1}{3}\overline{v^2}$ and $\overline{v_x} = \overline{v_y} = \overline{v_z} = 0$

$$\Rightarrow \overline{\Delta v^2} = 6 \cdot \tfrac{1}{3}\overline{v^2} = 2 \cdot \overline{v^2}\,.$$

b) Here all absolute values v of the velocity $\boldsymbol{v}$ have the same value $\Rightarrow \overline{\Delta v} = \sqrt{2}v$.

7.10 The molecular density at $p = 10^5$ Pa and $T = 20\,°\mathrm{C}$ is $n = 2.5 \cdot 10^{19}/\mathrm{cm}^3$

$$\Rightarrow \sigma_{Ar} = \frac{1}{n\Lambda} = \frac{10^{-25} \cdot 10^7}{2.5 \cdot 1.5}$$
$$= 2.6 \cdot 10^{-19}\,\mathrm{m}^2 = 26\,\text{Å}^2\,.$$

If both collision partners are moving, the mean time between two collisions is $\tau = \Lambda/\overline{\Delta v}$ where $\overline{\Delta v}$ is the mean relative velocity. We have the following numerical values:

$$\sigma_{N_2} = 31 \cdot 10^{-16}\,\mathrm{cm}^2\,,$$
$$\overline{\Delta v}_{Ar} = \sqrt{2\overline{v}} = 565\,\mathrm{m/s}$$
$$\Rightarrow \tau_{Ar} = \Lambda/\overline{\Delta v} = 2.6 \cdot 10^{-10}\,\mathrm{s}\,,$$
$$\tau_{N_2} = 1.8 \cdot 10^{-10}\,\mathrm{s}\,.$$

7.11 The density is (as in 7.10) $n = 2.5 \cdot 10^{19}/\mathrm{cm}^3$.
a) $N = \dfrac{M}{m_{He}} = \dfrac{0.1}{6.68 \cdot 10^{-27}} = 1.5 \cdot 10^{25}$, where N is the total number of the He-atoms in the container.
b) $\sigma_{He-He} = 10 \cdot 10^{-16}\,\mathrm{cm}^2$

$$\Rightarrow \Lambda = \frac{1}{n\sigma} = 4 \cdot 10^{-7}\,\mathrm{m}\,.$$

c) The sum $\sum_i S_i$ is:

$$\sum_i S_i = \sum_i N_i v_i\,\Delta t = N\overline{v}\Delta t$$
$$= 1.5 \cdot 10^{25} \cdot 1260\,\mathrm{m} \qquad \text{for} \quad \Delta t = 1\,\mathrm{s}$$
$$= 6.3 \cdot 10^{19}\ \text{light seconds}$$
$$= 2 \cdot 10^{12}\ \text{light years}\,.$$

7.12 Assume, two atoms with velocities v_1 and v_2 pass the disc at $t = 0$. Their arrival times at the detector are

$$t_1 = \frac{L}{v_1}\,; \quad t_2 = \frac{L}{v_2} \Rightarrow \Delta t = L\frac{\Delta v}{v_1 v_2}$$

with $\Delta v = v_1 - v_2$.
If one atom passes the disc at the beginning of the opening time Δt_0, the other atom at the end, the time difference between the arrival times at the detector is

$$\Delta t_{\max} = \Delta t + \Delta t_0\,.$$

The time spectrum $N(t)$ of the arriving atoms with the velocity distribution $N(v) = N \cdot f(v)$ can be obtained as follows: With $v = L/t$ and $\mathrm{d}v = -(L/t^2)\mathrm{d}t = (v/t)\mathrm{d}t$ we get the distribution function

$$f(v)\,\mathrm{d}v = \frac{1}{t} v f\left(\frac{L}{t}\right)\mathrm{d}t\,.$$

The function $f(v) \propto v^2 \mathrm{e}^{-v^2/v_W^2}$ (see (7.30)) is then converted to

$$f(v,t) \propto \frac{L^3}{t^4}\mathrm{e}^{-L^2/t^2 v_W^2}\mathrm{d}t\,.$$

If the time profile of the velocity selector is $g(t)$ the time dependence of the detector signal is

$$S(t) = \int\limits_{-\infty}^{+\infty} g(t') f(t-t')\,\mathrm{d}t'\,.$$

If the opening time Δt_0 of the selector were infinitely short ($\Delta t_0 \to 0$), the difference Δt of the arrival times of the atoms at the detector (because of their different velocities) would be

$$\Delta t = \frac{t^2}{L}\,\Delta v = \frac{L}{v_W^2}\,\Delta v = \frac{1}{600^2} = 1.6\,\mathrm{ms}\,.$$

Taking into account the finite opening time, the convolution $S(t)$ gives a time profile with a half width Δt which is for a rectangular opening time profile $g(t)$ with $\Delta t_0 = 1\,\mathrm{ms}$ approximately $\Delta t \approx 2.5\,\mathrm{ms}$.

7.13 $\frac{m}{2}v_0^2 > G\frac{M_\mathrm{E} m}{R+h} \Rightarrow v_0 > \sqrt{\frac{2GM_\mathrm{E}}{R+h}}$

$$\Rightarrow v_0(h) = v_0(h=0)\sqrt{\frac{1}{1+h/R}}$$

$$\approx v_0(0)\left(1 - \frac{1}{2}h/R\right)\,.$$

For $h = 100\,\mathrm{km}$, $\Rightarrow v_0(h) = 0.992 v_0(0) = 11.1\,\mathrm{km/s}$.
a) If half of all molecules within the Maxwell distribution has a velocity $v > \overline{v}$

$$\Rightarrow v > \overline{v} = \sqrt{\frac{8kT}{\pi m}} = 11.1\,\mathrm{km/s}$$

$$\Rightarrow T = 1.6 \cdot 10^5\,\mathrm{K}\,.$$

7.14 The density of the outside air in a height of 50 m at $T = 300\,\mathrm{K}$ is

$$\varrho = \varrho_0 \mathrm{e}^{-\varrho_0 g h/p_0}\,,$$

with $\varrho_0 = 1.29\,\mathrm{kg/m^3}$, $p_0 = 10^5\,\mathrm{N/m^2} \Rightarrow \varrho = 1.28\,\mathrm{kg/m^3}$.
The exhaust gases must have a temperature $T > T_0$.
Because the pressure at the upper end of the smokestack is at the same temperature the same for the exhaust gases and the outside air it follows

$$\Rightarrow \varrho_1/\varrho_2 = T_2/T_1 \Rightarrow T_2 = 452\,\mathrm{K}\,.$$

For the outside air is $p_1 = p_0 \mathrm{e}^{-\varrho_1 g h/p_0}$ and for the exhaust gases inside of the smokestack $p_2 = p_0' \cdot \mathrm{e}^{-\varrho_2 g h/p_0'}$. With $p_0' = p_2(h=0)$ and $p_1(h) = p_2(h)$ we obtain with the approximation $\mathrm{e}^x \approx 1 + x$

$$p_0' - \varrho_2 g h = p_0 - \varrho_1 g h$$

$$\Rightarrow \Delta p_0 = p_0 - p_0' = \Delta\varrho g h$$

$$= (1.28 - 0.85) \cdot 9.81 \cdot 50\,\mathrm{Pa} = 211\,\mathrm{Pa}\,.$$

7.15 $\varrho_0(\mathrm{He}) = 0.178\,\mathrm{kg/m^3}$ at $p_0 = 1$ bar.
$\Rightarrow \varrho_\mathrm{He}(1.5\,\mathrm{bar}) = 0.267\,\mathrm{kg/m^3}$.
From $m_\mathrm{He} + m_\mathrm{Bal} = V\varrho_\mathrm{Air} \Rightarrow$

$$V = \frac{m_\mathrm{Bal}}{\varrho_\mathrm{Air} - \varrho_\mathrm{He}(1.5\,\mathrm{bar})}$$

$$= \frac{0.01}{1.023}\,\mathrm{m^3} = 9.8 \cdot 10^{-3}\,\mathrm{m^3}\,.$$

7.16 a) $\frac{1}{2}m \cdot \langle v^2\rangle = \frac{3}{2}kT = 3.1 \cdot 10^{-16}\,\mathrm{J} \mathrel{\hat{=}} 1.9 \cdot 10^3\,\mathrm{eV}$.
The ionization energy of the H-atom is 13.5 eV. At a density of $5 \cdot 10^{29}\,/\mathrm{m^3}$ the mean distance between the protons is $1.25 \cdot 10^{-10}\,\mathrm{m}$. The mean potential energy, due to the Coulomb repulsion, is $E_\mathrm{p} \approx 1.8 \cdot 10^{-18}\,\mathrm{J}$ which is small compared to the mean kinetic energy at a temperature of 15 million Kelvin. This means that the matter in the central part of the sun can be safely regarded as ideal gas.

b) $\overline{v} = \sqrt{\frac{8kT}{\pi m}}\overline{v}_\mathrm{p} = 5.6 \cdot 10^5\,\mathrm{m/s}$;
$\overline{v}_\mathrm{el} = 2.4 \cdot 10^7\,\mathrm{m/s} = 0.08\,c$.
c) $p = nkT = 1 \cdot 10^{14}\,\mathrm{Pa} \mathrel{\hat{=}} 10^9\,\mathrm{atm}$.

7.17 $M_\mathrm{Atm} = \frac{4\pi R^2 \cdot 1.013 \cdot 10^5\,\mathrm{N}}{9.81\,\mathrm{m/s^2}} = 5.3 \cdot 10^{18}\,\mathrm{kg}$.
The comparison with the earth mass $M_\mathrm{E} = 6 \cdot 10^{24}\,\mathrm{kg}$ shows that $M_\mathrm{Atm} \approx 10^{-6}\,M_\mathrm{E}$.

7.18 $M_\mathrm{B} + \varrho_\mathrm{He} \cdot V = \varrho_\mathrm{Air} \cdot V$

$$\Rightarrow V = \frac{M_\mathrm{B}}{\varrho_\mathrm{Air} - \varrho_\mathrm{He}}\,.$$

a) $h = 0$, $T = 300\,\mathrm{K} \Rightarrow p_\mathrm{Air} = 1\,\mathrm{bar}$, $p_\mathrm{He} = 1.1\,\mathrm{bar} \Rightarrow \varrho_\mathrm{Air} = 1.23\,\mathrm{kg/m^3}$, $\varrho_\mathrm{He}(p = 1.1\,\mathrm{bar}) = 0.196\,\mathrm{kg/m^3}$

$$\Rightarrow V = \frac{300\,\mathrm{kg}}{(1.23 - 0.196)\,\mathrm{kg/m^3}} = 290\,\mathrm{m^3}\,.$$

b) $h = 20\,\text{km}$, $T = 217\,\text{K}$, $p = 5.5 \cdot 10^{-2}\,\text{bar}$

$$\Rightarrow \varrho_{\text{Air}} = 0.9\,\text{kg/m}^3\,, \quad \varrho_{\text{He}}(p = 0.055\,\text{bar}) = 0.042\,\text{kg/m}^3$$

$$\Rightarrow V = \frac{300}{0.09 - 0.042}\,\text{m}^3 = 6250\,\text{m}^3\,.$$

The balloon has to expand considerably. On the ground it has only 5% of its maximum volume.

7.19 a) If the pressure at the upper end of the atmosphere (which is here assumed to have a sharp edge) should be $p_1 = 10\,\text{bar}$, the pressure at the bottom must be $p_0 = 11\,\text{bar}$. We assume as mean pressure

$$\Rightarrow \varrho_{\text{Air}}(\overline{p} = 10.5\,\text{bar}\,, T = 300\,\text{K}) = 1.23 \cdot 10.5\,\text{kg/m}^3 = 12.9\,\text{kg/m}^3\,,$$

$$\Rightarrow \varrho \cdot g \cdot h = 10^5\,\text{Pa}$$

$$\Rightarrow h = \frac{10^5}{12.9 \cdot 9.81\,\text{m}} = 7.9 \cdot 10^2\,\text{m}\,,$$

b) The density of solid air at $T = 0\,\text{K}$ is $\varrho = 10^3\,\text{kg/m}^3$. $\Rightarrow h = 10\,\text{m}$.

14.8 Chapter 8

8.1 a) The force acting on the area A is according to (8.41b)

$$F_{\text{w}} = c_{\text{w}} \frac{\varrho}{2} u^2 \cdot A\,.$$

Numerical values: $A = 100\,\text{m}^2$, $\varrho_{\text{L}} = 1.225\,\text{kg/m}^3$, $u = 100\,\text{km/h} = 27.8\,\text{m/s}$.
$\Rightarrow F = 5.67 \cdot 10^4\,\text{N}$. This corresponds to a weight of 5.8 tons.
b) For a simple estimation we assume that the streamlines of the wind above the roof are following the roof profile. The air above the roof then passes through a path length $S_2 = 2 \cdot 6\,\text{m} = 12\,\text{m}$. In the same time the horizontal wind flow passes only a distance $S_1 = 2 \cdot 6\,\text{m} \cdot \sin(\alpha/2) = 11.6\,\text{m}$. The velocity is then $u_2 = 100\,\text{km/h} \cdot 12/11.6 = 103.4\,\text{km/h} = 28.7\,\text{m/s}$. From the Bernoulli equation

$$p = p_0 - \tfrac{1}{2}\varrho \cdot \overline{u}_2^2\,.$$

we can determine the pressure difference $\Delta p = p - p_0$. With p_0 (pressure below the roof) $= 10^5\,\text{N/m}^2$ and $\frac{1}{2}\varrho\overline{u}_2^2 = 531\,\text{N/m}^2$ the pressure p becomes $p = (10^5 - 531)\,\text{N/m}^2$ and the difference $531\,\text{N/m}^2$. The force is $F = A \cdot \Delta p$. With $A = 2L_y \cdot 6\,\text{m} \cdot \sin(\alpha/2) = 96.7\,\text{m}^2$ effective roof area (projection onto a horizontal plane) we get

$$F = 531 \cdot 96.7\,\text{N} = 5.1 \cdot 10^4\,\text{N}\,.$$

8.2 The buoyancy depends not only on the wing profile but also on the stalling angle (Fig. 8.43). When a plane flies upside down the buoyancy is much smaller but can be still larger than zero if the stalling angle is correctly chosen.

8.3 The mean free path length $\Lambda = 1/(n \cdot \sigma)$ in liquids with typical densities $n = 3 \cdot 10^{28}\,/\text{m}^3$ and $\sigma = 10^{-19}\,\text{m}^2 \Rightarrow \Lambda = 3 \cdot 10^{-10}\,\text{m}$. The boundary layer where molecules diffuse from neighbouring layers is therefore very thin.
The appearance of curls at large velocities is not caused by diffusion but by macroscopic turbulence (Convection).

8.4 The following relations apply:

$$\textbf{grad}\,(\boldsymbol{a} \cdot \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\,\boldsymbol{a} + (\boldsymbol{a} \cdot \nabla)\,\boldsymbol{b} + \boldsymbol{a} \times (\nabla \times \boldsymbol{b}) + \boldsymbol{b} \times (\nabla \times \boldsymbol{a})$$

$$\Rightarrow \textbf{grad}\,(\boldsymbol{u} \cdot \boldsymbol{u}) = \textbf{grad}\,\overline{u}^2 = 2 \cdot (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{u} + 2 \cdot \boldsymbol{u} \times (\nabla \times \boldsymbol{u})\,.$$

The last equation can be verified in component representation. For the x-component the left hand side can be written as:

$$\frac{\partial}{\partial x}\left(u_x^2 + u_y^2 + u_z^2\right) = 2u_x \frac{\partial u_x}{\partial x} + 2u_y \frac{\partial u_y}{\partial x} + 2u_z \frac{\partial u_z}{\partial x}\,. \tag{14.5}$$

For the components on the right hand side is

$$2\left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_x + 2\left[u_y(\text{curl}\,\boldsymbol{u})_z - u_z(\text{curl}\,\boldsymbol{u})_y\right]\,. \tag{14.6}$$

The second bracket [ ] in Eq. 14.6 is in the component representation

$$u_y\left(\frac{\partial}{\partial x}u_y - \frac{\partial}{\partial y}u_x\right) - u_z\left(\frac{\partial}{\partial z}u_x - \frac{\partial}{\partial x}u_z\right)\,.$$

Inserting this into (14.5) the right hand side gives the same expression as the left hand side.
Analogous results are obtained for the y- and z-component.

8.5 The pressure at the height h is

$$p(h) = \varrho \cdot g \cdot (H - h) + p_0\,.$$

At the exit of the pipe the Bernoulli equation yields

$$\Delta p = p(h) - p_0 = \tfrac{1}{2}\varrho u_x^2$$
$$u_x^2 = 2g(H - h)\,.$$

The trajectory of the liquid stream is a parabola. The initial velocity is

$$v = \{u_x, u_y = 0, u_z = 0\}\,.$$

The drop time can be obtained from $h = (1/2)\,gt^2 \Rightarrow t = \sqrt{2h/g}$.

a) The point of impinge is

$$P = \{x_i = u_x \cdot t\,; \quad y = z = 0\} = \left\{2\sqrt{h(H-h)}, 0, 0\right\} .$$

The velocity at P is

$$\boldsymbol{v}(P) = \{u_x, u_z = gt\}$$
$$|\boldsymbol{v}| = \sqrt{u_x^2 + u_z^2} = \sqrt{2gH} .$$

This is the same velocity as for body falling vertically from the height H.
b) According to the Hagen–Poiseuille Law is:

$$-\frac{\mathrm{d}V}{\mathrm{d}t} = -\pi R^2 \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\pi r^4}{8\eta L}\Delta p$$

with $\Delta p = \varrho g H + p_0 - p_0$

$$\Rightarrow \frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{r^4}{R^2}\frac{\varrho g H}{8\eta L} \Rightarrow H = H_0 \mathrm{e}^{-at}$$

with $H_0 = H(t=0)$ and $a = \dfrac{r^4 \varrho g}{8R^2 L \cdot \eta}$.

8.6 The probe in Fig. 8.10c measures the total pressure

$$p_0 = p + \tfrac{1}{2}\varrho u^2 = \varrho g h$$
$$= 10^3 \cdot 9.81 \cdot 1.5 \cdot 10^{-1}\,\mathrm{Pa} = 1470\,\mathrm{Pa} .$$

The results of the measurements in Fig. 8.10a give $p = 10\,\mathrm{mbar} = 10^3\,\mathrm{Pa}$
$\Rightarrow 0.5\,\varrho u^2 = 470\,\mathrm{Pa} \Rightarrow u = 0.97\,\mathrm{m/s}$.

8.7 If the funnel is filled up to the height H the radius R of the water surface is $R = H \cdot \tan(\alpha/2)$. The volume of the water is then

$$V = \tfrac{1}{3}\pi R^2 H = \tfrac{1}{3}\pi H^3 \tan^2(\alpha/2) = \tfrac{1}{9}\pi H^3 ,$$

because $\tan^2 30° = 1/3$.
a) The reduction of the water volume per time unit is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}H}\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{3}\pi H^2 \frac{\mathrm{d}H}{\mathrm{d}t} .$$

On the other side the Hagen–Poiseuille Law demands

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\pi r^4}{8\eta L}\Delta p$$

with $r = d/2$ and $\Delta p = \varrho g H$

$$\Rightarrow \frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{3}{8}\frac{r^4 \varrho g}{\eta L H} \Rightarrow H\,\mathrm{d}H = -a\,\mathrm{d}t$$

with $a = \frac{3}{8}\frac{r^4 \varrho g}{\eta L} \approx 7.2 \cdot 10^{-4}\,\mathrm{m^2 s^2}$.
Integration gives:

$$H^2 = -2at + H_0^2$$

with $H_0 = H(t=0)$

$$\Rightarrow H = \sqrt{H_0^2 - 2at} .$$

b)
$$\frac{\mathrm{d}M}{\mathrm{d}t} = \varrho\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{3}\pi a H \varrho$$
$$= -\frac{1}{3}\pi a \varrho \sqrt{H_0^2 - 2at}$$
$$\Rightarrow M(t) = \tfrac{1}{9}\pi\varrho\left(H_0^2 - 2at\right)^{3/2} .$$

c) The time when all of the water has streamed out of the funnel (i. e. $H(t) = 0$) is

$$T = H_0^2/2a .$$

With $H_0 = 0.3\,\mathrm{m}$, $r = 2.5 \cdot 10^{-3}\,\mathrm{m}$, $L = 0.2\,\mathrm{m}$, $\eta = 1.0 \cdot 10^{-3}\,\mathrm{Pa\,s} \Rightarrow T = 62.5\,\mathrm{s}$.
d) With 4 litre water $\Rightarrow H_0 = (9V/\pi)^{1/3} = 0.225\,\mathrm{m}$.
The time to fill the container with $V = 4\,\mathrm{l}$ completely is with $a = 7.2 \cdot 10^{-4}$ $T = 35\,\mathrm{s}$. If the outflowing water in the funnel is continuously substituted by pouring water into the funnel in order to keep the water level always constant at $H = H_0$, the time to fill the $4\,\mathrm{l}$ container is obtained by:

$$V = \frac{1}{3}\pi a H_0 \cdot t$$
$$\Rightarrow t = \frac{3 \cdot 4 \cdot 10^{-3}}{\pi \cdot 7.2 \cdot 10^{-4} \cdot 0.225\,\mathrm{s}} = 23.6\,\mathrm{s} .$$

8.8
$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi R^4}{8\eta L}\Delta p \quad \text{with} \quad \Delta p = \varrho g(\Delta h + L\sin\alpha)$$
$$= 1.5 \cdot 10^{-4}(0.1 + \sin\alpha)\,\mathrm{m^3/s}.$$

The mean flow velocity is

$$\overline{u} = \frac{1}{A}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{\pi r^2}\frac{\mathrm{d}V}{\mathrm{d}t} = 7.6(0.1 + \sin\alpha)\,\mathrm{m/s} .$$

The Reynold's number is

$$\mathrm{Re} = 2300 = \frac{\varrho r u_c}{\eta} .$$

This gives the critical velocity

$$\overline{u}_c = \frac{\eta \mathrm{Re}}{\varrho r} = 0.92\,\mathrm{m/s} .$$

The inclination angle α is then for $\overline{u} = \overline{u}_c$

$$\sin\alpha = 0.021 \Rightarrow \alpha = 1.2° .$$

8.9
$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi R^4}{8\eta L}\varrho g\,\Delta h = 10^{-3}\,\mathrm{m^3}; \quad \Delta h = 20\,\mathrm{m}$$
$$\Rightarrow R = \left(\frac{10^{-3} \cdot 8\eta L}{\pi\varrho g \cdot \Delta h}\right)^{1/4} = 6 \cdot 10^{-3}\,\mathrm{m} = 6\,\mathrm{mm}$$
$$\Rightarrow d = 1.2\,\mathrm{cm}$$
$$\Rightarrow u = 8.8\,\mathrm{m/s} .$$

This is already above the critical velocity, which means that d has to be larger because the flow resistance is for $\overline{u} > u_c$ larger than obtained from the Hagen–Poiseuille law.

8.10 The total force acting on the ball is

$$F = am = m^* g - 6\pi\eta r v \quad \text{with} \quad m^* = (\varrho_K - \varrho_l)\frac{4}{3}\pi r^3$$
$$\Rightarrow \frac{dv}{dt} = \frac{m^*}{m} g - \frac{6\pi\eta r v}{m} .$$

Rearrangement, division by m^* and multiplication by m yields

$$\frac{dv}{g - (6\pi\eta r v/m^*)} = \frac{m^*}{m}\, dt ,$$

with the abbreviations

$$b = \frac{6\pi\eta r}{g m^*} \quad \text{and} \quad c = g\frac{m^*}{m}$$
$$\Rightarrow \frac{dv}{1 - bv} = c\, dt .$$

Integration gives

$$-\frac{1}{b}\ln(1 - bv) = ct + C_1$$
$$\Rightarrow v = \frac{1}{b}\left(1 - e^{-bC_1} e^{-bct}\right) .$$

Since $v(0) = v_0 \Rightarrow e^{-bC_1} = 1 - v_0 b$

$$\Rightarrow v(t) = \frac{1}{b}\left(1 + (v_0 b - 1)e^{-bct}\right)$$
$$\Rightarrow z(t) = \frac{1}{b}t - \frac{v_0 b - 1}{b^2 c} e^{-bct} .$$

8.11 Division of (8.36a)by ϱ and applying the differential operator $\mathbf{rot} = \nabla\times$ onto both sides yields

$$\frac{\partial}{\partial t}\mathbf{rot}\,\boldsymbol{u} + \nabla \times (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$$
$$= -\frac{1}{\varrho}\nabla \times (\nabla p) - \nabla \times \boldsymbol{g} + \frac{\eta}{\varrho}\nabla \times \operatorname{div}\mathbf{grad}\,\boldsymbol{u} .$$

Now we use the relations $\nabla\times\nabla p = 0$ and $\nabla\times\nabla\cdot(\nabla\boldsymbol{u}) = \mathbf{0}$. If the influence of gravity can be neglected ($\Rightarrow \boldsymbol{g} = \mathbf{0}$) we obtain with $\Omega = \mathbf{rot}\,\boldsymbol{u}$ the relation

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \tfrac{1}{2}\mathbf{grad}\,u^2 - \boldsymbol{u}\times\mathbf{rot}\,\boldsymbol{u}$$
$$= \tfrac{1}{2}\nabla u^2 + (\Omega \times \boldsymbol{u}) .$$

Vector multiplication with ∇ gives with $\nabla \times \nabla = 0$

$$\nabla \times (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) .$$

Then (8.36a) converts to

$$\frac{\partial\Omega}{\partial t} + \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = 0 .$$

14.9 Chapter 9

9.1 Through the capillary flows per second the air mass $\varrho \cdot dV/dt \propto p \cdot dV/dt$. At the high pressure side this is $p_1 \cdot dV_1/dt$ and at the low pressure side $p_2 \cdot dV_2/dt$. It is

$$p_1\frac{dV_1}{dt} = p_2\frac{dV_2}{dt} .$$

The pumped-out volume is V_2. According to Hagen–Poiseuille we get

$$p_2\frac{dV}{dt} = \frac{\pi R^4}{8\eta L}(p_1 - p_2)\frac{p_1 + p_2}{2}$$

with $p_1 = 10^5$ Pa, $p_2 = 10^{-1}$ Pa

$$\Rightarrow p_2\frac{dV}{dt} = 4.25 \cdot 10^{-3}\,\text{m}^3\,\text{Pa/s} .$$

In order to maintain a pressure of 10^{-3} hPa $= 10^{-1}$ Pa the throughput of the vacuum pump must be at least $dV/dt = 4.25 \cdot 10^{-2}\,\text{m}^3/\text{s} = 42.5\,\text{l/s}$.

9.2 The force acting on each hemisphere is

$$F = \pi \cdot \left(\frac{d}{2}\right)^2 \Delta p = 2.5 \cdot 10^4\,\text{N} .$$

One has to pull on each hemisphere with this force in order to separate the two hemispheres.

9.3 For $p = 10^{-5}$ hPa is $n = 2.5 \cdot 10^{17}\,/\text{m}^3$.

$$\Lambda = 6\,\text{m} ,\ \tau = \frac{\Lambda}{\overline{v}} \approx 1.2 \cdot 10^{-2}\,\text{s} \quad \text{for} \quad \overline{v} = 500\,\text{m/s}$$
$$Z_1 = n \cdot \sigma \cdot \overline{v}_{\text{rel}} \approx n\sigma\sqrt{2}\cdot\overline{v} \quad \text{with} \quad \sigma = 10^{-14}\,\text{cm}^2$$

The number Z_1 of collisions between the molecules is

$$Z_1 \approx 180\,\text{s}^{-1} .$$

The number Z_2 of collisions per sec with the wall is

$$Z_2 = \tfrac{1}{4}n\overline{v} \approx 3 \cdot 10^{19}\,\text{m}^{-2}\text{s}^{-1} .$$

Onto the whole container wall with $A = 3.26\,\text{m}^2$ impinge $9.8 \cdot 10^{19}$ molecules per second $\Rightarrow Z_1/Z_2 = 1.8 \cdot 10^{-18}$; $\sum s_i = n \cdot V \cdot \Lambda/\tau = n \cdot V \cdot \overline{v} = 5 \cdot 10^{19}\,\text{m/s}$.

9.4 The number of collisions per second and per m^2 with the wall is $Z_2 = \frac{1}{4}n \cdot \overline{v}$. At a pressure $p = 10^{-7}$ hPa $\Rightarrow n = 2.5 \cdot 10^{15}\,\text{m}^{-3}$. With $\overline{v} = 500\,\text{m/s} \Rightarrow Z_2 = 3 \cdot 10^{17}\text{m}^{-2}\,\text{s}^{-1}$. A complete monolayer on the wall is achieved for Z collisions, where

$$Z = \frac{1\,\text{m}^2}{(\text{surface area per molecule})}$$
$$= \frac{1}{0.15 \cdot 0.2 \cdot 10^{-18}} = 3.3 \cdot 10^{19} .$$

Since the number of wall colissions per sec and m^2 is $Z_2 = 3 \cdot 10^{17}$, it follows that after about 100 s the wall is completely covered by a monolayer.

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9.5 The suction capacity $\mathrm{d}V/\mathrm{d}t$ of a mechanical pump at the pressure $p_1 = 0.1$ hPa must be equal to the suction capacity of the diffusion pump at the pressure $p_2 = 10^{-6}$ hPa.

$$p_2 \cdot 3000\,\mathrm{l/s} = p_1 (\mathrm{d}V/\mathrm{d}t)_{\text{mech. pump}}$$

$$\Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{p_2}{p_1} \cdot 3000\,\mathrm{l/s} = 3 \cdot 10^{-2}\,\mathrm{l/s} = 0.1\,\mathrm{m^3/h}\ .$$

It is, however, advisable to use a larger mechanical pump. Because the diffusion pump reaches its full suction capacity already at a pressure of 10^{-4} hPa, where the mechanical pump needs a suction capacity of $\mathrm{d}V/\mathrm{d}t = 10^{-3} \cdot 3000\,\mathrm{l/s} = 10\,\mathrm{m^3/h}$ in order to prevent the rise of the pressure p_1 above 10^{-1} hPa.

9.6 When passing through the gas the intensity of the electron beam decreases according to $I = I_0 \mathrm{e}^{-n \cdot \sigma \cdot x}$. The number of produced ions is then equal to the difference $(I_0 - I)/q$, where $q = -\mathrm{e} = -1.6 \cdot 10^{-19}$ C is the electron charge. For $n \cdot \sigma \cdot x \ll 1$ we obtain

$$\frac{I_0 - I}{\mathrm{e}} = \frac{I_0}{\mathrm{e}} n\sigma x$$
$$= \frac{10^{-2}}{1.6 \cdot 10^{-19}} \cdot 2.5 \cdot 10^{15} \cdot 10^{-18} \cdot 2 \cdot 10^{-2}\,\mathrm{s^{-1}}$$
$$= 3 \cdot 10^{12}\,\mathrm{ions/s}\ .$$

The ion current is then 0.5 µA.

9.7 The mean free path length Λ at a pressure $p = 10^{-2}$ hPa is $\Lambda = 6 \cdot 10^{-3}$ m (see Tab. 9.1) and therefore comparable to the distance $d = 1$ cm between the hot wire and he wall. This case is between the limiting cases $\Lambda \gg d$ and $\Lambda \ll d$. For $\Lambda \gg d$ we get from (7.49)

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \kappa F \Delta T = 52\,\mathrm{mW}\ .$$

For $\kappa = n_1 \cdot \overline{v}(f/2)k = 4.4\,\mathrm{N\,m^{-1}\,K^{-1}}$ $F = 2\pi r_1 \cdot l = 7.8 \cdot 10^{-5}\,\mathrm{m^2}$ und $\Delta T = 150$ K.
For $\Lambda \ll d$ we obtain

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \lambda \cdot F \frac{\mathrm{d}t}{\mathrm{d}x} = 60\,\mathrm{mW}\ .$$

If we choose for $\Lambda \approx d$ the average of the two values we get

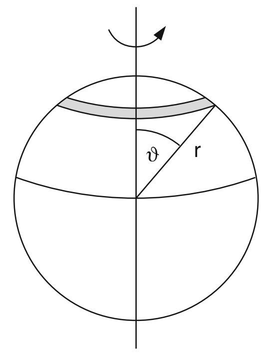
$$\frac{\mathrm{d}W}{\mathrm{d}t} \approx 56\,\mathrm{mW}\ .$$

The electric power input is

$$P_\mathrm{e} = U \cdot I = 1\,\mathrm{W}\ .$$

Only 5.6% of the input power are transported by heat conduction through the gas. The major part (94.4%) are lost due to heat radiation and heat conduction through the mountings.

9.8 We assume that every molecule that hits the wall sticks there for some time and then evaporates again. For a ball at rest



with a temperature equal to that of the gas the velocity distribution of the evaporating molecules is equal to that of the impinging molecules. Therefore there is no net angular momentum transfer. This is different for the rotating ball. Molecules that impinge onto a strip around the latitude ϑ get the rotational velocity

$$\overline{u} = \omega \cdot r \cdot \sin\vartheta\ ,$$

while resting on the rotating sphere. We choose the z-axis into the direction of the rotational axis. The velocity perpendicular to the rotational axis is for the impinging molecules $v_\perp = (v_x^2 + v_y^2)^{1/2}$ and the evaporating molecules have the velocity $v'_\perp = (v_x^2 + v_y^2 + u^2)^{1/2}$.
The number of molecules impinging per sec onto the surface element $\mathrm{d}A = 2\pi r^2 \sin\vartheta\,\mathrm{d}\vartheta$ (the grey strip in the Figure) is according to (7.47)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{n}{4}\overline{v}\,\mathrm{d}A\ .$$

Each evaporating molecule wins the additional momentum $m \cdot \overline{u}$ in tangential direction. The total number impinging on $\mathrm{d}A$ gets the additional momentum per sec

$$\Delta p(\vartheta) = \frac{n}{4}\overline{v} m \overline{u}\,\mathrm{d}A$$
$$= \frac{n}{4}\overline{v} m \omega r \sin\vartheta \cdot 2\pi r^2 \sin\vartheta\,\mathrm{d}\vartheta\ .$$

The torque transferred to the ball by these molecules is then with $F = \mathrm{d}p/\mathrm{d}t$

$$\mathrm{d}D(\vartheta) = \frac{n}{2}\pi \overline{v} m \omega r^4 \sin^3\vartheta\,\mathrm{d}\vartheta\ .$$

Integration over all ϑ gives with

$$\int_{-\pi/2}^{+\pi/2} \sin^3\vartheta\,\mathrm{d}\vartheta = \frac{4}{3}\ ,$$

the torque

$$D = \tfrac{2}{3}\pi m \overline{v} \omega n r^4$$
$$= \tfrac{1}{2} V_{\text{sphere}} m \overline{v} \omega n r \quad \text{with} \quad V = M/\varrho$$

$$D = -\frac{d}{dt}L = -I\frac{d\omega}{dt}$$
$$\Rightarrow \frac{d\omega}{dt} = -\frac{D}{(2/5)Mr^2} = -a\omega$$

with

$$a = \frac{5nm\overline{v}}{4r\varrho} = \frac{10}{\pi}\frac{p}{r\varrho\overline{v}} \approx 3.18\frac{p}{r\varrho\overline{v}} ,$$

where the relations $p = (1/3)\,n \cdot m \cdot \overline{v}^2$ and $\overline{v}^2 = (3kT)/m$ have been used.

$$\Rightarrow \frac{d\omega}{\omega} = -a\,dt \Rightarrow \omega = \omega_0 e^{-at} .$$

For $\omega = 0.99\,\omega_0$ we get $e^{-at} = 0.99$

$$\Rightarrow t = \frac{1}{a}\ln\frac{100}{99} = \frac{0.01}{a} .$$

Numerical example:

$$r = 1 \cdot 10^{-3}\,\text{m},$$
$$\varrho = 5 \cdot 10^3\,\text{kg/m}^3,$$
$$\overline{v} = 5 \cdot 10^2\,\text{m/s},$$
$$p = 10^{-3}\,\text{hPa} = 10^{-1}\,\text{Pa}$$
$$\Rightarrow a = 1.3 \cdot 10^{-4}\,\text{s}^{-1} \Rightarrow t = 78\,\text{s}.$$

14.10 Chapter 10

10.1 The ratio of E_{kin}/E_{pot} is generally larger for liquids than for solids. Therefore, the atoms move in the upper part of the interaction potential $V(r_{ik})$ between neighbouring atoms. Here the slope of the attractive part of the potential is smaller, therefore, the mean distance $\langle r_{ik}\rangle$ increases faster with increasing energy than in the lower part of the potential (Fig. 6.1).

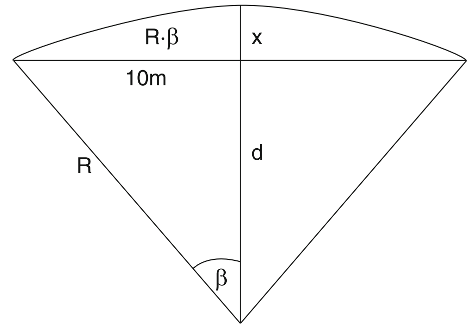
10.2 a)
$$\Delta L = \alpha \cdot L \cdot \Delta T$$
$$= 16 \cdot 10^{-6} \cdot 20 \cdot 40\,\text{m}$$
$$= 1.28 \cdot 10^{-2}\,\text{m} = 1.28\,\text{cm} .$$

b) The maximum distortion can be obtained from the figure below as

$$x = R - d = R(1 - \cos\beta) .$$

Since the length L increases by $\Delta L = 1.28\,\text{cm}$, the half length of the circular arc is

$$R \cdot \beta = (10 + 0.64 \cdot 10^{-2}) = 10.0064\,\text{m}$$
$$\frac{10}{R} = \sin\beta .$$



Division by $R \cdot \sin\beta$ gives

$$\Rightarrow \frac{\beta}{\sin\beta} = 1.00064 \quad \Rightarrow \frac{1}{1 - \frac{1}{6}\beta^2} = 10.0064\,\text{m}$$
$$\Rightarrow \beta = 0.00623 \mathrel{\hat{=}} 3.57°$$
$$\Rightarrow R = \frac{10\,\text{m}}{\sin 3.57°} = 160.6\,\text{m}$$
$$\Rightarrow x = 160.6 \cdot (1 - \cos 3.57°) = 0.31\,\text{m} .$$

c) If the distortion should be prevented, the necessary pressure onto the rail in the longitudinal direction can be obtained from the relation

$$\Rightarrow \frac{F}{A} = E \cdot \frac{\Delta L}{L} = 200 \cdot 10^9\,\text{N/m}^2 \cdot \frac{0.0128}{20}$$
$$= 0.128\,\text{GPa}$$
$$F = A \cdot E \cdot \frac{\Delta L}{L} = 2.56 \cdot 10^6\,\text{N} .$$

10.3 The heat energy of one mole is

$$Q = \frac{f}{2} \cdot R \cdot T .$$

a) For helium is $f = 3 \Rightarrow Q = \frac{3}{2}R \cdot T$
The heating energy is then

$$W = 10 \cdot t\,\text{W s}$$
$$= \tfrac{3}{2}R(100 - 20)\,\text{K} + 10\,\text{W s/K} \cdot 80\,\text{K} ,$$

where the last term takes into account the heating of the container wall. With $R = 8.3\,\text{J/(K} \cdot \text{mol)}$ we get

$$t = \frac{120 \cdot 8.31 + 800}{10}\,\text{s} = 180\,\text{s} = 3\,\text{min} .$$

For N_2-molecules at the temperature $T > 300\,\text{K}$ is $f = 5/2$.

$$\Rightarrow t = \frac{200 \cdot 8.31 + 800}{10}\,\text{s} = 246\,\text{s} = 4.1\,\text{min} .$$

b) The heating up to 1000 °C takes for helium the time

$$t = \frac{980 \cdot 3R/2 + 9800}{10}\,\text{s} \approx 2200\,\text{s} \approx 37\,\text{min} ;$$

Chapter 14

for N_2 is f in the range from 20–500 °C $f = 5$; for $T > 500\,°\mathrm{C}$ is $f = 7/2$.

$$\mathrm{d}Q = U(T_2) - U(T_1) = R(\tfrac{7}{2}T_2 - \tfrac{5}{2}T_1) = 2.89 \cdot 10^4\,\mathrm{J}$$

$$\Rightarrow t = \frac{\mathrm{d}Q}{10}\,\mathrm{s} = 2.89 \cdot 10^3\,\mathrm{s} = 48\,\mathrm{min}\ .$$

10.4 If T is lower than the temperature T_S of the surrounding the temperature T will be approach T_S by heat conduction. After the mixing the temperature T_m is above T_S. The heat losses are proportional to the temperature difference $(T - T_S)$. Therefore the temperature decline

$$\mathrm{d}T/\mathrm{d}t = -a \cdot (T - T_S)\ .$$

The time-dependence of $T(t)$ after the mixture at $t = t_1$ is then

$$\frac{\mathrm{d}T}{T - T_S} = -a(t - t_1) \Rightarrow T - T_S = C\,\mathrm{e}^{-a(t-t_1)}\ .$$

If the mixing process occurs in a very short time at $t = t_1$ it is $T(t = t_1) = T_m \Rightarrow C = T_m - T_S \Rightarrow T(t) = T_S + (T_m - T_S) \cdot \mathrm{e}^{-a\cdot(t-t1)}$.
If the real measured temperature curve $T(t)$ is replaced by the dashed curve in Fig. 10.12b in such a way, that the areas A_1 and A_2 are equal, the true mixing temperature is obtained. The dashed curve represents the ideal case of an infinitesimal short mixing process, where the heat losses are zero during the mixing process.

10.5 1 Mole air (N_2/O_2-mixture) has a mass of about 29 g and contains $6 \cdot 10^{23}$ molecules. For the lift of the container by 10 cm in the gravity field of the earth the energy

$$E = m \cdot g \cdot h = 0.129\,\mathrm{kg} \cdot 9.81\,\mathrm{m/s^2} \cdot 0.1\,\mathrm{m} = 0.13\,\mathrm{N \cdot m}$$

is required.
The thermal energy of the gas at room temperature is

$$E_{\mathrm{th}}^{\mathrm{gas}} = (5/2)R \cdot T = 6.2 \cdot 10^3\,\mathrm{N \cdot m}\ ,$$

and that of the container with the specific heat c is

$$E_{\mathrm{th}}^{\mathrm{cont}} = m_C \cdot c \cdot T = 0.1 \cdot 10^3 \cdot 300 = 3 \cdot 10^4\,\mathrm{N \cdot m}$$

$$\Rightarrow E_{\mathrm{tot}} = 3.6 \cdot 10^4\,\mathrm{N \cdot m}\ .$$

The energy additionally required for the lift is therefore very small compared to the thermal energy. And the cooling after the lift would be only $\Delta T = 1 \cdot 10^{-3}$ K. Nevertheless this lift is extremely improbable, not because of energetic reasons but because of statistical reasons:
For the lift the z-component of the momentum must be at least $p_z = m \cdot v_{0z} = m \cdot \sqrt{2gh} \Rightarrow p_z > 0.18\,\mathrm{kg \cdot m/s}$. The mean velocity component $\langle v_z \rangle$ of all molecules, which is $\langle v_z \rangle = 0$ at thermal equilibrium, must be $\langle v_z \rangle > v_0 = p_z/m_{\mathrm{gas}} = 6.2\,\mathrm{m/s}$. The probability that a molecule has a velocity component $v_z > v_0$ is given by the integral

$$W(v_z > v_0) = \int_{v_0}^{\infty} \mathrm{e}^{-v_z^2 m/2kT}\mathrm{d}v_z \Big/ \int_{-\infty}^{+\infty} \mathrm{e}^{-v_z^2 m/2kT}\mathrm{d}v_z\ .$$

With $x^2 = \frac{1}{2}m \cdot v_z^2/kT$ and $x(v_0) \ll 1 \Rightarrow$

$$\int_{x_0}^{\infty} = -\int_0^{x_0} + \int_0^{\infty}$$

$$\text{with} \int_0^{x_0} \mathrm{e}^{-x^2}\mathrm{d}x \approx \int_0^{x_0} (1 - x^2)\mathrm{d}x = x_0 - \frac{x_0^3}{3}\ .$$

This gives with $v_0 = 6.2\,\mathrm{m/s} \Rightarrow W(v_z > v_0) = 0.49$.
The probability that all $6 \cdot 10^{23}$ molecules have at the same time the velocity component $v_z > v_0$ is then

$$W = 0.49^{-6 \cdot 10^{23}} < 10^{-10^{-23}}\ ,$$

and therefore practically zero.

10.6 a) The entropy change ΔS for an isobaric temperature rise from $T_0 = 273$ K to $T_1 = 500$ K is

$$\Delta S_{\mathrm{isobaric}} = \nu \left(C_V \ln \frac{T_1}{T_0} + R \ln \frac{V_1}{V_0} \right)\ ,$$

where $\nu = 1/22.4$ is the mole fraction.
With $V_1/V_0 = T_1/T_0$ for $p = \mathrm{const}$ and $C_p = R + C_V$ we obtain

$$\Delta S_{\mathrm{isobaric}} = \nu C_p \ln(T_1/T_0)$$

with $C_p = 21\,\mathrm{J/(K \cdot mol)}$

$$\Rightarrow \Delta S_{\mathrm{isobaric}} = \frac{21}{22.4} \ln \frac{500}{273} = 0.57\,\mathrm{J/K}\ .$$

b) isochoric heating:

$$\Delta S_{\mathrm{isochoric}} = \nu C_V \ln \frac{T_1}{T_0}$$

with $C_V = 12.7\,\mathrm{J/(K \cdot mol)}$

$$\Rightarrow \Delta S_{\mathrm{isochoric}} = 0.34\,\mathrm{J/K}\ .$$

10.7 With $M = \varrho \cdot V$ the critical mole volume is

$$V_c = \frac{0.044\,\mathrm{kg}}{\varrho_c} = \frac{0.044}{46} = 9.56 \cdot 10^{-4}\,\mathrm{m^3} = 0.956\,\mathrm{l}\ .$$

The mole volume is compressed from 22.4 l at standard conditions of an ideal gas to 0.956 l.

From the general gas equation of an ideal gas it follows:

$$V_c = \frac{RT_c}{p_c} = 0.33 \cdot 10^{-3}\,\mathrm{m}^3 .$$

This shows that eigen-volume and internal pressure of a real gas around the critical point cause considerable deviations from the ideal gas. From Eq. 10.129 we obtain the van-der-Waals constants

$$b = \frac{1}{3}V_c \quad \Rightarrow \quad b = 0.32 \cdot 10^{-3}\,\mathrm{m}^3 ,$$
$$a = 3p_c V_c^2 \quad \Rightarrow \quad a = 20.8\,\mathrm{N \cdot m^4} .$$

Under standard conditions ($p = 1$ bar, $T = 273$ K) the internal pressure is for 1 mole

$$\frac{a}{V^2} = 4.1 \cdot 10^4\,\mathrm{N/m^2}$$
$$\widehat{=}\ 41\%\ \text{of normal pressure!}$$

The eigen-volume of CO_2 molecules is $b/4 = 8 \cdot 10^{-5}\,\mathrm{m}^3$ and the relative correction $b/V = \frac{0.32 \cdot 10^{-3}}{22.4 \cdot 10^{-3}} = 1.4\%$.

10.8

$$\Delta S_1 = mc_v \ln \frac{323.15}{273.15}\,\mathrm{J/K}$$
$$= 4.18 \cdot 10^3 \ln 1.183\,\mathrm{J/K}$$
$$= 689\,\mathrm{J/K} ;$$

$$\Delta S_2 = mc_v \ln \frac{T_m^2}{T_1 T_2}$$
$$= 0.5 \cdot 4.18 \cdot 10^3 \cdot \ln \frac{323.15^2}{273.15 \cdot 373.15}$$
$$= 49.62\,\mathrm{J/K} .$$

10.9 The theoretically possible maximum efficiency for $T_1 = 600\,°\mathrm{C}$ and $T_2 = 100\,°\mathrm{C}$ is

$$\eta = \frac{T_1 - T_2}{T_1} = \frac{500\,\mathrm{K}}{873\,\mathrm{K}} = 0.57 .$$

The heat delivered at 100 °C amounts therefore to 43% of the heat reveived at 600 °C. When using the technique of "cogeneration of heat and mechanical power" the heat delivered at 100 °C can be partly used for heating of buildings. The efficiency increase for cooling down to 30 °C is

$$\varepsilon = \frac{100 - 30}{373} = 18.8\% .$$

However, this saves only part of this efficiency increase, because when the additional heat energy available by cooling from 100 °C down to 30 °C can be used for driving a gas-turbine connected to an electric generator, this can deliver additional electric power. The theoretical efficiency of the power station increases then from $\eta = 57\%$ to $\eta = 570/873 = 0.65 = 65\%$. In order to prevent the water vapour to condensate at temperatures below 100 °C one has to decrease the pressure in the expansion chamber. This demands additions energy for the expansion against the external pressure.

10.10 a)

$$m_1 c_1 (T_1 - T_m) = m_2 c_2 (T_m - T_2)$$
$$\Rightarrow T_m = \frac{m_1 c_1 T_1 + m_2 c_2 T_2}{m_1 c_1 + m_2 c_2} .$$

Numerical values:

$$m_1 = 1\,\mathrm{kg}, \quad c_1 = 470\,\mathrm{J/(kg \cdot K)},$$
$$m_2 = 10\,\mathrm{kg}, \quad c_2 = 4.17 \cdot 10^3\,\mathrm{J/(kg \cdot K)}$$
$$\Rightarrow T_m = 23.2\,°\mathrm{C} \ \widehat{=}\ 296.34\,\mathrm{K} .$$

b)

$$\Delta S_1 = +mc \ln \frac{T_m}{T_1} = 1 \cdot 470 \ln \frac{296.34}{573.15}$$
$$= -310\,\mathrm{J/K}$$
$$\Delta S_2 = 10 \cdot 4.1 \cdot 10^3 \ln \frac{296.34}{293.15}$$
$$= +445\,\mathrm{J/K}$$
$$\Delta S = \Delta S_1 + \Delta S_2 = +135\,\mathrm{J/K} .$$

10.11 a) A mass $M = 2 \cdot 5\,\mathrm{kg} = 10\,\mathrm{kg}$ exerts a pressure onto the area $0.1\,\mathrm{m} \cdot 10^{-3}\,\mathrm{m} = 10^{-4}\,\mathrm{m}^2$

$$p = \frac{98}{10^{-4}}\,\mathrm{N/m^2} = 9.8 \cdot 10^5\,\mathrm{Pa} .$$

Since the increase of the melting temperature T_m under the pressure p at $T = -8\,°\mathrm{C}$ is given by $\mathrm{d}T_m/\mathrm{d}p = 10^{-7}\,°\mathrm{C/Pa}$ the resulting temperature increase is $\mathrm{d}T_m = 10^{-7} \cdot 9.8 \cdot 10^5\,°\mathrm{C} = 0.1\,°\mathrm{C}$. Ice at $T = -8\,°\mathrm{C}$ therfore cannot melt solely due to the pressure.
b) The heat conduction is, according to (10.35)

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \lambda \cdot A \cdot \frac{\mathrm{d}T}{\mathrm{d}x} .$$

With $A = \pi r^2 = \pi \cdot 0.25 \cdot 10^{-6}\,\mathrm{m}^2 = 7.8 \cdot 10^{-7}\,\mathrm{m}^2$, $\mathrm{d}x = 5\,\mathrm{cm} = 0.05\,\mathrm{m}$, $\mathrm{d}T = 35\,°\mathrm{C} = 35\,\mathrm{K}$ and $\lambda = 67\,\mathrm{W/m^{-1} \cdot K^{-1}}$, for steel we get

$$dQ/\mathrm{d}t = 0.037\,\mathrm{W} .$$

This heat energy flows into the horizontal part of the wire and in the surrounding volume of the ice. Since the surface $2\pi rL$ of the wire with radius r and length L is large compared to its volume, we can assume that nearly all of the heat energy flows into the ice. If the wire should melt the ice, it has to increase the temperature up to T_m and in addition it has to supply the heat of fusion. The energy balance requires:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = (c_{\mathrm{Ice}} \cdot \Delta T + W_m)\varrho \cdot \frac{\mathrm{d}V}{\mathrm{d}t} .$$

This gives the velocity with which the wire melts through the ice block:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{L \cdot d \cdot \varrho} \cdot \frac{\mathrm{d}Q/\mathrm{d}t}{c \cdot \Delta T + W_m} .$$

Chapter 14

Inserting the numerical values gives: $c = 2.1\,\mathrm{kJ/(kg\cdot K)}$, $\varrho = 0.9\cdot 10^3\,\mathrm{kg/m^3}$, $L = 0.1\,\mathrm{m}$, $d = 10^{-3}\,\mathrm{m}$, $W_\mathrm{m} = 333\,\mathrm{kJ/kg}$, $\Delta T = 8\,\mathrm{K}$.
This finally yields the result:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 10^{-6}\,\mathrm{m/s}\,.$$

The observed velocity is higher by one order of magnitude. The reason for this is the heat, transferred directly from the warm wire at the edges of the horizontal part into the ice. This increases $\mathrm{d}Q/\mathrm{d}t$ considerably.

10.12 During the compression of V_1 to $V_2 < V_1$ the work supplied to the system is

$$\mathrm{d}W_1 = \int_{V_1}^{V_2} p\,\mathrm{d}V\,.$$

With $\mathrm{d}S = \mathrm{d}Q/T = 0 \Rightarrow \mathrm{d}Q = 0 \Rightarrow$

$$p\cdot V^\kappa = \text{const} = C_1 \Rightarrow$$

$$\mathrm{d}W_1 = -\int_{V_1}^{V_2} \frac{C_1}{V^\kappa}\,\mathrm{d}V = \frac{C_1}{\kappa-1}\left(\frac{1}{V_2^{\kappa-1}} - \frac{1}{V_1^{\kappa-1}}\right)$$
$$= \frac{C_1}{(\kappa-1)\cdot V_2^{\kappa-1}}\left(1 - \frac{1}{(V_1/V_2)^{\kappa-1}}\right)\,.$$

During the transition $3\to 4$ the system delivers the work

$$\mathrm{d}W_2 = \frac{C_2}{\kappa-1}\left(\frac{1}{V_1^{\kappa-1}} - \frac{1}{V_2^{\kappa-1}}\right)\,.$$

The gain of energy during one cycle is then

$$\Delta W = \frac{C_1 - C_2}{(\kappa-1)V_2^{\kappa-1}}\left(1 - \frac{1}{(V_1/V_2)^{\kappa-1}}\right)\,.$$

For the isochoric processes $2\to 3$ and $4\to 1$ is $\Delta W = 0$.
The heat energy supplied to the ssterm is for 1 mole

$$Q_1 = C_V(T(3) - T(1))\,.$$

For isentropic processes is $\mathrm{d}Q = 0$, $T\cdot V^{\kappa-1} = \text{const} = C_1$ for the point 2 and C_2 for the point 3.
With $\kappa = C_p/C_V \Rightarrow (\kappa-1)\cdot C_V = C_p - C_V = R$. The efficiency then becomes

$$\eta = \frac{\Delta W}{Q_1} = 1 - \frac{1}{(V_1/V_2)^{\kappa-1}}\,.$$

Example: $V_1 = 10\,V_2$, $\kappa = 1.4 \Rightarrow \eta = 0.6$.
Note that η does not depend on the temperature but only on the compression ration V_1/V_2.

10.13 We treat at first the stationary case with $\Delta T = 0$

$$\Rightarrow T = T_0 + (T_1 - T_0)\mathrm{e}^{-\alpha_1 x}$$
$$\Rightarrow \frac{\partial^2 T}{\partial x^2} = \alpha_1^2(T_1 - T_0)\mathrm{e}^{-\alpha_1 x}\,. \tag{14.7}$$

The stationary heat conduction equation (10.38a) gives with $\partial T/\partial t = 0$

$$\frac{\lambda}{\varrho\cdot c}\frac{\partial^2 T}{\partial x^2} = h^*\cdot(T - T_0)\,.$$

Inserting of (14.7) yields

$$\frac{\partial^2 T}{\partial x^2} = \frac{\varrho c h^*}{\lambda}(T_1 - T_0)\mathrm{e}^{-\alpha_1 x}$$
$$\Rightarrow \alpha_1^2 = \frac{\varrho c h^*}{\lambda} \Rightarrow \alpha_1 = \sqrt{\frac{\varrho c h^*}{\lambda}} = \sqrt{\frac{\varrho\cdot h}{m\cdot\lambda}}\,.$$

For $\Delta T \neq 0 \Rightarrow T = T_0 + (T_1 - T_0)\mathrm{e}^{-\alpha_1 x} + \Delta T\mathrm{e}^{-\alpha_2 x}\cos(\omega t - kx)$,

$$\frac{\partial T}{\partial t} = -\Delta T\cdot\mathrm{e}^{-\alpha_2 x}\cdot\omega\sin(\omega t - kx)\,,$$
$$\frac{\partial^2 T}{\partial x^2} = \alpha_1^2(T_1 - T_0)\mathrm{e}^{-\alpha_1 x} + (\alpha_2^2 - k^2)\Delta T\mathrm{e}^{-\alpha_2 x}$$
$$\cdot\cos(\omega t - kx) - 2\alpha_2\Delta T\mathrm{e}^{-\alpha_2 x}\cdot k\cdot\sin(\omega t - kx)\,.$$

Inserting into (10.38a) gives the stated relations by comparing the coefficients of sin and cos.

14.11 Chapter 11

11.1 $F = D\cdot x$. With $F = 1\,\mathrm{N}$ and $x = 0.05\,\mathrm{m} \Rightarrow D = 20\,\mathrm{N/m}$. $\Rightarrow T = 2\pi\sqrt{m/D} = 1.4\,\mathrm{s}$.

11.2 a) Approximation $m \ll M = \mu\cdot L$.
Velocity of the transverse wave

$$v_\mathrm{Ph} = \sqrt{F/\mu} = \sqrt{mg/\mu}\,.$$

Running time of the wave over the distance z:

$$t_1 = \frac{z}{v_\mathrm{Ph}} = z\sqrt{\mu/mg}\,.$$

Falling time of the ball:

$$t_2 = \sqrt{2z/g}\,.$$

For $t_1 = t_2$ the ball overtakes the wave pulse.

$$\Rightarrow z = 2m/\mu\,,$$

$\Rightarrow$ with $M = \mu\cdot L$ this gives

$$z = \frac{2m}{M}L\,.$$

This shows, that for $m > M \Rightarrow z > L$ the ball cannot overtake the wave.
b) More accurate calculation for an arbitrary ratio m/M:
For the distance z below the suspension point $z = 0$ the force

$$F = \mu(L - z)g + mg$$

acts on the rope due to the weight of rope + mass m. The phase velocity of the wave is

$$v_{Ph}(z) = [(L - z + m/\mu)g]^{1/2} .$$

It decreases with increasing z (due to the decreasing weight force) according to

$$\frac{dv}{dz} = -\frac{g}{2v(z)} .$$

With

$$\frac{dv}{dt} = \frac{dv}{dz} \cdot \frac{dz}{dt} = -\frac{g}{2v} v = -\frac{g}{2}$$
$$\Rightarrow v(t) = v(z=0) - \tfrac{1}{2} g t .$$

The distance z that the wave propagates during the time t is then

$$z_1(t) = v_0 t - \tfrac{1}{4} g t^2 .$$

The ball falls in this time the distance $z_2(t) = \frac{1}{2} g t^2$. The meeting point is t at $z_1 = z_2$

$$\Rightarrow v_0 t_1 - \frac{1}{4} g t_1^2 = \frac{1}{2} g t_1^2$$
$$\Rightarrow t_1 = \frac{4}{3} v_0 / g = \frac{4}{3} \sqrt{\frac{\mu L + m}{g\mu}} .$$

This gives the meeting point

$$z_m = \frac{1}{2} g t_1^2 = \frac{8}{9} \left(L + \frac{m}{\mu} \right) = \frac{8}{9} L \left(1 + \frac{m}{M} \right) .$$

11.3 According to (11.36) the power supplied to the oscillating system is

$$P = m \cdot \gamma \omega^2 A_2^2 = \frac{(F_0^2/m) \cdot \gamma \cdot \omega^2}{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2} .$$

It is proportional to the square of the imaginary part b in equation (11.27c), because only the friction consumes energy. The real part a in (11.27b) determines the phase shift φ, because $\tan\varphi = b/a$, but does not consume energy. For $b = 0$ and $\omega \neq \omega_0$ is $\varphi = 0$.
For $b = 0$ and $\omega = \omega_0$ no stationary oscillation is possible. The amplitude A of the forced oscillation increases until $A = \infty$ (resonance catastrophe). In this case energy is supplied by the exciter which increases the amplitude until the oscillating system is destroyed.

11.4 The pressure at equilibrium is $p = 4\varepsilon/r$ with $\varepsilon = \sigma$. Changing the radius r changes the pressure by

$$dp = \frac{dp}{dr} dr = -\frac{4\varepsilon}{r^2} dr .$$

Therefore the restoring force is

$$dF = 4\pi r^2 dp = -16\pi\varepsilon dr \Rightarrow \boldsymbol{F} = -16\pi\varepsilon \boldsymbol{r}.$$

With the restoring constant $D = 16\pi\varepsilon$ the oscillation period becomes

$$T = 2\pi\sqrt{m/D} = \tfrac{1}{2}\sqrt{\pi m/\varepsilon} = \pi r \sqrt{\varrho d/ep}$$

with the mass $m = 4\pi r^2 \cdot \varrho \cdot d$ and the thickness d of the skin of the soap bubble.
(The small change of the air pressure inside the bubble due to the small change of the volume is negligible).

11.5 The phase velocity is

$$v_{Ph} = \sqrt{E/\varrho} = 5.2 \cdot 10^3 \text{ m/s} .$$

The wavelength is

$$\lambda = v_{Ph}/\nu = 0.52 \text{ m} .$$

The maximum change of the length appears between maxima and minima of the longitudinal wave. Therefore we obtain

$$(\Delta L/L)_{max} = 2A/(\lambda/2) = 4A/\lambda$$
$$\Rightarrow \Delta L/L = 7.7 \cdot 10^{-4}$$
$$\Rightarrow \sigma_{max} = E \left(\frac{\Delta L}{L} \right)_{max} = 1.7 \cdot 10^8 \text{ N/m}^2 .$$

This is below the tensile strength by a factor 9.

11.6 The intensity of the sound wave is

$$I = \frac{1}{2} \frac{\Delta p_0^2}{\varrho v_{Ph}} \Rightarrow \Delta p_0 = \sqrt{2\varrho v_{Ph} I} .$$

At the hearing limit is $I = 10^{-12}$ W/m^2. With $\varrho = 1.25$ kg/m^3, $v_{Ph} = 300$ m/s is

$$\Delta p_0 = 2.74 \cdot 10^{-5} \text{ N/m}^2 .$$

With $\Delta p_0 = v_{Ph} \cdot \varrho \cdot \omega \cdot \xi_0$ the oscillation amplitude becomes

$$\xi_0 = 1.2 \cdot 10^{-11} \text{ m} = 0.12 \text{ Å} .$$

The amplitude is therefore smaller as one atomic diameter. The acoustic particle velocity u_0 is

$$u_0 = \omega \xi_0 = 7 \cdot 10^{-8} \text{ m/s} .$$

This is small compared to the thermal velocity $\langle v \rangle = 5 \cdot 10^2$ m/s of the molecules.

11.7 a) The surface of the liquid at rest is $z = 0$ in both sides of the U-tube. For a change $\Delta z = z_0 - z = -z$ in one side of the tube rises the liquid in the other side by $+z$. The restoring force is then for an ideal liquid (no friction)

$$F = -2z\varrho gA = m\ddot{z}$$
$$\Rightarrow z(t) = \Delta z \sin\left(\sqrt{\frac{2\varrho gA}{m}}\, t\right) = \Delta z \sin \omega t$$
$$\Rightarrow \omega = 3.5\,\mathrm{s}^{-1} \Rightarrow T = \frac{2\pi}{\omega} = 1.8\,\mathrm{s}\,.$$

The velocity v is then

$$v = \dot{z} = \omega\,\Delta z \cos \omega t\,.$$

With $\Delta z = 0.1$ m is follows: $v_{\max} = \omega \cdot \Delta z = 0.35$ m/s. The acceleration is

$$a = \ddot{z} = -\omega^2\,\Delta z \sin \omega t \Rightarrow a_{\max} = 1.23\,\mathrm{m/s^2}\,.$$

b) Taking into account friction:
According to Hagen–Poiseuille the velocity profile is

$$u(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)\,.$$

Defining a mean velocity $\overline{u}$, averaged over the cross section $\pi \cdot r^2$ we obtain with (8.31) and $F_\mathrm{f} = \Delta p \cdot \pi r^2$ the friction force on a liquid column of length L

$$F_\mathrm{f} = 8\pi\eta L\overline{u} \quad \text{with} \quad \overline{u} = \frac{1}{\pi R^2}\int_0^R \overline{u}(r) 2\pi r\,\mathrm{d}r\,.$$

In the equation of motion

$$m \cdot \ddot{z} - b \cdot \dot{z} - 2\varrho g\pi R^2 z = 0\,.$$

With $\mathrm{d}z/\mathrm{d}t = \overline{u}$, $b = 8\pi\eta L$ the damping constant

$$\gamma = b/(2m) = 4\pi\eta L/m = 4 \cdot 10^{-2}\,\mathrm{s}^{-1}\,,$$

where

$$L = \frac{m}{\varrho\pi R^2} = 1.6\,\mathrm{m} \quad \text{and} \quad \eta = 10^{-3}\,\mathrm{Pa\cdot s}$$

have been inserted. After the time $\tau = 25$ s the oscillation amplitude has decreased to $1/e$ of its initial value.

11.8 The intensity distribution of the wave, diffracted by the slit is

$$I(\alpha) = I_0 \frac{\sin^2(\pi d \sin\alpha/\lambda)}{(\pi d \sin\alpha/\lambda)^2}\,,$$

where α is the diffraction angle.
a) For $I(\alpha)/I_0 = 0.5$ is

$$\frac{\sin^2 x}{x^2} = 0.5 \Rightarrow x \approx 1.4 \Rightarrow \sin\alpha = 1.4\lambda/(\pi d)\,.$$

With $\lambda = c/\nu = 330/(2 \cdot 10^3) = 0.165$ m und $d = 0.5$ m $\Rightarrow \sin\alpha = 0.147 \Rightarrow \alpha = 9.4°$

$$\Rightarrow \Delta s = 2 \cdot 20 \cdot \tan\alpha = 6\,\mathrm{m}\,.$$

b) For $I(\alpha)/I_0 = 0.05 \Rightarrow x \approx 2.5 \Rightarrow \alpha = 17.0°$

$$\Rightarrow \Delta s = 10.9\,\mathrm{m}\,.$$

11.9 Conservation of energy demands for the intensities of the waves:

$$I_\mathrm{e} = I_\mathrm{r} + I_\mathrm{d}$$

with $I = \frac{1}{2}\varrho v_\mathrm{Ph} u_0^2$ (u_0 = sound particle velocity).

$$\Rightarrow \tfrac{1}{2}\varrho_1 v_{\mathrm{Ph}1} u_{0e}^2 = \tfrac{1}{2}\varrho_1 v_{\mathrm{Ph}1} u_{0r}^2 + \tfrac{1}{2}\varrho_2 v_{\mathrm{Ph}2} u_{0d}^2\,.$$

With the wave resistance $z = \varrho \cdot v_\mathrm{Ph}$ this can be written as

$$z_1\left(u_{0\mathrm{e}}^2 - u_{0\mathrm{r}}^2\right) = z_2 u_{0\mathrm{d}}^2\,.$$

At the boundary is $u_{0\mathrm{e}} + u_{0\mathrm{r}} = u_{0\mathrm{d}}$

$$\Rightarrow \overline{u}_{0\mathrm{r}} = u_{0\mathrm{e}}\frac{z_1 - z_2}{z_1 + z_2} \quad \text{and} \quad u_{0\mathrm{d}} = u_{0\mathrm{e}}\frac{2z_1}{z_1 + z_2}$$
$$\Rightarrow I_\mathrm{r} = \frac{1}{2} z_1 u_{0\mathrm{r}}^2 = I_\mathrm{e}\left(\frac{z_2 - z_1}{z_2 + z_1}\right)^2$$
$$= RI_\mathrm{e}\,, \quad R = \text{reflection coefficient}$$
$$I_\mathrm{d} = \frac{1}{2} z_2 u_{0\mathrm{d}}^2 = 4I_\mathrm{e}\frac{z_1 z_2}{(z_1 + z_2)^2}$$
$$= TI_\mathrm{e}\,, \quad T = \text{transmission coefficient}\,.$$

With the numerical values

$$\varrho_\mathrm{Air} = 1.29\,\mathrm{kg/m^3}\,, \quad \varrho_\mathrm{Water} = 10^3\,\mathrm{kg/m^3}\,,$$
$$v_\mathrm{Ph}^\mathrm{Air} = 334\,\mathrm{m/s}\,, \quad v_\mathrm{Ph}^\mathrm{Water} = 1480\,\mathrm{m/s}$$

we obtain

$$Z_1 = 1.29 \cdot 334\,\mathrm{kg/(m^2 \cdot s)}\,,$$
$$Z_2 = 10^3 \cdot 1480\,\mathrm{kg/(m^2 \cdot s)}$$
$$\Rightarrow R = 99.88\%\,; \quad T = 0.12\%\,.$$

11.10

$$\xi = \xi_1 + \xi_2$$
$$= 2A\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}z\right)\cos\left(\omega_m t - k_m z\right)$$
$$= 2A\cos(85t - 0.25z)\cos(715t - 1.75z)\,;$$
$$v_{1\mathrm{Ph}} = \frac{\omega_1}{k_1} = \frac{800}{2}\,\mathrm{m/s} = 400\,\mathrm{m/s}\,,$$
$$v_{2\mathrm{Ph}} = \frac{\omega_2}{k_2} = \frac{630}{1.5}\,\mathrm{m/s} = 420\,\mathrm{m/s}\,,$$
$$v_\mathrm{G} = \frac{\Delta\omega}{\Delta k} = \frac{170}{0.5}\,\mathrm{m/s} = 340\,\mathrm{m/s}\,.$$

11.11 For large values of λ the term $2\pi\sigma/(\varrho\lambda)$ in equation (11.86) can be neglected. This gives

$$v_{\text{Ph}} = \sqrt{\frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}} .$$

For $\lambda = 500\,\text{m}$ and $h > \lambda$ is $\tanh(2\pi h/\lambda) \approx 1$.

$$\Rightarrow v_{\text{Ph}} = 28\,\text{m/s} .$$

For $\lambda = 0.5\,\text{m}$ is $g \cdot \lambda/2\pi \approx 0.78\,\text{m}^2/\text{s}^2$ and with $\sigma = 7.25\,\text{J/m}^2$ is $2\pi\sigma/(\varrho \cdot \lambda) \approx 9.1 \cdot 10^{-2}\,\text{m}^2/\text{s}^2$

$$\Rightarrow v_{\text{Ph}} = 0.93\,\text{m/s} .$$

11.12 The frequency of the fundamental oscillation is

$$\nu_0 = \frac{1}{2L}\sqrt{F/\mu} = \frac{1}{2L}\sqrt{\sigma/\varrho} ,$$
$$\lambda_0 = 2L \Rightarrow v_{\text{Ph}} = \nu_0\lambda_0 = \sqrt{\sigma/\varrho} .$$

With $\sigma = 3 \cdot 10^{10}\,\text{N/m}^2 = 3 \cdot 10^4\,\text{N/mm}^2$ und $\varrho = 7.8 \cdot 10^3\,\text{kg/m}^3$, $L = 1\,\text{m}$

$$\Rightarrow \nu_0 = 10^3\,\text{s}^{-1}$$
$$\Rightarrow T_0 = \frac{1}{\nu_0} = 1\,\text{ms}$$
$$\Rightarrow \lambda_0 = 2\,\text{m} , \quad v_{\text{Ph}} = 2 \cdot 10^3\,\text{m/s} .$$

11.13 The weight $m \cdot g$ is compensated by the elastic restoring force $F_{\text{E}} = \pi r^2 E\frac{\Delta L}{L}$

$$m \cdot g = -k\,\Delta L \quad \text{with} \quad k = \pi r^2 E/L .$$
$$\Rightarrow k = \frac{m \cdot g}{\Delta L} .$$

The oscillation period is

$$T = 2\pi\sqrt{m/k} = 2\pi\sqrt{\Delta L/g} .$$

It is independent of m as long as the mass of the rope is negligible.
With $\Delta L = 2 \cdot 10^{-3}\,\text{m} \Rightarrow T = 0.09\,\text{s}$.
A pendulum with the length L has the oscillation period $T = 2\pi\sqrt{L/g}$. In our example the period would be $T = 2.84\,\text{s}$, i.e. 30 times as long.

11.14 A force F acting onto the end of the flat spring bends the end by a distance (see (6.20))

$$\Delta \boldsymbol{s} = -\frac{4L^3}{Ed^3b}\boldsymbol{F}$$
$$\Rightarrow \boldsymbol{F} = -k\,\Delta\boldsymbol{s} \quad \text{with} \quad k = \frac{Ed^3b}{4L^3} .$$

The oscillation period of the spring without additional mass is

$$\omega_0 = \sqrt{4k/m_{\text{F}}} = 2\sqrt{k/m_{\text{F}}} .$$

Here the fourfold restoring force constant has to be inserted because the mean deflection of the spring is

$$\Delta \boldsymbol{s} = \frac{1}{L}\int\limits_{x=0}^{L} \Delta s(x)\,\text{d}x = \frac{1}{4}\Delta s(L) .$$

With $m_{\text{F}} = \varrho \cdot b \cdot d \cdot L$ and $k = \frac{Ed^3b}{4L^3}$ we obtain

$$\omega_0 = \frac{d}{L^2}\sqrt{E/\varrho} .$$

The frequency deceases with $1/L^3$!
With $\omega_0 = 2\pi \cdot 100\,\text{s}^{-1}$, $L = 0.1\,\text{m}$, $E = 2 \cdot 10^{11}\,\text{N/m}^2$ and $\varrho = 7.8 \cdot 10^3\,\text{kg} \Rightarrow d = 0.63\,\text{mm}$.
If a mass m is attached to the end of the spring, the frequency becomes

$$\omega = \sqrt{k/(m + m_{\text{F}}/4)}$$

$$\Rightarrow \frac{\omega_0}{\omega} = 2\sqrt{m + m\frac{\text{F}/4}{m_{\text{F}/4}}} = 2\sqrt{1 + \frac{4m}{m_{\text{F}}}} .$$

11.15 Without waves the equilibrium position is given by: buoyancy = weight,

$$\Rightarrow aqL\varrho_1 \cdot g = m \cdot g \quad \Rightarrow m = aLq\varrho_1 .$$

If the buoy is immersed by Δz below its equilibrium position and then released it performs an oscillation because of the restoring force $F = -q\varrho_1 g\Delta z$

$$z = \Delta z \sin\omega_0 t \quad \text{with} \quad \omega_0^2 = \frac{q\varrho_1 g}{m} = \frac{g}{aL} .$$

If waves are present the water surface at the location of the buoy is

$$z = z_0 + h \cdot \sin\omega_0 t \quad \text{with} \quad \omega_0 = \frac{2\pi}{T} \quad \text{and} \quad z_0 = 0 .$$

The waves generate an additional periodic buoyancy

$$\Delta F_{\text{B}} = h \cdot q \cdot g \cdot \varrho_1 \cdot \sin\omega t ,$$

which results in a forced oscillation

$$\Delta z = A\sin(\omega t + \varphi) .$$

The oscillation amplitude is then, according to (11.26) when we neglect friction

$$A(\omega) = \frac{hg/(aL)}{(\omega_0^2 - \omega^2)} .$$

Numerical values: $a \cdot L = 30\,\text{m}$, $h = 2\,\text{m}$, $T = 5\,\text{s} \Rightarrow \omega = 1.25\,\text{s}^{-1}$, $\omega_0 = 0.57\,\text{s}^{-1} \Rightarrow A(\omega) = 0.525\,\text{m}$.
Without waves (plane water surface) the fraction $(1-a)L$ of the buoy are above the water surface, at the wave peak only $x = [(1-a)L - (2m - 0.525m)]$. If the buoy should be just under water at the wave peak we must set $x = 0$. This gives $L = 32.475\,\text{m}$.

11.16 The radial part of the Laplace-operator is

$$\Delta_r = \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} .$$

If it is applied to $\xi = (A/r)e^{i(kr-\omega t)}$, we obtain

$$\Delta_r \xi = \left[-\frac{2}{r^3} + \frac{ik}{r^2} + \frac{2}{r^3} - \frac{2ik}{r^2} - \frac{k^2}{r}\right] Ae^{i(kr-\omega t)}$$
$$= -\frac{k^2}{r} Ae^{i(kr-\omega t)} = -k^2 \xi ,$$
$$\frac{\partial^2 \xi}{\partial t^2} = -\omega^2 \xi \quad \Rightarrow \frac{\partial^2 \xi}{\partial t^2} = \frac{\omega^2}{k^2}\Delta_r \xi \quad \Rightarrow c = \frac{\omega}{k} .$$

11.17 a) $\nu = \nu_0 \dfrac{1}{1-\overline{u}/v_{Ph}}$; From $\nu/\nu_0 = 1.12246$

$$\Rightarrow u_Q = v_{Ph}\left(1 - \frac{1}{1.12246}\right) = 0.1091 v_{Ph}$$

with $v_{Ph} = 330\,\text{m/s}$;

$$\overline{u}_Q \approx 36\,\text{m/s} = 130\,\text{km/h} .$$

b) $\nu = \nu_0\left(1 + \dfrac{\overline{u}_B}{v_{Ph}}\right) \Rightarrow 1 + \frac{u_B}{v_{Ph}} = 1.12246$

$$\Rightarrow \overline{u}_B = 0.12246 v_{Ph} \approx 145\,\text{km/h} .$$

11.18 We choose $x = 0$ as equilibrium position. When the block is shifted to an elongation $x_0 > 0$, is potential energy is

$$E_{pot_0} = \int_0^{x_0} 2D_0 x\,dx = D_0 x_0^2 .$$

a) After releasing the block it slides until the reverse point x_1 and loses on the way the energy by friction

$$E_f = f_1 mg(x_0 - x_1) \quad \text{with} \quad x_1 < 0 .$$
$$\Rightarrow D_0 x_1^2 = D_0 x_0^2 - f_1 mg(x_0 - x_1)$$
$$\Rightarrow x_1 = \frac{f_1 mg}{D_0} - x_0 < 0 .$$

The absolute values of the elongations are

$$|x_1| = |x_0| - f_1 mg/D_0 .$$

For the general reverse points we obtain

$$|x_n| = |x_{n-1}| - f_1 mg/D_0 = |x_{n-1}| - 0.059\,\text{m} ,$$
$$|x_n| = |x_0| - n f_1 mg/D_0 = |x_0| - n \cdot 0.059\,\text{m} .$$

The distances between the revers points decrease linearly with n. The motion of the block is a damped, but not harmonic oscillation.

b) The block sticks at the nth reverse point, if here the restoring force is smaller than the static friction coefficient.

$$\Rightarrow 2D_0|x_n| < f_0 \cdot m \cdot g \quad \Rightarrow n > \frac{D_0 x_0}{f_1 mg} - \frac{f_0}{2f_1} .$$

Inserting the numerical values gives

$$n > \frac{100 \cdot 0.22}{0.3 \cdot 2 \cdot 9.81} - \frac{0.9}{2 \cdot 0.3} = 2.3 ,$$

i. e. the block sticks at least at the 3rd reverse point, if it reaches it at all. In order to check this we determine its start energy at the 2nd reverse point:

$$E_p = D_0 x_2^2 = D_0(x_0 - 2f_1 mg/D_0)^2 .$$

It should be larger than the friction energy $f_1 \cdot m \cdot g|x_3 - x_2|$, if it should reach the reverse point x_3. Inserting the numerical values gives: $E_p(x_1) = 1.05\,\text{N}\cdot\text{m}, f_1 \cdot m \cdot g|x_3 - x_2| = 0.346\,\text{N}\cdot\text{m}$. This proves that the block reaches x_3 but sticks there.

c) The total energy is

$$E = \tfrac{1}{2}D_0 x_0^2 = \tfrac{1}{2}D_0 x^2 + \tfrac{1}{2}mv^2 + f_1 \cdot mg \cdot (x_0 - x)$$
$$\Rightarrow v^2 = (D_0/m)\left(x_0^2 - x^2\right) - 2f_1 g(x_0 - x) .$$

The block is released at x_0. The time T at which it reaches x_1 is

$$T = \int_{t=0}^{t_0} dt + \int_{t_0}^{t_1} dt \qquad \text{where} \qquad \begin{array}{l} t_0 = t(x=0) \\ t_1 = t(x = x_1) \end{array}$$
$$-dx = v\,dt \quad \Rightarrow dt = -dx/v$$
$$\Rightarrow T = -\int_{x_0}^{0} \frac{dx}{v} - \int_0^{x_1} \frac{dx}{v} = \int_{x_1}^{x_0} \frac{dx}{v} .$$

Inserting of

$$v = \sqrt{(D_0/m)(x_0^2 - x^2) - 2f_1 g(x_0 - x)}$$

gives with the substitutions

$$z = (x_0 - x) \Rightarrow x_0^2 - x^2 = (x_0 - x)\cdot(x_0 + x)$$
$$= z \cdot (2x_0 - z) ; \quad dx = -dz$$
$$\Rightarrow x = x_0 \;\Rightarrow z = 0 ; \quad x = x_1 \;\Rightarrow z = z_1 = x_0 - x_1$$
$$a = D_0/m ; \quad b = 2 \cdot a \cdot x_0 - 2f_1 g$$
$$T = \int_{z=z_1}^{0} \frac{dz}{\left(-az^2 + bz\right)^{1/2}} .$$

The integral can be solved analytically and gives

$$T = -\frac{1}{\sqrt{a}}\left(\arcsin\frac{-2az + b}{b}\right)\Bigg|_{z1}^{0} .$$

The revers point x_1 can be obtained from

$$x_1 = \frac{f_1 mg}{D_0} - x_0 < 0 \,.$$

With $x_0 = 0.22\,\text{m} \Rightarrow x_1 = -0.161\,\text{m} \Rightarrow z_1 = 0.22 + 0.16 = 0.38\,\text{m}$

$$\Rightarrow T = +\frac{1}{\sqrt{D_0/m}}\left[\arcsin 1 - \arcsin\left(1 - \frac{2az_1}{b}\right)\right] .$$

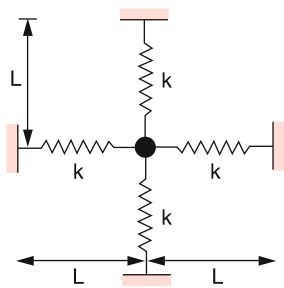
14.12 Chapter 12

12.1 a) $$m \cdot \ddot{x} + 2kx\left[1 + \frac{L}{(L^2 + x^2)^{1/2}}\right] = 0 \,.$$

b) $$m \cdot \ddot{x} + 4kx - (k/L^2)x^3 = 0 \,.$$

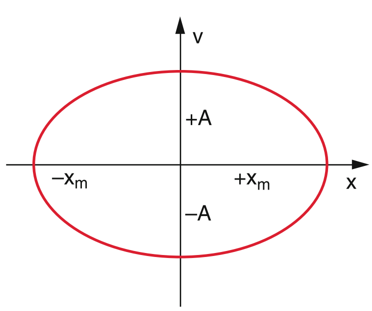
c) $$m \cdot \ddot{x} + 4kx = 0\,; \quad \omega_0 = (4k/m)^{1/2} \,.$$

For $x_0/L = \varepsilon$ is $\omega = \omega_0 \cdot K(\varepsilon) \approx \omega_0(1 - \varepsilon^2/2)$, where $K(\varepsilon)$ is the elliptical integral, that has been discussed in Sect. 2.9.7).



12.2 $$\ddot{x} + ax + bx^3 = 0\,, \quad a, b > 0$$

$$v = \frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d(v^2)}{dx} \,.$$



Inserting and integration over x and multiplication with the factor 2 yields the equation

$$v^2 + ax^2 + \frac{b}{2}x^4 = \text{const} = ax_0^2 + \frac{b}{2}x_0^4 = A \,.$$

The function $v(x)$ represents in an x-v-diagram a closed curve (called phase-space trajectory) where $x_0 = x(t = 0)$ and $v(t = 0) = 0$.

$$v(x) = \pm\left[A - ax^2 - \frac{b}{2}x^4\right]^{1/2} ,$$

which is periodically traversed. For $v = 0$ we obtain the intersection points of $v(x)$ with the x-axis

$$x_m = \pm\left[-\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} + \frac{2A}{b}}\right]^{1/2} .$$

The time for one circulation is obtained from

$$v = \frac{dx}{dt} \Rightarrow dt = \frac{dx}{v(x)}$$

$$\Rightarrow T = 4\int_{x=0}^{x_m} \frac{dx}{\left[A - ax^2 - \frac{b}{2}x^4\right]^{1/2}} ,$$

because the path from $x = 0$ until x_m is traversed in the time $T/4$. The elliptical integral can be found in integral tables.

12.3 a) With $y = x/x_0$ and $t^* = \omega_0 t$ we get

$$\frac{d^2x}{dt^2} = x_0\frac{d^2y}{dt^2} \quad \text{and} \quad dt = \frac{1}{\omega_0}dt^*$$

$$\Rightarrow \frac{d^2x}{dt^2} = \omega_0^2 x_0 \frac{d^2y}{dt^{*2}} \,.$$

The differential equation then transformes into

$$\frac{d^2y}{dt^{*2}} + \frac{k_1 x_0}{\omega_0^2 x_0 m}y + \frac{k_2 x_0^2}{\omega_0^2 x_0 m}y^2 = 0 \,.$$

This yields with $\omega_0^2 = k_1/m$ and $\varepsilon = (k_2/k_1)x_0$

$$\frac{d^2y}{dt^{*2}} + y + \varepsilon y^2 = 0$$

with the initial conditions: $y(0) = 1$ and $\dot{y}(0) = 0$.
b) With $t^* = \omega \cdot t$ we obtain the equations

$$\frac{d^2y}{dt^{*2}} + y + \varepsilon y^2 = 0\,, \tag{14.8a}$$

$$\frac{d^2y}{dt^2} + \omega^2 y + \varepsilon\omega^2 y^2 = 0\,, \tag{14.8b}$$

$$\frac{d^2y}{dt^2} + \omega^2 y[1 + \varepsilon y] = 0\,. \tag{14.8c}$$

The last equation can be solved by series expansion with regard to ε, because the term $(1 + \varepsilon)$ can be regarded as perturbation term of the unperturbed equation with $\varepsilon = 0$.

We make the ansatz

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \ldots$$
$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots\ .$$

Inserting this into the differential equation (14.8) and ordering the term according to the power exponents of ε, the different "coefficients" of ε^n have to be zero, if the equation should hold for arbitrary values of ε.
For the ε-free terms with ε^0 one obtains the unperturbed equation for the oscillation

$$\frac{\mathrm{d}^2 y_0}{\mathrm{d}t^2} + \omega_0^2 y_0 = 0 \quad \Rightarrow \quad y_0 = A_0 \cdot \cos\omega_0 t\ .$$

For the term with ε^1 this gives

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + \omega_0^2(y_0^2 + y_1) + 1\omega_0\omega_1 y_0 = 0\ .$$

Inserting y_0 from the previous equation we get

$$\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} + \omega_0^2(A_0^2\cos^2\omega_0 t + y_1) + 2\omega_0 y_1 A_0\cos\omega t = 0\ .$$

Proceeding to the quadratic term with ε^2 one obtains after some efforts

$$\omega(\varepsilon) = \omega_0\left(1 - \frac{5A^2}{12}\varepsilon^2 + O(\varepsilon^3)\right)\ .$$

For $\varepsilon = 0.1 \Rightarrow \omega = \omega_0(1 - 4.17\cdot 10^{-3}A_0^2)$.

12.4 Fixpoints occur for $\mathrm{d}x/\mathrm{d}t = 0$. From the two equations of the problem we get for the fix points
a) $F = (x_{F1}, x_{F2}) = (0,0)$.
b) $F = (x_{F1}, x_{F2}) = (\lambda_3/\lambda_2, \lambda_1/\lambda_2)$.
They are stable, if for an elongation Δx from the fix point the conditions hold

$$\dot{x}_1 > 0 \quad \text{for} \quad \Delta x_i < 0\ ; \quad i = 1,2\ ,$$
$$x_1 < 0 \quad \text{for} \quad \Delta x_i > 0\ ; \quad i = 1,2\ .$$

If the stability condition only holds for one direction (for instance x_1 but not for x_2) than the fix point is stable for shifts in the direction of x_1 but not for a shift in the direction of x_2.
In analogy to a saddle point on a curved surface this fix point is called saddle point. If the stability condition does not hold for both directions, the fix point is unstable.
Which of the cases applies for x_1 and x_2 can be obtained as follows:
Addition of the two equations yields

$$\dot{x}_1 + \dot{x}_2 = \lambda_1 x_1 - \lambda_3 x2\ .$$

The stability depends on the sign of λ_1 and λ_3. That of λ_2 is not significant.
We distinguish between the 4 cases listed in the table. The arrows indicate the motion of the point $F = (0,0)$ at a displacement. If the arrows point towards F the fix point F is stable against displacements in this direction, if they point away from F it is unstable in this direction.

λ_1	λ_3	$F(0,0)$	
> 0	< 0	Saddle	↓ ← F → x_1 ↑ x_2
> 0	< 0	Unstable	↑ ← F → ↓
< 0	> 0	Stable	↓ → F ← ↑
< 0	< 0	Saddle	↑ → F ← ↓

For the second fix point $F = (\lambda_3/\lambda_2, \lambda_1/\lambda_2)$ no stable position exists. For $(\lambda_1 > 0, \lambda_3 < 0)$ and $(\lambda_1 < 0, \lambda_3 > 0)$ saddle-points exist. For the other two possible cases F moves on an elliptical path.

12.5 In Sect. 2.9.7 it was shown that the oscillation period T of the undamped pendulum is given by

$$T = \frac{2}{\pi}T_0 F(\varphi_0)$$

with

$$F(\varphi_0) = \int\limits_0^{\pi/2} \frac{\mathrm{d}\xi}{\sqrt{1 - \sin^2(\varphi_0/2)\sin^2\xi}}\ .$$

As can be seen in tables of elliptical integrals [2.6a, 2.6b], is

$$F\left(\varphi_0 = \frac{\pi}{4}\right) = 1.63\ ; \quad F\left(\varphi_0 = \frac{\pi}{2}\right) = 1.84\ ;$$
$$F\left(\varphi_0 = \frac{3}{4}\pi\right) = 2.4\ ; \quad F\left(\varphi_0 = \frac{\pi}{2}\right) = \infty\ .$$

The oscillation periods of the undamped pendulum are then

$$T(\varphi) = a\cdot T_0 \quad \text{with} \quad \begin{array}{lll} a = 1.038 & \text{for} & \varphi = \pi/4\ , \\ a = 1.17 & \text{for} & \varphi = \pi/2\ , \\ a = 1.53 & \text{for} & \varphi = \frac{3}{4}\pi \\ a = \infty & \text{for} & \varphi = \pi\ . \end{array}$$

For the last case the pendulum reaches the metastable position in the upper reversal point where it could stay (without perturbation) infinitely long.
The equation for the damped oscillation can be solved only approximately, when using the expansion $\sin\varphi = \varphi - \frac{1}{6}\varphi^3 + \ldots$

The resulting equation

$$\ddot{\varphi} + \omega_0^2\varphi + \gamma\dot{\varphi} - \frac{\omega_0^2}{6}\varphi^3 = 0$$

can be solved for the case $\gamma/\omega_0 \ll 1$ with the ansatz $\varphi = \varphi_0 + \varepsilon\varphi_1 + \ldots$
This gives the oscillation period

$$T = T_0 \frac{4F(\varphi_0)}{\sqrt{1+\frac{1}{6}A^2}} \quad \text{with} \quad T_0 = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}} ,$$

where $F(\varphi_0)$ is again the elliptical integral and $A = \varphi(t = 0)$ the initial amplitude (see Probl. 12.2).

12.6

$$z(t) = \left[\frac{b}{a} + \left(\frac{1}{z_0} - \frac{b}{a}\right) \mathrm{e}^{-a(t-t_0)}\right]^{-1}$$

with $z(t = 0) = z_0$, $z(t \to \infty) = a/b$.
The doubling time for the case $a = b$:

$$T = \frac{1}{a} \ln \frac{2 - 2z_0}{1 - 2z_0} .$$

This shows that z doubles for $0 < z_0 < 0.5$ within a finite time, but for $z_0 = 0.5$ only after an infinite time. For $z_0 > 0.5$ there is no doubling at all.

12.7 The numerical analysis of the equation

$$x_{n+1} = 3.1x_n(1 - x_n)$$

gives for $x_0 = 0.5$

$$\lim_{n\to\infty} x_{2n} = 0.5580\ldots = x_{\mathrm{F1}} ,$$

$$\lim_{n\to\infty} x_{2n+1} = 0.7646\ldots = x_{\mathrm{F2}} .$$

For $x_0 = 1/4$ the two quantities x_{F1} and x_{F2} interchange their values remain, however, independent of the initial value x_0.
For $a = 3.3$ one obtains

$$x_{\mathrm{F1}} = 0.4794\ldots , \quad x_{\mathrm{F2}} = 0.8236\ldots .$$

The Ljapunov exponent is

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln|f'(x_i)| .$$

With $f(x_n) = ax_n(1 - x_n) \Rightarrow f'(x_n) = a - 2ax_n$

$$\Rightarrow \lambda = \ln a + \tfrac{1}{2}\ln|1 - 2x_{\mathrm{F1}}| + \tfrac{1}{2}\ln|1 - 2x_{\mathrm{F2}}| .$$

For $a = 3.1 \Rightarrow \lambda = -0.264$.

For $a = 3.3 \Rightarrow \lambda = -0.619$.

12.8 For each iteration the length of a triangle side is cut in halves, but the number of sides becomes threefold larger. The total length therefore becomes

$$L_n = \left(\tfrac{3}{2}\right)^n L_0 \quad \text{and} \quad N(L/2) = 3N(L) .$$

In Eq. 12.54 is $\lambda = 1/2$ and $\lambda^\kappa = 3$.

$$\Rightarrow \kappa = -\frac{\ln 3}{\ln 2} \Rightarrow d = -\kappa = 1.58496\ldots .$$

12.9 The condition for the boundary curve is: $\dot{r} = 0$.

$$\Rightarrow (a - r^2)r = 0 \quad \Rightarrow r_1 = 0 , \quad r_2 = \sqrt{a} .$$

The first solution $r_1 = 0$ gives the stable fix point. The second solution $r_2 = \sqrt{a}$ and $\varphi = \omega_0 t$ gives as boundary curve a circle with radius $\sqrt{a}$ and centre $r = 0$, which is traversed with constant angular velocity.

12.10

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x} = -2a(x - x_0) - 3b(x - x_0)^2 .$$

The equation of motion: $F = m\ddot{x}$

$$\Rightarrow \ddot{x} + \frac{2a - 6bx_0}{m}x + \frac{3b}{m}x^2 + \frac{-2ax_0 + 3bx_0^2}{m} = 0$$

$$\ddot{x} + Ax + Dx^2 + C = 0 .$$

Besides the minimum at $x = x_0$, $V(x)$ has a maximum

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 2a(x - x_0) + 3b(x - x_0)^2 = 0$$

$$\Rightarrow x_{\max} = x_0 - \frac{2a}{3b} .$$

The maximum amplitude is then $2a/3b$. For $|x - x_0| > 2a/b$ no periodic motion is possible.