Supplement

13

Chapter 13

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13.1 Vector Algebra and Analysis

13.1.1 Definition of Vectors

A **vector** is an oriented line segment. Its length is the **magnitude of the vector**. Vectors are denoted in this textbook by bold letters.

Two vectors are equal, if they have the same direction and magnitude. The magnitude of a vector is a pure number (**scalar**), independent of the direction of the vector.

> Since a parallel displacement of a vector in space does not change its direction nor its magnitude, all parallel vectors with the same magnitude are equal, independent of the coordinates of their starting points.

The starting point of a vector is also called **point of origin**.

A vector starting from the origin and ending at a point P is called **position vector**, because it defines the position of P in space.

Multiplying a vector with a scalar number changes its length but not its direction.

13.1.2 Representation of Vectors

Every vector in a three-dimensional space can be represented by three linear independent **basis vectors**. The selection of these basis vectors depends on the chosen coordinate system.

13.1.2.1 Cartesian Coordinates

When we plot a vector $\boldsymbol{r}$ in a Cartesian coordinate system (x, y, z) with the point of origin $(0, 0, 0)$ it ends at the point $P(x, y, z)$ with the coordinates x, y, z (Fig. 13.1). These coordinates are the projection of the vector onto the three coordinate axes. They are called the components of the vector.

The component representation of the vector r is

$$\boldsymbol{r} = \{x, y, z\} \,. \tag{13.1a}$$

A vector $\boldsymbol{r}$ is uniquely defined by its components, because its magnitude (written as $|\boldsymbol{r}|$) is

$$r = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2} \,, \tag{13.1b}$$

as can be derived by Fig. 13.1 and the theorem of Pythagoras.

The direction of a vector is defined by its components. It can be also represented by the three angles α, β, γ against the coordinate axes. It is

$$\cos\alpha = x/r \,, \quad \cos\beta = y/r \,, \quad \cos\gamma = z/r \,.$$

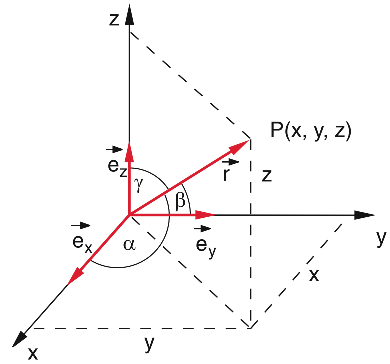

Figure 13.1 Cartesian coordinate system

A vector with the length $L = 1$ ($\sqrt{x^2 + y^2 + z^2} = 1$) is called **unit vector**. It is often represented by

$$\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}| = \hat{\boldsymbol{e}} \,. \tag{13.1c}$$

Special unit vectors are the three vectors

$$\hat{\boldsymbol{e}}_1 = \{1, 0, 0\} \,; \quad \hat{\boldsymbol{e}}_2 = \{0, 1, 0\} \,; \quad \hat{\boldsymbol{e}}_3 = \{0, 0, 1\} \,. \tag{13.1d}$$

Every vector $\boldsymbol{r} = \{x, y, z\}$ can be written as linear combination of the three basis vectors

$$\boldsymbol{r} = x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2 + z\hat{\boldsymbol{e}}_3 \,. \tag{13.2}$$

13.1.2.2 Spherical Coordinates

The position vector $\boldsymbol{r}$ pointing from the origin $(0, 0, 0)$ to the point $P(r, \vartheta, \varphi)$ is defined in spherical coordinates (also called polar coordinates) by its length $r = |\boldsymbol{r}|$ and the angles ϑ and φ that define uniquely its direction (Fig. 13.2).

The conversion to Cartesian coordinates is given by

$$x = r \cdot \sin\vartheta \cdot \cos\varphi \,,$$
$$y = r \cdot \sin\vartheta \cdot \sin\varphi \,,$$
$$z = r \cdot \cos\vartheta \,.$$

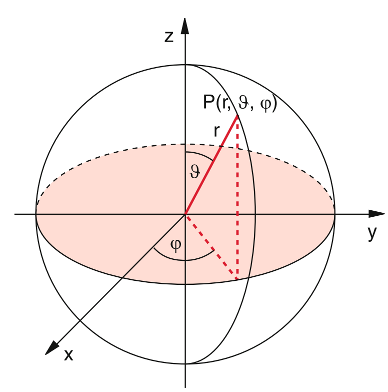

Figure 13.2 Spherical coordinates r, ϑ, φ

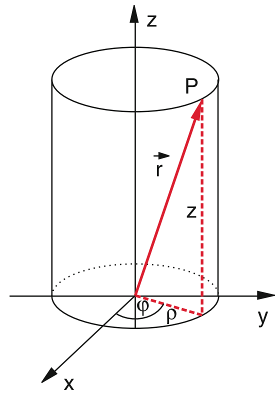

Figure 13.3 Cylindrical coordinates ϱ, φ, z

13.1.2.3 Cylindrical Coordinates

The point $P(\varrho, \varphi, z)$ in Fig. 13.3 is defined in cylindrical coordinates $(\varrho, \varphi.z)$ by the vector $\boldsymbol{r} = \{\varrho, \varphi, z\}$ where ϱ gives the distance from the z-axis, z the distance from the x–y-plane and φ the angle of the projection of $\boldsymbol{r}$ onto the x–y-plane against the x-axis. The conversion to Cartesian coordinates is:

$$x = \varrho \cdot \cos\varphi \ ,$$
$$y = \varrho \cdot \sin\varphi \ ,$$
$$z = z \ .$$

The length of the vector r is

$$|\boldsymbol{r}| = \sqrt{\varrho^2 + z^2} \ .$$

The direction of r is defined by the angle φ and the ratio z/ϱ.

13.1.3 Polar and Axial Vectors

The transformation $x \to -x$; $y \to -y$; $z \to -z$ (mirror imaging of the coordinate system) transforms the position vector $\boldsymbol{r} \to -\boldsymbol{r}$. Therefore r is called a **polar vector**.

Besides these polar vectors which are defined by their length and their direction, there are also vectors that define apart from direction and magnitude a sense of rotation.

Example

Magnitude and orientation of a surface element can be characterized by the *normal vector* $\boldsymbol{A}$ perpendicular to the surface element (Fig. 13.4). The magnitude of the vector gives the area of the surface element and its direction the orientation of the surface element. In order to define uniquely on which of the two sides of the surface element the vector $\boldsymbol{A}$ starts, its direction is defined such, that it forms a right hand screw (like a corkscrew) when the surface element is counterclockwise circulated.

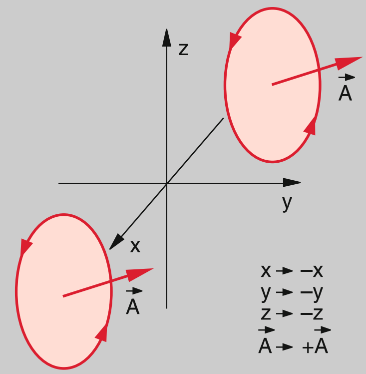

Figure 13.4 Surface normal vector $\boldsymbol{A}$ as axial vector, that is characterized by its direction of orientation. At the coordinate transformation $\boldsymbol{r} \to -\boldsymbol{r}$, $\boldsymbol{A}$ keeps its orientation as a right hand helix

Under the coordinate transformation $x \to -x$, $y \to -y$, $z \to -z$ the sense of rotation of A is preserved, i. e. $\boldsymbol{A} \to +\boldsymbol{A}$ but not $\boldsymbol{A} \to -\boldsymbol{A}$, i. e. $\boldsymbol{A}$ forms again a right-handed screw. Such vectors are called **axial vectors**. Examples are the angular momentum vector $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ (see Sect. 2.8 and 13.1.5.3).

13.1.4 Addition and Subtraction of Vectors

Definition

Vectors are added by adding their components (Fig. 13.5). The sum of the two vectors $\boldsymbol{a} = \{a_1, a_2, a_3\}$ and $\boldsymbol{b} = \{b_1, b_2, b_3\}$ is the vector

$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b} = \{a_1 + b_1, a_2 + b_2, a_3 + b_3\} \ . \qquad (13.3)$$

According to this rule each vector can be written as the sum of its component vectors

$$\boldsymbol{a} = \{a_1, a_2, a_3\} = a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_3\hat{\boldsymbol{e}}_3 \ .$$

The graphical representation of vector addition shown in Fig. 13.5, illustrates that the sum vector $\boldsymbol{c}$ is the diagonal in the parallelogram of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

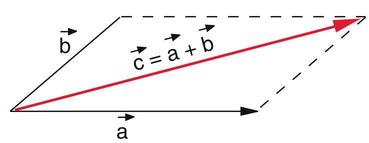

Figure 13.5 Addition of vectors

Problem

Show that this graphical representation fulfils the rules of vector addition.

13.1.5 Multiplication of Vectors

13.1.5.1 Multiplication of a Vector with a Scalar

The vector $\boldsymbol{a}$ is multiplied by a scalar c by multiplying each component of $\boldsymbol{a}$ by c.

$$c \cdot \boldsymbol{a} = c \cdot \{a_1, a_2, a_3\} = \{c \cdot a_1, c \cdot a_2, c \cdot a_3\} ,$$
$$|c \cdot \boldsymbol{a}| = |c| \cdot |\boldsymbol{a}| .$$

13.1.5.2 The Scalar Product

The scalar product of two vectors

$$\boldsymbol{a} = \{a_1, a_2, a_3\} \quad \text{and} \quad \boldsymbol{b} = \{b_1, b_2, b_3\}$$

with the angle α between them is defined as the scalar

$$c = \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\alpha . \tag{13.4a}$$

It is the product of the projection $|\boldsymbol{b}| \cdot \cos\alpha$ of $\boldsymbol{b}$ on $\boldsymbol{a}$ times the amount $|\boldsymbol{a}|$ of $\boldsymbol{a}$. For $\alpha = 90°$ is the scalar product zero (Fig. 13.6).

Two vectors $\neq 0$ are perpendicular to each other only if their scalar product is zero.

For the three unit vectors $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3$ hold the relation

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_k = \delta_{ik} ,$$

where δ_{ik} is the Kronecker symbol which is defined by

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k , \\ 0 & \text{for } i \neq k . \end{cases}$$

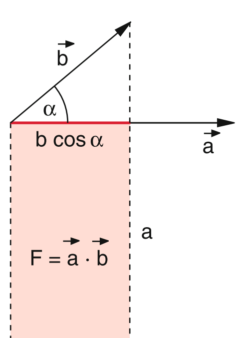

Figure 13.6 The scalar product $\boldsymbol{a} \cdot \boldsymbol{b}$ represents the area $A = \boldsymbol{a} \cdot \boldsymbol{b} = a \cdot b \cdot \cos\alpha$

The scalar product can be also expressed by the vector components. For

$$\begin{aligned} \boldsymbol{a} &= a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_3\hat{\boldsymbol{e}}_3 , \\ \boldsymbol{b} &= b_1\hat{\boldsymbol{e}}_1 + b_2\hat{\boldsymbol{e}}_2 + b_3\hat{\boldsymbol{e}}_3 , \end{aligned} \tag{13.4b}$$

the scalar product becomes

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (a_1\hat{\boldsymbol{e}}_1 + a_2\hat{\boldsymbol{e}}_2 + a_3\hat{\boldsymbol{e}}_3) \cdot (b_1\hat{\boldsymbol{e}}_1 + b_2\hat{\boldsymbol{e}}_2 + b_3\hat{\boldsymbol{e}}_3) , \\ &= a_1b_1 + a_2b_2 + a_3b_3 \quad \text{since} \quad \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_k = \delta_{ik} . \end{aligned} \tag{13.5}$$

13.1.5.3 The Vector Product

Definition

The vector product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$

- that is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$,
- that forms a right handed screw when $\boldsymbol{a}$ is rotated toward $\boldsymbol{b}$ on the shortest way,
- that has the magnitude $|\boldsymbol{c}| = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \sin\alpha$, where α is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

The vector $\boldsymbol{c}$ defines besides magnitude and direction also the orientation. It is therefore an **axial vector**.

Note, that

$$(\boldsymbol{a} \times \boldsymbol{b}) = -(\boldsymbol{b} \times \boldsymbol{a}) .$$

The absolute value $|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \sin\alpha$ of the vector product $\boldsymbol{a} \times \boldsymbol{b}$ is equal to the area of the parallelogram, formed by $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 13.7). The vector product can be therefore regarded as the surface normal of the parallelogram generated by the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\boldsymbol{c} = (\boldsymbol{a} \times \boldsymbol{b})$ is an axial vector, because under reflection of all coordinates at the origin we have $\boldsymbol{a} \to -\boldsymbol{a}, \boldsymbol{b} \to -\boldsymbol{b} \Rightarrow \boldsymbol{c} \to \boldsymbol{c}$ (Fig. 13.7b).

For the unit vectors we get the relations:

$$\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_3 , \quad \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1 , \quad \hat{\boldsymbol{e}}_3 \times \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2 .$$

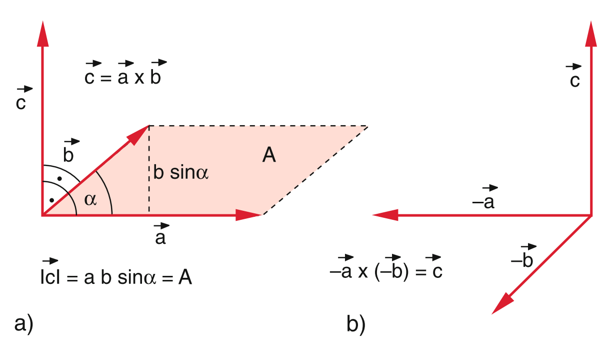

Figure 13.7 The vectorial product as normal vector to the area $|\boldsymbol{a} \times \boldsymbol{b}|$

We therefore get the component representation of the vector product

$$(\boldsymbol{a} \times \boldsymbol{b}) = \{a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1\}\ . \quad (13.6)$$

Check this relation by multiplication of the six vector components.

This component representation can be abbreviated by the symbolic determinant notation

$$\boldsymbol{c} = \begin{vmatrix} \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{\boldsymbol{e}}_1 + (a_3b_1 - a_1b_3)\hat{\boldsymbol{e}}_2 + (a_1b_2 - a_2b_1)\hat{\boldsymbol{e}}_3\ . \quad (13.7)$$

13.1.5.4 Multiple Products

Scalar Products of a Polar and an Axial Vector

The dot product (scalar product) $d = \boldsymbol{c}(\boldsymbol{a} \times \boldsymbol{b})$ of the polar vector $\boldsymbol{c}$ and the axial vector $(\boldsymbol{a} \times \boldsymbol{b})$ gives a scalar number d that transforms into $-\boldsymbol{c}$ when all coordinates are reflected at the origin, because the axial vector $(\boldsymbol{a} \times \boldsymbol{b})$ does not change its sign, while the scalar number $\boldsymbol{c}$ does. The number d is called a **pseudo-scalar**.

The product $d = |\boldsymbol{c}| \cdot |\boldsymbol{a} \times \boldsymbol{b}| \cdot \cos\beta$ describes the volume of the parallel-epiped (oblique angled cuboid) which is formed by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ (Fig. 13.8).

This scalar triple product can be written as a determinant

$$\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\ . \quad (13.8)$$

Vector Product of a Polar and an Axial Vector

$$\boldsymbol{d} = \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b})\ .$$

Since the vector $(\boldsymbol{a} \times \boldsymbol{b})$ is perpendicular to $\boldsymbol{a}$ and to $\boldsymbol{b}$ and the vector $\boldsymbol{d}$ is perpendicular to $(\boldsymbol{a} \times \boldsymbol{b})$, $\boldsymbol{d}$ must lie in the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$ which we choose as the x–y-plane. It therefore can be described as a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\boldsymbol{d} = x \cdot \boldsymbol{a} + y \cdot \boldsymbol{b}\ , \quad (x \text{ and } y \text{ are real numbers})\ .$$

Inserting the components of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ gives, following the rules given above for the components,

$$x = c_1b_1 + c_2b_2 + c_3b_3\ ,$$
$$y = -c_1a_1 - c_2a_2 - c_3a_3\ .$$

These relations give the vector equation

$$\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a} - (\boldsymbol{c} \cdot \boldsymbol{a})\boldsymbol{b}\ . \quad (13.9)$$

Since the vector product changes its sign when the sequence of the factors are interchanged, we get the relations

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = -\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b})\ ,$$
$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \neq \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{c}) \times \boldsymbol{b}\ .$$

Neither the commutative law nor the associative law are valid for the triple vector products.

From Eq. 13.9 follows

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}\ . \quad (13.10)$$

Scalar Product of Two Axial Vectors

From the relations above we can conclude

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})\ , \quad (13.11a)$$

$$(\boldsymbol{a} \times \boldsymbol{b})^2 = a^2b^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2\ . \quad (13.11b)$$

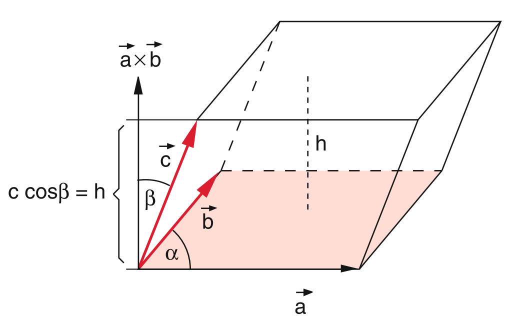

Figure 13.8 Scalar triple products $\boldsymbol{d} = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$ represents the volume of the parallel-epiped generated by vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$

13.1.6 Differentiation of Vectors

13.1.6.1 Vector-Fields

If it is possible to attribute to each space point $P(x, y, z)$ a vector $\boldsymbol{a} = \{a_x, a_y, a_z\}$ the entity of all vectors $\boldsymbol{a} = \boldsymbol{a}(x, y, z)$ is called a vector field. Each component of $\boldsymbol{a}$ is a function of the coordinates (x, y, z):

$$a_x = f_1(x, y, z)\ ; \quad a_y = f_2(x, y, z) \quad \text{and} \quad a_z = f_3(x, y, z)\ .$$

This means that length and direction of the vector a depend on the coordinates (x, y, z). If the components depend additionally on the time t, $\boldsymbol{a}(x, y, z)$ represents a time-dependent vector field. If $\boldsymbol{a}$ does not depend on time, the field is called stationary or static.

Examples

1. The velocity of particles in a fluid, flowing through a pipe with locally variable cross section represents a vector field. If the pressure is time dependent, the vector field $\boldsymbol{v} = \boldsymbol{f}(x, y, z, t)$ is non-stationary.

2. The force on a mass in the gravitation field of the earth depends on the distance from the earth centre. The force field is stationary because the force is independent of time. ◀

13.1.6.2 Scalar Differentiation of a Vector

We assume that the position vector $\boldsymbol{r}(x, y, z, t)$ is a continuous function $f(t)$ of time t, i. e. its components are continuous function of time. The variation of r with time is determined by the corresponding variation of the components. The equation (Fig. 13.9)

$$\frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

is the abbreviation for the three equations for the components

$$\frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

with corresponding equation for y and z.

For the limes $\Delta t \to 0$ the equation converges towards

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\mathrm{d}x}{\mathrm{d}t} , \quad \text{ect. for } y \text{ and } z .$$

The time derivative of the vector r then becomes

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \dot{\boldsymbol{r}} \stackrel{\text{Def}}{=} \{\dot{x}, \dot{y}, \dot{z}\} . \tag{13.12}$$

The derivative of a vector with respect to a scalar (e. g. the time t) is formed by differentiating all three components.

For the differentiation of products of vectors, the same rules are valid as for scalar quantities:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a} \cdot \boldsymbol{b}) = \dot{\boldsymbol{a}} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \dot{\boldsymbol{b}} \tag{13.13a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a} \times \boldsymbol{b}) = (\dot{\boldsymbol{a}} \times \boldsymbol{b}) + (\boldsymbol{a} \times \dot{\boldsymbol{b}}) . \tag{13.13b}$$

Note, that the succession of the factors in the product is essential because $(\boldsymbol{a} \times \boldsymbol{b}) = -(\boldsymbol{b} \times \boldsymbol{a})$.

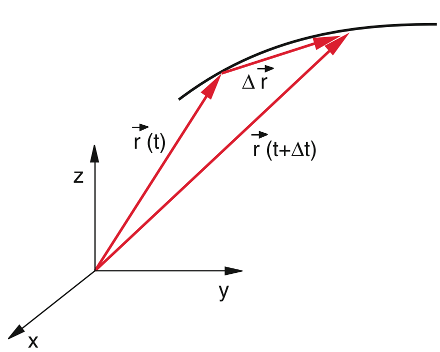

Figure 13.9 Differentiation of a vector with respect to time

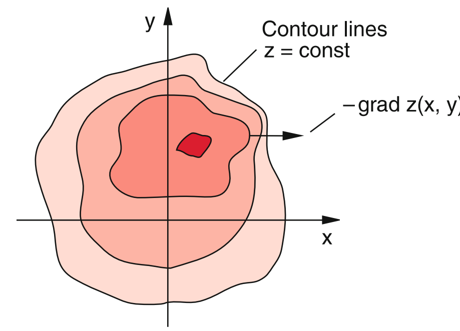

Figure 13.10 The gradient ∇f is a vector, perpendicular to the contour lines $f = z(x, y) =$ const. The *dark red spot* indicates the top of the mountain

13.1.6.3 The Gradient of a Scalar Quantity

The partial derivative $\partial f/\partial x$ of a scalar function $f(x, y, z)$ gives the change of f per unit length in x-direction, while y and z are kept constant. For example, for a surface $z = f(x, y)$ the expression $(\partial f/\partial x)_P$ gives the slope of the surface in the x-direction at the point P. Analogue expressions apply for $\partial f/\partial y$ and $\partial f/\partial z$.

The vector

$$\mathbf{grad} f = \nabla f = \left\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\} , \tag{13.14}$$

with the partial derivatives as components is called the **gradient** of the function $f(x, y, z)$. It is denoted b the symbol ∇ (nabla, which is the symbol for an Egyptian musical string instrument, similar to our harp).

The differential operator is then expressed by the vector

$$\nabla = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} , \tag{13.15}$$

which gets its significance only when applied to the scalar function $f(x, y, z)$. One can formally write $\mathbf{grad} f$ as the product of the differential operator ∇ and the function f.

The total change of the function $f(x, y, z)$ obtained when passing from a point $P(x, y, z) = P(\boldsymbol{r})$ into an arbitrary direction $\mathrm{d}\boldsymbol{n}$ to the neighbouring point $P(\boldsymbol{r} + d\boldsymbol{n})$ is

$$(\mathrm{d}f)_n = \frac{\partial f}{\partial x}\mathrm{d}n_x + \frac{\partial f}{\partial y}\mathrm{d}n_y + \frac{\partial f}{\partial z}\mathrm{d}n_z = \mathrm{d}\boldsymbol{n} \cdot \nabla f . \tag{13.16}$$

$(\mathrm{d}f_n)$ becomes maximum, when $\mathrm{d}\boldsymbol{n}$ is parallel to $\mathbf{grad} f$.

The gradient ∇f gives the direction of the maximum change of $f(x, y, z)$ (Fig. 13.10).

Example

For the surface $z = f(x, y)$ which gives the height z as a function of the coordinates x and y, ∇z is always perpendicular to the contour lines $z =$ const. The gradient is therefore the tangent vector to the trajectory of maximum slope. ◀

13.1.6.4 The Divergence of a Vector Field

The scalar product of the vector ∇ with a vector function $\boldsymbol{u}(x, y, z)$ (for example the locally varying velocity field of a fluid flow) is called the divergence of the vector field.

$$\operatorname{div} \boldsymbol{u}(x, y, z) = \nabla \cdot \boldsymbol{u} \tag{13.17}$$

According to the definition of the Nabla-operator in (13.15) this is equal to

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \,. \tag{13.18}$$

As has been shown in Sect. 8.3 the divergence of a volume element

$$\operatorname{div} \boldsymbol{u} \cdot \mathrm{d}V = \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S}$$

gives the vector flux passing per sec through the surface dS that surrounds the volume element dV. It is therefore also called the source function of the vector field $\boldsymbol{u}(x, y, z)$.

13.1.6.5 The Curl of a Vector Field

The vector product

$$\nabla \times \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{u} \tag{13.19}$$

of the vector ∇ with the vector $\boldsymbol{u}(x, y, z)$ is the **curl** of the vector field $\boldsymbol{u}(x, y, z)$.

According to the algorithm for vector products we obtain for the components of $\nabla \times u$

$$\begin{aligned}
(\nabla \times \boldsymbol{u})_x &= \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \,, \\
(\nabla \times \boldsymbol{u})_y &= \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \,, \\
(\nabla \times \boldsymbol{u})_z &= \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \,.
\end{aligned} \tag{13.20}$$

As has been shown in Chap. 8 is $\mathbf{curl}\, \boldsymbol{u}$ (also written as $\mathbf{rot}\, \boldsymbol{u}$) a measure for the rotation of a vortex in a fluid flow with the velocity field $\boldsymbol{u}(x, y, z)$.

13.1.6.6 Second Derivatives

With the nabla operator ∇ higher derivatives of scalar functions $f(x, y, z)$ or of vector fields $\boldsymbol{u}(x, y, z)$ can be written in a clear way:

- $\nabla \cdot (\nabla f) = \operatorname{div} \mathbf{grad} f$

$$\begin{aligned}
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f \,,
\end{aligned} \tag{13.21}$$

  where the symbol Δ is the **Laplace operator**.

- $\nabla(\nabla \cdot \boldsymbol{u}) = \mathbf{grad} \operatorname{div} \boldsymbol{u}$ is a vector with the three components

$$\begin{aligned}
\nabla(\nabla \cdot \boldsymbol{u})_x &= \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) \\
&= \frac{\partial}{\partial x}(\operatorname{div} \boldsymbol{u}) \,,
\end{aligned} \tag{13.22}$$

  and similar equations for the y- and z-components.

- $\nabla \times (\nabla \times \boldsymbol{u}) = \mathbf{curl\, curl}\, \boldsymbol{u}$.

  From the rules in Sect. 13.1.5 we obtain

$$\begin{aligned}
\nabla \times (\nabla \times \boldsymbol{u}) &= \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \cdot (\nabla \boldsymbol{u}) \\
&= \mathbf{grad} \operatorname{div} \boldsymbol{u} - \operatorname{div} \mathbf{grad}\, \boldsymbol{u} \,.
\end{aligned} \tag{13.23}$$

  This is a vector equation because $\nabla \cdot \nabla \boldsymbol{u}$ is the scalar product of the vector ∇ with the tensor $\nabla \boldsymbol{u}$ (see below). The equation for the x-components is

$$\begin{aligned}
&(\nabla \times \nabla \times \boldsymbol{u})_x \\
&\quad = \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - \Delta u_x \,.
\end{aligned} \tag{13.23a}$$

  Similar equations hold the y- and z-component.

- Besides the gradient of a scalar field there is also a vector gradient $\nabla \boldsymbol{u}$, which can be written in tensor form as

$$\nabla \boldsymbol{u} = \begin{pmatrix}
\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\
\frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\
\frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z}
\end{pmatrix} \,. \tag{13.24}$$

  The product $\nabla \cdot \nabla \boldsymbol{u}$ gives a vector with the components

$$\nabla \cdot \nabla \boldsymbol{u} = \{\Delta u_x, \Delta u_y, \Delta u_z\} \,, \tag{13.25}$$

  where Δ is the Laplace operator.

- $$(\nabla \times \nabla f) = \mathbf{curl\, grad} f \equiv 0 \,, \tag{13.26}$$

  which can be proved with (13.14) and (13.20) for functions f that have a continuous second derivative.
  Finally we consider the product div $\mathbf{rot}\, \boldsymbol{u}$

- $$\nabla \cdot (\nabla \times \boldsymbol{u}) = \operatorname{div} \mathbf{curl}\, \boldsymbol{u} \equiv 0 \,, \tag{13.27}$$

  because

$$\begin{aligned}
\nabla \cdot (\nabla \times \boldsymbol{u}) &= \frac{\partial}{\partial x}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \\
&\quad + \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \equiv 0 \,.
\end{aligned}$$

13.2 Coordinate Systems

The mathematical description of a physical process can be often essentially simplified when choosing the optimum coordinate system.

13.2.1 Cartesian Coordinates

The coordinate system consists of three coordinate axes (x, y, z) that are perpendicular to each other.

The coordinate planes

$$x = \text{const} \ , \quad y = \text{const} \ , \quad z = \text{const}$$

are planes perpendicular to the x, resp. y or z-axis. The intersection lines between two coordinate planes give the coordinate axes.

The intersection line between the(x–y)-plane ($z = \text{const}$) and the (x–z)-plane ($y = \text{const}$) is the x-axis. The y-axis is the intersection of the (x–y)-plane ($z = \text{const}$) with the (y–z)-plane ($x = \text{const}$), while the z-axis is the intersection of the (x–z) and the (y–z)-planes.

The vector $\boldsymbol{r}$ from the origin to the point $P(x, y, z)$ has the components

$$\boldsymbol{r} = \{x, y, z\} \ .$$

The three orthogonal unit vectors pointing int the three coordinate axes are

$$\hat{\boldsymbol{e}}_1 = \{1, 0, 0\} \ , \ \hat{\boldsymbol{e}}_2 = \{0, 1, 0\} \ , \ \hat{\boldsymbol{e}}_3 = \{0, 0, 1\} \ ,$$
$$\rightarrow \boldsymbol{r} = x \cdot \hat{\boldsymbol{e}}_1 + y \cdot \hat{\boldsymbol{e}}_2 + z \cdot \hat{\boldsymbol{e}}_3 \ .$$

The line element $\mathrm{d}\boldsymbol{r}$ on a curve between the points $P(\boldsymbol{r})$ and $P(\boldsymbol{r} + \mathrm{d}\boldsymbol{r})$ is

$$\mathrm{d}\boldsymbol{r} = \{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z\} \ . \tag{13.28}$$

The velocity of a point moving along the curve is

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \{\dot{x}, \dot{y}, \dot{z}\} \ . \tag{13.29}$$

The acceleration is then

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \dot{\boldsymbol{v}} = \{\ddot{x}, \ddot{y}, \ddot{z}\} \ . \tag{13.30}$$

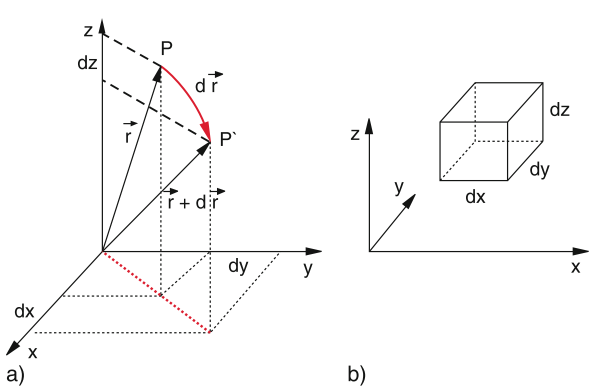

Figure 13.11 **a** Line element $\mathrm{d}\boldsymbol{r}$ and its components $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$; **b** volume element $\mathrm{d}V = \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z$

A volume element $\mathrm{d}V$ consists of the side edges $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ and has the volume

$$\mathrm{d}V = \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z \ . \tag{13.31}$$

13.2.2 Cylindrical Coordinates

When we use in the x–y-plane polar coordinates but keep for the z-direction the Cartesian z-coordinate we get **cylindrical coordinates** (ϱ, φ, z) (Fig. 13.12). They are useful for the description of problems with rotational symmetry (calculation of bodies with cylindrical symmetry, two-atomic molecules, fluid flow through circular pipes etc).

A space point $P(\varrho, \varphi, z)$ is described in cylindrical coordinates by its three coordinates ϱ, φ and z, where ϱ is the distance from the z-axis, φ the angle between the x-axis and the projection of P onto the x–y-plane and z its distance from the x–y-plane.

The conversion to cartesian coordinates is

$$\left.\begin{aligned} x &= \varrho \cos\varphi \\ y &= \varrho \sin\varphi \\ z &= z \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} \varrho &= \sqrt{x^2 + y^2} \\ \varphi &= \arctan(y/x) \\ z &= z \ . \end{aligned} \tag{13.32}$$

The coordinate planes are

- $\varrho = \text{const}$
  $=$ rotational cylinder surface around the z-axis,
- $\varphi = \text{const}$
  $=$ planes through the z-axis,
- $z = \text{const}$
  $=$ planes perpendicular to the z-axis.

The coordinate lines are

- **ϱ-lines** ($\varphi = \text{const}$, $z = \text{const}$)
  $=$ straight lines through the z-axis parallel to the x–y-plane,
- **φ-lines** ($\varrho = \text{const}$, $z = \text{const}$)
  $=$ horizontal circles around the z-axis,
- **z-lines** ($\varphi = \text{const}$, $\varrho = \text{const}$)
  $=$ straight lines parallel to the z-axis.

The three unit vectors form for each point $P(\varrho, \varphi, z)$ an orthogonal tripod

$$\begin{aligned} \hat{\boldsymbol{e}}_\varrho &= \{\cos\varphi, \sin\varphi, 0\} \\ \hat{\boldsymbol{e}}_\varphi &= \{-\sin\varphi, \cos\varphi, 0\} \\ \hat{\boldsymbol{e}}_z &= \{0, 0, 1\} \ . \end{aligned} \tag{13.33}$$

The line element $\mathrm{d}\boldsymbol{s}$ (Fig. 13.12) has the three components

$$\mathrm{d}\boldsymbol{s} = \{\mathrm{d}\varrho, \varrho\mathrm{d}\varphi, \mathrm{d}z\} \ . \tag{13.34}$$

The velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{s}/\mathrm{d}t$ therefore has the components

$$\boldsymbol{v} = \{\dot{\varrho}, \varrho\dot{\varphi}, \dot{z}\} = \dot{\varrho}\hat{\boldsymbol{e}}_\varrho + \varrho\dot{\varphi}\hat{\boldsymbol{e}}_\varphi + \dot{z}\hat{\boldsymbol{e}}_z \ , \tag{13.35}$$

and the acceleration is

$$\begin{aligned} \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= \ddot{\varrho}\hat{\boldsymbol{e}}_\varrho + \dot{\varrho}\frac{\mathrm{d}\hat{\boldsymbol{e}}_\varrho}{\mathrm{d}t} + \dot{\varrho}\dot{\varphi}\hat{\boldsymbol{e}}_\varphi + \varrho\ddot{\varphi}\hat{\boldsymbol{e}}_\varphi \\ &+ \varrho\dot{\varphi}\frac{\mathrm{d}\hat{\boldsymbol{e}}_\varphi}{\mathrm{d}t} + \ddot{z}\hat{\boldsymbol{e}}_z + \dot{z}\frac{\mathrm{d}\hat{\boldsymbol{e}}_z}{\mathrm{d}t} \ . \end{aligned} \tag{13.36}$$

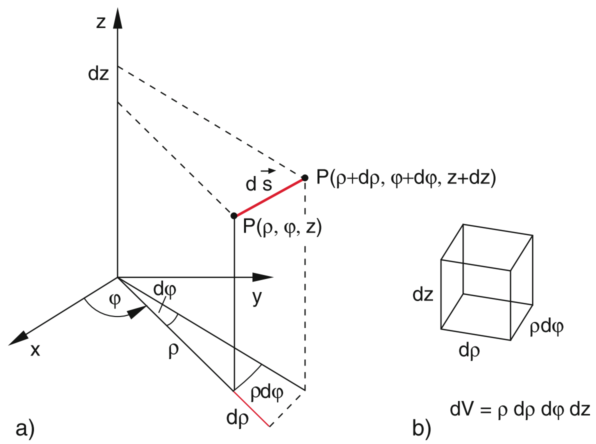

Figure 13.12 **a** Line element $\mathrm{d}\boldsymbol{s} = \{\mathrm{d}\varrho, \varrho\mathrm{d}\varphi, \mathrm{d}z\}$ in cylindrical coordinates; **b** volume element $\mathrm{d}V = \varrho\mathrm{d}\varrho\mathrm{d}\varphi\mathrm{d}z$

Inserting (13.33) gives

$$\boldsymbol{a} = (\ddot{\varrho} - \varrho\dot{\varphi}^2)\hat{\boldsymbol{e}}_\varrho + (2\dot{\varrho}\dot{\varphi} + \varrho\ddot{\varphi})\hat{\boldsymbol{e}}_\varphi + \ddot{z}\hat{\boldsymbol{e}}_z \,. \tag{13.37a}$$

The absolute value of the acceleration is then

$$a = |\boldsymbol{a}|\sqrt{\left(\ddot{\varrho} - \varrho\dot{\varphi}^2\right)^2 + \left(2\dot{\varrho}\dot{\varphi} + \varrho\ddot{\varphi}\right)^2 + \ddot{z}^2} \,. \tag{13.37b}$$

The surface element on the cylinder surface is

$$\mathrm{d}S = \varrho \cdot \mathrm{d}\varphi \cdot \mathrm{d}z \,, \tag{13.38}$$

and the volume element

$$\mathrm{d}V = \mathrm{d}\varrho \cdot \mathrm{d}S = \varrho \cdot \mathrm{d}\varrho \cdot \mathrm{d}\varphi \cdot \mathrm{d}z \,. \tag{13.39}$$

13.2.3 Spherical Coordinates

They are useful for all spherical symmetric problems, i. e. if the calculated quantities depend solely on the distance r from the centre.

Example

Motion of particles in a central force field. ◀

The position vector from the origin to the point $P(r; \vartheta, \varphi)$ is defined by its length r and the angles ϑ and φ. (Fig. 13.13).

The conversion relations between spherical and Cartesian coordinates are

$$\left.\begin{aligned} x &= r\sin\vartheta\cos\varphi \\ y &= r\sin\vartheta\sin\varphi \\ z &= r\cos\vartheta \end{aligned}\right\} \qquad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \vartheta &= \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi &= \arctan(y/x) \,. \end{aligned}$$

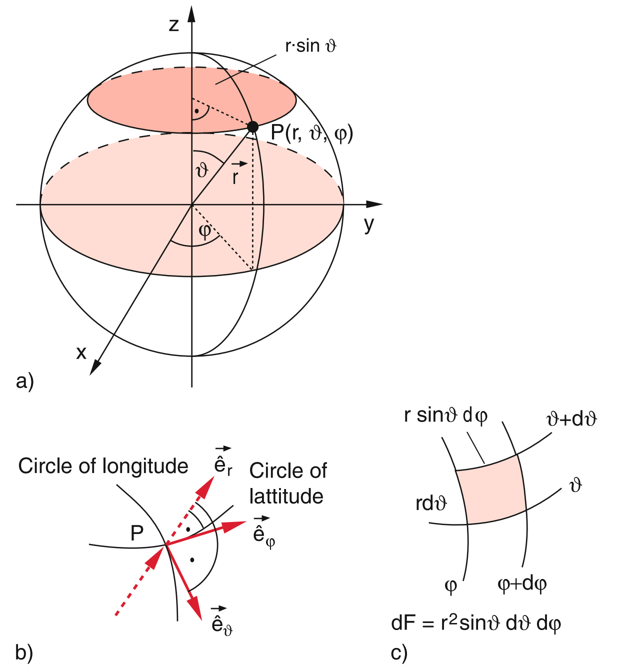

Figure 13.13 **a** Spherical coordinates; **b** orthogonal tripod of the unit vectors $\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\vartheta, \hat{\boldsymbol{e}}_\varphi$ at the point P; **c** surface element $\mathrm{d}S = r^2 \cdot \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$ on the surface of the sphere

Example

The unit sphere has in spherical coordinates the equation $r = 1$, in Cartesian coordinates $x^2 + y^2 + z^2 = 1$, in cylindrical coordinates $\varrho^2 + z^2 = 1$. ◀

The coordinate surfaces are:

- $r = \text{const}$: concentric spheres around $r = 0$,
- $\vartheta = \text{const}$: rotational cones around the z-axis with the peak at the origin,
- $\varphi = \text{const}$ planes through the z-axis.

The coordinate lines are:

- r-lines ($\varphi = \text{const}$, $\vartheta = \text{const}$): straight lines through the origin,
- ϑ-lines ($r = \text{const}$, $\varphi = \text{const}$): longitudinal circles (meridians),
- φ-lines ($r = \text{const}$, $\vartheta = \text{const}$): parallel circles around the z-axis (circles of lattitude).

The unit vectors in the point $P(r, \vartheta, \varphi)$ are (Fig. 13.13b)

$$\begin{aligned} \hat{\boldsymbol{e}}_r &= \{\sin\vartheta\cos\varphi; \sin\vartheta\sin\varphi; \cos\vartheta\} \,, \\ \hat{\boldsymbol{e}}_\vartheta &= \{\cos\vartheta\cos\varphi; \cos\vartheta\sin\varphi; -\sin\vartheta\} \,, \\ \hat{\boldsymbol{e}}_\varphi &= \{-\sin\varphi; \cos\varphi; 0\} \,. \end{aligned} \tag{13.40}$$

$\hat{\boldsymbol{e}}_r$ points into the $\boldsymbol{r}$-direction, $\hat{\boldsymbol{e}}_\vartheta$ is tangent to the longitudinal circles (meridians) in the point P and $\hat{\boldsymbol{e}}_\varphi$ is tangent to the circles of lattitude in P (Fig. 13.13b).

The line elements of the coordinate lines are

$$dr, \quad r \cdot d\vartheta, \quad r \cdot \sin\vartheta \, d\varphi \; . \tag{13.41a}$$

The line element of an arbitrary curve in the threedimensional space is then

$$ds = \{dr, r d\vartheta, r \cdot \sin\vartheta \, d\varphi\} \; . \tag{13.41b}$$

A surface element on the surface of the sphere is

$$dA = r^2 \sin\vartheta \, d\vartheta \, d\varphi \; . \tag{13.42}$$

A volume element is

$$dV = r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi \; . \tag{13.43}$$

The velocity of a point mass m on the trajectory $s(t)$ is according to (13.41b)

$$\boldsymbol{v} = \frac{d\boldsymbol{s}}{dt} = \left\{\dot{r}, r \cdot \dot{\vartheta}, r \cdot \sin\vartheta \, \dot{\varphi}\right\} \; . \tag{13.44}$$

The acceleration $\boldsymbol{a} = d\boldsymbol{v}/dt$ is obtained by differentiation of (13.44). This gives

$$\begin{aligned} \boldsymbol{a} = \ddot{r}\,\hat{\boldsymbol{e}}_r + \dot{r}\,\frac{d\hat{\boldsymbol{e}}_r}{dt} + \left(\dot{r}\dot{\vartheta} + r\ddot{\vartheta}\right)\hat{\boldsymbol{e}}_\vartheta + r\,\dot{\vartheta}\,\frac{d\hat{\boldsymbol{e}}_\vartheta}{dt} \\ + \left(\dot{r}\sin\vartheta\,\dot{\varphi} + r\cos\vartheta\,\dot{\vartheta}\,\dot{\varphi} + r\sin\vartheta\,\ddot{\varphi}\right)\hat{\boldsymbol{e}}_\varphi \\ + r\sin\vartheta\,\dot{\varphi}\,\frac{d\hat{\boldsymbol{e}}_\varphi}{dt} \; . \end{aligned} \tag{13.45}$$

This can be written (using (13.40)) as linear combination of $\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\vartheta, \hat{\boldsymbol{e}}_\varphi$.

13.3 Complex Numbers

The solution of the quadratic equation $x^2 + 1 = 0$ gives $x_{1,2} = \pm\sqrt{-1}$, which do not belong to the real numbers, for which the square of a real number x must be always ≥ 0.

Numbers x with $x^2 < 0$ are named imaginary numbers. Their unit element is $i = +\sqrt{-1}$.

Similar to the assignment of real numbers to the axis of real numbers (generally the x-axis) the imaginary numbers are assigned to the y-axis, called the imaginary axis.

The two axis define a plane, called the complex plane. One attributes to each point $P(x, y)$ in the complex plane a complex number

$$z = x + iy \; , \tag{13.46}$$

where the first number x is the real part and the second number y the imaginary part of the complex number (Fig. 13.14).

Introducing the unit vectors $\hat{\boldsymbol{e}}_x$ and $\hat{\boldsymbol{e}}_y$ in the complex plane, each point $P(x, y)$ in this plane with the position vector $\boldsymbol{r}$ can be characterized by

$$\boldsymbol{r} = x \cdot \hat{\boldsymbol{e}}_x + iy\hat{\boldsymbol{e}}_y \; . \tag{13.47}$$

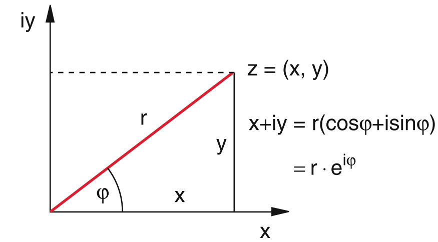

Figure 13.14 Representation of a complex number z as a point in the complex plane (x, iy)

The absolute value of the complex number z is

$$|z| = \sqrt{x^2 + y^2} \; . \tag{13.48}$$

It represents the distance $r = |\boldsymbol{r}|$ of $P(x, y)$ from the origin $(0, 0)$. The number

$$z^* = x - iy \tag{13.49}$$

is the conjugate complex of $z = x + iy$.

13.3.1 Calculation rules of Complex Numbers

The following rules for addition, multiplication and division of complex numbers are analogous to those of two-dimensional vectors $\boldsymbol{r} = \{x, y\}$.

Addition

$$z_1 + z_2 = \{x_1, y_1\} + \{x_2, y_2\} \stackrel{\text{Def}}{=} \{x_1 + x_2, y_1 + y_2\} \; . \tag{13.50}$$

Two complex numbers are added by adding the real parts and the imaginary parts. When z and z^* are added, this gives

$$z + z^* = \{x + iy\} + \{x - iy\} = 2x \; , \tag{13.51}$$

i.e. twice the real part.

Multiplication

$$\begin{aligned} z_1 \cdot z_2 &= \{x_1, y_1\} \cdot \{x_2, y_2\} = (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= (x_1 \cdot x_2 - y_1 \cdot y_2) + i \cdot (x_1 y_2 + x_2 y_1) \end{aligned} \tag{13.52}$$

The product

$$z \cdot z^* = (x + iy) \cdot (x - iy) = (x^2 + y^2) = |z|^2 \tag{13.53}$$

gives the square of the absolute value $|z|$.

Division

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} .$$

Multiplication of numerator and denumerator with $(x_2 - iy_2)$ gives

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{(x_1 + iy_1)\cdot(x_2 - iy_2)}{x_2^2 + y_2^2} \\ &= \frac{(x_1x_2 + y_1y_2) + \mathrm{i}(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} = a + ib .\end{aligned}$$

This gives again a complex number with the real part

$$a = \frac{(x_1x_2 + y_1y_2)}{x_2^2 + y_2^2}$$

and the imaginary part

$$b = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} .$$

13.3.2 Polar Representation

Often the representation of a complex number in polar coordinates r and φ is more convenient (Fig. 13.14).

With

$$x = r\cdot\cos\varphi ; \quad y = r\cdot\sin\varphi$$

we obtain

$$\begin{aligned}&z = x + iy = r(\cos\varphi + \mathrm{i}\sin\varphi) = r\cdot\mathrm{e}^{\mathrm{i}\varphi} \\ &z^* = x - iy = r\cdot\mathrm{e}^{-\mathrm{i}\varphi} \\ &\Rightarrow z\cdot z^* = r^2 ; \quad |z| = \sqrt{x^2 + y^2} = r .\end{aligned}$$

From Fig. 13.14 we see that

$$\tan\varphi = \frac{iy}{x} = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} \quad \Rightarrow \varphi = \arctan\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} .$$

Note: The polar representation is not unambiguous, because all angles $\varphi_n = \varphi_0 + n\cdot 2\pi$ $(n = 1, 2, 3, \ldots)$ represent the same complex number z. The representation with $n = 0$ is called the principal value.

From $z = r\cdot\mathrm{e}^{\mathrm{i}\varphi} \Rightarrow \ln z = \ln r + \mathrm{i}(\varphi_0 + 2n\pi)$.

The polar representation facilitates multiplication and division of complex numbers

$$\begin{aligned}z_1\cdot z_2 &= r_1\cdot r_2\cdot\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}\cdot\mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)} .\end{aligned}$$

For arbitrary n one obtains

$$z^n = (r\cdot\mathrm{e}^{\mathrm{i}\varphi})^n = r^n\cdot\mathrm{e}^{\mathrm{i}n\varphi} .$$

Complex numbers are raised to higher powers n by calculatuing the n-th power of r and multiplying φ by n. In the same way we see the relation

$$\sqrt[n]{z} = z^{1/n} = \sqrt[n]{r}\cdot\mathrm{e}^{\mathrm{i}\varphi/n} .$$

The general rule for complex numbers can be formulated as (see mathematics text books):

> The set of complex numbers $z = (x, y)$ forms a body which includes the real numbers $(x, 0)$ as subset.

13.4 Fourier-Analysis

In mathematical textbooks it is proved, that every continuously differentiable function $f(x)$ can be written as infinite series of basis functions $g(x)$, if the $g(x)$ represent a complete set.

We choose as basis functions the trigonometric functions $\sin(nx)$ and $\cos(nx)$ $(n = 0, 1, 2, \ldots)$.

The Fourier-theorem states:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right] . \tag{13.54}$$

(13.54) is called "Fourier-Series". By multiplication of (13.54) with $\cos(mx)$ or $\sin(mx)$ respectively and integration over x from $x = 0$ to $x = 2\pi$ one obtains because of

$$\begin{aligned}&\int_0^{2\pi}\cos(nx)\cos(mx)\,\mathrm{d}x = \begin{cases}0 & \text{for } m \neq n \\ \pi & \text{for } n = m \neq 0\end{cases} \\ &\int_0^{2\pi}\cos(nx)\sin(mx)\,\mathrm{d}x = 0 \quad \text{for } m \gtreqless n \\ &\int_0^{2\pi}\sin(nx)\sin(mx)\,\mathrm{d}x = \begin{cases}0 & \text{for } n \neq m \\ \pi & \text{for } n = m \neq 0\end{cases}\end{aligned} \tag{13.55}$$

the coefficients a_n and b_n as

$$\begin{aligned}&a_0 = b_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x ; \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,\mathrm{d}x ; \\ &b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,\mathrm{d}x .\end{aligned} \tag{13.56}$$

It is therefore possible to determine the coefficients a_n, b_n in the Fourier-Series (13.54) by integration of $f(x)$.

For $x = \omega \cdot t$ the coefficients give the amplitudes of the contributions $n\omega$ to the total function $f(\omega, t)$ (Fourier-Analysis) with $x = \omega t$ and $T = 2\pi/\omega$ (13.55) transfers to

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,\mathrm{d}t\,, \\ b_n &= \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,\mathrm{d}t\,. \end{aligned} \tag{13.57}$$