Systems of Point Masses; Collisions

4

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In the preceding chapters we have discussed the motion of a single particle and its trajectory under the influence of external forces. In this chapter we will deal with systems of many particles, where besides possible external forces also interactions between the particles play an important role.

4.1 Fundamentals

At first we introduce several expressions and definitions of fundamental terms and notations for systems of many particles.

4.1.1 Centre of Mass

We consider N point masses with position vectors r_i and define as the centre of mass the point with the position vector

$$\boldsymbol{R}_\mathrm{S} = \frac{\sum_i m_i \boldsymbol{r}_i}{\sum_i m_i} = \frac{1}{M}\sum_i m_i \boldsymbol{r}_i \;, \tag{4.1}$$

where $M = \sum m_i$ is the total mass of all N particles (Fig. 4.1).

When the masses m_i move with the velocities $\boldsymbol{v}_i = \mathrm{d}\boldsymbol{r}_i/\mathrm{d}t$ we define the velocity v_S of the centre of mass as

$$\boldsymbol{v}_\mathrm{S} = \frac{\mathrm{d}\boldsymbol{R}_\mathrm{S}}{\mathrm{d}t} = \frac{1}{M}\sum_i m_i \boldsymbol{v}_i \;. \tag{4.2a}$$

With the momenta $\boldsymbol{p}_i = m_i \cdot \boldsymbol{v}_i$ (4.2a) can be also expressed by the total momentum $\boldsymbol{P} = \sum \boldsymbol{p}_i$ as

$$\boldsymbol{P} = M\boldsymbol{v}_\mathrm{S} \;. \tag{4.2b}$$

If no external forces are acting on the particles, we need to regard only internal forces, i. e. interactions between the particles. Such a system without external forces is called a *closed system.*

From the Newtonian law $\boldsymbol{F}_{ik} = -\boldsymbol{F}_{ki}$ it follows: $\sum_i \sum_{k\neq i} \boldsymbol{F}_{ik} = \boldsymbol{0}$. **In a closed system the vector sum of all forces is zero.**

With $F_i = \sum_{k\neq i} F_{ik}$ and $F_i = \mathrm{d}\boldsymbol{p}_i/\mathrm{d}t$ the total momentum of the system

$$\boldsymbol{P} = \sum \boldsymbol{p}_i = \textbf{const} \;. \tag{4.3}$$

Since P is the momentum of the centre of mass we can state:

> The centre of mass of a closed system moves with constant momentum. This implies that its velocity does not change.

If an external total force $\boldsymbol{F} \neq \boldsymbol{0}$ acts onto the system we can write

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}\sum \boldsymbol{p}_i = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} \;, \tag{4.4}$$

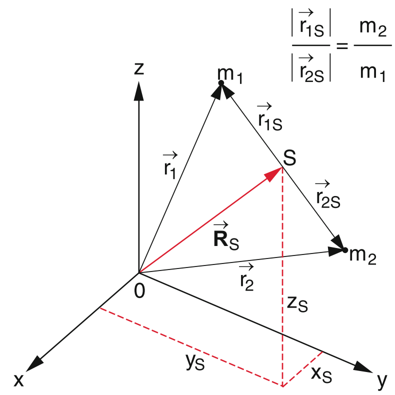

Figure 4.1 Definition of center of mass

With the acceleration of the centre of mass $\boldsymbol{a}_\mathrm{S} = d\boldsymbol{v}_\mathrm{S}/\mathrm{d}t$ we obtain

$$\boldsymbol{F} = M\boldsymbol{a}_\mathrm{S} \;. \tag{4.5}$$

> The centre of mass of an arbitrary system of particles moves in the same way as a body with the total mass $M = \sum m_i$ would move under the action of the external force $\boldsymbol{F}$.

Often it is useful to choose a coordinate system with the centre of mass as origin, which moves with the velocity $\boldsymbol{v}_\mathrm{S}$ of the centre of mass against the fixed laboratory system. Such a system is called the **centre of mass system (CM-system)**.

The position vectors $\boldsymbol{r}_i$ in the lab-system are related to the position vectors $\boldsymbol{r}_{i\mathrm{S}}$ in the CM-system (Fig. 4.1) by

$$\boldsymbol{r}_i = \boldsymbol{r}_{i\mathrm{S}} + \boldsymbol{R}_\mathrm{S} \;. \tag{4.6a}$$

Inserting into (4.1) gives

$$\begin{aligned}\sum_i m_i \boldsymbol{r}_{i\mathrm{S}} &= \sum_i m_i \left(\boldsymbol{r}_i - \boldsymbol{R}_\mathrm{S}\right)\\ &= \sum_i m_i \boldsymbol{r}_i - \boldsymbol{R}_\mathrm{S}\sum_i m_i = \boldsymbol{0}\;,\end{aligned}$$

$$\sum m_i \boldsymbol{r}_{i\mathrm{S}} = \boldsymbol{0} \tag{4.6b}$$

This implies that in the CM-system the position vector $\boldsymbol{R}_\mathrm{S}$ of the centre-of-mass is $\boldsymbol{R}_\mathrm{S} = (1/M)\sum m_i \boldsymbol{r}_{i\mathrm{S}} = \boldsymbol{0}$.

The relation between the velocity $\boldsymbol{v}_i$ in the lab-system and $\boldsymbol{v}_{iS}$ in the CM-system is

$$\boldsymbol{v}_i = \boldsymbol{v}_{iS} + \boldsymbol{V}_S \,, \tag{4.6c}$$

which can be verified by differentiation of (4.6a). For the momenta we therefore get

$$\sum_i m_i \boldsymbol{v}_{iS} = \sum_i \boldsymbol{p}_{iS} = \boldsymbol{0} \,. \tag{4.6d}$$

The sum of all momenta in the CM-system is always zero.

For a closed system of two masses m_1 and m_2 the total kinetic energy in the lab-system is

$$\begin{aligned} E_{\text{kin}} &= \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 \\ &= \tfrac{1}{2}\left(m_1 v_{1S}^2 + m_2 v_{2S}^2\right) + \tfrac{1}{2}(m_1 + m_2) V_S^2 \\ &\quad + (m_1 \boldsymbol{v}_{1S} + m_2 \boldsymbol{v}_{2S}) \cdot \boldsymbol{V}_S \,. \end{aligned} \tag{4.7a}$$

The last term is zero because $\boldsymbol{p}_{1S} + \boldsymbol{p}_{2S} = \boldsymbol{0}$ and we obtain:

$$E_{\text{kin}} = E_{\text{kin}}^{(S)} + \tfrac{1}{2} M V_S^2 \,. \tag{4.7b}$$

In the Lab-system the kinetic energy of a closed system can be written as the sum of $E_{\text{kin}}^{(S)}$ in the CM-system plus the kinetic energy of the total mass M concentrated in the center of mass S (translational energy of the system).

The total motion of the closed system can be divided into a uniform motion of S with the constant velocity V_S and a relative motion of the two particles against S.

4.1.2 Reduced Mass

We consider two particles with masses m_1 and m_2 which interact with each other due to the forces $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$. Without other external forces the equations of motion read:

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = \frac{\boldsymbol{F}_{12}}{m_1} \,; \quad \frac{\mathrm{d}\boldsymbol{v}_2}{\mathrm{d}t} = \frac{\boldsymbol{F}_{21}}{m_2} \,. \tag{4.8a}$$

Subtraction yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \boldsymbol{F}_{12} \,, \tag{4.8b}$$

where $\boldsymbol{v}_{12} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ is the relative velocity of the two particles.

Introducing the **reduced mass**

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \,, \tag{4.9}$$

and rewrite Eq. 4.8b we get

$$\boldsymbol{F}_{12} = \mu \frac{\mathrm{d}\boldsymbol{v}_{12}}{\mathrm{d}t} \,. \tag{4.10}$$

This means: For the relative motion of the two particles the equation of motion is completely analogous to Newton's equation (2.18a) for a single particle with the mass μ. This shows the usefulness of defining the reduced mass.

The kinetic energy E_{kin}^{S} of the two particles in the CM-system

$$\begin{aligned} E_{\text{kin}}^{(S)} &\overset{\text{Def}}{=} \sum_i \frac{m_i}{2} v_{iS}^2 \\ &= \frac{1}{2} \sum m_i v_i^2 - \frac{1}{2} M V_S^2 \,. \end{aligned} \tag{4.11a}$$

is the difference of E_{kin} in the lab-system and the kinetic energy of the CM.

Inserting $\boldsymbol{v}_S = (1/M) \sum m_i \boldsymbol{v}_i$ gives with (4.9)

$$E_{\text{kin}}^{(S)} = \frac{1}{2} \mu v_{12}^2 \,. \tag{4.11b}$$

The kinetic energy of a closed system of two particles in the CM-system equals the kinetic energy of a single parrtivle with the reduced mass μ which moves with the relative velocity $\boldsymbol{v}_{12}$.

This important relations can be summarized as:

The relative motion of two particles under the influence of their mutual interaction $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$ can be reduced to the motion of a single particle with the reduced mass μ driven by the force $\boldsymbol{F}_{12}$.

This is illustrated in Fig. 4.2 where two masses $m_1 = m$ and $m_2 = 1.5m$ move around their centre of mass S which moves itself with the velocity V_S. An example of such a system is a double-star system, where two stars with different masses circulate around their common CM (see Vol. 4).

4.1.3 Angular Momentum of a System of Particles

We consider two point masses m_1 and m_2 with their mutual interaction forces

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$$

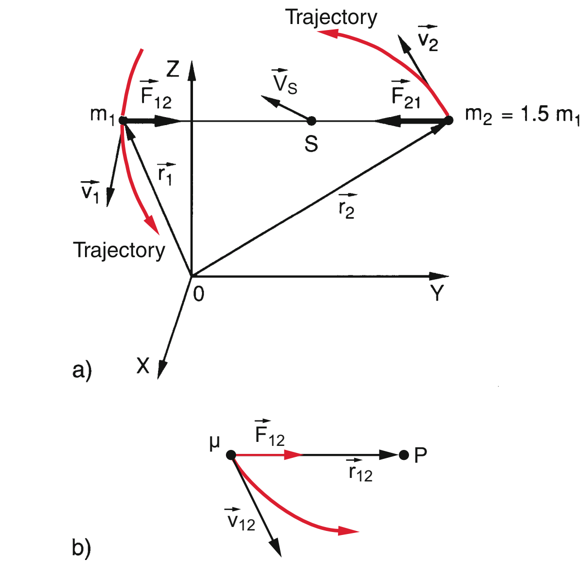

Figure 4.2 **a** Velocity V_S of the CM of a system of two masses with velocities v_i; **b** Reduction of the relative motion of two masses m_i to the motion of a single particle with the reduced mass μ under the action of the force F_{12}

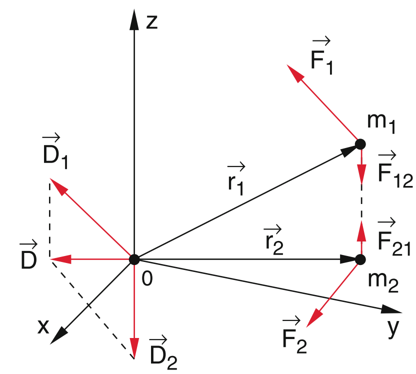

Figure 4.3 Torques acting on a system of two masses under the influence of external forces

and the external forces $\boldsymbol{F}_1$ acting on m_1 and $\boldsymbol{F}_2$ acting on m_2. The torques on the two masses with respect to the origin 0 of the coordinate system are

$$\boldsymbol{D}_1 = \boldsymbol{r}_1 \times (\boldsymbol{F}_1 + \boldsymbol{F}_{12}) \ ,$$
$$\boldsymbol{D}_2 = \boldsymbol{r}_2 \times (\boldsymbol{F}_2 + \boldsymbol{F}_{21}) \ ,$$

and the total torque of the system is then (Fig. 4.3)

$$\boldsymbol{D} = (\boldsymbol{r}_1 \times \boldsymbol{F}_1) + (\boldsymbol{r}_2 \times \boldsymbol{F}_2) + (\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{F}_{12} \ .$$

Since the direction of the internal forces $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$ lies in the direction of the connecting line $\boldsymbol{r}_{12} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)$ the last term vanishes and the total torque

$$\boldsymbol{D} = (\boldsymbol{r}_1 \times \boldsymbol{F}_1) + (\boldsymbol{r}_2 \times \boldsymbol{F}_2) \tag{4.12}$$

becomes the vector sum of the torques on the individual particles. Without external forces the total torque on the system is zero!

The total angular momentum L of the system with respect to the origin 0 is

$$\boldsymbol{L} = (\boldsymbol{r}_1 \times \boldsymbol{p}_1) + (\boldsymbol{r}_2 \times \boldsymbol{p}_2) \ , \tag{4.13}$$

and we obtain, analogous to the Eq. 2.48 for a single particle:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = (\boldsymbol{r}_1 \times \boldsymbol{F}_1) + (\boldsymbol{r}_2 \times \boldsymbol{F}_2) = \boldsymbol{D} \ . \tag{4.14}$$

The derivation of these equations and the situations discussed for a system of two particles can be readily generalized to a system of many particles. This gives the important statement:

The time derivative of the total angular momentum of a system of particles referred to an arbitrary point is equal to the total torque exerted onto the system and referred to the same point.

For the special case that no external forces are present the torque is zero and therefore the angular momentum L is constant.

The total angular momentum of a closed system of particles is constant.

Using CM-coordinates we can divide the angular momentum (4.13) according to (4.6a)

$$\begin{aligned}\boldsymbol{L} = {} & m_1\,(\boldsymbol{r}_{1S} + \boldsymbol{R}_S) \times (\boldsymbol{v}_{1S} + \boldsymbol{V}_S) \\ & + m_2\,(\boldsymbol{r}_{2S} + \boldsymbol{R}_S) \times (\boldsymbol{v}_{2S} + \boldsymbol{V}_S) \ .\end{aligned}$$

For many particles this reads

$$\begin{aligned}\boldsymbol{L} &= \sum_i m_i\,(\boldsymbol{r}_{iS} + \boldsymbol{R}_S) \times (\boldsymbol{v}_{iS} + \boldsymbol{V}_S) \\ &= M\,(\boldsymbol{r}_S \times \boldsymbol{V}_S) + \sum_i m_i\,(\boldsymbol{r}_{iS} \times \boldsymbol{v}_{iS}) \\ &\quad + \sum_i m_i\,(\boldsymbol{R}_S \times \boldsymbol{v}_{iS}) + \sum_i m_i\,(\boldsymbol{r}_{iS} \times \boldsymbol{V}_S) \ .\end{aligned}$$

The terms $\sum_i m_i(\boldsymbol{R}_S \times \boldsymbol{v}_S)$ and $\sum_i m_i(\boldsymbol{r}_{iS} \times \boldsymbol{v}_S)$ are zero according to (4.6d) and (4.6b) and it follows:

$$\boldsymbol{L} = M\,(\boldsymbol{r}_S \times \boldsymbol{V}_S) + \sum_i m_i\,(\boldsymbol{r}_{iS} \times \boldsymbol{v}_{iS}) \ . \tag{4.14a}$$

The first term

$$\boldsymbol{L}_{0S} = M\,(\boldsymbol{R}_S \times \boldsymbol{V}_S) \tag{4.15a}$$

is the angular momentum of the total mass contracted in the CM referred to the origin of the coordinate system. The second term gives the total angular momentum referred to the CM.

For a system of two particles we can transform $\boldsymbol{L}_S$ because of $\sum_i m_i \boldsymbol{v}_{iS} = 0$ into

$$\begin{aligned} \boldsymbol{L}_S &= \sum \boldsymbol{L}_{iS} = (\boldsymbol{r}_{1S} \times \boldsymbol{p}_{1S}) + (\boldsymbol{r}_{2S} \times \boldsymbol{p}_{2S}) \\ &= (\boldsymbol{r}_{1S} - \boldsymbol{r}_{2S}) \times \boldsymbol{p}_{1S} = \boldsymbol{r}_{12} \times \mu \boldsymbol{v}_{12} \,, \end{aligned} \tag{4.15b}$$

(with $\boldsymbol{p}_{iS} = \mu \boldsymbol{v}_{12}$). This follows from (4.6d) and (4.10). We can therefore state:

The angular momentum $\boldsymbol{L}_S$ of a system of two particles in the CM is equal to the angular momentum of a single particle with the reduced mass μ and the position vector $\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2$.

Examples

1. The relative motion of the earth-moon system around their common center of mass S (Fig. 4.4) can be reduced to the motion of a single body with reduced mass $\mu = m_E \cdot m_{Mo}/(m_E + m_{Mo}) \approx 0.99 m_{Mo}$ in the central gravitational force field between earth and moon around the centre M of the earth. The centre of mass is located inside the earth 4552 km away from the centre M because the mass of the moon $m_{Mo} \approx 0.01 m_E$ is small compared with the earth mass. In the CM-system earth and moon describe nearly circular elliptical orbits around the common CM with radii

$$r_E = (m_{Mo}/(m_E + m_{Mo}))\, r_{EMo} \approx 0.01 r_{EMo}$$

and

$$r_{Mo} = (m_E/(m_E + m_{Mo}))\, r_{EMo} \approx 0.99 r_{EMo} \,,$$

where r_{EMo} is the distance between earth and moon. In a coordinate system which is referred to the centre of our galaxy the lunar orbit is a complicated curve, shown in Fig. 4.4b where the deviations from the path of the CM are exaggerated in order to elucidate the situation. This complicated motion can be composed of

a) the motion of the moon around the CM of the earth-moon system
b) the motion of the CM around the centre of mass of the solar system, which is located inside the sun, because $M_\odot > 10^3 \cdot \sum m_{\text{Planets}}$.
c) the motion of the CM of the solar system around the centre of our galaxy.
d) The exact calculation of the lunar orbit has to take into account the simultaneous gravitational attraction of the moon by the earth and the sun, which changes with time because of the changing relative position of the three bodies. Because of this "perturbation" the lunar orbit is not exactly an ellipse around the CM. Although there is no analytical solution for the exact orbit, very good numerical approximations have been developed [4.1b].

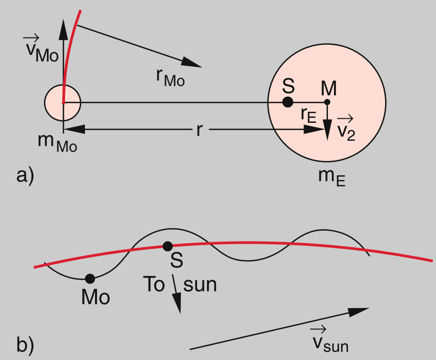

Figure 4.4 **a** Motion of the moon in the CM-system earth–moon. **b** Motion of the moon and the CM in the galactic coordinate system where the sun also moves

2. The hydrogen atom is a two-body system of an electron with mass m_e and proton with mass m_p. Because $m_p = 1836 m_e$ the reduced mass is $\mu = 0.99946 m_e \approx m_e$. In a classical picture proton and electron circulate around the CM. With the mean distance r_{pe} between proton and electron the CM lies $(1/1836) r_{pe}$ from the centre of the proton. The motion of the two particles can be separated into the translation of the CM with the velocity V_S and the motion of a particle with mass μ with the relative velocity v_{pe} around the CM. The total kinetic energy of the H-atom is then:

$$E_{kin} = \tfrac{1}{2}\left(m_p + m_e\right) V_S^2 + \tfrac{1}{2}\mu v_{pe}^2 \,.$$

For velocities of the H-atom which correspond to thermal energies at room temperature the first term ($\approx$ 0.03 eV) is very small compared to the second term of the "internal" energy ($\approx$ 10 eV). ◀

4.2 Collisions Between Two Particles

This section is of great importance for the understanding of many phenomena in Atomic and Nuclear Physics, because an essential part of our knowledge about the structure and dynamics of atoms and nuclei arises from investigations of collision processes.

When two particles approach each other they are deflected due to the interaction forces between them. The deflection occurs in the whole spatial range where the forces are noticeable (Fig. 4.5). Due to this interaction both particles change their momentum and often also their energy. However, conservation laws demand that momentum and energy of the total system are always preserved.

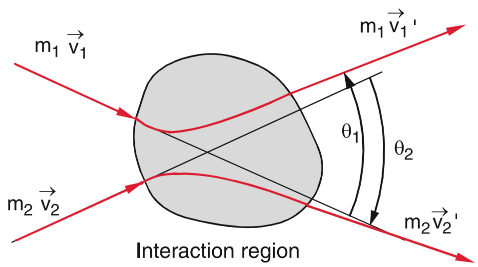

Figure 4.5 Schematic illustration of a collision with the asymptotic scattering angles θ_1 an θ_2

The exact form of the trajectory of the particles inside the interaction zone can be determined only if the exact interaction potential is known. However, it is possible to make definite statements about magnitude and direction of the particle momenta after the collision in a great distance from the interaction zone. These statements are based solely on the conservation of momentum and energy. We will illustrate this in more detail in the following section.

4.2.1 Basic Equations

Although the total energy of the two colliding partners is preserved during collisions, part of the translational energy is often converted into other forms of energy, as for instance potential energy or thermal energy. From (4.3) it follows however, that the total momentum of the collision partners is always retained.

The basic equations for collision processes between two particles with velocities v which are small compared to the velocity c of light (non-relativistic collisions) can be written as:

conservation of momentum (Fig. 4.6)

$$\boldsymbol{p}_1' + \boldsymbol{p}_2' = \boldsymbol{p}_1 + \boldsymbol{p}_2 \tag{4.16}$$

conservation of energy

$$\frac{p_1'^2}{2m_1'} + \frac{p_2'^2}{2m_2'} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + U \tag{4.17}$$

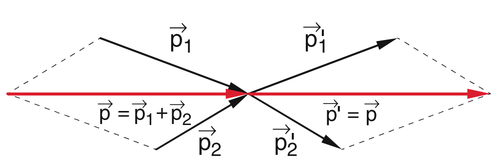

Figure 4.6 Conservation of total momentum at the collision of two particles

where p_i' is the momentum of particle i after the collision and U is that part of the initial energy that had been converted into internal energy of one or both of the collision partners and is therefore missing in the kinetic energy after the collision ($U < 0$). If internal energy of the colliding partners has been transferred into kinetic energy we get $U > 0$.

The Eq. 4.16 and 4.17 describe the collision process completely in that sense, that relations between magnitude and direction of the individual momenta of the particles after the collision can be determined, if they are known before the collision.

Depending on the magnitude of U we distinguish between three cases:

- $\boldsymbol{U = 0}$, **elastic collisions**. The total kinetic energy is preserved, while the kinetic energy of the individual particles generally changes.
- $\boldsymbol{U < 0}$, **inelastic collisions**. The total kinetic energy after the collision is smaller than before. Part of the initial kinetic energy has been converted into internal energy of the collision partners.
- $\boldsymbol{U > 0}$, **superelastic collisions** (sometimes called collisions of the second kind). At least one of the collision partners had internal energy before the collision which was transferred into kinetic energy during the collision.
  The kinetic energy after the collision is larger than before the collision.

During **reactive collisions** (for instance during chemical reactions or in high energy collisions) new particles can be produced and the masses of the collision partners may change. An example is the reaction

$$H_2 + Cl_2 \longrightarrow HCl + HCl\,.$$

These reactive collisions are treated later.

Note:

- While the kinetic energy is only preserved in elastic collisions the total momentum is preserved for all kinds of collisions (Fig. 4.6).
- Inelastic, super-elastic and reactive collisions can only occur, if at least one of the collision partners has an internal structure. This means that it must consist of at least two particles, which are bound together. Examples are atoms (consisting of nuclei and electrons) or nuclei (consisting of protons and neutrons). Part of the kinetic energy of the collision partners then can be transferred into the increase of the internal energy (potential or kinetic energy of the constituents). For collision partners consisting of many particles (for example solids) the increase of kinetic energy of the constituents can be defined as an increase of the temperature (see Sect. 7.3) which is then called "*thermal energy*" (see Sect. 10.1).

4.2.2 Elastic Collisions in the Lab-System

The description of collision processes can be essentially simplified when the appropriate coordinate system is chosen. For many situations one of the collision partners, for instance m_2, is at rest before the collision. We choose its position as the origin of our coordinate system, which is fixed relative to the laboratory system. In this system is therefore $\boldsymbol{p}_2 = \boldsymbol{0}$ (Fig. 4.7). We assume that the masses do not change during the collision ($m_1 = m_1'$, $m_2 = m_2'$). With $U = 0$ for elastic collisions we obtain from (4.16) and (4.17)

$$\boldsymbol{p}_1 = \boldsymbol{p}_1' + \boldsymbol{p}_2' = \boldsymbol{p}' \ , \tag{4.16a}$$

$$\frac{p_1^2}{2m_1} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2} \ . \tag{4.17a}$$

We choose the direction of the initial momentum $\boldsymbol{p}_1$ as the x-direction (Fig. 4.8) $\Rightarrow \boldsymbol{p}_1 = \{p_1, 0, 0\}$. The angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ points into the z-direction. Because the angular momentum is constant the motion of the collision partners is restricted to the x-y-plane. The endpoint of the vector $\boldsymbol{p}_2'$ is the point $P(x, y)$. From Fig. 4.8 we derive the relations:

$$x^2 + y^2 = p_2'^2 ,$$
$$(p_1 - x)^2 + y^2 = p_1'^2 \ .$$

Inserting into (4.17a) yields

$$\frac{p_1^2}{2m_1} = \frac{(p_1 - x)^2 + y^2}{2m_1} + \frac{x^2 + y^2}{2m_2} \ .$$

Rearranging gives with the reduced mass $\mu = m_1 \cdot m_2 / (m_1 + m_2)$ the equation

$$(x - \mu v_1)^2 + y^2 = (\mu v_1)^2 \tag{4.18}$$

of a circle in the x-y-plane with the radius $r = \mu v_1$ and the centre $M = \{\mu v_1, 0\}$. This implies that the endpoints of all possible vectors p_2' which fulfil energy-and momentum conservation have to lie on the circle around M, if they start from the origin $\{0, 0\}$ (Fig. 4.9).

The angles θ_1 and θ_2 are the deflection angles of the two collisions partners. The maximum deflectionangle $\theta_1^{\max}$ of the impinging particle is reached, when $\boldsymbol{p}_1'$ is the tangent to the circle. For $m_1 > m_2 \rightarrow p_1 = m_1 v_1 > 2\mu v_1$, which means that $|p_1| > 2r$. The magnitude of the momentum of the impinging particle is larger than the diameter of the circle. From Fig. 4.9 we can then conclude the relation

$$\sin \theta_1^{\max} = \frac{\mu v_1}{(m_1 - \mu) v_1} = \frac{\mu}{m_1 - \mu} = \frac{m_2}{m_1} \ . \tag{4.19}$$

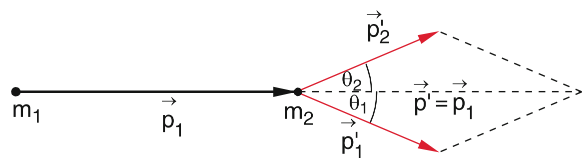

Figure 4.7 Collision of a particle with mass m_1 and momentum p_1 with a mass m_2 at rest, drawn in the Lab-system

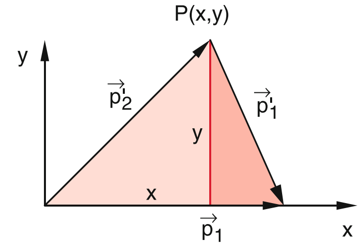

Figure 4.8 Illustration of (4.18)

Examples

1. $m_1 = 1.1 m_2 \Rightarrow \mu = 0.52 m_2 \Rightarrow \sin \theta_1^{\max} = 0.91$
   $\Rightarrow \theta_1^{\max} = 65°$.
2. $m_1 = 2 m_2 \Rightarrow \mu = 0.67 m_2 \Rightarrow \sin \theta_1^{\max} = 0.5$
   $\Rightarrow \theta_1^{\max} = 30°$.
3. $m_1 = 100 m_2 \Rightarrow \mu = 0.99 m_2$
   $\Rightarrow \theta_1^{\max} = 0.6°$. ◄

Special Case: Central Collisions

If $\boldsymbol{p}_2'$ has the same direction as $\boldsymbol{p}_1$ the deflection angle becomes $\theta_2 = 0$ (central or collinear collision). All vectors $\boldsymbol{p}_1, \boldsymbol{p}_1'$ and $\boldsymbol{p}_2$ are collinear and coincide in Fig. 4.9 with the x-axis. We obtain

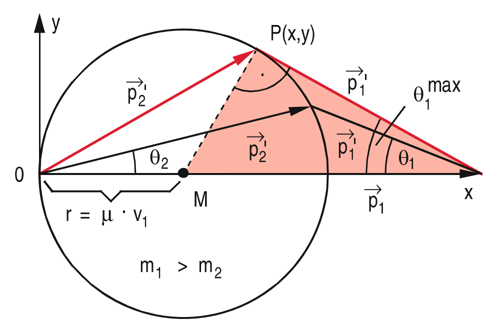

Figure 4.9 Momentum diagram of elastic collisions for $m_1 > m_2$. All possible endpoints of the vector $\boldsymbol{p}_2'$ are located on the circle with radius μv_1 around M

from Fig. 4.9:

$$\begin{aligned} p_1 &= 2\mu v_1 + p_1' \\ \Rightarrow m_1 v_1' &= m_1 v_1 - 2\frac{m_1 m_2}{m_1 + m_2} v_1 \\ \Rightarrow v_1' &= \frac{m_1 - m_2}{m_1 + m_2} v_1 \; ; \\ p_2' &= 2\mu v_1 \\ \Rightarrow v_2' &= 2\frac{\mu}{m_2} v_1 = \frac{2m_1}{m_1 + m_2} v_1 \; . \end{aligned} \tag{4.20}$$

The momentum of the pushed particle gets its maximum value $p_2' = 2\mu v_1$ for collinear collisions.

Also the kinetic energy, transferred from m_1 to m_2 during a collinear elastic collision reaches its maximum value

$$\begin{aligned} \Delta E_{\text{kin}} &= \frac{p_2'^2}{2m_2} \le \Delta E_{\text{kin}}^{\max} = \frac{2m_1^2 m_2}{(m_1 + m_2)^2} v_1^2 \\ \Delta E_{\text{kin}}^{\max} &= 4\frac{m_1 m_2}{M^2} E_1 = \frac{4\mu^2}{m_1 m_2} E_1 \; , \end{aligned} \tag{4.21}$$

which equals the fraction $4\mu^2/(m_1 m_2)$ of the initial energy E_1 of the impinging mass m_1. In Fig. 4.10 this maximum transferred fraction is shown as a function of the mass ratio m_1/m_2.

For $m_1 = m_2$ it is $v_1' = 0$ and $v_2' = v_1$. The two masses exchange their momentum during the collision, i. e. after the collision m_1 is at rest and m_2 moves with the momentum $\boldsymbol{p}_2' = \boldsymbol{p}_1$.

For equal masses $m_1 = m_2$ the energy of the incident particle is completely transferred to the resting mass m_2 during a collinear collision.

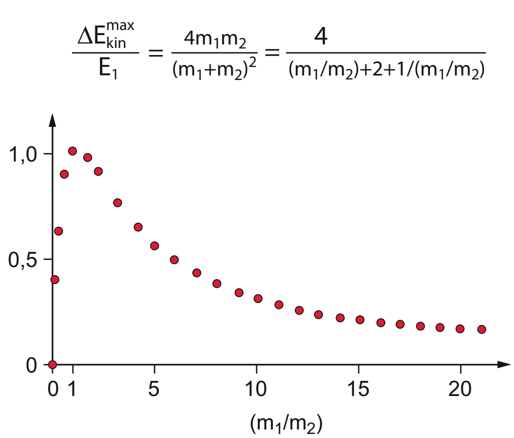

Figure 4.10 Maximum energy transfer $\Delta E = E - E_1'$ for a collinear elastic collison of a particle with mass m_1 onto a mass m_2 at rest for different ratios m_1/m_2

Special cases of non-collinear collisions

We will now discuss the general case of non-collinear collisions and illustrate it by some important special cases of the mass ratio m_1/m_2.

- $\boldsymbol{m_1 = m_2 = m \quad \Rightarrow \quad \mu = \frac{1}{2}m.}$

Equation 4.18 gives for the radius of the circle in Fig. 4.9 $r = \frac{1}{2}mv_1$, which implies that the momentum $p_1 = mv_1$ of the incident particle is the diameter of the circle (Fig. 4.11). For non-collinear collisions the momenta $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$ **after the collision** are perpendicular to each other according to the theorem of Thales. For the deflection angles it follows $\theta_1 + \theta_2 = \pi/2$.

The paths of the two particles are perpendicular to each other after the non-collinear collision, i. e. $\boldsymbol{p}_1' \perp \boldsymbol{p}_2'$.

Example

For the deceleration of neutrons in nuclear reactors a material with many hydrogen atoms is the best choice. Because the protons have nearly the same mass as neutrons.

- $\boldsymbol{m_1 \ll m_2 \quad \Rightarrow \quad \mu \approx m_1}$

The radius of the circle in Fig. 4.9 becomes for the limiting case $m_1/m_2 \to 0$ equal to the momentum $p_1 = m_1 v_1$ of the incident particle (Fig. 4.12a). The magnitude of $\boldsymbol{p}_1$ does not change during the collision ($|p_1| = |p_1'|$) but all directions of $\boldsymbol{p}_1'$ are possible. The scattering angle θ_1 can take all values in the range $-\pi \le \theta_1 \le +\pi$.

The maximum momentum transfer onto m_2 is

$$|p_2'|_{\max} = 2r = 2p_1 \; .$$

The maximum transferred energy is

$$\Delta E_{\text{kin}}^{\max} = \frac{(2p_1)^2}{2m_2} = \frac{4p_1^2}{2m_1}\frac{m_1}{m_2} = 4\frac{m_1}{m_2}E_1 \; . \tag{4.22}$$

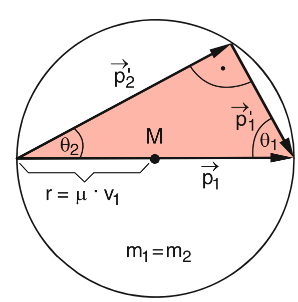

Figure 4.11 Elastic collision between particles of equal mass

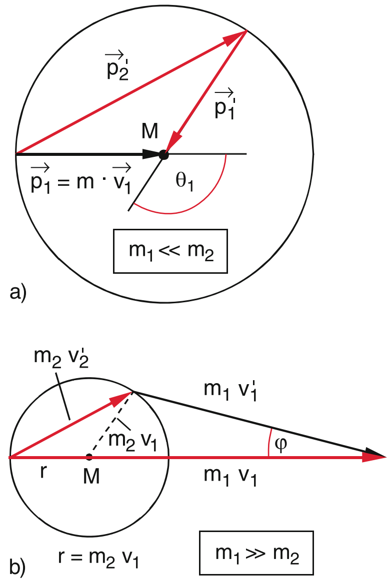

Figure 4.12 Elastic collision for $m_1 \ll m_2$ (**a**) and $m_1 \gg m_2$ (**b**)

In collisions of a small mass m_1 against a large mass m_2 the maximal transferred fraction of the initial kinetic energy is $4(m_1/m_2)$.

Examples

1. *Impact of a particle onto a solid wall.*

$$m_2 = \infty \Rightarrow \Delta E_{\text{kin}}^{\text{max}} = 0 \quad \text{but: } \boldsymbol{p}_2' = -2\boldsymbol{p}_1 \ .$$

During the elastic collision of a particle with a solid wall the particle is elastically reflected and $\boldsymbol{p}_1' = -\boldsymbol{p}_1$. Therefore twice the initial momentum is transferred to the wall but no energy!

2. *Collision of an electron with a proton at rest.* $m_1 = m_2/1836$. The maximal transferred energy occurs in central collisions and is then $\Delta E_{\text{kin}}^{\text{max}} = 4(m_1/m_2)E_1 = 0.00218\,E_1$. ◀

■ $\boldsymbol{m_1 \gg m_2 \quad \Rightarrow \quad \mu \approx m_2}$.

In this case the radius of the circle in Fig. 4.9 is $r = m_2 v_1$ (Fig. 4.12b). For central collisions is

$$m_2 v_2' = 2r = 2m_2 v_1 \Rightarrow v_2' = 2v_1 \ ,$$

and the transferred energy is

$$\Delta E_{\text{kin}} = \frac{m_2}{2} v_2'^2 = 4\frac{m_2}{m_1} E_1 \ . \tag{4.23}$$

For non-collinear collisions the energy transferred to m_2 is smaller. The maximum deflection angle $\varphi = \theta_1^{\text{max}}$ of the incident mass m_1 is according to (4.19)

$$\sin\varphi = \frac{m_2}{m_1} \ .$$

Example

In collisions of α-particles (helium nuclei) with electrons at most the fraction $\Delta E_1 = 0.00054 E_1$ of the initial energy E_1 can be transferred to the electron. The maximum deflection angle of the α-particles is $\varphi \approx \sin\varphi = 1.36 \cdot 10^{-4}$rad $= 0.48'$. When α-particles pass through matter the electron shell of the atoms contributes to the deflection only a tiny part. Most of the deflection is caused by the atomic nuclei (see Rutherford scattering in Vol. 3). ◀

4.2.3 Elastic Collisions in the Centre-of Mass system

When none of the collision partners is resting, the description of the collision process is often simpler in the CM-system than in the lab-system. Since, however, the observation of the process always occurs in the lab-system the measured results must be transformed into the CM-system in order to compare them with the predictions calculated in the CM-system. The relations between position vectors and velocities in the two systems is illustrated in Fig. 4.13b and the results are compiled in Tab. 4.1.

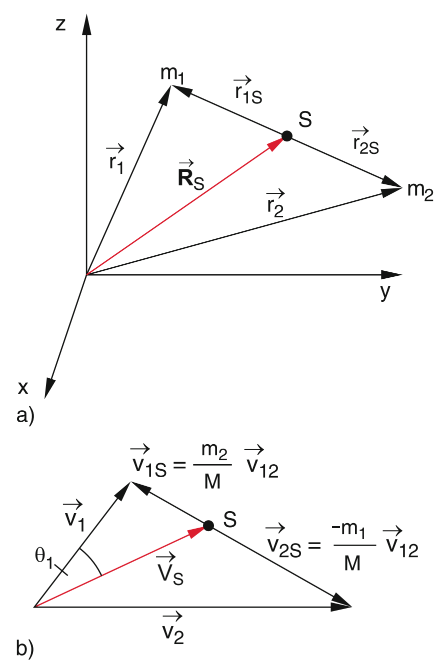

Figure 4.13 Graphical representation of the relations between **a** Lab- and CM-coordinates and **b** Lab- and CM-velocities

Table 4.1 Compilation of quantities relevant for collisions in the lab-system and the CM-system

$M = m_1 + m_2 =$ total mass
$\mu = \frac{m_1 \cdot m_2}{m_1+m_2} =$ reduced mass
$\boldsymbol{R}_S = \frac{1}{M}(m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2) =$ position vector of CM
$\boldsymbol{V}_S = \frac{1}{M}(m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2) =$ velocity of CM
$\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2 =$ relative distance
$\boldsymbol{v}_{12} = \boldsymbol{v}_1 - \boldsymbol{v}_2 =$ relative velocity
$\boldsymbol{r}_{iS} = \boldsymbol{r}_i - \boldsymbol{R}_S =$ position vector of i-th particle in the CM-system
$\boldsymbol{v}_{iS} = \boldsymbol{v}_i - \boldsymbol{V}_S =$ velocity of i-th particle in the CM-system
$\boldsymbol{p}_{iS} = m_i\boldsymbol{v}_{iS} =$ momentum of i-th particle in the CM-system
$\sum \boldsymbol{p}_{iS} = 0$
$\Theta_i =$ deflection angle of i-th particle in the lab-system
$\vartheta_i =$ deflection angle of i-th particle in the CM-system

Note: We will denote the center of mass with the index S.

Since the total momentum in the CM-system is always zero, we can write for two particles 1 and 2

$$\boldsymbol{p}_{1S} = -\boldsymbol{p}_{2S} \quad \text{and} \quad \boldsymbol{p}'_{1S} = -\boldsymbol{p}'_{2S} \,.$$

The sum of the momenta of the collision partners before the collision and after the collision is in the CM-system always zero.

From the energy conservation (4.17) it therefore follows:

$$\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)p'^2_{1S} = \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)p^2_{1S} + U \,,$$

which can be written when using the reduced mass μ

energy conservation in the S-system

$$\frac{p'^2_{1S}}{2\mu} = \frac{p^2_{1S}}{2\mu} + U \,. \qquad (4.24)$$

For elastic collisions ($U = 0$) in the CM-system is $p^2_{1S} = p'^2_{1S}$ and $p^2_{2S} = p'^2_{2S}$. This means:

In the CM-system each collision partner retains in elastic collisions its kinetic energy.

In the CM-system the result of an elastic collision is merely a turn of the momentum vectors which are always pointing into the opposite direction (Fig. 4.14).

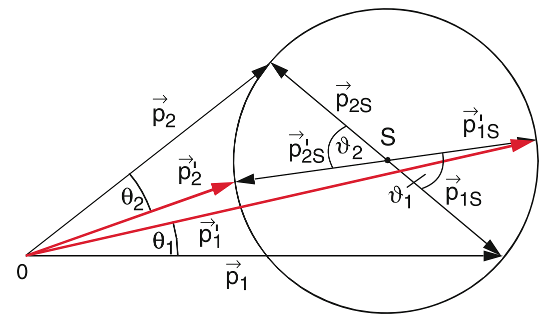

Figure 4.14 In the CM-system an elastic collision is represented by a turn of the momentum vectors without changing their length

Note: in order to distinguish the deflection angles θ in the lab-system written as capital letters, from those in the CM we will label all deflection angles in the CM-system by lower case letters ϑ.

Example

Deceleration of neutrons (mass m_1, velocity v_1) in elastic collisions by atomic nuclei (mass m_2) at rest. The CM-velocity is

$$\boldsymbol{V}_S = \boldsymbol{V}'_S = \frac{m_1\boldsymbol{v}_1}{m_1+m_2} = \frac{\boldsymbol{v}_1}{1+A} \quad \text{with} \quad A = m_2/m_1 \,.$$

◀

The velocity of the two particles in the CM-system is according to Fig. 4.15

$$\boldsymbol{v}_{1S} = \boldsymbol{v}_1 - \boldsymbol{V}_S = \frac{A\boldsymbol{v}_1}{1+A} \,;$$
$$\boldsymbol{v}_{2S} = \boldsymbol{0} - \boldsymbol{V}_S = -\frac{\boldsymbol{v}_1}{1+A} \,;$$
$$\boldsymbol{v}'_{1S} = \boldsymbol{v}'_1 + \boldsymbol{v}_{2S} = \boldsymbol{v}'_1 - \boldsymbol{V}_S \,.$$

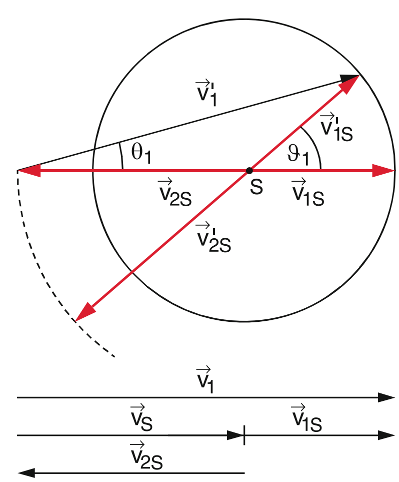

Figure 4.15 Determination of energy transfer at elastic collisions

Squaring gives with $\boldsymbol{v}'_{1S} \cdot \boldsymbol{V}_S = v'_{1S} \cdot V_S \cdot \cos\vartheta_1$

$$v_1'^2 = v_{1S}'^2 + V_S^2 + 2v'_{1S}v_S\cos\vartheta_1 ,$$

where ϑ_1 is the angle between $\boldsymbol{v}'_{1S}$ and $\boldsymbol{V}_S$. Because $\boldsymbol{V}_S \parallel \boldsymbol{v}_{1S}$ it follows that ϑ_1 is also the angle between v'_{1S} and v_{1S}, i. e. the deflection angle of m_1 in the CM-system. Inserting the relations above for V_S we obtain

$$v_{1S}'^2 = v_1^2 \frac{A^2 + 2A\cos\vartheta_1 + 1}{(1+A)^2} .$$

The ratio of the kinetic energies of the neutron after and before the collision is then

$$\left(\frac{E'_{kin}}{E_{kin}}\right) = \frac{v_1'^2}{v_1^2} = \frac{A^2 + 2A\cos\vartheta_1 + 1}{(1+A)^2} .$$

For central collisions is $\vartheta_1 = \pi$ and the ratio becomes

$$\left(\frac{E'_{kin}}{E_{kin}}\right)^{\text{central}} = \left(\frac{A-1}{A+1}\right)^2 .$$

For the transferred energy $\Delta E = E'_{kin} - E_{kin}$ we then obtain

$$\frac{\Delta E}{E_{kin}} = \frac{4A}{(A+1)^2} = \frac{4m_1m_2}{(m_1+m_2)^2} .$$

For $m_1 = m_2$ the transferred energy $\Delta E/E$ takes on its maximum value $\Delta E/E = 1$. This means that the neutron can transfer its total kinetic energy if it suffers a central collision with a proton.

4.2.4 Inelastic Collisions

For inelastic collisions part of the initial kinetic energy is transferred into internal energy U of the collision partners. Such collisions are only possible, if at least one of the partners has a variable internal sub-structure, which means that it has to be composed of two or more particles. For point masses there are no inelastic collisions!

For inelastic collisions momentum conservation remains valid (4.16) and also energy conservation with $U < 0$ (4.17). In the limiting case of *maximal inelastic collisions* the two collision partners stick together after the collision and move with the CM-velocity.

$$\boldsymbol{V}_S = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1+m_2} . \tag{4.25}$$

From (4.17) and (4.25) we obtain for the maximum fraction of the kinetic energy, which is transferred into internal energy

$$\begin{aligned} U &= \frac{1}{2}(m_1+m_2)V_S^2 - \frac{1}{2}\left(m_1v_1^2 + m_2v_2^2\right) \\ &= -\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 = -\frac{1}{2}\mu v_{12}^2 , \end{aligned} \tag{4.26a}$$

which is identical to the kinetic energy of the two particles in the CM-system (see (4.11b)).

In a completely inelastic collision, where the two particles stick together after the collision, just the kinetic energy of the two particles in the CM-system is converted into internal energy of one or both collision partners.

From (4.26) it follows that only in cases where the two collision partners have equal but opposite momenta ($m_1\boldsymbol{v}_1 = -m_2\boldsymbol{v}_2 \Rightarrow \boldsymbol{V}_S = \boldsymbol{0}$) the total kinetic energy can be converted into internal energy. The two particles then stick together and are at rest, their total momentum is zero before and after the collision. For all other collisions $|U| < |U_{max}|$. Therefore the general rule is:

For all inelastic collisions not more than $\Delta E = \frac{1}{2}\mu \cdot v_{12}^2$ of the initial kinetic energy can be converted into internal energy. At least the proportion

$$\tfrac{1}{2}(m_1+m_2)V_S^2 = \tfrac{1}{2}MV_S^2 \tag{4.26b}$$

of the CM-motion remains as kinetic energy of the collision partners.

Examples

1. A glider with mass m_1 on an air-track hits a second glider with mass m_2 at rest ($v_2 = 0$). The two colliding ends are covered with plasticine, which causes the two gliders to stick together after the collision and they move with the CM-velocity

   $$\boldsymbol{V}_S = \frac{m_1}{m_1+m_2}\boldsymbol{v}_1 .$$

   The kinetic energy after the collision is

   $$E'_{kin} = \frac{m_1+m_2}{2}V_S^2 = \frac{m_1^2}{2(m_1+m_2)}v_1^2 ,$$

   and the energy converted into the plasticine energy is

   $$U = E'_{kin} - E_{kin} = -\frac{m_2}{m_1+m_2}E_{kin} .$$

   For $m_1 = m_2$ this gives

   $$U = -\tfrac{1}{2}E_{kin} .$$

2. A neutron n with velocity v_1 impinges on a proton p at rest. This produces a deuteron d = np.

   $$\mathrm{n} + \mathrm{p} \longrightarrow \mathrm{d} .$$

   Because of $m_1 = m_2$ the deuteron moves with the CM-velocity $V_S = \frac{1}{2}v_1$ and has therefore half of the initial kinetic energy $E'_{kin} = E_{kin}$ of the incident neutron. The other half is converted into internal energy of the deuteron, which is excited into a higher energy state, that can decay by emission of γ-radiation. ◀

Summarizing the results: In inelastic collisions of particles with equal masses where one collision partner is at rest at most half of the kinetic energy of the incident particle can be converted into internal energy

$$|U| \leq |U_{\max}| = \frac{1}{2}\frac{m}{2}v_1^2 \,. \tag{4.27a}$$

The amount $U_{\max} - U$ remains as kinetic energy of the collision partners in addition to the kinetic energy $\frac{1}{2}MV_S^2$ of the CM-motion.

Special Cases

- If a particle with mass m_1 suffers a totally inelastic collision with a wall ($m_2 \gg m_1 \to \mu \approx m_1$) it remains adsorbed at the wall and transfers its kinetic energy completely to the wall, which heats up. ($U = -E_{\text{kin}}, E'_{\text{kin}} = 0$).
- If two equal masses collide head-on with $\boldsymbol{p}_1 = -\boldsymbol{p}_2$ the total momentum after the collision must be zero. With $v_1^2 = v_2^2 = v^2$ the increase of internal energy is

$$U = -\frac{1}{2}(m_1 + m_2)v^2 \,,$$

as in the first case the total kinetic energy is converted into internal energy. These two special cases are illustrated in Fig. 4.16 and compared with the corresponding elastic collisions.

Examples

1. Collisional excitation of mercury atoms by electron impact (Franck-Hertz-Experiment). Because of $m_{\text{Hg}} \gg m_e$ the reduced mass $m \approx m_e$. From (4.26) we can conclude that nearly the total kinetic energy of the electron can be converted into excitation energy of the Hg-atoms.
2. A heavy particle with mass $m_1 = 100m_2$ collides with a particle of mass m_2. Now $\mu = 0.99m_2$ and $U = (0.99/100)m_2v_1^2/2$. This implies that only about 1% of the kinetic energy is converted into internal energy U.

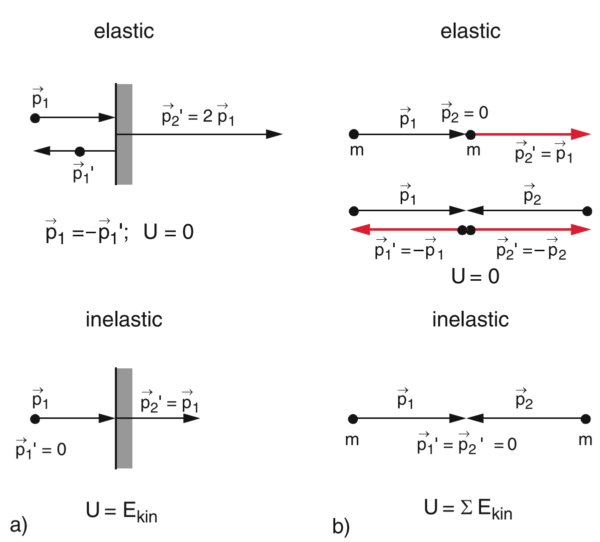

Figure 4.16 Comparison of elastic and completely inelastic collisions: **a** particle against a wall, **b** collision between two equal masses

4.2.5 Newton-Diagrams

The measurements of deflection angles at collisions between atoms or molecules is performed in the laboratory-system. The determination of the interaction potential derived from these deflection angles is, however, much easier in the CM-system. The relations between the relevant parameters in the two systems (velocities, deflection angles, energy transfer) for arbitrary elastic or inelastic collisions can be visualized with the help of **Newton diagrams**, which connects the velocities in the lab- system with those in the CM-system (Fig. 4.17). The parameters used in the following are listed in Tab. 4.1.

With the relations

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{R}_S + (m_2/M)\,\boldsymbol{r}_{12} \quad \text{and} \\ \boldsymbol{r}_2 &= \boldsymbol{R}_S - (m_1/M)\,\boldsymbol{r}_{12} \,, \end{aligned} \tag{4.28}$$

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{V}_S + (m_2/M)\,\boldsymbol{v}_{12} \quad \text{and} \\ \boldsymbol{v}_2 &= \boldsymbol{V}_S - (m_1/M)\,\boldsymbol{v}_{12} \,, \end{aligned} \tag{4.29}$$

which can be derived from Fig. 4.13, the kinetic energy can be separated into the two parts

$$E_{\text{kin}} = \underbrace{\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2}_{E_{\text{kin}}\text{ in lab frame}} = \underbrace{\tfrac{1}{2}MV_S^2}_{E_{\text{kin}}\text{ of CM-motion}} + \underbrace{\tfrac{1}{2}\mu v_{12}^2}_{E_{\text{kin}}\text{ of relative motion in the CM-system}} \,. \tag{4.30}$$

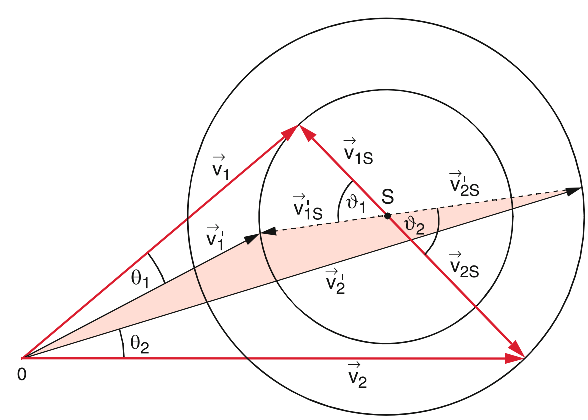

Figure 4.17 Newton diagram of elastic collision between two particles

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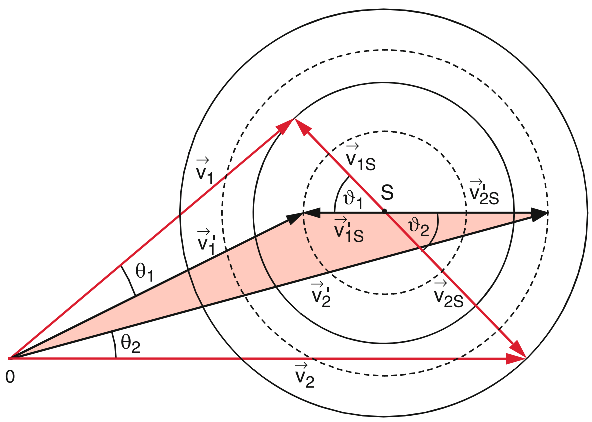

Figure 4.18 Newton-diagram of inelastic collisions between two particles

Since for elastic collisions the kinetic energy of each partner in the CM-system is preserved, the vector of the relative velocity v_{12} retains its magnitude but turns around the centre of mass S where the end of the vector describes a circle with the radii $v_{1S} = (m_2/M)v_{12}$ and $v_{2S} = (m_1/M)v_{12}$. The deflection angles ϑ_1 in the CM-system can be determined graphically from the deflection angles θ_1 in the lab-system.

In particular the maximum deflection angle θ_1^{max} can be determined readily. It appears when v_1' is the tangent to the Newton circle.

For inelastic collisions (Fig. 4.18) part of the kinetic energy $\frac{1}{2}\mu v_{12}^2$ is converted into excitation energy, which means that v_{12}' becomes smaller. However, still the centre-of-mass S divides the connecting line between the endpoints of the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in the ratio m_1/m_2 of the two masses. The endpoints of v_{12}' are now located on a circle with smaller radius (dashed circles in Fig. 4.18).

For both elastic and inelastic collisions the range of possible deflection angles and the maximum deflection angles can be determined from the Newton diagrams. Therefore such diagrams are very useful for the planning of experiments, because they tell us, in which deflection ranges one must look for scattered particles for given initial conditions [4.2].

4.3 What Do We Learn from the Investigation of Collisions?

The deflection of a particle A during the collision with another particle B is due to the momentum transfer

$$\Delta p = \int_{-\infty}^{+\infty} \boldsymbol{F}\,\mathrm{d}t\,, \tag{4.31}$$

which is caused by the force F acting between A and B while passing by each other. The momentum change Δp experienced

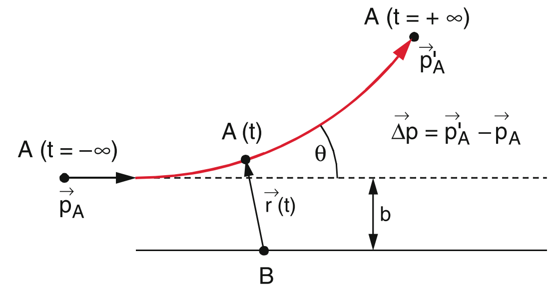

Figure 4.19 Illustration of impact parameter

by A is, of course, compensated by the change $-\Delta p$ experienced by B, because the total momentum has to be conserved.

Since $\boldsymbol{F}(\boldsymbol{r}) = -\nabla E_p$ the force $\boldsymbol{F}$ is a measure for the potential energy $E_p(r)$ of the interaction between A and B which depends on the distance r between A and B. The deflection of A therefore depends on the impact parameter b in Fig. 4.19, which is a measure of the closest approach between A and B. It is defined in the following way:

For large distances between A and B the force F is negligible and the incident particle A will follow a straight line. If there would be no interaction between A and B the incident particle A would follow this straight line and pass B at the closest distance b. This line is parallel to the straight line through B where B is resting in the origin of the coordinate system. To each impact parameter b belongs a certain deflection angle θ in the lab-system resp. ϑ in the CM-system., which depends on the interaction potential $V(r)$ between A and B.

4.3.1 Scattering in a Spherical Symmetric Potential

In Sect. 4.1.3 it was shown, that the relative motion of two particles around the CM caused by the mutual interaction force $\boldsymbol{F}(r)$ can be reduced to the motion of a single particle with the reduced mass μ in the spherical symmetric potential with its origin at the position of one of the two particles (usually the one with the larger mass). If the force $F(r)$ is known, the deflection angle ϑ in the CM-system can be determined from Eq. 4.3 and the re-

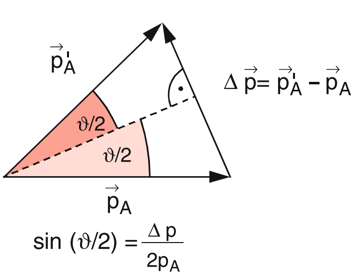

Figure 4.20 Relation between momentum change Δp and deflection angle ϑ in the CM-system

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lation $\sin(\vartheta/2) = \frac{1}{2}\frac{\Delta p}{p_A}$ (Fig. 4.20). In Sect. 4.2.5 it was shown how the angle θ measured in the lab-system can be transformed into the angle ϑ in the CM-system.

The deflection of particles in a potential is called *potential scattering*. We will illustrate this potential scattering and its treatment by some examples.

Examples

1. *Collision of two hard spheres with radii r_1 and r_2 (Fig. 4.21).*
If the impact parameter b is larger the sum $r_1 + r_2$ no collision takes place. The hard sphere A moves on a straight line and passes B without deflection. For $b \leq r_1 + r_2$ the colliding partner A is reflected at the surface of B (Fig. 4.21a). In order to determine the deflection angle ϑ of A we decompose the momentum $\boldsymbol{p}_1$ of A into a component p_r parallel to the connecting line M_1M_2 at the touch of the two spheres and a component p_t in the tangential direction perpendicular to p_r (Fig. 4.21b). We assume the surface of the spheres as frictionless. Then no rotation of the spheres can be excited and the component p_t does not change during the collision. For the component p_r we can conclude from (4.20) for central collisions

$$p_r' = \frac{m_1 - m_2}{m_1 + m_2}\boldsymbol{p}_r \,. \tag{4.32a}$$

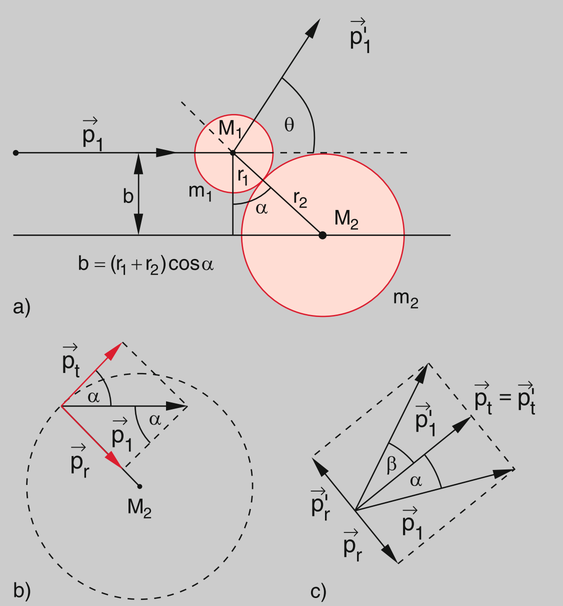

Figure 4.21 Determination of the deflection function for collisions between two hard spheres. **a** Definition of impact parameter, **b** decomposition of impact momentum, **c** momentum vector addition

For $m_2 \gg m_1$ this gives $p_r' = -p_r$. In this case is in Fig. 4.21c $\beta = \alpha$ and the deflection angle becomes $\theta = 2\alpha$.

From Fig. 4.21b one can deduce from $b = r_1 + r_2$ for $\theta = 2\alpha$ the dependence of the deflection angle on the impact parameter b

$$\theta(b) = 2\arccos\frac{b}{r_1 + r_2}\,. \tag{4.32b}$$

For $b = 0$ is $\theta = \pi$ i.e. A is reflected back., for $b > r_1 + r_2$ is $\theta = 0$. The function $\theta(b)$ is called *deflection function*. Its curve depends on the interaction potential $(V(r))$. For the collision of two hard spheres the potential is a step function (Fig. 4.22a) and the deflection function is the monotonic curved shown in Fig. 4.22b for $m_2 \gg m_1$.
For the general case of arbitrary ratios m_1/m_2 we obtain from Fig. 4.21c:

$$\Theta = \alpha + \beta$$

with $p_r/p_t = \tan\alpha$ it follows

$$\tan\beta = p_r'/p_r = -(m_1 - m_2)/(m_1 + m_2)\,.$$

While (4.32b) is only strictly valid for $m_2 = \infty$, the deflection function

$$\vartheta(b) = \arccos\frac{b}{r_1 + r_2} \tag{4.32c}$$

in the CM-system is correct for arbitrary ratios m_1/m_2.

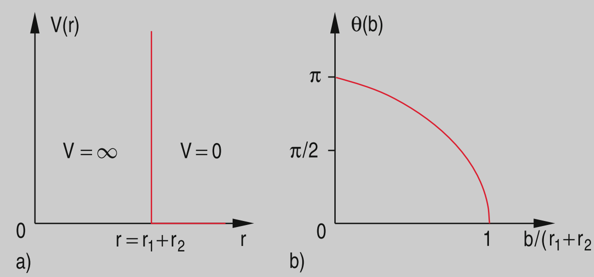

Figure 4.22 **a** Potential $V(r)$ for hard spheres; **b** deflection function for hard sphere collisions

2. *Scattering of a particle in a potential $V(r) \propto 1/r$.*
This important case applies for instance for the Coulomb-scattering of charged particles (electrons or α-particles) on atomic nuclei (see Vol. 3) or the Kepler-orbit of comets in the gravitational field of the sun.
For a potential $V(r)$ with the potential energy $E_p = a/r$ the force between the interacting particles A and B with masses m_1 and m_2 is

$$F = -\operatorname{grad} E_p = \frac{a}{r^2}\hat{\boldsymbol{r}}\,. \tag{4.33}$$

For $a > 0$ a repulsion between the particles occur, for $a < 0$ an attraction. The angular momentum in the CM-system is according to (4.15)

$$\boldsymbol{L} = \boldsymbol{r} \times \mu \cdot \boldsymbol{v} \quad \text{with} \quad \mu = \frac{m_1 m_2}{m_1 + m_2}\,,$$

where r is the distance between A and B and v is the relative velocity.

Since $\boldsymbol{L}$ is in the central potential temporally constant the orbit of the particle remains in the plane $\perp \boldsymbol{L}$, which we choose as the x-y-plane (Fig. 4.23). The particle A is incident parallel to the x-axis with an impact parameter b and the initial velocity v_0. It is convenient to use polar coordinates for the description of its trajectory. The magnitude of $\boldsymbol{L}$ is then

$$L = |\boldsymbol{r} \times \mu \boldsymbol{v}| = \mu r^2 \frac{d\varphi}{dt} = \mu v_0 b \,, \qquad (4.34)$$

where the last term describes the angular momentum of A for large distances $r \to \infty$ referred to the particle B which sits at the origin $r = 0$. It should be emphasized that we use the CM-system for our description. In the lab-system B does not stay at $r = 0$ but moves around the common centre of mass. We compose the force $F(r)$ of the components F_x and F_y. For the deflection of A only the component F_y is responsible. From Fig. 4.23 we see that

$$F_y = \frac{a \sin\varphi}{r^2} = \mu \frac{dv_y}{dt} \,. \qquad (4.35)$$

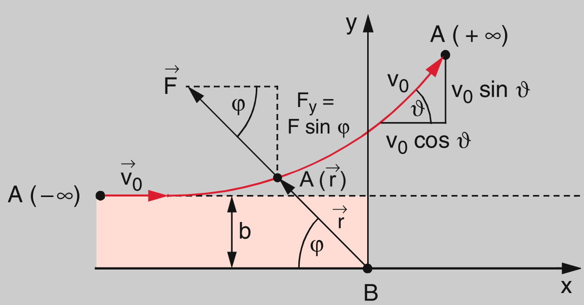

Figure 4.23 Scattering of a particle in the potential $V(r)$ with r = distance $\overline{AB}$

From (4.34) and (4.35) we obtain

$$\frac{dv_y}{dt} = \frac{a \sin\varphi}{\mu v_0 b} \frac{d\varphi}{dt} \,. \qquad (4.36)$$

The total deflection of A during its path through the potential $V(r)$ is obtained by integration over the whole range from $r = -\infty$ to $r = \infty$.

For $A(-\infty)$ we have $v_y = 0$ and $\varphi = 0$, for $A(+\infty)$ is $v_y = v_0 \cdot \sin\vartheta$ with $\vartheta = \pi - \varphi_{max}$. For the elastic potential scattering the magnitude v_0 of the velocity remains constant. Therefore the integration of (4.36) yields

$$\int dv_y = \frac{a}{\mu v_0 b} \int_0^{\pi-\vartheta} \sin\varphi \, d\varphi$$
$$\to v_0 \sin\vartheta = \frac{a}{\mu v_0 b}(1 + \cos\vartheta) \,.$$

With the equation $(1 + \cos\vartheta)/\sin\vartheta = \cot(\vartheta/2)$ the relation between deflection angle ϑ and impact parameter b for scattering in the potential with energy $E_p = a/r$ becomes

$$\cot\left(\frac{\vartheta}{2}\right) = \frac{\mu v_0^2}{a} b = \frac{2E_{kin}}{a} b \,. \qquad (4.37a)$$

The ratio a/b gives the potential energy of the interaction between the particles A and B at a distance $r = b$. Inserting this into (4.37a) gives the result

$$\cot\left(\frac{\vartheta}{2}\right) = \frac{2E_{kin}}{E_p(b)} \,. \qquad (4.37b)$$

The deflection angle ϑ in the cm-system is determined by the ratio of twice the kinetic energy and the potential energy at a distance $r = b$ between the interacting particles. The deflection function $\vartheta(b)$ is shown in Fig. 4.24. For $b = 0$ is $\cot(\vartheta/2) = \infty \to \vartheta = \pi$. The particle A is scattered back into the $-x$-direction. The turning point which is the closest approach r_0 can be obtained from $E_{kin} = E_p \to \mu v_0^2/2 = a/r_0$. This gives $r_0 = 2a/(\mu v_0^2)$.

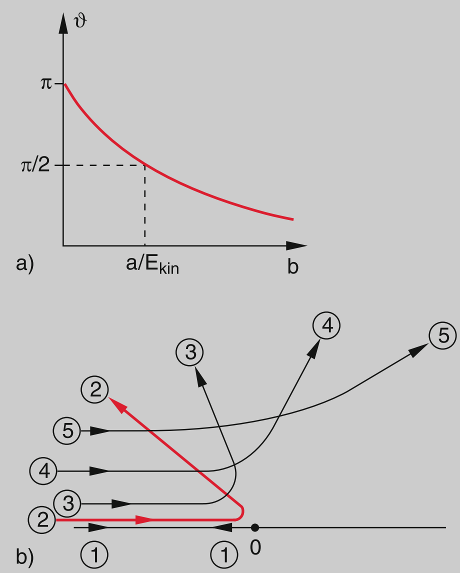

Figure 4.24 Deflection function $\vartheta(b)$ (**a**) and trajectories of a particle in a potential $V(r) \propto 1/r$ for different impact parameters but constant initial energy. Each deflection angle corresponds to a different ratio $f = 2E_{kin}/E_{pot}(b)$. ①: $\vartheta = \pi$; ②: $\vartheta = \frac{3}{4}\pi$; ③: $\vartheta = 105°$, $f = 0.76$; ④: $\vartheta = 60°$, $f = 1.7$; ⑤: $\vartheta = 30°$, $f = 3.7$

For the gravitational potential is $a = -Gm_1m_2$ (see (2.52)) and we get from (4.37b) with $M = m_1 + m_2$ the result

$$\cot\left(\frac{\vartheta}{2}\right) = -\frac{v_0^2}{GM} b \,. \qquad (4.37c)$$

The deflection angle depends only on the masses, the initial velocity v_0 and the impact parameter b. For a

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comet is $m_1 \ll m_2 = M_\odot$. The total mass M is then with a very good approximation $M = M_\odot$. The mass of the comet does not affect the deflection angle.
According to (2.60) the trajectories of the particle m_1 for $E = E_{kin} + E_p > 0$ are hyperbolas. In Fig. 4.24b some of these hyperbolas are shown for a repulsive potential ($a > 0$) and different impact parameters b. For the interaction between two positively charged particles with charges q_1 and q_2 is $a = (1/4\pi\varepsilon_0) \cdot q_1 \cdot q_2$ (see Vol. 2). ◀

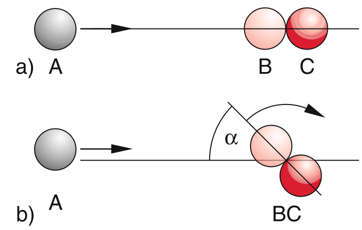

Figure 4.26 Collinear (**a**) and noncollinear (**b**) collision, where angular momentum of the relative motion is transferred to rotational angular momentum of the molecule BC

4.3.2 Reactive Collisions

Reactive collisions provide the basis of all chemical reactions. A simple example is the reaction

$$A + BC \to ABC \to AB + C\,, \tag{4.38}$$

where an atom A with the velocity v_A collides with a molecule BC (velocity v_{BC}), forms a complex ABC, which can decay into the fragments AB + C (Fig. 4.25).

Momentum conservation is also valid for reactive collisions. The momentum of the right side in equation (4.38) must be therefore the same as on the left side. The kinetic energy is, however, in general not conserved because part of this energy may be converted into internal energy ($U < 0$). In cases where the reactants on the left side are already excited, this internal energy may be also transferred to kinetic energy ($U > 0$, superelastic collisions). The measurement of velocities and deflection angles after the collision gives information about the energy balance of the reaction and the interaction potential between the reactants, if the initial conditions are known. The potential is in general no longer spherical symmetric but depends on the spatial orientation of the molecule BC against the momentum direction of A. The reaction probability can differ considerably for collinear collisions, (Fig. 4.26a) from that for non-collinear collisions where the internuclear axis of BC is inclined by the angle α against the momentum direction of A (Fig. 4.26b).

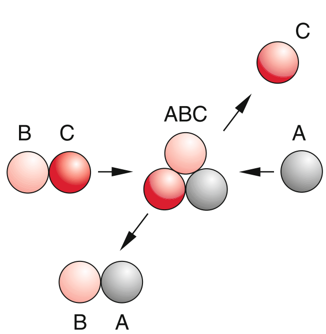

Figure 4.25 Schematic representation of a reactive collision, where a collision complex is formed that decays again

Often the reactants A and BC fly in two perpendicular collimated molecular beams. In the intersection volume of the two beams the reactants collide. For this arrangement the directions of the reactants are known and their velocities can be selected by velocity selectors which interrupt the beams (see Sect. 7.4.1). The initial conditions are then well known (apart from the often unknown internal energies).

Note, that the masses are generally not constant for reactive collisions, because the reduced mass $\mu(A + BC)$ before the collision differs in general from $\mu(AB + C)$ after the collision.

If the kinetic energy E_2 of the reaction products is smaller than the kinetic energy E_1 of the reactants, the reaction is called *endotherm*. One has to put energy into the system in order to make the reaction possible. If energy is released in the reaction it is called *exotherm*. In this case the kinetic energy of the reaction products is larger than that of the reactants. Measurements of the velocities of reactants and reaction products can decide which type applies to the investigated reaction.

The energy balance is illustrated by the potential diagram of Fig. 4.27. Often the reactants have to overcome a potential barriers in order to start the reaction. In this case a minimum initial energy is necessary even for exothermic reactions.

The heights of the potential barrier and with it the reaction probability depends on the internal energy (vibrational- rota-

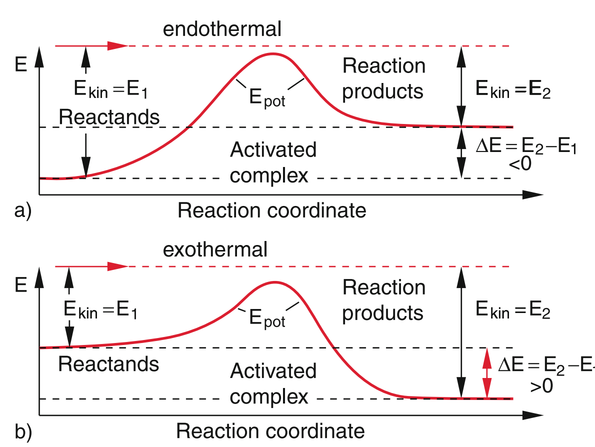

Figure 4.27 Diagram of potential energy as function of the reaction coordinate for **a** endothermic and **b** exothermic reactive collisions

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tional or even electronic energy) of the reaction partners. For the measurement of these internal energy several spectroscopic techniques have been developed which allow to determine the excitation state of the partners involved in the reaction.

An ideal experiments should allow to measure all relevant parameters of a collision process such as the internal energies, the deflection angles and the velocities of all particles. Such modern techniques are discussed in Vol. 3.

4.4 Collisions at Relativistic Energies

Up to now we have used the Newtonian laws (energy- and momentum conservation) for the description of collision processes and we have assumed that the masses of the reaction partners are constant (besides in reactive collisions). This is justified as long as the velocities of the collision partners are small compared with the velocity c of light (see Chap. 3).

For the investigation of interactions between elementary particles and atomic nuclei, higher energies of the collision partners are required. Such energies, where the velocity of particles comes close to the velocity of light can be realized in particle accelerators and storage rings (see Vol. 4). We will now discuss, how the rules governing collisions at relativistic energies (the domain of high energy physics) must be formulated.

4.4.1 Relativistic Mass Increase

We regard two particles A and B which have equal masses $m_1 = m_2 = m$, if they are at rest. We assume that A and B move in a system S with velocities

$$\boldsymbol{v}_1 = \{v_{x1}, -v_{y1}\} \quad \text{and} \quad \boldsymbol{v}_2 = \{0, v_{y2}\}$$

against each other where $v_{y1} = v_{y2}$ (Fig. 4.28a).

The particle B should suffer an elastic striking collision with A such that during the collision the x-component v_{x1} of A remains constant but the y-component is reversed. The velocity of A after the collision is then $\boldsymbol{v}'_1 = \{v_{x1}, v_{y1}\}$. The magnitude of its velocity $|\boldsymbol{v}_1| = \sqrt{v_{x1}^2 + v_{y1}^2}$ is therefore also preserved. Because the momentum must be constant the velocity of B after the collision must be

$$\boldsymbol{v}'_2 = \{0, -v_{y2}\} .$$

We assume that $v_y \ll v_x$. For the magnitudes of the velocities this implies

$$v_1 = (v_{x1}^2 + v_{y1}^2)^{(1/2)} \approx v_{x1} \quad \text{and} \quad v_2 \ll v_1 .$$

Now we describe this collision between A and B in a system S^* which moves against S with the velocity $v = v_{x1}$ into the x-direction (Fig. 4.28b). In this system we get for the velocities of A and B:

$$v_{x1}^* = 0 \quad \text{but} \quad v_{x2}^* = -v_{x1} ,$$

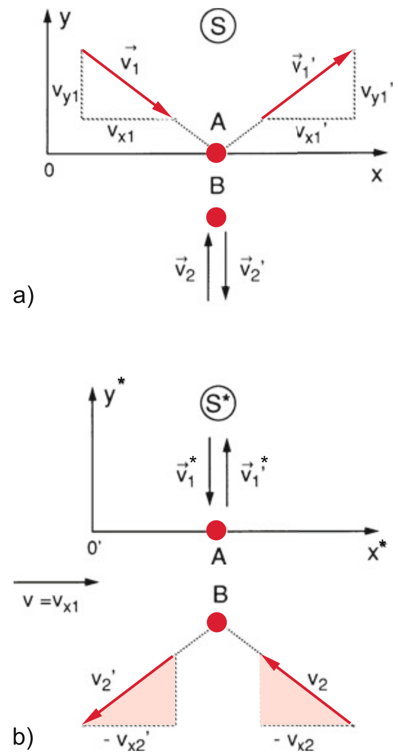

Figure 4.28 Grazing elastic collision between A and B at relativistic velocities. **a** In the system S has A a large and B a small velocity since $V_y \ll V_x$. **b** In the sytem S^*, that moves relativ to S the situation is reversed

which implies that the roles of A and B are just interchanged.

According to Eq. 3.28 for the transformation of velocities when changing from system S to S^* the observer O^* in S^* measures the velocity component

$$v_y^* = \frac{v_y/\gamma}{1 - v_x v/c^2} . \tag{4.39}$$

Since in the system S for the particle A holds: $v_{x1} = v'_{x1} \neq 0$, for B, however, $v_{x2} = v'_{x2} = 0$ the observer O^* measures for the two particles different y-components of the velocities

$$v_{y1}^* = \frac{v_{y1}/\gamma}{1 - v_{x1}v/c^2} = \frac{v_{y1}/\gamma}{1 - v^2/c^2} = \gamma v_{y1} , \quad \text{since } v \approx v_{x1} , \tag{4.40a}$$

$$v_{y2}^* = \frac{v_{y2}/\gamma}{1 - v_{x2}v/c^2} = v_{y2}/\gamma , \tag{4.40b}$$

while for the observer O in S the velocity component of A is v_{y1} and for B it is v_{y2}.

In both inertial systems S and S^* the conservation of total momentum holds, since the physical laws are independent of the chosen inertial system (see Sect. 3.2). This yields for the y-component of the total momentum

$$m_A v_{y1} - m_B v_{y2} = m_A^* v_{y1}^* - m_B^* v_{y2}^* = 0 . \tag{4.41}$$

For $m_A = m_A^*$ and $m_B = m_B^*$ this condition cannot be fulfilled, i. e. the conservation of momentum would fail, because according to (4.40) v_{y1}/v_{y1}^* is different from v_{y2}/v_{y2}^*. We are therefore forced (if we will not give up the well proved conservation of momentum) to assume that the mass of a particle is changing with its velocity. For the limiting case $v_{x1} \gg v_{y1} \approx 0$ we can write:

$$v_A \approx v_{x1} = v\,; \quad v_A^* \approx 0\,,$$
$$v_B \approx 0\,; \quad v_B^* \approx v_{x1} \approx v\,.$$

We therefore get with $m(v = 0) = m_0$ for (4.41)

$$m(v)v_{y1} + m_0 v_{y2} = 0\,, \tag{4.42a}$$
$$m_0 v_{y1}^* + m(v)v_{y2}^* = 0\,, \tag{4.42b}$$

with (4.40) this gives

$$\frac{(m(v))^2}{m_0^2} = \frac{v_{y2}}{v_{y2}^*} \cdot \frac{v_{y1}^*}{v_{y1}} = \gamma^2$$

$$\Rightarrow m(v) = \gamma m_0 = \frac{m_0}{\sqrt{1 - v^2/c^2}}\,. \tag{4.43}$$

The mass $m(v)$ of a moving particle increases with its velocity v. the mass $m_0 = m(v = 0)$ is called its **rest-mass**. This mass increase is noticeable only for large velocities [4.3].

Examples

1. For $v = 0.01c \Rightarrow m = 1.00005m_0$. The relative mass increase $\Delta m/m = (m - m_0)/m_0 \approx 5 \cdot 10^{-5}$.
2. For $v = 0.9c \Rightarrow m = 2.29m_0$.
3. For $v = 0.99c \Rightarrow m = 7m_0$. ◀

In Fig. 4.29 the increase of the mass $m(v)$ is plotted against the normalized velocity v/c. This illustrates also, that the maximum velocity of a particle with $m_0 \neq 0$ is always smaller than the velocity of light because for $v \to c$ it follows from (4.43) $m(v) \to \infty$.

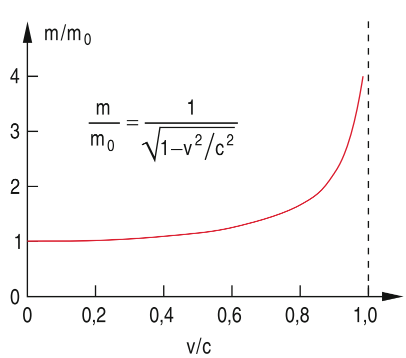

Figure 4.29 Dependence of a mass m on the ratio v/c

4.4.2 Force and Relativistic Momentum

The work, which has to be spent for the acceleration of a mass is used with increasing velocity more and more for the increase of the mass and less for the increase of the velocity.

The Newton-equation (2.18) between force and momentum with the inclusion of the relativistic mass increase (4.43) is

$$\begin{aligned} F &= \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{v}m_0}{\sqrt{1 - v^2/c^2}}\right) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{m_0}{\sqrt{1 - v^2/c^2}}\right)\boldsymbol{v} + m\boldsymbol{a}\,. \end{aligned} \tag{4.44a}$$

This gives with $\mathrm{d}/\mathrm{d}t = (\mathrm{d}v/\mathrm{d}t) \cdot (\mathrm{d}/\mathrm{d}v)$

$$\begin{aligned} F &= \frac{m_0\left(v/c^2\right)a}{(1 - v^2/c^2)^{3/2}}\boldsymbol{v} + m\boldsymbol{a} \\ &= \gamma^3 m_0 a\left[\frac{v^2}{c^2}\hat{\boldsymbol{e}}_v + \left(1 - \frac{v^2}{c^2}\right)\hat{\boldsymbol{e}}_a\right]\,, \end{aligned} \tag{4.44b}$$

where $\hat{\boldsymbol{e}}_v$ and $\hat{\boldsymbol{e}}_a$ are unit vectors in the direction of $\boldsymbol{v}$ and $\boldsymbol{a}$.

These equations show that for large velocities v the force $\boldsymbol{F}$ is no longer parallel to the acceleration $\boldsymbol{a}$ but has a component in the direction of $\boldsymbol{v}$. For $v \ll c$ the first term in (4.44b) can be neglected and we obtain the classical Newton equation $\boldsymbol{F} = m \cdot \boldsymbol{a}$.

In order to keep the Newton equation $F = \mathrm{d}p/\mathrm{d}t$ the relativistic momentum

$$\boldsymbol{p}(v) = m(v) \cdot \boldsymbol{v} = \gamma \cdot m_0 \cdot \boldsymbol{v} \tag{4.45a}$$

has to be used which has the magnitude

$$p = \beta\gamma m_0 c \qquad (\beta = v/c)\,. \tag{4.45b}$$

For the relativistic momentum the conservation law (4.41) is fulfilled for all velocities.

We will now discuss, how the components of the force are transformed for a transition from a system S where a particle has a velocity v and a mass $m = \gamma m_0$ into a system S^* which moves with the velocity $U = +v$ against S. In S^* is therefore $v^* = 0$ and $m^* = m_0$. We choose the axes of the coordinate system such that $\boldsymbol{v} = \{v_x, 0, 0\}$. It follows then from (4.44) with $\hat{\boldsymbol{e}}_v = \hat{\boldsymbol{e}}_a$

$$F_x = \frac{\mathrm{d}p_x}{\mathrm{d}t} = \gamma^3 m_0 a_x\,. \tag{4.45c}$$

In S^* the x-component of the acceleration a becomes

$$a_x^* = \gamma^3 a_x$$

as can be seen from (3.26) and (3.28). Therefore the component of the force in the system S^* becomes

$$F_x^* = m_0 \cdot a_x^* = \gamma^3 m_0 \cdot a_x \equiv F_x\,. \tag{4.45d}$$

We obtain the remarkable result that the x-components in the two systems which move against each other in the x-direction, are equal!

This no longer true for the components perpendicular to the relative motion of the two systems, because we get for $v_y \ll v_x$ the result

$$F_y^* = m_0 \cdot a_y^* = \gamma^2 m_0 a_y = \gamma \cdot F_y \,, \tag{4.45e}$$

and therefore obtain for the ratio

$$\frac{F_x}{F_y} = \gamma^2 \cdot \frac{a_x}{a_y} \,. \tag{4.45f}$$

This shows again that for $\gamma \neq 1$ the force $\boldsymbol{F}$ is no longer parallel to the acceleration $\boldsymbol{a}$ as in the nonrelativistic case.

4.4.3 The Relativistic Energy

In classical mechanics the kinetic energy of a particle

$$E_{\text{kin}} = \tfrac{1}{2} m v^2$$

is different in diverse inertial systems which move against each other, because the velocity v is different.

In order to obey energy conservation when changing from one system to the other the total energy of a particle has to be defined in such a way, that it is Lorentz-invariant. i. e. that it does not change for transformations into different inertial systems (see Sect. 3.3).

We will at first present an intuitively accessible description, which is based on a "Gedanken-experiment" of Einstein. We regard in Fig. 4.30 a box with length L and mass M. We assume, that at the time $t_1 = 0$ a light pulse with energy E is emitted from the left side of the box which travels with the velocity of light c to the right. According to results of classical physics the momentum of the light pulse is $\boldsymbol{p} = (E/c)\hat{e}$ (see Vol. 2). Because of the conservation of momentum the left wall and with it the total box suffers a recoil $-\boldsymbol{p}$ into the left direction. This results in a velocity

$$\boldsymbol{v} = -\boldsymbol{p}/M = (E/Mc)\hat{e}$$

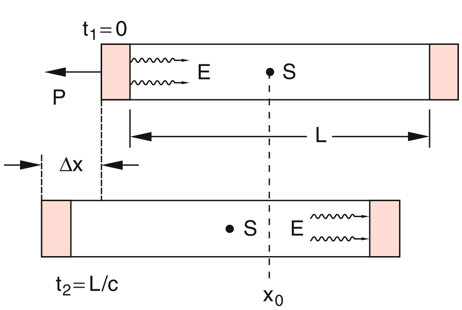

Figure 4.30 Einstein's "Gedanken-experiment" illustrating $E = mc^2$

of the box. For $v \ll c$ the light pulse reaches the right wall of the box at a time $t_2 = L/c$ and is absorbed by the wall. This transfers the momentum $\boldsymbol{p} = +(E/c)\hat{e}$ to the right wall. The total momentum transferred to the box is therefore zero and the box is again at rest. However, during the time t_2 the box has moved to the right by a distance

$$\Delta x = -v \cdot t_2 = -E \cdot \frac{L}{Mc^2} \,. \tag{4.46}$$

Since the box plus light pulse represent a closed system where no external forces act onto the system, the centre of mass cannot have moved. The CM of the box certainly has moved by Δx into the $-x$-direction. Therefore the light pulse must have transported mass into the $+x$-direction in order to leave the CM of the total system (box + light pulse) at rest. If we attribute a mass m to the light pulse with energy E the CM of the total system stays at rest, if

$$m \cdot L - M \cdot \Delta x = 0 \,. \tag{4.47}$$

This gives with (4.46) the result

$$m = E/c^2 \Rightarrow E = mc^2 \,. \tag{4.48a}$$

According to this consideration each mass m is correlated to the energy $E = m \cdot c^2$. Mass and energy are proportional to each other.

When we insert the rest mass m_0 from (4.43) we obtain from (4.48a) the energy of a mass m that moves with the velocity v

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} = m_0 c^2 + (m - m_0)c^2 \,. \tag{4.48b}$$

This energy E can be composed of two parts:

The rest energy $m_0 c^2$ which a particle at rest must have, and the kinetic energy

$$E_{\text{kin}} = (m(v) - m_0)c^2 \,, \tag{4.49a}$$

which is here described as the increase of its mass $m(v)$. If we expand the square root in (4.48b) according to

$$\frac{1}{\sqrt{1 - v^2/c^2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots \,,$$

the kinetic energy becomes

$$E_{\text{kin}} = \frac{1}{2} m_0 v^2 + \frac{3}{8} m_0 \frac{v^4}{c^2} + \cdots \,. \tag{4.49b}$$

In the limiting case $v \ll c$ we can neglect the higher order terms in (4.49b) and obtain the classical result

$$E_{\text{kin}} = \tfrac{1}{2} m v^2 \,.$$

This shows that the classical expression for the kinetic energy is an approximation for $v \ll c$. Since in daily life the condition

$v \ll c$ is always fulfilled, the relativistic expression is important only for cases where v approaches c as in high energy physics or astrophysics.

Squaring (4.48b) and multiplying both sides with c^2 gives

$$E^2 = \frac{m_0^2 c^6}{c^2 - v^2} = m_0^2 c^4 + m^2 c^2 v^2 \,. \tag{4.50}$$

Inserting (4.45) for the relativistic momentum yields

$$E^2 = m_0^2 c^4 + p^2 c^2 \,.$$

This gives the relativistic relation between total energy E and momentum p

$$E = c\sqrt{m_0^2 c^2 + p^2} \,. \tag{4.51}$$

For $v \ll c$ the square root can be expanded and gives the result

$$E_{\text{kin}} = E - m_0 c^2 \approx \frac{p^2}{2m} = \frac{1}{2} m_0 v^2$$

with the classical momentum $p = m_0 \cdot v$.

4.4.4 Inelastic Collisions at relativistic Energies

The relativistic energy and its conservation can be illustrated by the example of a collinear completely inelastic collision (Fig. 4.31). We regard two particles A and B with equal masses m which fly against each other width velocities $v_1 = \{v_1, 0, 0\}$ and $v_2 = \{-v_1, 0, 0\}$, measured in the system S. In a completely inelastic collision their total kinetic energy is converted into internal energy of the collision partners (see Sect. 4.2.4). After the collision they form a compound particle AB with the velocity $v = 0$ (Fig. 4.31 upper part).

In a system S^*, which moves with the velocity $v = v_1$ against S the particle A has the velocity $v_1^* = 0$, the compound particle AB which rests in S has in S^* the velocity $u = -v_1$. The particle

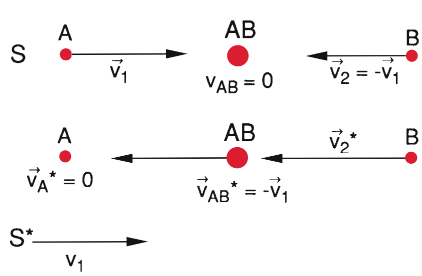

Figure 4.31 Description of a collinear completely inelastic collision in two different inertial systems S and S^*

B has in S^* according to the relativistic addition of velocities (3.28) the velocity

$$v_2^* = \frac{v_2 - v}{1 - v_2 v / c^2} = \frac{-2v}{1 + v^2/c^2} \,, \tag{4.52}$$

where $v_2 = -v_1$ is the velocity of B in S and $v = v_1$ is the velocity of S^* against S.

The conservation of momentum demands for the collision described in S^*

$$m\left(v_2^*\right) \cdot v_2^* = Mu = -Mv_1 \,, \tag{4.53}$$

where M is the mass of the compound AB with the velocity $u = -v_1$ measured in S^*.

Conservation of energy requires, when dividing by c^2

$$m\left(v_2^*\right) + m_0 = M \,. \tag{4.54}$$

Inserting from (4.53) the relation $M = -m(v_2^*) \cdot v_2^*/v$ into (4.54) we obtain

$$\frac{m\left(v_2^*\right)}{m_0} = -\frac{v}{v_2^* + v} \,. \tag{4.55}$$

Equation 4.52 gives the relation between v and v_2^*

$$v = -\frac{c^2}{v_2^*}\left[1 + \left(1 - \frac{v_2^{*2}}{c^2}\right)^{1/2}\right] ; \tag{4.56}$$

inserting this into (4.55) gives the mass ratio

$$\frac{m\left(v_2^*\right)}{m_0} = \left(1 - \frac{v_2^{*2}}{c^2}\right)^{-1/2} = \gamma\left(v_2^*\right) \,, \tag{4.57}$$

and therefore again the general relation

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \gamma(v) m_0 \,, \tag{4.58}$$

which has been already derived in Sect. 4.4.1.

4.4.5 Relativistic Formulation of Energy Conservation

In order to show, that the relativistic energy $E = m \cdot c^2$ is conserved we must discuss the relativistic formulation of the Newton equation $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$. Thereto we replace the classical position vector r by the Lorentz four-vector

$$\mathcal{R} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z + ict\hat{\boldsymbol{e}}_t = \boldsymbol{r} + ict\hat{\boldsymbol{e}}_t \,, \tag{4.59}$$

defined in the four-dimensional Minkowski space (x, y, z, ict) (see Sect. 3.6.2), where the unit vector $\hat{e}_t$ is perpendicular to the three spatial axes. From (4.59) one can derive that $\mathcal{R}^2 = r^2 - c^2 t^2$.

This gives the total differential

$$\left(\mathrm{d}\boldsymbol{\mathcal{R}}^2\right) = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\mathrm{d}t^2 = -c^2\mathrm{d}\tau^2 \;, \tag{4.60}$$

where we have used as abbreviation the differential

$$\begin{aligned} \mathrm{d}\tau &= \sqrt{\mathrm{d}t^2 - \frac{1}{c^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)} \\ &= \mathrm{d}t\sqrt{1 - \frac{v^2}{c^2}} = \mathrm{d}t/\gamma \end{aligned} \tag{4.61}$$

of the "*eigen-time*" τ, which approaches the classical time differential $\mathrm{d}t$ for $v \ll c$.

The differentiation of (4.59) gives the four-vector of the velocity

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{\mathcal{R}}}{\mathrm{d}\tau} &= \frac{\mathrm{d}x}{\mathrm{d}\tau}\hat{\boldsymbol{e}}_x + \frac{\mathrm{d}y}{\mathrm{d}\tau}\hat{\boldsymbol{e}}_y + \frac{\mathrm{d}z}{\mathrm{d}\tau}\hat{\boldsymbol{e}}_z + ic\frac{\mathrm{d}t}{\mathrm{d}\tau}\hat{\boldsymbol{e}}_t \\ &= \frac{\boldsymbol{v} + ic\hat{\boldsymbol{e}}_t}{\sqrt{1 - v^2/c^2}} \;. \end{aligned} \tag{4.62}$$

The four-momentum is defined as

$$\mathcal{P} = m_0\frac{\mathrm{d}\boldsymbol{\mathcal{R}}}{\mathrm{d}\tau} = m_0\frac{\boldsymbol{v} + ic\hat{\boldsymbol{e}}_t}{\sqrt{1 - v^2/c^2}} \;. \tag{4.63}$$

In analogy to the Newton equation $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ we define the four-force (also called the Minkowski-force)

$$\begin{aligned} \mathcal{F} &= \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\tau} = m_0\frac{\mathrm{d}^2\boldsymbol{\mathcal{R}}}{\mathrm{d}\tau^2} \\ &= \gamma\left[\frac{\mathrm{d}}{\mathrm{d}t}(m\cdot\boldsymbol{v}) + ic\frac{\mathrm{d}}{\mathrm{d}t}\left(m\hat{\boldsymbol{e}}_t\right)\right] \\ &= \gamma\left[\boldsymbol{F} + i\frac{\mathrm{d}}{\mathrm{d}t}\left(mc\hat{\boldsymbol{e}}_t\right)\right] \;. \end{aligned} \tag{4.64}$$

Using these definitions we can derive the relativistic energy conservation law. We multiply (4.64) with $\mathrm{d}\boldsymbol{\mathcal{R}}/\mathrm{d}\tau$

$$\begin{aligned} \left(\mathcal{F}\frac{\mathrm{d}\boldsymbol{\mathcal{R}}}{\mathrm{d}\tau}\right) &= m_0\left(\frac{\mathrm{d}^2\mathcal{R}}{\mathrm{d}\tau^2}\right)\cdot\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\tau} \\ &= \frac{m_0}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\tau}\right)^2 \;. \end{aligned} \tag{4.65}$$

According to (4.60) is $(\mathrm{d}\mathcal{R}/\mathrm{d}\tau)^2 = -c^2 = \mathrm{const}$. Therefore the right side of (4.65) is zero!

$$\mathcal{F}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\tau} = 0 \;. \tag{4.66}$$

This can be separated in a spatial and a temporal part, which gives

$$\begin{aligned} &\frac{1}{1 - v^2/c^2}\left(\boldsymbol{F}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\left(mc^2\right)\right) = 0 \\ &\Rightarrow \mathrm{d}\left(mc^2\right) = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \mathrm{d}W \;. \end{aligned} \tag{4.67}$$

The quantity $\mathrm{d}W$ represents the work, performed on the particle with mass m. For conservative forces $F = -\,\mathrm{grad}\,V$, which have a potential, is $\mathrm{d}W$ equal to the change of the potential energy E_p. Integration of (4.67) over the time yields

$$E_\mathrm{p} + mc^2 = \mathrm{const} = E \;, \tag{4.68a}$$

which corresponds to the classical energy conservation (2.38) if E_kin is replaced by mc^2.

For a particle with the velocity v we can write

$$mc^2 = m_0\gamma(v)\cdot c^2 = \frac{m_0c^2}{\sqrt{1 - (v/c)^2}} \;.$$

The equation (4.68a) of energy conservation then becomes

$$E_\mathrm{p} + \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} = E \;. \tag{4.68b}$$

4.5 Conservation Laws

In the foregoing sections we have discussed, that there are physical quantities which are conserved in closed systems, i. e. they do not change in the course of time.

As a reminder, please note,

that a closed system is a system which has no interaction with the outside, i. e. there are no external forces acting on the particles of the system, although the particles may interact with each other.

Such conserved quantities are the total momentum $\boldsymbol{p}$, the total energy E and the angular momentum $\boldsymbol{L}$ of a closed system. The conservation of these quantities is, because of its great importance, formulated in special conservation laws, which shall be summarized and generalized in the following sections.

4.5.1 Conservation of Momentum

For a single free particle (no forces acting on it) the momentum conservation reads:

The momentum $\boldsymbol{p} = m\cdot\boldsymbol{v}$ of a free particle is constant in time.

This is identical with the Newton postulate (Sect. 2.6).

Generalized for a system of particles this reads:

The total momentum of a closed system of particles which may interact with each other, does not change with time.

This can be also formulated as: If the vector sum of all external forces acting on a system of particles is zero, the total momentum of the system does not change with time. According to the 3. Newton's axiom *actio* = *reactio* the vector sum of all internal forces is anyway zero.

Note that the momentum of the individual particles can indeed change!

4.5.2 Energy Conservation

We have seen in Sect. 2.7 that in conservative force fields the sum of kinetic anf potential energy is constant. This energy conservation can be generalized to a system of particles and also further types of energy (internal energy, thermal energy, mass energy $E = mc^2$) can be included. The law of energy conservation in the general form is then:

The total energy of a closed system is constant in time, where the different forms of energy can be completely or partially converted into each other

For instance the kinetic energy of a particle can be converted into thermal energy at a collision with the wall, or the mass energy of electron and positron can be converted into radiation energy if the two particles collide.

4.5.3 Conservation of Angular Momentum

If the vector sum of all torques D_i which act on a system of particles is zero, the total angular momentum $\boldsymbol{L}$ of the system remains constant. This follows from the relation $\mathrm{d}\boldsymbol{L}/\mathrm{d}t = \sum \boldsymbol{D}_i$.

Note: For the definition of the angular momentum

$$\boldsymbol{L} = \sum(\boldsymbol{r}_i \times \boldsymbol{p}_i) ,$$

the reference point (generally the origin of the coordinate system from which the position vectors $\boldsymbol{r}_i$ start) has to be defined.

Since for a closed system $\sum \boldsymbol{D}_i = \boldsymbol{0}$ the conservation of angular momentum can be also formulated as

In a closed system the total angular momentum $\boldsymbol{L}$ remains constant in time.

4.5.4 Conservation Laws and Symmetries

A more detailed investigation of the real causes of the conservation laws reveals that these laws are based on symmetry properties of space and time [4.9]. In order to prove this, we introduce the Lagrange function $\mathcal{L}$

$$\begin{aligned}\mathcal{L}(\boldsymbol{r}_i, \boldsymbol{v}_i) &= \sum_i^N \frac{m_i}{2} v_i^2 - E_{\text{pot}}(\boldsymbol{r}_1, \boldsymbol{r}_2 \ldots \boldsymbol{r}_N) \\ &= E_{\text{kin}} - E_{\text{p}}\end{aligned} \tag{4.69}$$

of a closed system with N particles, which represents the difference of kinetic and potential energy. From (4.69) the relations

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i} = m_i \boldsymbol{v}_i = \boldsymbol{p}_i \tag{4.70a}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{r}_i} = -\frac{\partial E_{\text{p}}}{\partial \boldsymbol{r}_i} = \boldsymbol{F}_i \tag{4.70b}$$

follow immediately. This gives the equation of motion $F_i = m_i \cdot \mathrm{d}v_i/\mathrm{d}t$ in the general form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i}\right) = \frac{\partial \mathcal{L}}{\partial \boldsymbol{r}_i} . \tag{4.71}$$

Note: The Lagrange equation (4.71) can be derived quite general from a fundamental variation principle, called the *principle of minimum* action [4.8].

This principle also gives the definite justification for the following statements and their explanation.

1. The conservation of momentum is due to the homogeneity of space.

This homogeneity of space guarantees that the properties of a closed system do not change when all particles are shifted by an amount ε, which means that their position vectors changes from $\boldsymbol{r}$ to $\boldsymbol{r} + \boldsymbol{\varepsilon}$. Because of the homogeneity all masses and velocities remain unchanged.

The Lagrange function in a homogeneous space does not depend on the position vectors $\boldsymbol{r}_i$ i. e.

$$\sum \partial \mathcal{L}/\partial r_i = 0 .$$

From (4.71) we can conclude

$$\begin{aligned}&\sum_i \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i} = 0 \\ &\Rightarrow \sum \frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_i} = \sum \boldsymbol{p}_i = \boldsymbol{p} = \text{const} .\end{aligned} \tag{4.72}$$

Chapter 4

2. The conservation of energy follows from the homogeneity of time.

The homogeneity of time implies that the Lagrange function $\mathcal{L}$ does not explicitly depend on time. Which means that $\partial\mathcal{L}/\partial t = 0$.

The total derivation of $\mathcal{L}$ is

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \sum_{i=1}^{3N} \frac{\partial\mathcal{L}}{\partial x_i}\dot{x}_i + \sum \frac{\partial\mathcal{L}}{\partial \dot{x}_i}\ddot{x}_i .$$

If we replace according to (4.71) $\partial\mathcal{L}/\partial x_i$ by $\mathrm{d}/\mathrm{d}t(\partial\mathcal{L}/\partial\hat{e})$ we obtain

$$\begin{aligned}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} &= \sum \dot{x}_i \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}_i} + \sum \frac{\partial\mathcal{L}}{\partial\dot{x}_i}\ddot{x}_i = \sum \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_i}\dot{x}_i\right)\\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum \dot{x}_i\frac{\partial\mathcal{L}}{\partial\dot{x}_i} - \mathcal{L}\right) = 0 \qquad (4.73)\\ &\Rightarrow \sum \dot{x}_i \frac{\partial\mathcal{L}}{\partial\dot{x}_i} - \mathcal{L} = E = \text{const} ,\end{aligned}$$

which means that E is constant in time.

Finally the conservation of angular momentum follows from the isotropy of space, which means that no specific direction in space is preferred.

This isotropy implies that an arbitrary rotation of a closed system does not change the mechanical properties of the system. In particular the Lagrange function should not change when the system rotates by an angle $\partial\varphi$.

We introduce the vector $\boldsymbol{\delta\varphi}$ with the magnitude $\delta\varphi$ and the direction of the rotation axis. The change of the position vector r_i of the point P is (Fig. 4.32)

$$\boldsymbol{\delta r}_i = \boldsymbol{\delta\varphi} \times \boldsymbol{r}_i . \qquad (4.74a)$$

The velocity of P is then changing by

$$\delta v_i = \delta\boldsymbol{\varphi} \times \boldsymbol{v}_i . \qquad (4.74b)$$

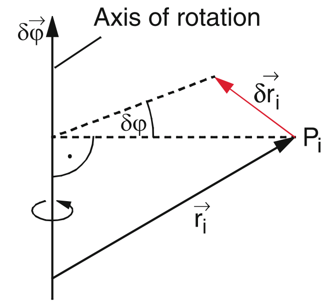

Figure 4.32 Definition of $\delta\varphi$ and δr

For the change $\delta\mathcal{L} = 0$ of the Lagrange function $\mathcal{L}$ we obtain

$$\delta\mathcal{L} = \sum_i \frac{\partial\mathcal{L}}{\partial\boldsymbol{r}_i}\delta\boldsymbol{r}_i + \frac{\partial\mathcal{L}}{\partial\boldsymbol{v}_i}\delta\boldsymbol{v} = 0 . \qquad (4.75)$$

With the relations

$$\partial\mathcal{L}/\partial\boldsymbol{v}_i = \boldsymbol{p}_i \quad \text{and} \quad \partial\mathcal{L}/\partial\boldsymbol{r}_i = \boldsymbol{F}_i = \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t}$$

we can write (4.75) in the form

$$\begin{aligned}&\sum_i \dot{\boldsymbol{p}}_i(\delta\boldsymbol{\varphi}\times\boldsymbol{r}_i) + \boldsymbol{p}_i(\delta\boldsymbol{\varphi}\times\boldsymbol{v}_i) = 0\\ &\Rightarrow \delta\boldsymbol{\varphi}\left[\sum_i((\boldsymbol{r}_i\times\dot{\boldsymbol{p}}_i) + (\boldsymbol{v}_i\times\boldsymbol{p}_i))\right] \qquad (4.76)\\ &= \delta\boldsymbol{\varphi}\frac{\mathrm{d}}{\mathrm{d}t}\sum(\boldsymbol{r}_i\times\boldsymbol{p}_i) = 0 .\end{aligned}$$

Since this must hold for arbitrary values of $\delta\varphi$ it follows

$$\sum(\boldsymbol{r}_i\times\boldsymbol{p}_i) = \boldsymbol{L} = \textbf{const} . \qquad (4.77)$$

Summary

- The centre of mass of a system of N point masses m_i with the position vectors $\boldsymbol{r}_i$ has the position vector

$$\boldsymbol{r}_S = \frac{1}{\sum m_i}\sum m_i\boldsymbol{r}_i = \frac{1}{M}\sum m_i\boldsymbol{r}_i .$$

- The coordinate system with the CM as origin is called the centre-of-mass system
- The vector sum of all momenta $m_i\boldsymbol{v}_i$ of the masses m_i in the CM-system is always zero.
- The reduced mass μ of two masses m_1 and m_2 is defined as

$$\mu = \frac{m_1 \cdot m_2}{m_1 + m_2} .$$

- The relative motion of two particles with the mutual interaction forces $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$ can be reduced to the motion of a single particle with the reduced mass μ which moves with the velocity $\boldsymbol{v}_{12} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ around the centre of m_1.

- A system of particles with masses m_i, where no external forces are present is called a **closed system**. The total momentum and the total angular momentum of a closed system are always constant, i. e. they do not change with time (conservation laws for momentum and angular momentum).
- In elastic collisions between two particles the total kinetic energy and the total momentum are conserved. For inelastic collisions the total momentum is also conserved but part of the initial kinetic energy is transferred into internal energy (e. g. potential energy or kinetic energy of the building blocks of composed collision partners). Inelastic collisions can only occur if at least one of the collision partners has a substructure, i. e. is compound of smaller entities.
- While for elastic collisions in the lab-system the kinetic energies E_i of the individual partners change (although the total energy is conserved), in the CM-system also the E_i are conserved.
- In inelastic collisions only the kinetic energy $\frac{1}{2}\mu v_{12}^2$ of the relative motion can be transferred into internal energy. At least the part $\frac{1}{2}Mv_S^2$ of the CM-motion must be preserved as kinetic energy of the collision partners.
- The collision between two particles with masses m_1 and m_2 can be reduced in the CM-system to the scattering of a single particle with reduced mass

$$\mu = \frac{m_1 \cdot m_2}{m_1 + m_2}$$

by a particle with mass $m\infty$ fixed in the CM. This can be also described by the scattering in a potential depending on the interaction force between the two particles.
- The deflection angle φ of the particle in the CM-system depends on the impact parameter b, the reduced mass μ, the initial kinetic energy $\frac{1}{2}\mu v_0^2$ and the radial dependence of the interaction potential.
- The evaluation of collisions at relativistic velocities v demands the consideration of the relativistic mass increase. Then also energy and momentum conservation remain valid.
- The conservation laws for energy, momentum and angular momentum can be ascribed to general symmetry principles, as the homogeneity of space and time and the isotropy of space.

Problems

Chapter 4

4.1 Two particles with masses $m_1 = m$ and $m_2 = 3m$ suffer a central collision. What are their velocities v_1' and v_2' after the collision if the two particles had equal but opposite velocities $v_1 = -v_2$ before the collision
a) For a completely elastic collision
b) For a completely inelastic collision?

4.2 A wooden block with mass $m_1 = 1\,\text{kg}$ hangs on a wire with length $L = 1\,\text{m}$. A bullet with mass $m_2 = 20\,\text{g}$ is shot with the velocity $v = 10^3\,\text{m/s}$ into the block and sticks there. What is the maximum deflection angle of the block?

4.3 A proton with the velocity v_1 collides elastically with a deuteron (nucleus consisting of proton and neutron) at rest. After the collision the deuteron flies under the angle of 45° against v_1. Determine
a) the deflection angle θ_1 of the proton
b) the CM-velocity
c) the velocities v_1' and v_2' of proton and deuteron after the collision.

4.4 A particle with mass $m_1 = 2\,\text{kg}$ has the velocity $\boldsymbol{v}_1 = \{3\hat{e}_x + 2\hat{e}_y - \hat{e}_z\}\,\text{m/s}$. It collides completely inelastic with a particle of mass $m_2 = 3\,\text{kg}$ and velocity $\boldsymbol{v}_2 = \{-2\hat{e}_x + 2\hat{e}_y + 4\hat{e}_z\}$. Determine
a) The kinetic energies of the two particle before the collision in the lab-system and the CM-system.
b) Velocity and kinetic energy of the compound particle after the collision.
c) Which fraction of the initial kinetic energy has been converted into internal energy? Calculate this fraction in the lab-system and the CM-system.

4.5 A mass $m_1 = 1\,\text{kg}$ with a velocity $v_1 = 4\,\text{m/s}$ collides with a mass $m_2 = 2\,\text{kg}$. After the collision m_1 moves with $v_1' = \sqrt{8\,\text{m/s}}$ under an angle of 45° against $\boldsymbol{v}_1$ and m_2 with $v_2' = \sqrt{2\,\text{m/s}}$ under an angle of $-45°$
a) What was the velocity v_2?
b) Which fraction of the initial kinetic energy has been converted into internal energy in the lab-system and the CM-system?
c) How large are the deflection angles ϑ_1 and ϑ_2 in the CM-system?

4.6 Two cuboids with masses $m_1 = 1\,\text{kg}$ and $m_2 < m_1$ slide frictionless on an air-track, which is blocked on both sides by a vertical barrier (Fig. 4.33). Initially m_1 is at rest and m_2 moves with constant velocity $v_2 = 0.5\,\text{m/s}$ to the left. After the collision with m_1 the mass m_2 is reflected to the right, collides with the barrier ($m = \infty$) and slides again to the left. We assume that all collisions are completely elastic.
a) What is the ratio m_1/m_2 if the two masses finally move to the left with equal velocities?
b) How large should m_2 be in order to catch m_1 before it reaches the left barrier?
c) Where collide the two masses at the second collision for $m_2 = 0.5\,\text{kg}$?

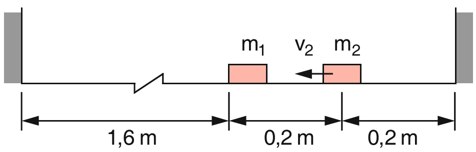

Figure 4.33 Illustrating Probl. 4.6

4.7 A steel ball with mass $m_1 = 1\,\mathrm{kg}$ hangs on a wire with $L = 1\,\mathrm{m}$, vertically above the left edge of a resting mass $m_2 = 5\,\mathrm{kg}$ which can slide without friction on a horizontal air-track. (Fig. 4.34). The steel ball with the wire is lifted by an angle $\varphi = 90°$ from the vertical into the horizontal position and then released. It collides elastically with the glider. What is the maximum angle φ of m_1 after the collision?

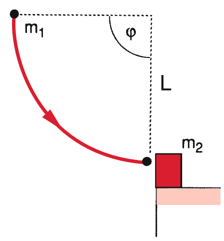

Figure 4.34 Illustration of Probl. 4.7

4.8 An elevator ascends with constant velocity $v = 2\,\mathrm{m/s}$. When its ceiling is still 30 m below the upper point A of the lift shaft a ball is released from A which falls freely down and hits elastically the ceiling of the elevator, from where it is elastically reflected upwards.

a) Where does it hit the elevator ceiling?
b) What is its maximum height after the reflection?
c) Where does it hit the elevator ceiling a second time?

4.9 An α-particle (nucleus of the He-atom) hits with the velocity $\boldsymbol{v}_1$ an oxygen nucleus at rest ($m_2 = 4m_1$). The α-particle is deflected by 64°, the oxygen nucleus by −51° against $\boldsymbol{v}_1$. The collision is completely elastic.

a) What is the ratio v_1'/v_2' of the velocities after the collision?
b) What is the ratio of the kinetic energies after the collision?

4.10 A particle has in a system S a kinetic energy of 6 GeV. and the momentum $P = 6\,\mathrm{GeV/c}$. What is its energy in a system S', where its momentum is measured as 5 GeV/c? What is the relative velocity of S' against S?

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