Chapter 22
Applications of Spinor QED

We have seen in Chapter 18 that inclusion of the quantized Maxwell field did not change the basic formalism of time-dependent perturbation theory, see equations (18.71–18.73), and this property also persists after promotion of the matter fields in the Hamiltonian to relativistic Klein-Gordon or Dirac fields. In the following sections we will use the Hamiltonian (21.94) of spinor quantum electrodynamics for the calculation of scattering processes. However, we first should generalize our previous results for the scattering matrix to the case of two free particles in the initial and final state.

22.1 Two-particle scattering cross sections

We have discussed events with one free particle in the initial or final state of a scattering event in the framework of potential scattering theory in Chapters 11 and 13, or in photon emission, absorption or scattering off bound electrons in Sections 18.6–18.9. The techniques that we have discussed so far cover many applications of scattering theory, but eventually we also wish to understand scattering involving two (quasi-)free particles in the initial and final states. Electron scattering off atomic nuclei, electron-electron scattering, electron-photon scattering, or electron-phonon scattering provide examples of these kinds of scattering events which happen all the time in materials. In these cases we are discussing scattering events with two particles in the initial or final states. We should therefore address the question how to generalize the equations from Sections 13.6 and 18.9, which dealt with the case of one free particle in the initial and final state.

Let us recall from Section 13.6 that with a free particle with wave vectors $\boldsymbol{k}$ and $\boldsymbol{k}'$ in the initial and final state, the scattering matrix element for a monochromatic perturbation $W(t) \sim \exp(-\mathrm{i}\omega t)$

$$S_{k',k} = \langle \boldsymbol{k}'|U_D(\infty,-\infty)|\boldsymbol{k}\rangle = -\mathrm{i}\mathcal{M}_{k',k}\delta(\omega(k')-\omega(k)-\omega)$$

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has dimension length^3 due to the length dimensions of the external states, and yields a differential scattering cross section

$$d\sigma_{\boldsymbol{k}\to\boldsymbol{k}'} = d^3\boldsymbol{k}'\frac{\left|S_{\boldsymbol{k}',\boldsymbol{k}}\right|^2}{Tdj(\boldsymbol{k})/d^3\boldsymbol{k}} = d^3\boldsymbol{k}'\frac{\left|\mathcal{M}_{\boldsymbol{k}',\boldsymbol{k}}\right|^2}{2\pi\,dj(\boldsymbol{k})/d^3\boldsymbol{k}}\delta(\omega(k') - \omega(k) - \omega).$$

Here we substituted the more precise notation $dj(\boldsymbol{k})/d^3\boldsymbol{k}$ for the incoming current density j_{in} per $\boldsymbol{k}$ space volume. Substitution of the current density for a free particle of momentum $\hbar\boldsymbol{k}$

$$\frac{d\boldsymbol{j}(\boldsymbol{k})}{d^3\boldsymbol{k}} = \frac{\boldsymbol{v}}{(2\pi)^3}$$

yields

$$vd\sigma_{\boldsymbol{k}\to\boldsymbol{k}'} = 4\pi^2\left|\mathcal{M}_{\boldsymbol{k}',\boldsymbol{k}}\right|^2\delta(\omega(k') - \omega(k) - \omega)d^3\boldsymbol{k}', \tag{22.1}$$

see also equation (18.115), where we found this equation for photon scattering off atoms or molecules.

Now suppose that we have two free particles with momenta $\boldsymbol{k}$ and $\boldsymbol{q}$ in the initial state, and they scatter into free particles with momenta $\boldsymbol{k}'$ and $\boldsymbol{q}'$ in the final state. We also assume that the scattering preserves total energy and momentum. The corresponding scattering matrix element

$$S_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}} = \langle\boldsymbol{k}',\boldsymbol{q}'|U_D(\infty,-\infty)|\boldsymbol{k},\boldsymbol{q}\rangle = -\mathrm{i}\mathcal{M}_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}}\delta(k' + q' - k - q) \tag{22.2}$$

has dimension length^6. This is consistent with the fact that $\left|S_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}}\right|^2$ is a transition probability density per volume units $d^3\boldsymbol{k}'d^3\boldsymbol{q}'d^3\boldsymbol{k}d^3\boldsymbol{q}$ in wave vector space.

For ease of the present discussion, we also assume that the scattering particles are different, like in electron-photon or electron-phonon scattering, and we will use markers e and γ to label quantities referring to the different particles. The notation is motivated from electron-photon scattering, but we will develop the formalism in this section with general pairs of particles of masses m_e and m_γ in mind.

Suppose the two particles have momentum 4-vectors

$$p_e = \hbar k = \hbar(\omega_e/c, \boldsymbol{k}), \quad p_\gamma = \hbar q = \hbar(\omega_\gamma/c, \boldsymbol{q})$$

relative to the laboratory frame in which we observe the collisions. The scattering rate will be proportional to the product

$$\frac{d\varrho_e(\boldsymbol{k})}{d^3\boldsymbol{k}}\frac{dj_\gamma(\boldsymbol{q})}{d^3\boldsymbol{q}} = \frac{d\varrho_e(\boldsymbol{k})}{d^3\boldsymbol{k}}\frac{d\varrho_\gamma(\boldsymbol{q})}{d^3\boldsymbol{q}}\tilde{v}_{e\gamma},$$

where

$$\tilde{v}_{e\gamma} = c^2\left|\frac{\boldsymbol{k}}{\omega_e} - \frac{\boldsymbol{q}}{\omega_\gamma}\right| \tag{22.3}$$

is the relative speed between the two particles that *we* assign from the point of view of our laboratory frame. The speed $\tilde{v}_{e\gamma}$ is usually replaced with another measure for relative speed between the two particles,

$$v_{e\gamma} = \frac{c^3}{\omega_e\omega_\gamma}\sqrt{(k\cdot q)^2 - \frac{m_e^2 m_\gamma^2 c^4}{\hbar^4}}, \tag{22.4}$$

which agrees with $\tilde{v}_{e\gamma}$ in laboratory frames in which the two momentum vectors $\boldsymbol{p}_e$ and $\boldsymbol{p}_\gamma$ are parallel or anti-parallel, or where the laboratory frame coincides with the rest frame of one of the two particles:

$$\tilde{v}_{e\gamma}^2 - v_{e\gamma}^2 = \frac{c^6}{\omega_e^2\omega_\gamma^2}\,|\boldsymbol{k}|^2\,|\boldsymbol{q}|^2\left(1-\cos^2\vartheta\right).$$

Here ϑ is the angle between $\boldsymbol{p}_e$ and $\boldsymbol{p}_\gamma$. E.g. in the rest frame of the e-particle, $\hbar k_e = \hbar(\omega_e/c, \mathbf{0}) = (m_e c, \mathbf{0})$, we find

$$v_{e\gamma} = \frac{c}{\omega_\gamma}\sqrt{\omega_\gamma^2 - \frac{m_\gamma^2 c^4}{\hbar^2}} = c^2\frac{|\boldsymbol{p}_\gamma|}{E_\gamma} = v_\gamma,$$

and in the center of mass frame of the two particles, $\boldsymbol{k} = -\boldsymbol{q}$, we also find the difference of particle velocities,

$$\begin{aligned}
v_{e\gamma}^2 &= c^2\frac{\hbar^4(\omega_e\omega_\gamma + c^2\boldsymbol{k}^2)^2 - m_e^2 m_\gamma^2 c^8}{\hbar^4\omega_e^2\omega_\gamma^2} \\
&= c^4\frac{2\hbar^2 c^2\boldsymbol{k}^4 + 2\hbar^2\omega_e\omega_\gamma\boldsymbol{k}^2 + c^4\boldsymbol{k}^2(m_e^2 + m_\gamma^2)}{\hbar^2\omega_e^2\omega_\gamma^2} = c^4\boldsymbol{k}^2\frac{\omega_e^2 + \omega_\gamma^2 + 2\omega_e\omega_\gamma}{\omega_e^2\omega_\gamma^2} \\
&= \left(c^2\frac{\boldsymbol{k}}{\omega_e} - c^2\frac{\boldsymbol{q}}{\omega_\gamma}\right)^2.
\end{aligned} \tag{22.5}$$

As a byproduct we also find another useful formula for the relative speed in the center of mass frame,

$$v_{e\gamma} = c^2|\boldsymbol{k}|\frac{\omega_e + \omega_\gamma}{\omega_e\omega_\gamma}. \tag{22.6}$$

Please keep in mind that (22.6) is the relative particle speed assigned to the two colliding particles by an observer at rest in the center of mass frame, but not the speed of one particle relative to the other particle as measured in the rest frame of one of the particles.

The differential cross section for two-particle scattering can then be defined through the equation

$$\begin{aligned} v d\sigma_{\boldsymbol{k},\boldsymbol{q}\to\boldsymbol{k}',\boldsymbol{q}'} &= v d^3\boldsymbol{k}' d^3\boldsymbol{q}' \frac{\left|S_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}}\right|^2}{VT(d\varrho_e/d^3\boldsymbol{k})(dj_\gamma/d^3\boldsymbol{q})} \\ &= d^3\boldsymbol{k}' d^3\boldsymbol{q}' \frac{\left|S_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}}\right|^2}{VT(d\varrho_e/d^3\boldsymbol{k})(d\varrho_\gamma/d^3\boldsymbol{q})}. \end{aligned} \tag{22.7}$$

In words, we divide the scattering rate $d^3\boldsymbol{k}' d^3\boldsymbol{q}' \left|S_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}}\right|^2 d^3\boldsymbol{k} d^3\boldsymbol{q}/T$ between wave vector volumes $d^3\boldsymbol{k} d^3\boldsymbol{q} \to d^3\boldsymbol{k}' d^3\boldsymbol{q}'$ by the number of scattering centers $Vd\varrho_e$ in the phase space volume $Vd^3\boldsymbol{k}$ and the incoming particle flux dj_γ in the wave vector volume $d^3\boldsymbol{q}$ to calculate $d\sigma_{\boldsymbol{k},\boldsymbol{q}\to\boldsymbol{k}',\boldsymbol{q}'}$.

If we substitute the scattering amplitude $\mathcal{M}_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}}$ for the scattering matrix element and use for the 4-dimensional δ function in momentum space the equation

$$\delta^4(0) = \lim_{k\to 0} \frac{1}{(2\pi)^4} \int d^4x\, \exp(\mathrm{i}k\cdot x) = \frac{cVT}{(2\pi)^4},$$

we find

$$v d\sigma_{\boldsymbol{k},\boldsymbol{q}\to\boldsymbol{k}',\boldsymbol{q}'} = d^3\boldsymbol{k}' d^3\boldsymbol{q}' \frac{c\left|\mathcal{M}_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}}\right|^2}{(2\pi)^4(d\varrho_e/d^3\boldsymbol{k})(d\varrho_\gamma/d^3\boldsymbol{q})} \delta^4(k'+q'-k-q). \tag{22.8}$$

The density per $\boldsymbol{x}$ space volume and per unit $d^3\boldsymbol{k}$ of $\boldsymbol{k}$ space volume for momentum eigenstates is

$$\frac{d\varrho}{d^3\boldsymbol{k}} = \frac{1}{(2\pi)^3}.$$

This yields

$$v d\sigma_{\boldsymbol{k},\boldsymbol{q}\to\boldsymbol{k}',\boldsymbol{q}'} = 4\pi^2 c \left|\mathcal{M}_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}}\right|^2 \delta^4(k'+q'-k-q) d^3\boldsymbol{k}' d^3\boldsymbol{q}'. \tag{22.9}$$

Note from equation (22.2) that the two-particle scattering amplitude $\mathcal{M}_{\boldsymbol{k}',\boldsymbol{q}';\boldsymbol{k},\boldsymbol{q}}$ has the dimension length^2 while the single-particle scattering amplitude $\mathcal{M}_{\boldsymbol{k}',\boldsymbol{k}}$ has dimension length^3/time due to the use of a δ function in frequencies rather than wave numbers in the single-particle case.

We can derive a single-particle scattering cross section from (22.9) by integrating over the final wave number of one of the two particles, e.g. $\boldsymbol{q}'$, while considering its initial wave number fixed, e.g. $\boldsymbol{q} = \boldsymbol{0}$. This yields

$$\begin{aligned} v d\sigma_{\boldsymbol{k}\to\boldsymbol{k}'} &= v \int d^3\boldsymbol{q}' \frac{d\sigma_{\boldsymbol{k},\boldsymbol{0}\to\boldsymbol{k}',\boldsymbol{q}'}}{d^3\boldsymbol{q}'} \\ &= 4\pi^2 c^2 \left|\mathcal{M}_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{k}';\boldsymbol{k},\boldsymbol{0}}\right|^2 \delta(\omega(\boldsymbol{k}') - \omega(\boldsymbol{k}) - \omega_q) d^3\boldsymbol{k}', \end{aligned}$$

with

$$\omega_q = \omega(\boldsymbol{q} = \boldsymbol{0}) - \omega(\boldsymbol{q}' = \boldsymbol{k} - \boldsymbol{k}')$$

and a resulting single particle scattering amplitude $\mathcal{M}_{\boldsymbol{k}',\boldsymbol{k}} = c\mathcal{M}_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{k}';\boldsymbol{k},\boldsymbol{0}}$.

Measures for final states with two identical particles

To explain the necessary modifications of the previous results if we have two identical particles in the final state, we first consider decay of a normalizable state $|i\rangle$ into two identical particles with momenta $\hbar\boldsymbol{k}_1$ and $\hbar\boldsymbol{k}_2$.

The initial state belongs to a set of orthonormal states, $\langle i|j\rangle = \delta_{ij}$. For the final states with two identical particles, we have to take into account that the decomposition of unity on identical 2-particle states is

$$\begin{aligned}
1_{\text{identical 2-particle states}} &= \frac{1}{2}\int d^3\boldsymbol{k}_1 \int d^3\boldsymbol{k}_2\, |\boldsymbol{k}_1,\boldsymbol{k}_2\rangle\langle\boldsymbol{k}_1,\boldsymbol{k}_2| \\
&= \frac{1}{2}\int d^3\boldsymbol{k}_1 \int d^3\boldsymbol{k}_2\, a^+(\boldsymbol{k}_1)a^+(\boldsymbol{k}_2)|0\rangle\langle 0|a(\boldsymbol{k}_2)a(\boldsymbol{k}_1).
\end{aligned}$$

If the scattering matrix allows only for decay into the two-particle states, unitarity $U_D^+U_D = 1$ (or equivalently $S^+S = 1$) implies

$$\frac{1}{2}\int d^3\boldsymbol{k}_1 \int d^3\boldsymbol{k}_2\, |\langle\boldsymbol{k}_1,\boldsymbol{k}_2|U_D(\infty,-\infty)|i\rangle|^2 = 1.$$

The proper probability density for the transition $|i\rangle \to |\boldsymbol{k}_1,\boldsymbol{k}_2\rangle$ is therefore

$$w_{i\to\boldsymbol{k}_1,\boldsymbol{k}_2} = \frac{1}{2}d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2\, |\langle\boldsymbol{k}_1,\boldsymbol{k}_2|U_D(\infty,-\infty)|i\rangle|^2\,.$$

Equation (22.9) for the two-particle scattering cross section must therefore be modified if the final state contains two identical particles,

$$\begin{aligned}
v d\sigma_{\boldsymbol{k}_1,\boldsymbol{k}_2\to\boldsymbol{k}'_1,\boldsymbol{k}'_2} &= 4\pi^2 c\left|\mathcal{M}_{\boldsymbol{k}'_1,\boldsymbol{k}'_2;\boldsymbol{k}_1,\boldsymbol{k}_2}\right|^2 \delta^4(k'_1 + k'_2 - k_1 - k_2) \\
&\quad \times\frac{1}{2}d^3\boldsymbol{k}'_1 d^3\boldsymbol{k}'_2,
\end{aligned} \tag{22.10}$$

and the total two-particle cross section is

$$\begin{aligned}
\sigma &= \int d^3\boldsymbol{k}'_1 \int d^3\boldsymbol{k}'_2\, \frac{d\sigma_{\boldsymbol{k}_1,\boldsymbol{k}_2\to\boldsymbol{k}'_1,\boldsymbol{k}'_2}}{d^3\boldsymbol{k}'_1 d^3\boldsymbol{k}'_2} \\
&= \frac{1}{2}\int d^3\boldsymbol{k}'_1 \int d^3\boldsymbol{k}'_2\, 4\pi^2\frac{c}{v}\left|\mathcal{M}_{\boldsymbol{k}'_1,\boldsymbol{k}'_2;\boldsymbol{k}_1,\boldsymbol{k}_2}\right|^2 \delta^4(k'_1 + k'_2 - k_1 - k_2).
\end{aligned} \tag{22.11}$$

However, if we want to derive an effective *single-particle* differential scattering cross section $d\sigma/d\Omega$ from $d\sigma_{\boldsymbol{k}_1,\boldsymbol{k}_2\to\boldsymbol{k}'_1,\boldsymbol{k}'_2}$ by integrating over the momentum of one particle and the magnitude of momentum of the second particle using the energy-momentum conserving δ function, we have to take into account that the particle observed in direction $d\Omega$ can be either one of the two scattered particles:

$$\frac{d\sigma}{d\Omega} = \int d^3\boldsymbol{k}'_1 \int_0^\infty d|\boldsymbol{k}'_2|\,|\boldsymbol{k}'_2|^2 \frac{d\sigma_{\boldsymbol{k}_1,\boldsymbol{k}_2\to\boldsymbol{k}'_1,\boldsymbol{k}'_2}}{d^3\boldsymbol{k}'_1 d^3\boldsymbol{k}'_2} + \int d^3\boldsymbol{k}'_2 \int_0^\infty d|\boldsymbol{k}'_1|\,|\boldsymbol{k}'_1|^2 \frac{d\sigma_{\boldsymbol{k}_1,\boldsymbol{k}_2\to\boldsymbol{k}'_1,\boldsymbol{k}'_2}}{d^3\boldsymbol{k}'_1 d^3\boldsymbol{k}'_2}.$$

In the center of mass frame this reduces to a factor of 2,

$$\frac{d\sigma}{d\Omega} = 2\int d^3\boldsymbol{k}'_2 \int_0^\infty d|\boldsymbol{k}'_1|\,|\boldsymbol{k}'_1|^2 \frac{d\sigma_{\boldsymbol{k}_1,\boldsymbol{k}_2\to\boldsymbol{k}'_1,\boldsymbol{k}'_2}}{d^3\boldsymbol{k}'_1 d^3\boldsymbol{k}'_2}. \tag{22.12}$$

If we then wish to calculate the total two-particle scattering cross section (22.11) from the single-particle differential cross section (22.12), we have to compensate with a factor 1/2,

$$\sigma = \frac{1}{2}\int d\Omega\,\frac{d\sigma}{d\Omega}. \tag{22.13}$$

In practice one is often only interested in the effective single-particle differential cross section (22.12) and the total two-particle scattering cross section σ. The factor 1/2 is then usually neglected in the differential two-particle cross section $d\sigma_{\boldsymbol{k}_1,\boldsymbol{k}_2\to\boldsymbol{k}'_1,\boldsymbol{k}'_2}/(d^3\boldsymbol{k}'_1 d^3\boldsymbol{k}'_2)$, so that the factor of 2 is not needed in the calculation of $d\sigma/d\Omega$ (22.12) because it has been absorbed in $d\sigma_{\boldsymbol{k}_1,\boldsymbol{k}_2\to\boldsymbol{k}'_1,\boldsymbol{k}'_2}/(d^3\boldsymbol{k}'_1 d^3\boldsymbol{k}'_2)$. However, the factor 1/2 is still needed in the calculation of the total two-particle cross section σ from $d\sigma/d\Omega$ according to equation (22.13).

22.2 Electron scattering off an atomic nucleus

As a first application of two-particle scattering, we discuss scattering of an electron off an atomic nucleus. We assume that the nucleus is also a fermion and that the electrons are not energetic enough to resolve the internal structure of the nucleus. In that case we can use an effective description of the nucleus through Dirac field operators for a particle of charge Ze and mass M for the nucleus.

The Coulomb gauge Hamiltonian (21.94) has the form

$$H = H_0 + H_{e\gamma} + H_{N\gamma} + H_C, \tag{22.14}$$

where the free part H_0 contains the kinetic and mass terms and we have separated the different interaction terms.

The electron-photon and nucleus-photon interaction terms are

$$H_{e\gamma} = ec\int d^3\boldsymbol{x}\,\overline{\psi}(\boldsymbol{x},t)\boldsymbol{\gamma}\cdot\boldsymbol{A}(\boldsymbol{x},t)\psi(\boldsymbol{x},t)$$

and

$$H_{N\gamma} = -Zec\int d^3\boldsymbol{x}\,\overline{\Psi}(\boldsymbol{x},t)\boldsymbol{\gamma}\cdot\boldsymbol{A}(\boldsymbol{x},t)\Psi(\boldsymbol{x},t),$$

respectively. The relevant part of the Coulomb interaction term is the term describing the interaction of the electron and the nucleus,

$$H_{eN} = -Z\frac{e^2}{4\pi\epsilon_0}\sum_{cc'}\int d^3\boldsymbol{x}\int d^3\boldsymbol{X}\,\frac{\psi_c^+(\boldsymbol{x},t)\Psi_{c'}^+(\boldsymbol{X},t)\Psi_{c'}(\boldsymbol{X},t)\psi_c(\boldsymbol{x},t)}{|\boldsymbol{x}-\boldsymbol{X}|},$$

where the sum is over 4-spinor indices. The relevant leading order matrix element contains two terms,

$$S_{fi} = -\mathrm{i}\mathcal{M}_{fi}\delta(k+K-k'-K') = S_{fi}^{(\gamma)} + S_{fi}^{(C)},$$

which correspond to photon exchange,

$$\begin{aligned}S_{fi}^{(\gamma)} &= \langle\boldsymbol{K}',S';\boldsymbol{k}',s'|\frac{Ze^2}{\hbar^2}\mathrm{T}\int d^4x\int d^4X\,\overline{\psi}(x)\boldsymbol{\gamma}\cdot\boldsymbol{A}(x)\psi(x)\\&\quad\times\overline{\Psi}(X)\boldsymbol{\gamma}\cdot\boldsymbol{A}(X)\Psi(X)|\boldsymbol{K},S;\boldsymbol{k},s\rangle,\end{aligned} \tag{22.15}$$

or Coulomb scattering,

$$\begin{aligned}S_{fi}^{(C)} &= \langle\boldsymbol{K}',S';\boldsymbol{k}',s'|\frac{\mathrm{i}Ze^2\mu_0c}{4\pi\hbar}\int d^4x\int d^3\boldsymbol{X}\\&\quad\times\sum_{cc'}\frac{\psi_c^+(\boldsymbol{x},t)\Psi_{c'}^+(\boldsymbol{X},t)\Psi_{c'}(\boldsymbol{X},t)\psi_c(\boldsymbol{x},t)}{|\boldsymbol{x}-\boldsymbol{X}|}|\boldsymbol{K},S;\boldsymbol{k},s\rangle.\end{aligned} \tag{22.16}$$

We first calculate the Coulomb contribution to the scattering amplitude. Evaluation of the operators yields

$$\begin{aligned}S_{fi}^{(C)} &= \mathrm{i}\frac{Ze^2\mu_0c}{8(2\pi)^7\hbar}\frac{u_{s'}^+(\boldsymbol{k}')\cdot u_s(\boldsymbol{k})u_{S'}^+(\boldsymbol{K}')\cdot u_S(\boldsymbol{K})}{\sqrt{E_e(\boldsymbol{k}')E_e(\boldsymbol{k})E_N(\boldsymbol{K}')E_N(\boldsymbol{K})}}\\&\quad\times\int d^4x\int d^3\boldsymbol{X}\,\frac{\exp[\mathrm{i}(\boldsymbol{K}-\boldsymbol{K}')\cdot\boldsymbol{X}+\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}]}{|\boldsymbol{x}-\boldsymbol{X}|}\\&\quad\times\exp[-\mathrm{i}(\omega_e(\boldsymbol{k})+\omega_N(\boldsymbol{K})-\omega_e(\boldsymbol{k}')-\omega_N(\boldsymbol{K}'))t].\end{aligned}$$

In the next step we use the Fourier decomposition of the Coulomb potential

$$\int d^3\boldsymbol{x}\,\frac{\exp(-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x})}{|\boldsymbol{x}|} = \frac{4\pi}{\boldsymbol{q}^2} \tag{22.17}$$

to find

$$\begin{aligned}S_{fi}^{(C)} &= \mathrm{i}\frac{Ze^2\mu_0c}{4(2\pi)^2\hbar}\frac{u_{s'}^+(\boldsymbol{k}')\cdot u_s(\boldsymbol{k})u_{S'}^+(\boldsymbol{K}')\cdot u_S(\boldsymbol{K})}{\sqrt{E_e(\boldsymbol{k}')E_e(\boldsymbol{k})E_N(\boldsymbol{K}')E_N(\boldsymbol{K})}}\\&\quad\times\frac{\delta(k+K-k'-K')}{|\boldsymbol{k}-\boldsymbol{k}'|^2}.\end{aligned} \tag{22.18}$$

For the evaluation of the photon exchange contribution (22.15), we first note that the photon operators between the photon vacuum states yield

$$\langle 0|\mathrm{T}\boldsymbol{A}(x)\otimes\boldsymbol{A}(X)|0\rangle = \frac{\hbar\mu_0 c}{(2\pi)^3}\int\frac{d^3\boldsymbol{q}}{2|\boldsymbol{q}|}\sum_\alpha \boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\otimes\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})$$

$$\times\Big[\Theta(t-T)\exp[\mathrm{i}q\cdot(x-X)]+\Theta(T-t)\exp[-\mathrm{i}q\cdot(x-X)]\Big]_{q^0=\omega_\gamma(\boldsymbol{q})/c=|\boldsymbol{q}|}$$

$$= \frac{\hbar\mu_0 c}{(2\pi)^3}\frac{1}{2\pi\mathrm{i}}\int d^4q\,\frac{\exp[\mathrm{i}q\cdot(x-X)]}{q^2-\mathrm{i}\epsilon}\left(\underline{1}-\frac{\boldsymbol{q}\otimes\boldsymbol{q}}{\boldsymbol{q}^2}\right).$$

Evaluation of the fermion operators yields

$$\langle \boldsymbol{K}',S';\boldsymbol{k}',s'|\overline{\psi}(x)\boldsymbol{\gamma}\psi(x)\otimes\overline{\Psi}(X)\boldsymbol{\gamma}\Psi(X)|\boldsymbol{K},S;\boldsymbol{k},s\rangle$$

$$=\frac{\overline{u}_{s'}(\boldsymbol{k}')\boldsymbol{\gamma}u_s(\boldsymbol{k})\otimes\overline{u}_{S'}(\boldsymbol{K}')\boldsymbol{\gamma}u_S(\boldsymbol{K})}{4(2\pi)^6\sqrt{E_e(\boldsymbol{k}')E_e(\boldsymbol{k})E_N(\boldsymbol{K}')E_N(\boldsymbol{K})}}\exp[\mathrm{i}(k-k')\cdot x+\mathrm{i}(K-K')\cdot X].$$

Assembling the pieces yields

$$S_{fi}^{(\gamma)} = \frac{Ze^2\mu_0 c}{4\mathrm{i}(2\pi)^2\hbar}\frac{\delta(k+K-k'-K')}{\sqrt{E_e(\boldsymbol{k}')E_e(\boldsymbol{k})E_N(\boldsymbol{K}')E_N(\boldsymbol{K})}}\frac{1}{(k-k')^2-\mathrm{i}\epsilon}$$

$$\times\overline{u}_{s'}(\boldsymbol{k}')\boldsymbol{\gamma}u_s(\boldsymbol{k})\left(\underline{1}+\frac{(\boldsymbol{k}-\boldsymbol{k}')\otimes(\boldsymbol{K}-\boldsymbol{K}')}{|\boldsymbol{k}-\boldsymbol{k}'|^2}\right)\overline{u}_{S'}(\boldsymbol{K}')\boldsymbol{\gamma}u_S(\boldsymbol{K}),$$

where $\boldsymbol{k}-\boldsymbol{k}' = -(\boldsymbol{K}-\boldsymbol{K}')$ from momentum conservation was used in the projection term. Substitution of the free Dirac equation for the external fermion states yields

$$S_{fi}^{(\gamma)} = \frac{Ze^2\mu_0 c}{4\mathrm{i}(2\pi)^2\hbar}\frac{\delta(k+K-k'-K')}{\sqrt{E_e(\boldsymbol{k}')E_e(\boldsymbol{k})E_N(\boldsymbol{K}')E_N(\boldsymbol{K})}}\frac{1}{(k-k')^2-\mathrm{i}\epsilon}$$

$$\times\Big(\overline{u}_{s'}(\boldsymbol{k}')\boldsymbol{\gamma}u_s(\boldsymbol{k})\cdot\overline{u}_{S'}(\boldsymbol{K}')\boldsymbol{\gamma}u_S(\boldsymbol{K})$$

$$+u_{s'}^+(\boldsymbol{k}')u_s(\boldsymbol{k})\frac{[E_e(\boldsymbol{k})-E_e(\boldsymbol{k}')][E_N(\boldsymbol{K})-E_N(\boldsymbol{K}')]}{\hbar^2c^2|\boldsymbol{k}-\boldsymbol{k}'|^2}u_{S'}^+(\boldsymbol{K}')u_S(\boldsymbol{K})\Big)$$

and energy conservation yields finally

$$S_{fi}^{(\gamma)} = \frac{Ze^2\mu_0 c}{4\mathrm{i}(2\pi)^2\hbar}\frac{\delta(k+K-k'-K')}{\sqrt{E_e(\boldsymbol{k}')E_e(\boldsymbol{k})E_N(\boldsymbol{K}')E_N(\boldsymbol{K})}}\frac{1}{(k-k')^2-\mathrm{i}\epsilon}$$

$$\times\Big(\overline{u}_{s'}(\boldsymbol{k}')\boldsymbol{\gamma}u_s(\boldsymbol{k})\cdot\overline{u}_{S'}(\boldsymbol{K}')\boldsymbol{\gamma}u_S(\boldsymbol{K})$$

$$-\overline{u}_{s'}(\boldsymbol{k}')\gamma^0u_s(\boldsymbol{k})\frac{(E_e(\boldsymbol{k})-E_e(\boldsymbol{k}'))^2}{\hbar^2c^2|\boldsymbol{k}-\boldsymbol{k}'|^2}\overline{u}_{S'}(\boldsymbol{K}')\gamma^0u_S(\boldsymbol{K})\Big). \qquad (22.19)$$

The sum $S_{fi} = S_{fi}^{(\gamma)} + S_{fi}^{(C)}$ contains a term

$$\frac{1}{|\boldsymbol{k}-\boldsymbol{k}'|^2}\left(1+\frac{(E_e(\boldsymbol{k})-E_e(\boldsymbol{k}'))^2}{\hbar^2c^2(k-k')^2-\mathrm{i}\epsilon}\right)=\frac{1}{(k-k')^2-\mathrm{i}\epsilon}.$$

This yields finally the scattering matrix element

$$\begin{aligned}S_{fi} &= -\mathrm{i}\mathcal{M}_{fi}\delta(k+K-k'-K')\\ &= \frac{Z\alpha}{4\pi\mathrm{i}}\frac{\overline{u}_{s'}(\boldsymbol{k}')\gamma^{\mu}u_s(\boldsymbol{k})\overline{u}_{S'}(\boldsymbol{K}')\gamma_{\mu}u_S(\boldsymbol{K})}{\sqrt{E_e(\boldsymbol{k}')E_e(\boldsymbol{k})E_N(\boldsymbol{K}')E_N(\boldsymbol{K})}}\frac{\delta(k+K-k'-K')}{(k-k')^2-\mathrm{i}\epsilon},\end{aligned} \quad (22.20)$$

where the fine structure constant $\alpha = \mu_0 ce^2/4\pi\hbar$ (7.61) was substituted.

The differential scattering cross section for electron-nucleus scattering is then given by (22.9)

$$v_{eN}d\sigma = 4\pi^2 c|\mathcal{M}_{fi}|^2\delta(k+K-k'-K')d^3\boldsymbol{K}'d^3\boldsymbol{k}'$$

with the relativistic expression for the relative velocity of the electron and the nucleus,

$$v_{eN} = \frac{c^3}{E_e(\boldsymbol{k})E_N(\boldsymbol{K})}\sqrt{\hbar^4(K\cdot k)^2 - m^2M^2c^4}, \quad (22.21)$$

where m is the electron mass. Integration over the momentum $\boldsymbol{K}'$ of the scattered nucleus and the magnitude k' of the scattered electron momentum yields

$$v_{eN}\frac{d\sigma}{d\Omega_{\boldsymbol{k}'}} = 4\pi^2 c|\mathcal{M}_{fi}|^2\left.\frac{k'^2}{|\partial f(\boldsymbol{k}')/\partial k'|}\right|_{f(\boldsymbol{k}')=0}, \quad (22.22)$$

where k' has to satisfy the condition

$$\begin{aligned}f(\boldsymbol{k}') = &\sqrt{(\boldsymbol{k}'-\boldsymbol{K}-\boldsymbol{k})^2+(Mc/\hbar)^2}+\sqrt{k'^2+(mc/\hbar)^2}\\ &-\sqrt{K^2+(Mc/\hbar)^2}-\sqrt{k^2+(mc/\hbar)^2}=0.\end{aligned} \quad (22.23)$$

Usually we are not interested in the scattering with fixed initial and final spin polarizations. Therefore we average over initial spins and sum over final spins to calculate the unpolarized scattering cross section,

$$|\mathcal{M}_{fi}|^2 \to \frac{1}{4}\sum_{s,s',S,S'}|\mathcal{M}_{fi}|^2. \quad (22.24)$$

We will further evaluate these expressions for the case of negligible momenta and momentum transfers compared to Mc. We can take this into account through the limit $M \to \infty$. The condition (22.23) then reduces to the elastic electron scattering condition $k = k'$ which yields

$$\frac{k'^2}{|\partial f(\boldsymbol{k}')/\partial k'|} = k\sqrt{k^2 + (mc/\hbar)^2}.$$

Furthermore, using the Dirac representation (21.36) of the γ matrices yields with equations (21.45, 21.46, 21.50) for the 4-spinors the limit

$$\lim_{M\to\infty} \frac{\overline{u}_{S'}(\boldsymbol{K}')\gamma^{\mu}u_S(\boldsymbol{K})}{\sqrt{E_N(\boldsymbol{K}')E_N(\boldsymbol{K})}} = 2\delta_{SS'}\eta^{\mu}{}_0. \tag{22.25}$$

Note that this equation is invariant under similarity transformations of the γ matrices and therefore holds in every representation. Furthermore, the speed v_{eN} (22.21) becomes the electron speed in the rest frame of the heavy nucleus, $v_{eN} = c^2\hbar k/E_e(\boldsymbol{k})$.

The spin polarized scattering amplitude following from (22.20) in the heavy nucleus limit is

$$\mathcal{M}_{fi} = -\frac{Z\alpha}{2\pi}\frac{\overline{u}_{s'}(\boldsymbol{k}')\gamma^0 u_s(\boldsymbol{k})}{\sqrt{E_e(\boldsymbol{k}')E_e(\boldsymbol{k})}}\frac{\delta_{SS'}}{(k-k')^2 - \mathrm{i}\epsilon}. \tag{22.26}$$

The remaining electron spin averaging is easily accomplished using equation (21.58),

$$\frac{1}{2}\sum_{ss'}\left|\overline{u}_{s'}(\boldsymbol{k}')\gamma^0 u_s(\boldsymbol{k})\right|^2 = \frac{c^2}{2}\mathrm{tr}\left[(mc - \hbar k\cdot\gamma)\gamma^0(mc - \hbar k'\cdot\gamma)\gamma^0\right].$$

The trace theorems for products of γ matrices in Appendix G, in particular equation (G.19) and equation (G.20) in the form

$$\mathrm{tr}\left(\gamma_\kappa\gamma^0\gamma_\mu\gamma^0\right) = 8\eta_\kappa{}^0\eta_\mu{}^0 + 4\eta_{\mu\kappa}, \tag{22.27}$$

and the vanishing traces of odd numbers of γ matrices yield

$$\begin{aligned}\frac{1}{2}\sum_{ss'}\left|\overline{u}_{s'}(\boldsymbol{k}')\gamma^0 u_s(\boldsymbol{k})\right|^2 &= 2m^2c^4 + 4\hbar^2c^2k^0k'^0 + 2\hbar^2c^2k\cdot k' \\ &= 2c^2(\hbar^2k^0k'^0 + \hbar^2\boldsymbol{k}\cdot\boldsymbol{k}' + m^2c^2).\end{aligned}$$

This yields the unpolarized differential scattering cross section for electrons in the field of a heavy nucleus,

$$\frac{d\sigma}{d\Omega_{\boldsymbol{k}'}} = \frac{Z^2\alpha^2}{2\hbar^2}\frac{2m^2c^2 + \hbar^2k^2(1+\cos\theta)}{k^4(1-\cos\theta)^2}, \tag{22.28}$$

where $k \equiv |\boldsymbol{k}|$. The Rutherford scattering formula (11.45) follows for $\hbar k \ll mc$. Electron scattering off a heavy nucleus is equivalent to scattering in an external Coulomb field $Ze/4\pi\epsilon_0 r$. This is known as *Mott scattering*[1]. From our calculation, we can easily understand why scattering off heavy nuclei is equivalent to scattering in an external Coulomb field. The Coulomb scattering matrix element (22.18) yields already the amplitude (22.26) in the heavy nucleus limit, while the photon exchange matrix element (22.19) vanishes in that limit. If we take into account that terms of the form $\overline{u}_{s'}(\boldsymbol{k}')\boldsymbol{\gamma}u_s(\boldsymbol{k})/\sqrt{E_e(\boldsymbol{k}')E_e(\boldsymbol{k})}$ are of order $\hbar^2|\boldsymbol{k}||\boldsymbol{k}'|/m^2c^2 \ll 1$ in the non-relativistic limit, we find that the ratio between the photon exchange amplitude and the Coulomb amplitude in the non-relativistic limit is of order

$$\left|\frac{S_{fi}^{(\gamma)}}{S_{fi}^{(C)}}\right| \simeq \frac{(E_e(\boldsymbol{k}) - E_e(\boldsymbol{k}'))^2}{\hbar^2 c^2 (k-k')^2} \simeq \frac{\hbar^2(\boldsymbol{k}^2 - \boldsymbol{k}'^2)^2}{4m^2c^2(\boldsymbol{k}-\boldsymbol{k}')^2}$$

$$= \frac{\boldsymbol{k}-\boldsymbol{k}'}{|\boldsymbol{k}-\boldsymbol{k}'|}\cdot\frac{\hbar^2(\boldsymbol{k}+\boldsymbol{k}')\otimes(\boldsymbol{k}+\boldsymbol{k}')}{4m^2c^2}\cdot\frac{\boldsymbol{k}-\boldsymbol{k}'}{|\boldsymbol{k}-\boldsymbol{k}'|} \ll 1, \qquad (22.29)$$

If we denote the average velocity of the incoming and the scattered electron with $\boldsymbol{v}_e$, equation (22.29) tells us that photon exchange is suppressed by about $\boldsymbol{p}_e^2/m^2c^2 = \boldsymbol{v}_e^2/c^2$ compared to the Coulomb interaction in the non-relativistic limit. That is the reason why Coulomb gauge is convenient for the description of systems with non-relativistic charged particles. We can use Coulomb potentials in the calculation of scattering events and bound states of the non-relativistic particles without worrying about photon exchange. The photon terms are only needed for photon absorption and emission, and for photon scattering. On the other hand, if we are primarily concerned with interactions of relativistic charged particles, then use of a Hamiltonian like (21.88) with covariantly gauged photons as in Section 21.6 is more efficient.

22.3 Photon scattering by free electrons

Photon scattering by free or quasifree electrons is also known as Compton scattering. The cross section for this process had been calculated in leading order by Klein and Nishina[2].

The electron-photon interaction term from (21.94) is

$$\mathcal{H}_{e\gamma} = ec\overline{\Psi}\boldsymbol{\gamma}\cdot\boldsymbol{A}\Psi. \qquad (22.30)$$

We denote the wave vectors of the incoming photon and electron with $\boldsymbol{q}$ and $\boldsymbol{k}$, respectively. The relevant second order matrix element for scattering of photons by

[1]N.F. Mott, Proc. Roy. Soc. London A 124, 425 (1929).

[2]O. Klein, Y. Nishina, Z. Phys. 52, 853 (1929).

free electrons is

$$\begin{aligned}
S_{fi} &= S_{\boldsymbol{k}',s';\boldsymbol{q}',\alpha'|\boldsymbol{k},s;\boldsymbol{q},\alpha} \\
&= -\frac{e^2c^2}{\hbar^2}\int d^3\boldsymbol{x}\int d^3\boldsymbol{x}'\int_{-\infty}^{\infty}dt\int_{-\infty}^{t}dt'\,\langle\boldsymbol{k}',s';\boldsymbol{q}',\alpha'|\exp\!\left(\frac{\mathrm{i}}{\hbar}H_0t\right) \\
&\quad\times\overline{\Psi}(\boldsymbol{x})\boldsymbol{\gamma}\cdot\boldsymbol{A}(\boldsymbol{x})\Psi(\boldsymbol{x})\exp\!\left(-\frac{\mathrm{i}}{\hbar}H_0(t-t')\right)\overline{\Psi}(\boldsymbol{x}')\boldsymbol{\gamma}\cdot\boldsymbol{A}(\boldsymbol{x}')\Psi(\boldsymbol{x}') \\
&\quad\times\exp\!\left(-\frac{\mathrm{i}}{\hbar}H_0t'\right)|\boldsymbol{k},s;\boldsymbol{q},\alpha\rangle \\
&= -\frac{e^2}{\hbar^2}\int d^4x\int d^4x'\,\Theta(t-t')\langle\boldsymbol{k}',s';\boldsymbol{q}',\alpha'|\overline{\Psi}(x)\boldsymbol{\gamma}\cdot\boldsymbol{A}(x)\Psi(x) \\
&\quad\times\overline{\Psi}(x')\boldsymbol{\gamma}\cdot\boldsymbol{A}(x')\Psi(x')|\boldsymbol{k},s;\boldsymbol{q},\alpha\rangle.
\end{aligned}$$

Here $\boldsymbol{A}(x)\equiv\boldsymbol{A}_D(\boldsymbol{x},t)$ and $\Psi(x)\equiv\Psi_D(\boldsymbol{x},t)$ are the freely evolving field operators (18.21, 21.51) in the interaction picture.

We can insert a decomposition of unity between the two vertex operators $\overline{\Psi}\boldsymbol{\gamma}\cdot\boldsymbol{A}\Psi$ with a fermionic and a photon factor,

$$1 = 1_f\otimes 1_\gamma.$$

The relevant parts in the photon factor have zero or two intermediate photons,

$$1_\gamma \Rightarrow |0\rangle\langle 0| + \frac{1}{2}\sum_{\beta,\beta'}\int d^3\boldsymbol{K}\int d^3\boldsymbol{K}'\,|\boldsymbol{K},\beta;\boldsymbol{K}',\beta'\rangle\langle\boldsymbol{K},\beta;\boldsymbol{K}',\beta'|,$$

while for the intermediate fermion states only states with one intermediate electron or with two intermediate electrons and a positron contribute,

$$\begin{aligned}
1_f \Rightarrow &\sum_\sigma\int d^3\boldsymbol{\kappa}\,b_\sigma^+(\boldsymbol{\kappa})|0\rangle\langle 0|b_\sigma(\boldsymbol{\kappa}) \\
&+\frac{1}{2}\sum_{\sigma,\sigma',\nu}\int d^3\boldsymbol{\kappa}\int d^3\boldsymbol{\kappa}'\int d^3\boldsymbol{\lambda}\,b_\sigma^+(\boldsymbol{\kappa})b_{\sigma'}^+(\boldsymbol{\kappa}')d_\nu^+(\boldsymbol{\lambda})|0\rangle \\
&\times\langle 0|d_\nu(\boldsymbol{\lambda})b_{\sigma'}(\boldsymbol{\kappa}')b_\sigma(\boldsymbol{\kappa}).
\end{aligned}$$

The full photon matrix element is

$$\begin{aligned}
\langle\boldsymbol{q}',\alpha'|\boldsymbol{A}(x)\otimes\boldsymbol{A}(x')|\boldsymbol{q},\alpha\rangle &= \frac{\hbar\mu_0c}{16\pi^3\sqrt{|\boldsymbol{q}||\boldsymbol{q}'|}}\Big(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\otimes\boldsymbol{\epsilon}_\alpha(\boldsymbol{q}) \\
&\times\exp\big[\mathrm{i}(q\cdot x'-q'\cdot x)\big]+\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\otimes\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\exp\big[\mathrm{i}(q\cdot x-q'\cdot x')\big]\Big),
\end{aligned}$$

where the first term arises from the term without intermediate photons and the second term arises from the term with two intermediate photons after integrating the intermediate photon momenta $\boldsymbol{K}$ and $\boldsymbol{K}'$.

Evaluation of the fermion matrix element with an electron in the intermediate state yields

$$\sum_\sigma \langle \boldsymbol{k}', s'|\overline{\Psi}(x)\boldsymbol{\gamma}\Psi(x)b^+_\sigma(\boldsymbol{\kappa})|0\rangle \otimes \langle 0|b_\sigma(\boldsymbol{\kappa})\overline{\Psi}(x')\boldsymbol{\gamma}\Psi(x')|\boldsymbol{k}, s\rangle$$
$$= \sum_\sigma \frac{\exp[\mathrm{i}\kappa\cdot(x-x') - \mathrm{i}k'\cdot x + \mathrm{i}k\cdot x']}{(2\pi)^6 4E(\boldsymbol{\kappa})\sqrt{E(\boldsymbol{k}')E(\boldsymbol{k})}}\overline{u}_{s'}(\boldsymbol{k}')\boldsymbol{\gamma}u_\sigma(\boldsymbol{\kappa}) \otimes \overline{u}_\sigma(\boldsymbol{\kappa})\boldsymbol{\gamma}u_s(\boldsymbol{k}).$$

We can substitute the sum over intermediate u spinors using equation (21.58),

$$\sum_\sigma \langle \boldsymbol{k}', s'|\overline{\Psi}(x)\boldsymbol{\gamma}\Psi(x)b^+_\sigma(\boldsymbol{\kappa})|0\rangle \otimes \langle 0|b_\sigma(\boldsymbol{\kappa})\overline{\Psi}(x')\boldsymbol{\gamma}\Psi(x')|\boldsymbol{k}, s\rangle$$
$$= \frac{\exp[\mathrm{i}\kappa\cdot(x-x') - \mathrm{i}k'\cdot x + \mathrm{i}k\cdot x']}{(2\pi)^6 4E(\boldsymbol{\kappa})\sqrt{E(\boldsymbol{k}')E(\boldsymbol{k})}}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$$
$$\times \overline{u}_{s'}(\boldsymbol{k}')\gamma^i\left(mc^2 - \hbar c\boldsymbol{\gamma}\cdot\boldsymbol{\kappa} - \gamma^0 E(\boldsymbol{\kappa})\right)\gamma^j u_s(\boldsymbol{k}).$$

Assembling the pieces so far then yields the amplitude with a single intermediate fermion,

$$S^{(1)}_{fi} = -\int \frac{d^3\boldsymbol{\kappa}}{E(\boldsymbol{\kappa})}\int d^4x \int d^4x' \, \frac{e^2\Theta(t-t')}{8(2\pi)^9\epsilon_0\hbar c\sqrt{|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k})E(\boldsymbol{k}')}}$$
$$\times \overline{u}_{s'}(\boldsymbol{k}')\Big[\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\left(mc^2 - \hbar c\boldsymbol{\kappa}\cdot\boldsymbol{\gamma} - \gamma^0 E(\boldsymbol{\kappa})\right)\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)$$
$$\times \exp\left[\mathrm{i}(\kappa - k' - q')\cdot x + \mathrm{i}(k + q - \kappa)\cdot x'\right]$$
$$+ \left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\left(mc^2 - \hbar c\boldsymbol{\kappa}\cdot\boldsymbol{\gamma} - \gamma^0 E(\boldsymbol{\kappa})\right)\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)$$
$$\times \exp\left[\mathrm{i}(\kappa - k' + q)\cdot x + \mathrm{i}(k - q' - \kappa)\cdot x'\right]\Big]u_s(\boldsymbol{k}). \tag{22.31}$$

The fermion matrix element with three intermediate fermions is

$$\frac{1}{2}\sum_{\sigma,\sigma',\nu} \langle \boldsymbol{k}', s'|\overline{\Psi}(x)\boldsymbol{\gamma}\Psi(x)b^+_\sigma(\boldsymbol{\kappa})b^+_{\sigma'}(\boldsymbol{\kappa}')d^+_\nu(\boldsymbol{\lambda})|0\rangle$$
$$\otimes \langle 0|d_\nu(\boldsymbol{\lambda})b_{\sigma'}(\boldsymbol{\kappa}')b_\sigma(\boldsymbol{\kappa})\overline{\Psi}(x')\boldsymbol{\gamma}\Psi(x')|\boldsymbol{k}, s\rangle$$
$$= -\sum_\nu \delta(\boldsymbol{\kappa}' - \boldsymbol{k}')\delta(\boldsymbol{\kappa} - \boldsymbol{k})\frac{\exp[\mathrm{i}\lambda\cdot(x-x') + \mathrm{i}k\cdot x - \mathrm{i}k'\cdot x']}{(2\pi)^6 4E(\boldsymbol{\lambda})\sqrt{E(\boldsymbol{k}')E(\boldsymbol{k})}}$$
$$\times \overline{v}_\nu(\boldsymbol{\lambda})\boldsymbol{\gamma}u_s(\boldsymbol{k}) \otimes \overline{u}_{s'}(\boldsymbol{k}')\boldsymbol{\gamma}v_\nu(\boldsymbol{\lambda}).$$

The last line has been simplified from an expression which is symmetric in the intermediate momenta $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ by taking into account that those momenta will be integrated.

We can substitute the sum over intermediate v spinors using equation (21.60),

$$\begin{aligned}
&\frac{1}{2}\sum_{\sigma,\sigma',\nu}\langle \boldsymbol{k}',s'|\overline{\Psi}(x)\boldsymbol{\gamma}\Psi(x)b^+_\sigma(\boldsymbol{\kappa})b^+_{\sigma'}(\boldsymbol{\kappa}')d^+_\nu(\boldsymbol{\lambda})|0\rangle\\
&\otimes\langle 0|d_\nu(\boldsymbol{\lambda})b_{\sigma'}(\boldsymbol{\kappa}')b_\sigma(\boldsymbol{\kappa})\overline{\Psi}(x')\boldsymbol{\gamma}\Psi(x')|\boldsymbol{k},s\rangle\\
&=\delta(\boldsymbol{\kappa}'-\boldsymbol{k}')\delta(\boldsymbol{\kappa}-\boldsymbol{k})\frac{\exp[\mathrm{i}\lambda\cdot(x-x')+\mathrm{i}k\cdot x-\mathrm{i}k'\cdot x']}{(2\pi)^6 4E(\boldsymbol{\lambda})\sqrt{E(\boldsymbol{k}')E(\boldsymbol{k})}}\\
&\times\boldsymbol{e}_i\otimes\boldsymbol{e}_j\overline{u}_{s'}(\boldsymbol{k}')\gamma^j\left(mc^2+\hbar c\boldsymbol{\gamma}\cdot\boldsymbol{\lambda}+\gamma^0E(\boldsymbol{\lambda})\right)\gamma^i u_s(\boldsymbol{k}).
\end{aligned}$$

If we substitute $\boldsymbol{\lambda}\to\boldsymbol{\kappa}$ (after integration over the intermediate electron momenta) for the wave vector of the intermediate positron, the contribution from three intermediate fermions to the scattering matrix element is

$$\begin{aligned}
S^{(3)}_{fi}&=-\int\frac{d^3\boldsymbol{\kappa}}{E(\boldsymbol{\kappa})}\int d^4x\int d^4x'\,\frac{e^2\Theta(t-t')}{8(2\pi)^9\epsilon_0\hbar c\sqrt{|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k})E(\boldsymbol{k}')}}\\
&\times\overline{u}_{s'}(\boldsymbol{k}')\Big[\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\left(mc^2+\hbar c\boldsymbol{\kappa}\cdot\boldsymbol{\gamma}+\gamma^0E(\boldsymbol{\kappa})\right)\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\\
&\times\exp\left[\mathrm{i}(\kappa+k+q)\cdot x-\mathrm{i}(\kappa+k'+q')\cdot x'\right]\\
&+\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\left(mc^2+\hbar c\boldsymbol{\kappa}\cdot\boldsymbol{\gamma}+\gamma^0E(\boldsymbol{\kappa})\right)\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\\
&\times\exp\left[\mathrm{i}(\kappa+k-q')\cdot x-\mathrm{i}(\kappa+k'-q)\cdot x'\right]\Big]u_s(\boldsymbol{k}).
\end{aligned}\tag{22.32}$$

We can simplify $S_{fi}=S^{(1)}_{fi}+S^{(3)}_{fi}$ by swapping $x\leftrightarrow x'$ in $S^{(3)}_{fi}$ and taking into account that

$$\begin{aligned}
&\int d^4\kappa\,\frac{(mc^2-\hbar c\gamma\cdot\kappa)f(\boldsymbol{\kappa})}{\kappa^2+(m^2c^2/\hbar^2)-\mathrm{i}\epsilon}\exp(\mathrm{i}\kappa\cdot x)\\
&=-\int d^3\boldsymbol{\kappa}\int d\kappa^0\,\frac{\exp(\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{x})(mc^2-\hbar c\gamma\cdot\kappa)f(\boldsymbol{\kappa})\exp(-\mathrm{i}\kappa^0ct)}{[\kappa^0-(\omega(\boldsymbol{\kappa})/c)+\mathrm{i}\epsilon][\kappa^0+(\omega(\boldsymbol{\kappa})/c)-\mathrm{i}\epsilon]}\\
&=2\pi\mathrm{i}c\Theta(t)\int d^3\boldsymbol{\kappa}\,\exp(\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{x})\frac{mc^2-\hbar c\boldsymbol{\kappa}\cdot\boldsymbol{\gamma}+\gamma^0E(\boldsymbol{\kappa})}{2\omega(\boldsymbol{\kappa})}f(\boldsymbol{\kappa})\exp(-\mathrm{i}\omega(\boldsymbol{\kappa})t)\\
&-2\pi\mathrm{i}c\Theta(-t)\int d^3\boldsymbol{\kappa}\,\exp(\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{x})\frac{mc^2-\hbar c\boldsymbol{\kappa}\cdot\boldsymbol{\gamma}-\gamma^0E(\boldsymbol{\kappa})}{-2\omega(\boldsymbol{\kappa})}f(\boldsymbol{\kappa})\exp(\mathrm{i}\omega(\boldsymbol{\kappa})t)\\
&=\mathrm{i}\pi c\Theta(t)\int\frac{d^3\boldsymbol{\kappa}}{\omega(\boldsymbol{\kappa})}(mc^2-\hbar c\gamma\cdot\kappa)f(\boldsymbol{\kappa})\exp(\mathrm{i}\kappa\cdot x)\Big|_{\kappa^0=\omega(\boldsymbol{\kappa})/c}\\
&+\mathrm{i}\pi c\Theta(-t)\int\frac{d^3\boldsymbol{\kappa}}{\omega(\boldsymbol{\kappa})}(mc^2+\hbar c\gamma\cdot\kappa)f(\boldsymbol{\kappa})\exp(-\mathrm{i}\kappa\cdot x)\Big|_{\kappa^0=\omega(\boldsymbol{\kappa})/c}.
\end{aligned}\tag{22.33}$$

This yields the total scattering matrix element in the form

$$
\begin{aligned}
S_{fi} = & \int d^4\kappa \int d^4x \int d^4x' \, \frac{\mathrm{i}e^2}{4(2\pi)^{10}\epsilon_0\hbar^2 c\sqrt{|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k})E(\boldsymbol{k}')}} \\
& \times \overline{u}_{s'}(\boldsymbol{k}')\Big[\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right) \frac{mc - \hbar\gamma\cdot\kappa}{\kappa^2 + (m^2c^2/\hbar^2) - \mathrm{i}\epsilon} \left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right) \\
& \times \exp\left[\mathrm{i}(\kappa - k' - q')\cdot x + \mathrm{i}(k + q - \kappa)\cdot x'\right] \\
& + \left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right) \frac{mc - \hbar\gamma\cdot\kappa}{\kappa^2 + (m^2c^2/\hbar^2) - \mathrm{i}\epsilon} \left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right) \\
& \times \exp\left[\mathrm{i}(\kappa - k' + q)\cdot x + \mathrm{i}(k - q' - \kappa)\cdot x'\right]\Big] u_s(\boldsymbol{k}).
\end{aligned}
$$

After performing the trivial integrations, we find

$$
\begin{aligned}
S_{fi} = & \, \delta(k' + q' - k - q) \frac{\mathrm{i}e^2}{16\pi^2\epsilon_0\hbar^2 c\sqrt{|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k})E(\boldsymbol{k}')}} \\
& \times \overline{u}_{s'}(\boldsymbol{k}')\Big[\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right) \frac{mc - \hbar\gamma\cdot(k+q)}{(k+q)^2 + (m^2c^2/\hbar^2) - \mathrm{i}\epsilon} \left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right) \\
& + \left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right) \frac{mc - \hbar\gamma\cdot(k-q')}{(k-q')^2 + (m^2c^2/\hbar^2) - \mathrm{i}\epsilon} \left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\Big] u_s(\boldsymbol{k}). \quad (22.34)
\end{aligned}
$$

The first contribution to the amplitude corresponds to absorption of a photon with wave vector q followed by emission of a photon with wave vector q', see Figure 22.1, while the second contribution to the amplitude corresponds to emission of the photon with wave vector q' before absorption of the photon with wave vector q as shown in Figure 22.2.

The denominators in (22.34) can be simplified by noting that $k^2 + (m^2c^2/\hbar^2) = 0$, $q^2 = q'^2 = 0$, and

$$
k\cdot q = \boldsymbol{k}\cdot\boldsymbol{q} - |\boldsymbol{q}|\sqrt{\boldsymbol{k}^2 + (m^2c^2/\hbar^2)} < 0.
$$

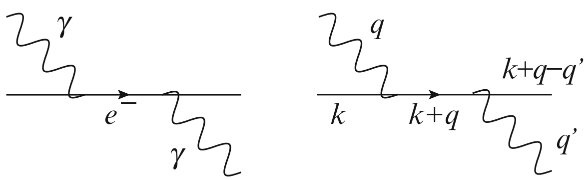

Fig. 22.1 Absorption of the incoming photon with momentum $\hbar\boldsymbol{q}$ before emission of the outgoing photon with momentum $\hbar\boldsymbol{q}'$. The virtual intermediate electron has 4-momentum $\hbar(k+q)$. The left panel uses particle labels and the right panel uses momentum labels

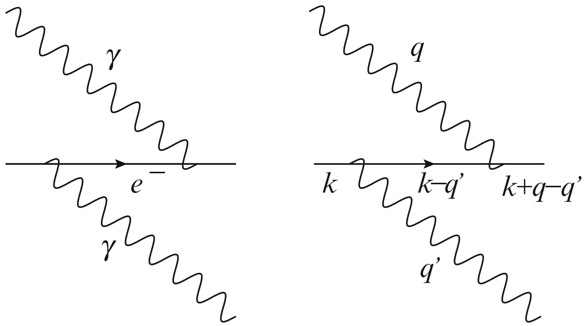

Fig. 22.2 Emission of the outgoing photon with momentum $\hbar \boldsymbol{q}'$ before absorption of the incoming photon with momentum $\hbar \boldsymbol{q}$. The virtual intermediate electron has 4-momentum $\hbar(k-q')$

This yields with the definition $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ (7.61) of Sommerfeld's fine structure constant the result

$$S_{fi} = \delta(k'+q'-k-q)\frac{\mathrm{i}\alpha}{8\pi\hbar\sqrt{|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k})E(\boldsymbol{k}')}}$$
$$\times\overline{u}_{s'}(\boldsymbol{k}')\Big[\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\frac{mc-\hbar\gamma\cdot(k+q)}{k\cdot q}\left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)$$
$$-\left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\frac{mc-\hbar\gamma\cdot(k-q')}{k\cdot q'}\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\Big]u_s(\boldsymbol{k}). \qquad (22.35)$$

The spin and helicity polarized differential scattering cross section then follows from (22.9),

$$v d\sigma_{\boldsymbol{k},s;\boldsymbol{q},\alpha\to\boldsymbol{k}',s';\boldsymbol{q}',\alpha'} = cd^3\boldsymbol{k}'d^3\boldsymbol{q}'\frac{\alpha^2\delta(k'+q'-k-q)}{16\hbar^2|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k})E(\boldsymbol{k}')}$$
$$\times\Big|\overline{u}_{s'}(\boldsymbol{k}')\Big[\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\frac{mc-\hbar\gamma\cdot(k+q)}{k\cdot q}\left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)$$
$$-\left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\frac{mc-\hbar\gamma\cdot(k-q')}{k\cdot q'}\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\Big]u_s(\boldsymbol{k})\Big|^2. \qquad (22.36)$$

Spin-polarized cross sections are usually of less physical interest than electron-photon cross sections which average over polarizations of initial electron states and sum over the polarizations of the final electron states[3],

$$d\sigma_{\boldsymbol{k};\boldsymbol{q},\alpha\to\boldsymbol{k}';\boldsymbol{q}',\alpha'} = \frac{1}{2}\sum_{s,s'}d\sigma_{\boldsymbol{k},s;\boldsymbol{q},\alpha\to\boldsymbol{k}',s';\boldsymbol{q}',\alpha'}.$$

[3]However, spin polarized cross sections for electron scattering will likely become important in the framework of spintronics and spin based quantum computing.

Use of the property (21.58)

$$\sum_s u_s(\boldsymbol{k})\overline{u}_s(\boldsymbol{k}) = mc^2 - \hbar c\gamma^\mu k_\mu\Big|_{k^0=\omega(\boldsymbol{k})/c}$$

and of the relations $(\overline{u})^+ = \gamma^0 u$, $\gamma_\mu^+ = \gamma^0\gamma_\mu\gamma^0$, yields

$$\begin{aligned}
v d\sigma_{\boldsymbol{k};\boldsymbol{q},\alpha\to\boldsymbol{k}';\boldsymbol{q}',\alpha'} &= d^3\boldsymbol{k}'d^3\boldsymbol{q}'\frac{\alpha^2c^3\delta(k'+q'-k-q)}{32\hbar^2|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k})E(\boldsymbol{k}')}\,\mathrm{tr}\Bigg[\left(mc-\hbar\gamma\cdot k'\right)\\
&\times\Bigg(\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\frac{mc-\hbar\gamma\cdot(k+q)}{k\cdot q}\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\\
&-\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\frac{mc-\hbar\gamma\cdot(k-q')}{k\cdot q'}\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\Bigg)\left(mc-\hbar\gamma\cdot k\right)\\
&\times\Bigg(\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\frac{mc-\hbar\gamma\cdot(k+q)}{k\cdot q}\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\\
&-\left(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\right)\frac{mc-\hbar\gamma\cdot(k-q')}{k\cdot q'}\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\Bigg)\Bigg].
\end{aligned}\tag{22.37}$$

This can be evaluated using the trace theorems for γ matrices from Appendix G. The full evaluation of $d\sigma_{\boldsymbol{k};\boldsymbol{q},\alpha\to\boldsymbol{k}';\boldsymbol{q}',\alpha'}$ needs in particular the trace theorems (G.20–G.22) for products of 4, 6 and 8 γ matrices.

We can simplify the evaluation in the rest frame of the initial electron,

$$k = \frac{1}{\hbar c}\begin{pmatrix}\sqrt{m^2c^4+\hbar^2\boldsymbol{k}^2}\\ \hbar c\boldsymbol{k}\end{pmatrix} \Rightarrow \frac{mc}{\hbar}\begin{pmatrix}1\\ \boldsymbol{0}\end{pmatrix}.$$

We can also use that $\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{q} = \epsilon_\alpha(\boldsymbol{q})\cdot q = 0$ implies

$$(\gamma\cdot q)\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right) = -\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)(\gamma\cdot q).$$

This reduces products according to

$$\begin{aligned}
&(mc-\hbar\gamma\cdot(k+q))\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)(mc-\hbar\gamma\cdot k)\\
&= \left(mc+mc\gamma^0-\hbar\gamma\cdot q\right)\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)mc\left(1+\gamma^0\right)\\
&= \left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\left(mc-mc\gamma^0+\hbar\gamma\cdot q\right)mc\left(1+\gamma^0\right)\\
&= m\hbar c\left(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)(\gamma\cdot q)\left(1+\gamma^0\right).
\end{aligned}$$

The resulting cross section in the rest frame of the electron before scattering is

$$
\begin{aligned}
d\sigma_{\mathbf{0};\boldsymbol{q},\alpha\to\boldsymbol{k}';\boldsymbol{q}',\alpha'} &= d^3\boldsymbol{k}'d^3\boldsymbol{q}'\frac{\alpha^2\hbar^2\delta(k'+q'-k-q)}{32m^2c|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k}')}\,\mathrm{tr}\Bigg[\big(mc-\hbar\gamma\cdot k'\big)\\
&\times\left(\big(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\big)\left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\frac{\gamma\cdot q}{|\boldsymbol{q}|}+\left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\big(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\big)\frac{\gamma\cdot q'}{|\boldsymbol{q}'|}\right)\big(1+\gamma^0\big)\\
&\times\left(\frac{\gamma\cdot q}{|\boldsymbol{q}|}\left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\big(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\big)+\frac{\gamma\cdot q'}{|\boldsymbol{q}'|}\big(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{\gamma}\big)\left(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\gamma}\right)\right)\Bigg].
\end{aligned}
\tag{22.38}
$$

Traces over products of an odd number of γ matrices vanish. The terms under the trace proportional to mc^2 contain products of six γ matrices, but two of these products vanish due to $(\gamma\cdot q)^2=-q^2=0$ and $(\gamma\cdot q')^2=-q'^2=0$. The remaining two terms involving six γ matrices turn out to yield the same result, such that the contribution to the trace term from products of six γ matrices is

$$
\begin{aligned}
tr_6 = \frac{8mc}{|\boldsymbol{q}||\boldsymbol{q}'|}\Big(&q\cdot q'-2(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))^2 q\cdot q'\\
&+2(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{q}')(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{q})\Big).
\end{aligned}
\tag{22.39}
$$

For the traces over products of eight γ matrices, we observe that those which contain the products $(\gamma\cdot q)\gamma^0(\gamma\cdot q)$ or $(\gamma\cdot q')\gamma^0(\gamma\cdot q')$ can be simplified to products of six γ matrices due to

$$
\gamma^\mu\gamma^0\gamma^\nu q_\mu q_\nu=\big(-2\eta^{\mu 0}\gamma^\nu-\gamma^0\gamma^\mu\gamma^\nu\big)\,q_\mu q_\nu=-2|\boldsymbol{q}|\gamma^\nu q_\nu.
$$

This yields for the sum of those terms which contain the products $(\gamma\cdot q)\gamma^0(\gamma\cdot q)$ or $(\gamma\cdot q')\gamma^0(\gamma\cdot q')$ the result

$$
\begin{aligned}
tr_{8a} &= \frac{8\hbar}{|\boldsymbol{q}|}\Big(2(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{k}')(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{q})-k'\cdot q\Big)\\
&\quad+\frac{8\hbar}{|\boldsymbol{q}'|}\Big(2(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{k}')(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{q}')-k'\cdot q'\Big),
\end{aligned}
$$

and after substitution of $k'=k+q-q'$,

$$
\begin{aligned}
tr_{8a} &= 16mc+\frac{8\hbar}{|\boldsymbol{q}|}\Big(q\cdot q'+2(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{q})^2\Big)\\
&\quad-\frac{8\hbar}{|\boldsymbol{q}'|}\Big(q\cdot q'+2(\boldsymbol{\epsilon}_{\alpha}(\boldsymbol{q})\cdot\boldsymbol{q}')^2\Big).
\end{aligned}
\tag{22.40}
$$

The traces over products of eight γ matrices which contain terms $(\gamma\cdot q)\gamma^0(\gamma\cdot q')$ or $(\gamma\cdot q')\gamma^0(\gamma\cdot q)$ can also be reduced to traces over products of six γ matrices by using the fact that γ^0 can only by contracted with one of the three γ matrices in products with 4-vectors. This yields after a bit of calculation and after substitution of $k' = k + q - q'$,

$$\begin{aligned} tr_{8b} = {} & 16mc\Big(2(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))^2 - 1\Big) - \frac{8mc}{|\boldsymbol{q}||\boldsymbol{q}'|}\Big(q\cdot q' \\ & - 2(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))^2 q\cdot q' + 2(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{q}')(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{q})\Big) \\ & - 16\frac{\hbar}{|\boldsymbol{q}|}(\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}')\cdot\boldsymbol{q})^2 + 16\frac{\hbar}{|\boldsymbol{q}'|}(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{q}')^2. \end{aligned} \tag{22.41}$$

The total trace term is therefore

$$\begin{aligned} tr &= tr_6 + tr_{8a} + tr_{8b} \\ &= 32mc(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))^2 + 8\hbar\left(\frac{1}{|\boldsymbol{q}|} - \frac{1}{|\boldsymbol{q}'|}\right) q\cdot q', \end{aligned}$$

and combining all the terms yields

$$\begin{aligned} d\sigma_{\mathbf{0};\boldsymbol{q},\alpha\to\boldsymbol{k}';\boldsymbol{q}',\alpha'} = {} & d^3\boldsymbol{k}'d^3\boldsymbol{q}'\frac{\alpha^2\hbar^2\delta(k'+q'-k-q)}{4m^2c|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k}')} \\ & \times\left[4mc(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))^2 + \hbar\left(\frac{1}{|\boldsymbol{q}|} - \frac{1}{|\boldsymbol{q}'|}\right) q\cdot q'\right]. \end{aligned} \tag{22.42}$$

The product $q\cdot q'$ is directly related to the photon scattering angle,

$$q\cdot q' = -|\boldsymbol{q}||\boldsymbol{q}'|\,(1-\cos\theta)\,.$$

However, energy and momentum conservation also imply

$$q\cdot q' = -\frac{1}{2}(q-q')^2 = -\frac{1}{2}(k'-k)^2 = \frac{m^2c^2}{\hbar^2} - \frac{mc}{\hbar}k'^0 = \frac{mc}{\hbar}\left(|\boldsymbol{q}'| - |\boldsymbol{q}|\right).$$

The relation between scattering angle and scattered photon wave number is therefore

$$\cos\theta = 1 - \frac{mc}{\hbar}\left(\frac{1}{|\boldsymbol{q}'|} - \frac{1}{|\boldsymbol{q}|}\right), \quad |\boldsymbol{q}'| = \frac{mc|\boldsymbol{q}|}{mc + \hbar|\boldsymbol{q}|\,(1-\cos\theta)}. \tag{22.43}$$

This is of course nothing but the Compton relation (1.11) for the wavelength of the scattered photon in terms of the scattering angle,

$$\lambda' = \lambda + \frac{h}{mc}\,(1-\cos\theta)\,.$$

The four-dimensional δ function

$$\delta(k' + q' - k - q) = \delta\left(\sqrt{\boldsymbol{k}'^2 + (m^2c^2/\hbar^2)} + |\boldsymbol{q}'| - (mc/\hbar) - |\boldsymbol{q}|\right)$$
$$\times\delta(\boldsymbol{k}' + \boldsymbol{q}' - \boldsymbol{q})$$

reduces the six-dimensional final state measure $d^3\boldsymbol{k}'d^3\boldsymbol{q}'$ to the two-dimensional measure $d\Omega(\hat{\boldsymbol{q}}') \equiv d\Omega$ over direction of the scattered photon after integration over $d^3\boldsymbol{k}'$ and $d|\boldsymbol{q}'|$. We include already the factor $|\boldsymbol{q}'|E(\boldsymbol{k}')$ in the denominator in equation (22.42) in the calculation:

$$\int d^3\boldsymbol{k}' \int_0^\infty d|\boldsymbol{q}'|\,\frac{|\boldsymbol{q}'|}{E(\boldsymbol{k}')} f(|\boldsymbol{q}'|)\delta(k' + q' - k - q)$$
$$= \int_0^\infty d|\boldsymbol{q}'|\,\frac{|\boldsymbol{q}'|f(|\boldsymbol{q}'|)}{c\sqrt{\hbar^2|\boldsymbol{q}'|^2 + \hbar^2|\boldsymbol{q}|^2 - 2\hbar^2|\boldsymbol{q}'||\boldsymbol{q}|\cos\theta + m^2c^2}}$$
$$\times\delta\left(\sqrt{|\boldsymbol{q}'|^2 + |\boldsymbol{q}|^2 - 2|\boldsymbol{q}'||\boldsymbol{q}|\cos\theta + (m^2c^2/\hbar^2)} + |\boldsymbol{q}'| - (mc/\hbar) - |\boldsymbol{q}|\right)$$
$$= \frac{1}{c}\left[\frac{|\boldsymbol{q}'|f(|\boldsymbol{q}'|)}{mc + \hbar|\boldsymbol{q}|(1-\cos\theta)}\right]_{|\boldsymbol{q}'|=mc|\boldsymbol{q}|/[mc+\hbar|\boldsymbol{q}|(1-\cos\theta)]}$$
$$= \frac{m|\boldsymbol{q}|}{[mc + \hbar|\boldsymbol{q}|\,(1-\cos\theta)]^2} f\left(\frac{mc|\boldsymbol{q}|}{mc + \hbar|\boldsymbol{q}|\,(1-\cos\theta)}\right).$$

This yields the Klein-Nishina cross section

$$d\sigma_{\boldsymbol{0};\boldsymbol{q},\alpha\to\boldsymbol{q}-\boldsymbol{q}';\boldsymbol{q}',\alpha'} = d\Omega\frac{\alpha^2\hbar^2}{4mc\,[mc + \hbar|\boldsymbol{q}|\,(1-\cos\theta)]^2}$$
$$\times\left(4mc(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))^2 + \frac{\hbar^2|\boldsymbol{q}|^2\,(1-\cos\theta)^2}{mc + \hbar|\boldsymbol{q}|\,(1-\cos\theta)}\right). \tag{22.44}$$

Averaging over the initial photon polarization and summing over the final polarization (18.119) yields the unpolarized differential cross section

$$d\sigma_{\boldsymbol{0};\boldsymbol{q}\to\boldsymbol{q}-\boldsymbol{q}';\boldsymbol{q}'} = \frac{1}{2}\sum_{\alpha,\alpha'} d\sigma_{\boldsymbol{0};\boldsymbol{q},\alpha\to\boldsymbol{q}-\boldsymbol{q}';\boldsymbol{q}',\alpha'} = d\Omega\frac{\alpha^2\hbar^2}{2mc\,[mc + \hbar|\boldsymbol{q}|\,(1-\cos\theta)]^2}$$
$$\times\left(mc(1+\cos^2\theta) + \frac{\hbar^2|\boldsymbol{q}|^2\,(1-\cos\theta)^2}{mc+\hbar|\boldsymbol{q}|\,(1-\cos\theta)}\right). \tag{22.45}$$

The resulting total cross section is

$$\sigma_{\mathbf{0};\boldsymbol{q}\to\boldsymbol{q}-\boldsymbol{q}';\boldsymbol{q}'} = \frac{\pi\alpha^2}{mc\hbar|\boldsymbol{q}|^3}\left[2\hbar|\boldsymbol{q}|\frac{2(mc)^3+8(mc)^2\hbar|\boldsymbol{q}|+9mc(\hbar|\boldsymbol{q}|)^2+(\hbar|\boldsymbol{q}|)^3}{(mc+2\hbar|\boldsymbol{q}|)^2}\right.$$
$$\left.-\left[2(mc)^2+2mc\hbar|\boldsymbol{q}|-(\hbar|\boldsymbol{q}|)^2\right]\ln\left(1+\frac{2\hbar|\boldsymbol{q}|}{mc}\right)\right]. \tag{22.46}$$

Photons below the hard X-ray regime satisfy $\hbar|\boldsymbol{q}| \ll mc$. This limit is also often denoted as the non-relativistic limit of Compton scattering because the kinetic energy imparted on the recoiling electron is small in this case,

$$\hbar^2(\boldsymbol{q}-\boldsymbol{q}')^2 \simeq 2\hbar^2\boldsymbol{q}^2\,(1-\cos\theta) \ll m^2c^2.$$

The cross section in the non-relativistic limit yields the Thomson cross section (18.120, 18.121) for photon scattering,

$$\frac{d\sigma_{\mathbf{0};\boldsymbol{q},\alpha\to\boldsymbol{q}-\boldsymbol{q}';\boldsymbol{q}',\alpha'}}{d\Omega} = \left(\frac{\alpha\hbar}{mc}\right)^2(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))^2,$$

$$\frac{d\sigma_{\mathbf{0};\boldsymbol{q}\to\boldsymbol{q}-\boldsymbol{q}';\boldsymbol{q}'}}{d\Omega} = \left(\frac{\alpha\hbar}{mc}\right)^2\frac{1+\cos^2\theta}{2},$$

$$\sigma_{\mathbf{0};\boldsymbol{q}\to\boldsymbol{q}-\boldsymbol{q}';\boldsymbol{q}'} = \frac{8\pi}{3}\left(\frac{\alpha\hbar}{mc}\right)^2 \equiv \sigma_T = 6.652\times10^{-9}\,\text{Å}^2 = 0.6652\,\text{barn}. \tag{22.47}$$

The unpolarized differential scattering cross section (22.45) for Compton scattering is displayed for various photon energies in Figure 22.3. Forward scattering is energy independent, but scattering in other directions is suppressed with energy.

The energy dependence of the total Compton scattering cross section (22.46) is displayed in Figure 22.4.

22.4 Møller scattering

The leading order scattering cross section for electron-electron scattering was calculated in the framework of quantum electrodynamics by C. Møller[4].

The Hamiltonian (21.94) in Coulomb gauge $\nabla\cdot\boldsymbol{A}=0$ for the photon field is

$$H = H_0 + H_I + H_C, \tag{22.48}$$

[4]C. Møller, Annalen Phys. 406, 531 (1932).

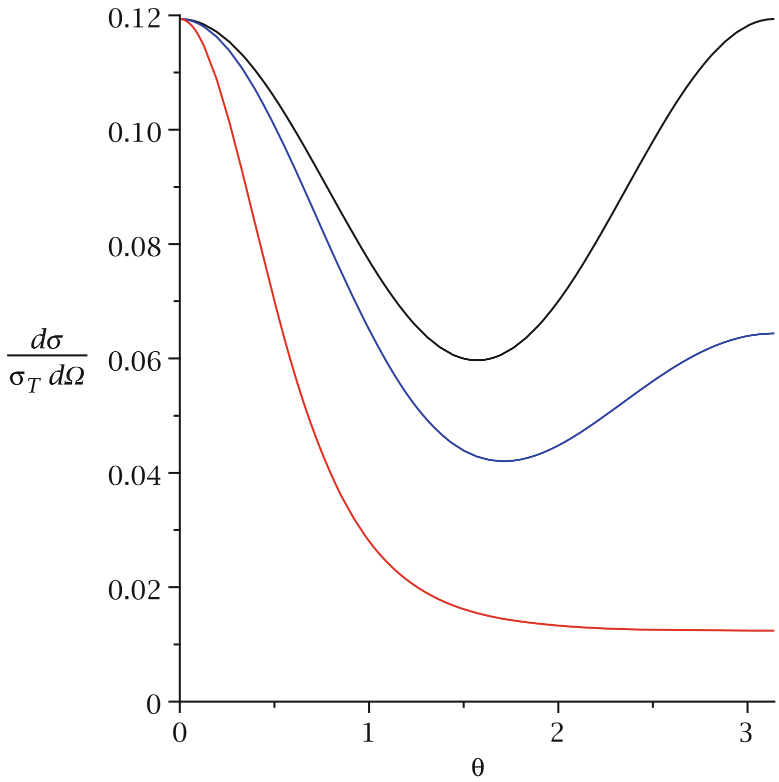

Fig. 22.3 The differential scattering cross section (22.45) for scattering angle $0 \le \theta \le \pi$. The energy of the incident photon is $E_\gamma = 0$ (top black curve), $E_\gamma = 0.2mc^2$ (center blue curve) and $E_\gamma = 2mc^2$ (lower red curve)

with the electron-photon interaction term

$$H_I \equiv H_{e\gamma} = ec \int d^3\boldsymbol{x}\, \overline{\Psi}(\boldsymbol{x}, t)\boldsymbol{\gamma} \cdot \boldsymbol{A}(\boldsymbol{x}, t)\Psi(\boldsymbol{x}, t)$$

and the Coulomb interaction term

$$H_C = \frac{e^2}{8\pi\epsilon_0} \sum_{ss'} \int d^3\boldsymbol{x} \int d^3\boldsymbol{x}'\, \Psi_s^+(\boldsymbol{x}, t)\Psi_{s'}^+(\boldsymbol{x}', t)\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}\Psi_{s'}(\boldsymbol{x}', t)\Psi_s(\boldsymbol{x}, t).$$

Note that the summation is over Dirac indices, which are related to spin projections through the corresponding u or v spinors.

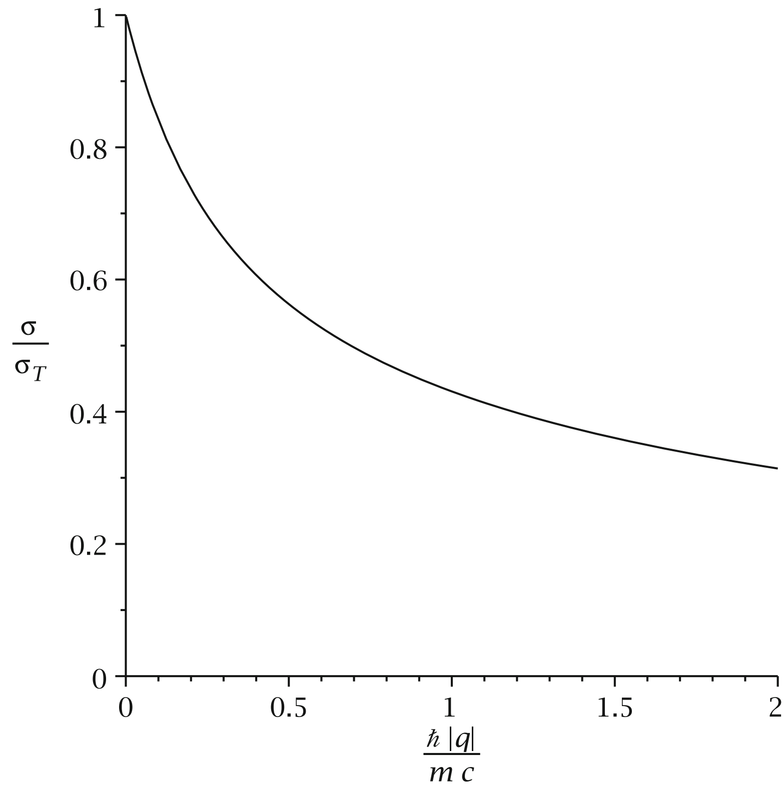

Fig. 22.4 The total Compton scattering cross section (22.46) in units of the Thomson cross section (22.47) for incident photon energy $0 < E_\gamma < 2mc^2$

The corresponding Hamiltonian on the states in the interaction picture is

$$H_D(t) = ec\sum_s \int d^3\boldsymbol{x}\,\overline{\Psi}_s(\boldsymbol{x},t)\boldsymbol{\gamma}\cdot\boldsymbol{A}(\boldsymbol{x},t)\Psi_s(\boldsymbol{x},t)$$
$$+\frac{e^2}{8\pi\epsilon_0}\sum_{ss'}\int d^3\boldsymbol{x}\int d^3\boldsymbol{x}'\,\Psi_s^+(\boldsymbol{x},t)\Psi_{s'}^+(\boldsymbol{x}',t)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\Psi_{s'}(\boldsymbol{x}',t)\Psi_s(\boldsymbol{x},t)$$

with the freely evolving field operators $\boldsymbol{A}(\boldsymbol{x},t)$ (18.21) and $\Psi(\boldsymbol{x},t)$ (21.51) of the interaction picture.

The scattering matrix element for electron-electron scattering

$$S_{fi} \equiv S_{\boldsymbol{k}_1',s_1';\boldsymbol{k}_2',s_2'|\boldsymbol{k}_1,s_1;\boldsymbol{k}_2,s_2}$$
$$= \langle \boldsymbol{k}_1',s_1';\boldsymbol{k}_2',s_2'|\mathrm{T}\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{\infty}dt\,H_D(t)\right)|\boldsymbol{k}_1,s_1;\boldsymbol{k}_2,s_2\rangle$$

becomes in leading order $\mathcal{O}(e^2)$

$$S_{fi} = S_{fi}^{(\gamma)} + S_{fi}^{(C)},$$

with the photon contribution

$$S_{fi}^{(\gamma)} = -\frac{e^2}{\hbar^2}\int d^4x \int d^4x'\,\Theta(t-t')\langle \boldsymbol{k}_1', s_1'; \boldsymbol{k}_2', s_2'|\overline{\Psi}(x)\boldsymbol{\gamma}\cdot\boldsymbol{A}(x)\Psi(x)$$
$$\times\overline{\Psi}(x')\boldsymbol{\gamma}\cdot\boldsymbol{A}(x')\Psi(x')|\boldsymbol{k}_1, s_1; \boldsymbol{k}_2, s_2\rangle$$

and the Coulomb term

$$S_{fi}^{(C)} = \frac{\mu_0 e^2}{8\pi\mathrm{i}\hbar}\langle \boldsymbol{k}_1', s_1'; \boldsymbol{k}_2', s_2'|\int d^4x \int d^4x'\,\Psi^+(x)\Psi^+(x')$$
$$\times\frac{\delta(ct-ct')}{|\boldsymbol{x}-\boldsymbol{x}'|}\Psi(x')\Psi(x)|\boldsymbol{k}_1, s_1; \boldsymbol{k}_2, s_2\rangle.$$

We evaluate $S_{fi}^{(\gamma)}$ first. Substitution of the relevant parts of the mode expansions yields (here we also use summation convention for the helicity and spin polarization indices)

$$S_{fi}^{(\gamma)} = -\frac{e^2}{\hbar^2}\frac{\hbar\mu_0 c}{8(2\pi)^9}\int d^4x \int d^4x'\,\Theta(t-t')\int\frac{d^3\boldsymbol{q}_1'}{\sqrt{E(\boldsymbol{q}_1')}}\int\frac{d^3\boldsymbol{q}_2'}{\sqrt{E(\boldsymbol{q}_2')}}\int\frac{d^3\boldsymbol{q}'}{\sqrt{|\boldsymbol{q}'|}}$$
$$\times\int\frac{d^3\boldsymbol{q}_1}{\sqrt{E(\boldsymbol{q}_1)}}\int\frac{d^3\boldsymbol{q}_2}{\sqrt{E(\boldsymbol{q}_2)}}\int\frac{d^3\boldsymbol{q}}{\sqrt{|\boldsymbol{q}|}}\exp[\mathrm{i}(q'+q_2'-q_1')\cdot x - \mathrm{i}(q+q_1-q_2)\cdot x']$$
$$\times\overline{u}(\boldsymbol{q}_1',\sigma)\boldsymbol{\gamma}u(\boldsymbol{q}_2',\sigma')\cdot\boldsymbol{\epsilon}_\beta(\boldsymbol{q}')\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\overline{u}(\boldsymbol{q}_1,s)\boldsymbol{\gamma}u(\boldsymbol{q}_2,s')\langle 0|b(\boldsymbol{k}_1',s_1')b(\boldsymbol{k}_2',s_2')$$
$$\times b^+(\boldsymbol{q}_1',\sigma)b(\boldsymbol{q}_2',\sigma')a_\beta(\boldsymbol{q}')a_\alpha^+(\boldsymbol{q})b^+(\boldsymbol{q}_1,s)b(\boldsymbol{q}_2,s')b^+(\boldsymbol{k}_2,s_2)b^+(\boldsymbol{k}_1,s_1)|0\rangle.$$

Elimination of the photon operators yields

$$S_{fi}^{(\gamma)} = \frac{\mu_0 e^2 c}{8(2\pi)^9\hbar}\int d^4x \int d^4x'\,\Theta(t-t')\int\frac{d^3\boldsymbol{q}_1'}{\sqrt{E(\boldsymbol{q}_1')}}\int\frac{d^3\boldsymbol{q}_2'}{\sqrt{E(\boldsymbol{q}_2')}}\int\frac{d^3\boldsymbol{q}}{|\boldsymbol{q}|}$$
$$\times\int\frac{d^3\boldsymbol{q}_1}{\sqrt{E(\boldsymbol{q}_1)}}\int\frac{d^3\boldsymbol{q}_2}{\sqrt{E(\boldsymbol{q}_2)}}\exp[\mathrm{i}(q+q_2'-q_1')\cdot x - \mathrm{i}(q+q_1-q_2)\cdot x']$$
$$\times\overline{u}(\boldsymbol{q}_1',\sigma)\boldsymbol{\gamma}u(\boldsymbol{q}_2',\sigma')\cdot\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\overline{u}(\boldsymbol{q}_1,s)\boldsymbol{\gamma}u(\boldsymbol{q}_2,s')\langle 0|b(\boldsymbol{k}_1',s_1')b(\boldsymbol{k}_2',s_2')$$
$$\times b^+(\boldsymbol{q}_1',\sigma)b^+(\boldsymbol{q}_1,s)b(\boldsymbol{q}_2',\sigma')b(\boldsymbol{q}_2,s')b^+(\boldsymbol{k}_2,s_2)b^+(\boldsymbol{k}_1,s_1)|0\rangle,$$

where fermionic operators were also re-ordered such that only the connected amplitude contributes. Evaluation of the fermionic operators yields

$$S_{fi}^{(\gamma)} = \frac{e^2}{8\hbar}\frac{\mu_0 c}{(2\pi)^9}\int d^4x\int d^4x'\int\frac{d^3\boldsymbol{q}}{|\boldsymbol{q}|}\,\frac{\Theta(t-t')\exp[\mathrm{i}q\cdot(x-x')]}{\sqrt{E(\boldsymbol{k}_1')E(\boldsymbol{k}_2')E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}}$$
$$\times\Big[\overline{u}(\boldsymbol{k}_2',s_2')\boldsymbol{\gamma}u(\boldsymbol{k}_1,s_1)\cdot\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\overline{u}(\boldsymbol{k}_1',s_1')\boldsymbol{\gamma}u(\boldsymbol{k}_2,s_2)$$
$$\times\Big(\exp[\mathrm{i}(k_1-k_2')\cdot x-\mathrm{i}(k_1'-k_2)\cdot x'] + \exp[\mathrm{i}(k_2-k_1')\cdot x-\mathrm{i}(k_2'-k_1)\cdot x']\Big)$$
$$-\overline{u}(\boldsymbol{k}_1',s_1')\boldsymbol{\gamma}u(\boldsymbol{k}_1,s_1)\cdot\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\overline{u}(\boldsymbol{k}_2',s_2')\boldsymbol{\gamma}u(\boldsymbol{k}_2,s_2)$$
$$\times\Big(\exp[\mathrm{i}(k_1-k_1')\cdot x-\mathrm{i}(k_2'-k_2)\cdot x']$$
$$+\exp[\mathrm{i}(k_2-k_2')\cdot x-\mathrm{i}(k_1'-k_1)\cdot x']\Big)\Big].$$

This yields after changing the integration variables $x \leftrightarrow x'$ in the second and fourth term

$$S_{fi}^{(\gamma)} = \frac{e^2}{8\hbar}\frac{\mu_0 c}{(2\pi)^9}\frac{1}{\sqrt{E(\boldsymbol{k}_1')E(\boldsymbol{k}_2')E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}}\int d^4x\int d^4x'\int\frac{d^3\boldsymbol{q}}{|\boldsymbol{q}|}$$
$$\times\Big(\overline{u}(\boldsymbol{k}_2',s_2')\boldsymbol{\gamma}u(\boldsymbol{k}_1,s_1)\cdot\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\overline{u}(\boldsymbol{k}_1',s_1')\boldsymbol{\gamma}u(\boldsymbol{k}_2,s_2)$$
$$\times\exp[\mathrm{i}(k_1-k_2')\cdot x+\mathrm{i}(k_2-k_1')\cdot x']$$
$$-\overline{u}(\boldsymbol{k}_1',s_1')\boldsymbol{\gamma}u(\boldsymbol{k}_1,s_1)\cdot\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\overline{u}(\boldsymbol{k}_2',s_2')\boldsymbol{\gamma}u(\boldsymbol{k}_2,s_2)$$
$$\times\exp[\mathrm{i}(k_1-k_1')\cdot x+\mathrm{i}(k_2-k_2')\cdot x']\Big)$$
$$\times\Big(\Theta(t-t')\exp[\mathrm{i}q\cdot(x-x')]+\Theta(t'-t)\exp[\mathrm{i}q\cdot(x'-x)]\Big).$$

We can use the following equation with $\omega(\boldsymbol{q}) = c\sqrt{\boldsymbol{q}^2 + (m^2c^2/\hbar^2)}$

$$\int d^4q\,\frac{f(\boldsymbol{q})\exp(\mathrm{i}q\cdot x)}{q^2+(m^2c^2/\hbar^2)-\mathrm{i}\epsilon} = -\int d^3\boldsymbol{q}\int d\omega\,\frac{cf(\boldsymbol{q})\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x})\exp(-\mathrm{i}\omega t)}{[\omega-\omega(\boldsymbol{q})+\mathrm{i}\epsilon][\omega+\omega(\boldsymbol{q})-\mathrm{i}\epsilon]}$$
$$= 2\pi c\mathrm{i}\Theta(t)\int d^3\boldsymbol{q}f(\boldsymbol{q})\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x})\frac{\exp(-\mathrm{i}\omega(\boldsymbol{q})t)}{2\omega(\boldsymbol{q})}$$
$$-2\pi c\mathrm{i}\Theta(-t)\int d^3\boldsymbol{q}f(\boldsymbol{q})\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x})\frac{\exp(\mathrm{i}\omega(\boldsymbol{q})t)}{-2\omega(\boldsymbol{q})}$$
$$= \mathrm{i}\pi c\,\Theta(t)\int\frac{d^3\boldsymbol{q}}{\omega(\boldsymbol{q})}f(\boldsymbol{q})\exp(\mathrm{i}q\cdot x)\Big|_{\omega=\omega(\boldsymbol{q})}$$
$$+\mathrm{i}\pi c\,\Theta(-t)\int\frac{d^3\boldsymbol{q}}{\omega(\boldsymbol{q})}f(\boldsymbol{q})\exp(-\mathrm{i}q\cdot x)\Big|_{\omega=\omega(\boldsymbol{q})} \qquad (22.49)$$

to find

$$S_{fi}^{(\gamma)} = \frac{\mu_0 e^2 c}{4\mathrm{i}\hbar(2\pi)^{10}} \frac{1}{\sqrt{E(\boldsymbol{k}_1')E(\boldsymbol{k}_2')E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}} \int d^4x \int d^4x' \int \frac{d^4q}{q^2 - \mathrm{i}\epsilon}$$
$$\times \Big(\overline{u}(\boldsymbol{k}_2', s_2')\boldsymbol{\gamma} u(\boldsymbol{k}_1, s_1) \cdot \boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\boldsymbol{\epsilon}_\alpha(\boldsymbol{q}) \cdot \overline{u}(\boldsymbol{k}_1', s_1')\boldsymbol{\gamma} u(\boldsymbol{k}_2, s_2)$$
$$\times \exp[\mathrm{i}(k_1 - k_2' + q)\cdot x + \mathrm{i}(k_2 - k_1' - q)\cdot x']$$
$$- \overline{u}(\boldsymbol{k}_1', s_1')\boldsymbol{\gamma} u(\boldsymbol{k}_1, s_1) \cdot \boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\boldsymbol{\epsilon}_\alpha(\boldsymbol{q}) \cdot \overline{u}(\boldsymbol{k}_2', s_2')\boldsymbol{\gamma} u(\boldsymbol{k}_2, s_2)$$
$$\times \exp[\mathrm{i}(k_1 - k_1' + q)\cdot x + \mathrm{i}(k_2 - k_2' - q)\cdot x'] \Big).$$

The integrations then yield

$$S_{fi}^{(\gamma)} = \frac{\mu_0 e^2 c}{16\mathrm{i}\pi^2\hbar} \frac{\delta(k_1 + k_2 - k_1' - k_2')}{\sqrt{E(\boldsymbol{k}_1')E(\boldsymbol{k}_2')E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}}$$
$$\times \left(\frac{\overline{u}(\boldsymbol{k}_2', s_2')\boldsymbol{\gamma} u(\boldsymbol{k}_1, s_1) \cdot \boldsymbol{\epsilon}_\alpha(\boldsymbol{k}_2' - \boldsymbol{k}_1)\boldsymbol{\epsilon}_\alpha(\boldsymbol{k}_2' - \boldsymbol{k}_1) \cdot \overline{u}(\boldsymbol{k}_1', s_1')\boldsymbol{\gamma} u(\boldsymbol{k}_2, s_2)}{(k_2' - k_1)^2 - \mathrm{i}\epsilon} \right.$$
$$\left. - \frac{\overline{u}(\boldsymbol{k}_1', s_1')\boldsymbol{\gamma} u(\boldsymbol{k}_1, s_1) \cdot \boldsymbol{\epsilon}_\alpha(\boldsymbol{k}_1' - \boldsymbol{k}_1)\boldsymbol{\epsilon}_\alpha(\boldsymbol{k}_1' - \boldsymbol{k}_1) \cdot \overline{u}(\boldsymbol{k}_2', s_2')\boldsymbol{\gamma} u(\boldsymbol{k}_2, s_2)}{(k_1' - k_1)^2 - \mathrm{i}\epsilon} \right).$$

Taking into account the energy-momentum conserving δ function, the transversal projectors can e.g. be written as

$$\boldsymbol{\epsilon}_\alpha(\boldsymbol{k}_1' - \boldsymbol{k}_1) \otimes \boldsymbol{\epsilon}_\alpha(\boldsymbol{k}_1' - \boldsymbol{k}_1) = \underline{1} + \frac{(\boldsymbol{k}_1' - \boldsymbol{k}_1) \otimes (\boldsymbol{k}_2' - \boldsymbol{k}_2)}{(\boldsymbol{k}_1' - \boldsymbol{k}_1)^2}.$$

The Dirac equation implies

$$\overline{u}(\boldsymbol{k}', s')\boldsymbol{\gamma} \cdot (\boldsymbol{k}' - \boldsymbol{k}) u(\boldsymbol{k}, s) = \frac{E(\boldsymbol{k}') - E(\boldsymbol{k})}{\hbar c} \overline{u}(\boldsymbol{k}', s')\gamma^0 u(\boldsymbol{k}, s).$$

This yields the photon exchange contribution to the electron-electron scattering matrix element,

$$S_{fi}^{(\gamma)} = \frac{\mu_0 c e^2}{16\mathrm{i}\pi^2\hbar} \frac{\delta(k_1 + k_2 - k_1' - k_2')}{\sqrt{E(\boldsymbol{k}_1')E(\boldsymbol{k}_2')E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}}$$
$$\times \left(\frac{\overline{u}(\boldsymbol{k}_2', s_2')\boldsymbol{\gamma} u(\boldsymbol{k}_1, s_1) \cdot \overline{u}(\boldsymbol{k}_1', s_1')\boldsymbol{\gamma} u(\boldsymbol{k}_2, s_2)}{(k_2' - k_1)^2 - \mathrm{i}\epsilon} \right.$$
$$- \frac{\overline{u}(\boldsymbol{k}_1', s_1')\boldsymbol{\gamma} u(\boldsymbol{k}_1, s_1) \cdot \overline{u}(\boldsymbol{k}_2', s_2')\boldsymbol{\gamma} u(\boldsymbol{k}_2, s_2)}{(k_1' - k_1)^2 - \mathrm{i}\epsilon}$$

$$-\frac{\overline{u}(\boldsymbol{k}_2',s_2')\gamma^0 u(\boldsymbol{k}_1,s_1)\overline{u}(\boldsymbol{k}_1',s_1')\gamma^0 u(\boldsymbol{k}_2,s_2)}{(k_2'-k_1)^2-\mathrm{i}\epsilon}\frac{[E(\boldsymbol{k}_2')-E(\boldsymbol{k}_1)]^2}{\hbar^2c^2(\boldsymbol{k}_2'-\boldsymbol{k}_1)^2}$$
$$+\frac{\overline{u}(\boldsymbol{k}_1',s_1')\gamma^0 u(\boldsymbol{k}_1,s_1)\overline{u}(\boldsymbol{k}_2',s_2')\gamma^0 u(\boldsymbol{k}_2,s_2)}{(k_1'-k_1)^2-\mathrm{i}\epsilon}\frac{[E(\boldsymbol{k}_1')-E(\boldsymbol{k}_1)]^2}{\hbar^2c^2(\boldsymbol{k}_1'-\boldsymbol{k}_1)^2}\Bigg). \quad (22.50)$$

For the evaluation of the Coulomb term, substitution of the mode expansions and evaluation of the operators in $S_{fi}^{(C)}$ yields

$$S_{fi}^{(C)}=\frac{\mu_0 ce^2}{8\pi\mathrm{i}\hbar}\frac{2}{4(2\pi)^6\sqrt{E(\boldsymbol{k}_1')E(\boldsymbol{k}_2')E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}}\int d^4x\int d^4x'\,\frac{\delta(ct-ct')}{|\boldsymbol{x}-\boldsymbol{x}'|}$$
$$\times\Big(u^+(\boldsymbol{k}_1',s_1')u(\boldsymbol{k}_1,s_1)u^+(\boldsymbol{k}_2',s_2')u(\boldsymbol{k}_2,s_2)\exp\big[\mathrm{i}(k_2'-k_2)\cdot x'+\mathrm{i}(k_1'-k_1)\cdot x\big]$$
$$-u^+(\boldsymbol{k}_2',s_2')u(\boldsymbol{k}_1,s_1)u^+(\boldsymbol{k}_1',s_1')u(\boldsymbol{k}_2,s_2)\exp\big[\mathrm{i}(k_1'-k_2)\cdot x'+\mathrm{i}(k_2'-k_1)\cdot x\big]\Big)$$
$$=\frac{\mu_0 ce^2}{16\pi\mathrm{i}\hbar}\frac{1}{(2\pi)^6\sqrt{E(\boldsymbol{k}_1')E(\boldsymbol{k}_2')E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}}$$
$$\times\int d^4x\int d^3\boldsymbol{x}'\,\frac{\exp\big[\mathrm{i}(k_1'+k_2'-k_1-k_2)\cdot x\big]}{|\boldsymbol{x}-\boldsymbol{x}'|}$$
$$\times\Big(\overline{u}(\boldsymbol{k}_1',s_1')\gamma^0 u(\boldsymbol{k}_1,s_1)\overline{u}(\boldsymbol{k}_2',s_2')\gamma^0 u(\boldsymbol{k}_2,s_2)\exp\big[\mathrm{i}(\boldsymbol{k}_2'-\boldsymbol{k}_2)\cdot(\boldsymbol{x}'-\boldsymbol{x})\big]$$
$$-\overline{u}(\boldsymbol{k}_2',s_2')\gamma^0 u(\boldsymbol{k}_1,s_1)\overline{u}(\boldsymbol{k}_1',s_1')\gamma^0 u(\boldsymbol{k}_2,s_2)\exp\big[\mathrm{i}(\boldsymbol{k}_1'-\boldsymbol{k}_2)\cdot(\boldsymbol{x}'-\boldsymbol{x})\big]\Big).$$

In the next step we use the Fourier decomposition (22.17) of the Coulomb potential to find

$$S_{fi}^{(C)}=\frac{\mu_0 ce^2}{16\pi^2\mathrm{i}\hbar}\frac{\delta(k_1'+k_2'-k_1-k_2)}{\sqrt{E(\boldsymbol{k}_1')E(\boldsymbol{k}_2')E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}}$$
$$\times\Bigg(\frac{\overline{u}(\boldsymbol{k}_1',s_1')\gamma^0 u(\boldsymbol{k}_1,s_1)\overline{u}(\boldsymbol{k}_2',s_2')\gamma^0 u(\boldsymbol{k}_2,s_2)}{(\boldsymbol{k}_2'-\boldsymbol{k}_2)^2}$$
$$-\frac{\overline{u}(\boldsymbol{k}_2',s_2')\gamma^0 u(\boldsymbol{k}_1,s_1)\overline{u}(\boldsymbol{k}_1',s_1')\gamma^0 u(\boldsymbol{k}_2,s_2)}{(\boldsymbol{k}_2'-\boldsymbol{k}_1)^2}\Bigg). \quad (22.51)$$

For the addition of $S_{fi}^{(\gamma)}$ and $S_{fi}^{(C)}$, we observe

$$\frac{1}{(\boldsymbol{k}'-\boldsymbol{k})^2}\left(\frac{[\omega(\boldsymbol{k}')-\omega(\boldsymbol{k})]^2}{c^2(k'-k)^2}+1\right)=\frac{1}{(k'-k)^2} \quad (22.52)$$

to find

$$\begin{aligned}S_{fi} &= \mathrm{i}\frac{\mu_0 ce^2}{16\pi^2\hbar}\frac{\delta(k_1' + k_2' - k_1 - k_2)}{\sqrt{E(\boldsymbol{k}_1')E(\boldsymbol{k}_2')E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}} \\ &\quad\times\left(\frac{\overline{u}(\boldsymbol{k}_1', s_1')\gamma^\mu u(\boldsymbol{k}_1, s_1)\overline{u}(\boldsymbol{k}_2', s_2')\gamma_\mu u(\boldsymbol{k}_2, s_2)}{(k_1' - k_1)^2}\right. \\ &\quad\left. - \frac{\overline{u}(\boldsymbol{k}_2', s_2')\gamma^\mu u(\boldsymbol{k}_1, s_1)\overline{u}(\boldsymbol{k}_1', s_1')\gamma_\mu u(\boldsymbol{k}_2, s_2)}{(k_2' - k_1)^2}\right) \\ &= -\mathrm{i}\mathcal{M}_{fi}\delta(k_1 + k_2 - k_1' - k_2'),\end{aligned} \tag{22.53}$$

where the last equation defines the scattering amplitude

$$\mathcal{M}_{fi} \equiv \mathcal{M}_{\boldsymbol{k}_1',s_1';\boldsymbol{k}_2',s_2'|\boldsymbol{k}_1,s_1;\boldsymbol{k}_2,s_2}$$

for Møller scattering.

The two contributions to the scattering amplitude can be interpreted as virtual photon exchange with virtual photon 4-momentum $k_1 - k_1'$ or $k_1 - k_2'$, respectively. This is shown in Figure 22.5.

The scattering amplitude (22.53) yields the spin polarized differential cross section (22.10)

$$\begin{aligned}vd\sigma_{\boldsymbol{k}_1',s_1';\boldsymbol{k}_2',s_2'|\boldsymbol{k}_1,s_1;\boldsymbol{k}_2,s_2} &= 4\pi^2 c\left|\mathcal{M}_{\boldsymbol{k}_1',s_1';\boldsymbol{k}_2',s_2'|\boldsymbol{k}_1,s_1;\boldsymbol{k}_2,s_2}\right|^2 \\ &\quad\times\delta(k_1 + k_2 - k_1' - k_2')\frac{1}{2}d^3\boldsymbol{k}_1' d^3\boldsymbol{k}_2',\end{aligned}$$

where

$$v = \frac{c^3}{E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)}\sqrt{(\hbar^2 k_1\cdot k_2)^2 - m^4c^4} \tag{22.54}$$

is the relative speed (22.4) between the two electrons with momentum 4-vectors $\hbar k_1$ and $\hbar k_2$.

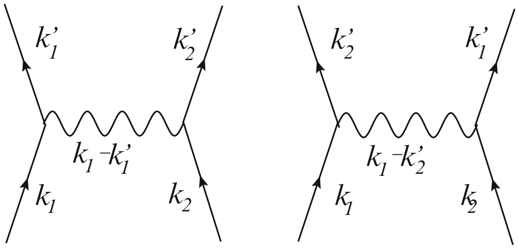

Fig. 22.5 Contributions to the Møller scattering amplitude (22.53)

The differential cross section is often averaged over initial spin states and summed over final spin states,

$$d\sigma_{\boldsymbol{k}'_1;\boldsymbol{k}'_2|\boldsymbol{k}_1;\boldsymbol{k}_2} = \frac{1}{4}\sum_{s_1,s_2,s'_1,s'_2} d\sigma_{\boldsymbol{k}'_1,s'_1;\boldsymbol{k}'_2,s'_2|\boldsymbol{k}_1,s_1;\boldsymbol{k}_2,s_2}.$$

The property (21.58)

$$\sum_s u(\boldsymbol{k},s)\overline{u}(\boldsymbol{k},s) = mc^2 - \hbar c\gamma^\mu k_\mu\Big|_{k^0=\omega(\boldsymbol{k})/c}$$

yields

$$\begin{aligned}
v d\sigma = {} & \frac{c}{2} d^3\boldsymbol{k}'_1 d^3\boldsymbol{k}'_2 \left(\frac{\mu_0 c e^2}{16\pi\hbar}\right)^2 \frac{\delta(k_1+k_2-k'_1-k'_2)}{E(\boldsymbol{k}'_1)E(\boldsymbol{k}'_2)E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)} \\
& \times\Bigg(\frac{1}{(k'_1-k_1)^4}\mathrm{tr}\big[\left(mc^2-\hbar c\gamma\cdot k'_1\right)\gamma^\mu\left(mc^2-\hbar c\gamma\cdot k_1\right)\gamma^\nu\big] \\
& \times\mathrm{tr}\big[\left(mc^2-\hbar c\gamma\cdot k'_2\right)\gamma_\mu\left(mc^2-\hbar c\gamma\cdot k_2\right)\gamma_\nu\big] \\
& + \frac{1}{(k'_2-k_1)^4}\mathrm{tr}\big[\left(mc^2-\hbar c\gamma\cdot k'_2\right)\gamma^\mu\left(mc^2-\hbar c\gamma\cdot k_1\right)\gamma^\nu\big] \\
& \times\mathrm{tr}\big[\left(mc^2-\hbar c\gamma\cdot k'_1\right)\gamma_\mu\left(mc^2-\hbar c\gamma\cdot k_2\right)\gamma_\nu\big] \\
& - \frac{2}{(k'_1-k_1)^2(k'_2-k_1)^2}\mathrm{tr}\big[\left(mc^2-\hbar c\gamma\cdot k_2\right)\gamma^\mu\left(mc^2-\hbar c\gamma\cdot k'_2\right) \\
& \times\gamma^\nu\left(mc^2-\hbar c\gamma\cdot k_1\right)\gamma_\mu\left(mc^2-\hbar c\gamma\cdot k'_1\right)\gamma_\nu\big]\Bigg),
\end{aligned} \tag{22.55}$$

where it is understood that all 4-momenta of electrons are on shell. The 4-momenta of the intermediate photons are then automatically off shell with dominant spacelike components, $(k'-k)^2 > 0$ (except in the zero momentum transfer limit $\boldsymbol{k}' = \boldsymbol{k}$).

The traces in equation (22.55) are readily evaluated using the contraction and trace theorems for γ matrices from Appendix G. This yields together with 4-momentum conservation $k'_1 + k'_2 = k_1 + k_2$ the result

$$\begin{aligned}
v d\sigma = {} & c d^3\boldsymbol{k}'_1 d^3\boldsymbol{k}'_2 \left(\frac{e^2 c}{4\pi\epsilon_0\hbar}\right)^2 \frac{\delta(k_1+k_2-k'_1-k'_2)}{E(\boldsymbol{k}'_1)E(\boldsymbol{k}'_2)E(\boldsymbol{k}_1)E(\boldsymbol{k}_2)} \\
& \times\Bigg(\frac{\hbar^4(k_1\cdot k_2)^2 + \hbar^4(k_1\cdot k'_2)^2 + 2m^2c^2\hbar^2 k_1\cdot k'_1 + 2m^4c^4}{(k'_1-k_1)^4}
\end{aligned}$$

$$+ \frac{\hbar^4(k_1 \cdot k_2)^2 + \hbar^4(k_1 \cdot k_1')^2 + 2m^2c^2\hbar^2 k_1 \cdot k_2' + 2m^4c^4}{(k_2' - k_1)^4}$$
$$+ 2\frac{\hbar^4(k_1 \cdot k_2)^2 + 2m^2c^2\hbar^2 k_1 \cdot k_2}{(k_1' - k_1)^2(k_2' - k_1)^2}\Bigg). \quad (22.56)$$

We further evaluate the cross section through integration over $d^3\boldsymbol{k}_2'$ and $d|\boldsymbol{k}_1'|$ in the center of mass frame $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{0}$ of the colliding electrons. If we integrate over the final states $d^3\boldsymbol{k}_2'$ of one of the electrons to get a single-electron differential cross section $d\sigma/d\Omega$, we have to include a factor of 2 because we could just as well observe the electron with momentum $\boldsymbol{k}_2'$ being scattered into the direction $d\Omega$, see equation (22.12).

It is convenient to define $\boldsymbol{k} = \boldsymbol{k}_1$, $\boldsymbol{k}' = \boldsymbol{k}_1'$. The integration with the energy-momentum δ function then yields

$$\int d^3\boldsymbol{k}_2' \int_0^\infty d|\boldsymbol{k}'|\, |\boldsymbol{k}'|^2 f(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}_2')\delta(\boldsymbol{k}_2' + \boldsymbol{k}')$$
$$\times\delta\left(2\sqrt{|\boldsymbol{k}'|^2 + (mc/\hbar)^2} - 2\sqrt{|\boldsymbol{k}|^2 + (mc/\hbar)^2}\right)$$
$$= \frac{|\boldsymbol{k}|}{2}\sqrt{|\boldsymbol{k}|^2 + (mc/\hbar)^2} f(\boldsymbol{k}, |\boldsymbol{k}|\hat{\boldsymbol{k}}', -|\boldsymbol{k}|\hat{\boldsymbol{k}}').$$

The scalar products in the center of mass frame are

$$k_1 \cdot k_2 = -2|\boldsymbol{k}|^2 - (mc/\hbar)^2,$$
$$k_1 \cdot k_1' = -|\boldsymbol{k}|^2(1 - \cos\theta) - (mc/\hbar)^2,$$
$$k_1 \cdot k_2' = -|\boldsymbol{k}|^2(1 + \cos\theta) - (mc/\hbar)^2,$$

where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. The relative speed (22.54) of the electrons in the center of mass frame is

$$v = \frac{2c\hbar|\boldsymbol{k}|}{\sqrt{\hbar^2|\boldsymbol{k}|^2 + m^2c^2}}.$$

The differential scattering cross section is then with the factor of 2 from equation (22.12), and using the fine structure constant $\alpha = e^2/(4\pi\epsilon_0\hbar c)$,

$$\frac{d\sigma}{d\Omega} = \alpha^2\frac{\hbar^4k^4(3 + \cos^2\theta)^2 + m^2c^2(4\hbar^2k^2 + m^2c^2)(1 + 3\cos^2\theta)}{4\hbar^2k^4(\hbar^2k^2 + m^2c^2)\sin^4\theta}. \quad (22.57)$$

This is symmetric under $\theta \to (\pi/2) - \theta$ with a minimum for scattering angle $\theta = \pi/2$ and divergences in forward and backward direction. This divergence in the zero momentum transfer limit is due to the vanishing photon mass, or in other words due to the infinite range of electromagnetic interactions. It is the same divergence which rendered the Rutherford cross section non-integrable.

Equation (22.57) looks fairly complicated, but in terms of energy it essentially entails that Møller scattering is suppressed with kinetic energy K like K^{-2}: The low energy result for $K \simeq \hbar^2 k^2/2m \ll mc^2$ is

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar c}{4K\sin^2\theta}\right)^2 (1 + 3\cos^2\theta),$$

and the high energy result for $K \simeq \hbar ck \gg mc^2$ is

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar c}{2K}\frac{3 + \cos^2\theta}{\sin^2\theta}\right)^2.$$

With respect to low-energy electron-electron scattering, we also note that the scattering matrix element (22.53) is dominated by the Coulomb contribution (22.51) if the electrons are non-relativistic, $\hbar|\boldsymbol{k}| \ll mc$, $\hbar|\boldsymbol{k}'| \ll mc$. The estimate (22.29) for the ratio of scattering amplitudes in the low energy limit applies here too, and this confirms again the domination of Coulomb interactions between non-relativistic charged particles.

22.5 Problems

22.1. Derive the relation (22.9) between the differential scattering cross section and the scattering amplitude in box normalization.

22.2. Calculate the differential scattering cross section for scattering of a relativistic charged scalar particle off a heavy nucleus. Assume that the heavy nucleus is non-relativistic and that the scalar particle cannot resolve its substructure, such that you can describe the nucleus with Schrödinger field operators.

22.3. Calculate the differential scattering cross section $d\sigma_{\boldsymbol{0};\boldsymbol{q},\alpha\to\boldsymbol{k}';\boldsymbol{q}'}/d\Omega$ for electron-photon scattering with polarized initial photons, i.e. sum over the polarizations of the scattered photons but do not average over the initial polarization.

22.4. Show that the differential cross sections (22.42, 22.44) for Compton scattering can also be written in the form

$$d\sigma_{\boldsymbol{0};\boldsymbol{q},\alpha\to\boldsymbol{k}';\boldsymbol{q}',\alpha'} = d^3\boldsymbol{k}' d^3\boldsymbol{q}' \frac{\alpha^2\hbar^2\delta(k' + q' - k - q)}{4m|\boldsymbol{q}||\boldsymbol{q}'|E(\boldsymbol{k}')} \times \left(4(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))^2 + \frac{|\boldsymbol{q}'|}{|\boldsymbol{q}|} + \frac{|\boldsymbol{q}|}{|\boldsymbol{q}'|} - 2\right), \quad (22.58)$$

$$\frac{d\sigma_{\boldsymbol{0};\boldsymbol{q},\alpha\to\boldsymbol{q}-\boldsymbol{q}';\boldsymbol{q}',\alpha'}}{d\Omega} = \left(\frac{\alpha\hbar|\boldsymbol{q}'|}{2mc|\boldsymbol{q}|}\right)^2 \left(4(\boldsymbol{\epsilon}_\alpha(\boldsymbol{q})\cdot\boldsymbol{\epsilon}_{\alpha'}(\boldsymbol{q}'))^2 + \frac{|\boldsymbol{q}'|}{|\boldsymbol{q}|} + \frac{|\boldsymbol{q}|}{|\boldsymbol{q}'|} - 2\right).$$

22.5. Calculate the kinetic energy imparted on the recoiling electron in Compton scattering as a function of $|\boldsymbol{q}|$ and θ.

22.6. Derive the scattering amplitude for electron-nucleus scattering using covariant quantization for the photon.

22.7. Derive the scattering amplitude for Compton scattering using covariant quantization for the photons.

22.8. Derive the scattering amplitude for Møller scattering using covariant quantization for the photon.