Chapter 6
Harmonic Oscillators and Coherent States

The harmonic oscillator is the general approximation for the dynamics of small fluctuations around a minimum of a potential. This is the reason why harmonic oscillators are very important model systems both in mechanics and in quantum mechanics. In addition there is another reason why we have to discuss the quantum harmonic oscillator in detail. For the discussion of quantum mechanical reactions between particles later on, we have to go beyond ordinary quantum mechanics and use a technique called second quantization or canonical quantum field theory. The techniques of second quantization are based on linear superpositions of infinitely many oscillators. Therefore it is important to have a very good understanding of oscillator eigenstates and of the calculational techniques involved with oscillation operators.

6.1 Basic aspects of harmonic oscillators

The classical motion of a particle in the three-dimensional isotropic potential

$$V(\boldsymbol{x}) = \frac{m}{2}\omega^2 \boldsymbol{x}^2$$

without external driving forces is described by the classical solution

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{X}\cos(\omega t) + \frac{\boldsymbol{P}}{m\omega}\sin(\omega t), \\ \boldsymbol{p}(t) &= \boldsymbol{P}\cos(\omega t) - m\omega \boldsymbol{X}\sin(\omega t), \end{aligned} \tag{6.1}$$

where $\boldsymbol{X} = \boldsymbol{x}(0)$, $\boldsymbol{P} = m\dot{\boldsymbol{x}}(0)$ are the values of location and momentum of the particle at time $t = 0$.

R. Dick, *Advanced Quantum Mechanics*, Graduate Texts in Physics,
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The corresponding Schrödinger equation is

$$\mathrm{i}\hbar\frac{d}{dt}|\psi(t)\rangle = \left(\frac{\mathbf{p}^2}{2m} + \frac{m}{2}\omega^2\mathbf{x}^2\right)|\psi(t)\rangle$$

or after substitution of energy-time Fourier transformation (5.12),

$$E|\psi(E)\rangle = \left(\frac{\mathbf{p}^2}{2m} + \frac{m}{2}\omega^2\mathbf{x}^2\right)|\psi(E)\rangle. \tag{6.2}$$

The corresponding differential equation in $\boldsymbol{x}$ representation

$$E\langle\boldsymbol{x}|\psi(E)\rangle = \left(-\frac{\hbar^2}{2m}\Delta + \frac{m}{2}\omega^2\boldsymbol{x}^2\right)\langle\boldsymbol{x}|\psi(E)\rangle \tag{6.3}$$

can be decomposed into three one-dimensional problems through separation of the spatial variables. The separation ansatz

$$\langle\boldsymbol{x}|\psi(E)\rangle = \prod_{i=1}^{3}\langle x_i|\psi_i(E_i)\rangle \tag{6.4}$$

yields

$$E = \sum_{i=1}^{3} E_i, \tag{6.5}$$

where the three energy values E_i and wave functions $\langle x_i|\psi(E_i)\rangle$ have to satisfy the one-dimensional equation

$$E\langle x|\psi(E)\rangle = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\langle x|\psi(E)\rangle + \frac{m}{2}\omega^2x^2\langle x|\psi(E)\rangle. \tag{6.6}$$

Indeed, the results of Section 5.5 imply that the solutions of the three-dimensional equation (6.3) will always have the separated form (6.4, 6.5).

6.2 Solution of the harmonic oscillator by the operator method

The one-dimensional oscillator equation (6.6) is in representation free notation

$$E|\psi(E)\rangle = \left(\frac{\mathrm{p}^2}{2m} + \frac{m}{2}\omega^2\mathrm{x}^2\right)|\psi(E)\rangle. \tag{6.7}$$

There exists a powerful and elegant method to solve equation (6.7) through a transformation from the self-adjoint operators x and p to mutually adjoint operators a and a^+. The substitutions

$$a = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\mathrm{x} + \mathrm{i}\frac{\mathrm{p}}{\sqrt{m\omega}}\right), \quad a^+ = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\mathrm{x} - \mathrm{i}\frac{\mathrm{p}}{\sqrt{m\omega}}\right), \tag{6.8}$$

yield the commutation relation

$$[a, a^+] = 1, \tag{6.9}$$

the inverse transformation

$$\mathrm{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^+\right), \quad \mathrm{p} = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^+\right), \tag{6.10}$$

and the Hamiltonian in the form

$$H = \frac{1}{2}\hbar\omega(aa^+ + a^+a) = \hbar\omega\left(a^+a + \frac{1}{2}\right).$$

The equations

$$[H, a] = -\hbar\omega a, \;\; [H, a^+] = \hbar\omega a^+ \tag{6.11}$$

and

$$H|\psi(E)\rangle = E|\psi(E)\rangle, \tag{6.12}$$

imply that the operator a decreases energy eigenvalues and the operator a^+ increases energy eigenvalues in units of $\hbar\omega$,

$$Ha|\psi(E)\rangle = (E - \hbar\omega)a|\psi(E)\rangle, \quad Ha^+|\psi(E)\rangle = (E + \hbar\omega)a^+|\psi(E)\rangle. \tag{6.13}$$

The operator a is therefore denoted as an *annihilation operator* or *lowering operator*, while a^+ is a *creation operator* or a *raising operator*. Together, they are also known as *ladder operators*.

Stability of the system requires existence of a lowest energy state $|\Omega\rangle$. This state must be annihilated by the operator a since otherwise $a|\Omega\rangle$ would be a state of lower energy,

$$\exists|\Omega\rangle\colon\; a|\Omega\rangle = 0, \;\; \Rightarrow \;\; H|\Omega\rangle = \frac{1}{2}\hbar\omega|\Omega\rangle.$$

The standard notation for this lowest energy state or vacuum state is $|\Omega\rangle = |0\rangle$.

The excited energy eigenstates are then

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle, \tag{6.14}$$

and the corresponding energy eigenvalues follow from

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$$

as

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

These relations are equivalent to

$$a^+a|n\rangle = n|n\rangle,$$

and therefore a^+a returns the level number of a state. The operator a^+a is denoted as an *occupation number operator*, or *number operator* for short, because this operator enumerates how many energy quanta $\hbar\omega$ are contained in an energy level.

For an explanation of the normalization of the states (6.14), we note that (4.32) implies that the adjoint of the state $(a^+)^n|0\rangle$ is $\langle 0|a^n$, and therefore the inner product of the state is $\langle 0|a^n(a^+)^n|0\rangle$. We can evaluate this product by using the property

$$[a, (a^+)^n] = n(a^+)^{n-1}, \tag{6.15}$$

which is easily proved by induction. We use (6.15) in the second step of the following calculation (and then in $n-1$ additional steps to arrive at the final result),

$$\begin{aligned}\langle 0|a^n(a^+)^n|0\rangle &= \langle 0|a^{n-1}[a, (a^+)^n]|0\rangle = n\langle 0|a^{n-1}(a^+)^{n-1}|0\rangle \\ &= n!\langle 0|0\rangle = n!\end{aligned} \tag{6.16}$$

On a more formal level, the proof of (6.16) would also involve an induction step with respect to n.

We have for arbitrary states $|\psi\rangle$

$$\langle 0|a^+|\psi\rangle = (\langle\psi|a|0\rangle)^+ = 0$$

and therefore the projector $\langle 0|a^+$ annihilates every state,

$$\langle 0|a^+ = 0.$$

6.3 Construction of the states in the x-representation

We construct the expansion coefficients $\langle x|n\rangle$ of the states with respect to the $|x\rangle$-basis. In the first step we construct the components $\langle x|0\rangle$ of the ground state $|0\rangle$. The equation $a|0\rangle = 0$ in x representation,

$$\langle x|a|0\rangle = \left(\sqrt{\frac{m\omega}{2\hbar}}x + \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right)\langle x|0\rangle = 0,$$

yields

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right),$$

and from this the x-components of the higher states can be calculated in the following way:

$$\begin{aligned}\langle x|n\rangle &= \frac{1}{\sqrt{n!}}\langle x|(a^+)^n|0\rangle \\ &= \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\sqrt{\frac{m\omega}{2\hbar}}x - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right)^n \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\ &= \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right). \qquad (6.17)\end{aligned}$$

The functions $H_n(x)$ are the Hermite polynomials

$$H_n(x) = \exp\left(\frac{1}{2}x^2\right)\left(x - \frac{d}{dx}\right)^n \exp\left(-\frac{1}{2}x^2\right), \qquad (6.18)$$

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \ldots$$

Properties of Hermite polynomials are discussed in Appendix D.

The general state of the one-dimensional harmonic oscillator in the x-representation is

$$\langle x|\psi(t)\rangle = \sum_{n\geq 0}\langle x|n\rangle\langle n|\psi(t)\rangle = \sum_{n\geq 0}\langle x|n\rangle\langle n|\psi(0)\rangle \exp\left(-\mathrm{i}\left(n+\frac{1}{2}\right)\omega t\right),$$

and the general normalizable state of a 1-dimensional system with Hamiltonian H (here $\dot{H} = 0$) in the x-representation can be expanded in oscillator eigenstates

$$\langle x|\psi(t)\rangle = \sum_{n\geq 0}\langle x|n\rangle\langle n|\psi(t)\rangle = \sum_{n\geq 0}\langle x|n\rangle\langle n|\exp(-\mathrm{i}Ht/\hbar)|\psi(0)\rangle$$
$$= \sum_{n\geq 0}\langle x|\exp(-\mathrm{i}Ht/\hbar)|n\rangle\langle n|\psi(0)\rangle.$$

These expansions are particular examples of the completeness relations of Sturm-Liouville eigenfunctions discussed in Appendix C. They hold pointwise for every continuous square integrable function $\langle x|\psi(t)\rangle$, and they hold for the derivatives as long as the derivatives are continuous (otherwise they remain valid in the mean).

Oscillator eigenstates in k space and bilinear relations for Hermite polynomials

The k space representations of the oscillator energy eigenstates can be constructed in the same way as the x representations. The equation

$$\langle k|a|0\rangle = \mathrm{i}\left(\sqrt{\frac{m\omega}{2\hbar}}\frac{d}{dk} + \sqrt{\frac{\hbar}{2m\omega}}k\right)\langle k|0\rangle = 0$$

yields

$$\langle k|0\rangle = \left(\frac{\hbar}{\pi m\omega}\right)^{\frac{1}{4}}\exp\left(-\frac{\hbar}{2m\omega}k^2\right),$$

and from this the k-components of the higher states can be calculated,

$$\langle k|n\rangle = \frac{1}{\sqrt{n!}}\langle k|(a^+)^n|0\rangle$$
$$= \frac{\mathrm{i}^n}{\sqrt{n!}}\left(\frac{\hbar}{\pi m\omega}\right)^{\frac{1}{4}}\left(\sqrt{\frac{m\omega}{2\hbar}}\frac{d}{dk} - \sqrt{\frac{\hbar}{2m\omega}}k\right)^n\exp\left(-\frac{\hbar}{2m\omega}k^2\right)$$
$$= \frac{(-\mathrm{i})^n}{\sqrt{2^n n!}}\left(\frac{\hbar}{\pi m\omega}\right)^{\frac{1}{4}}H_n\left(\sqrt{\frac{\hbar}{m\omega}}k\right)\exp\left(-\frac{\hbar}{2m\omega}k^2\right).$$

From this and the previous result we find an expression for the decomposition of plane waves,

$$\langle x|k\rangle = \frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}kx) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{2^n n!}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)H_n\left(\sqrt{\frac{\hbar}{m\omega}}k\right)$$
$$\times\exp\left[-\frac{1}{\hbar\omega}\left(\frac{\hbar^2k^2}{2m} + \frac{m}{2}\omega^2x^2\right)\right]. \qquad (6.19)$$

This reads in scaled variables

$$\frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}KX) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{2^n n!}H_n(X)\,H_n(K)\exp\left(-\frac{1}{2}\left(K^2+X^2\right)\right).$$

This equation follows from the Mehler formula (D.8) in the limit $z \to \mathrm{i}$.

For comparison, we must also have

$$\langle x|y\rangle = \delta(x-y) = \sqrt{\frac{m\omega}{\pi\hbar}}\sum_{n=0}^{\infty}\frac{1}{2^n n!}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}y\right) \times\exp\left(-\frac{m\omega}{2\hbar}\left(x^2+y^2\right)\right), \tag{6.20}$$

or in scaled variables

$$\delta(X-Y) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{1}{2^n n!}H_n(X)\,H_n(Y)\exp\left(-\frac{1}{2}\left(X^2+Y^2\right)\right).$$

This equation follows from the Mehler formula (D.8) in the limit $z \to 1$ and using $\lim_{\kappa\to\infty}\kappa\exp\left(-\kappa^2x^2\right) = \sqrt{\pi}\delta(x)$.

6.4 Lemmata for exponentials of operators

Exponentials

$$\exp(A) \equiv \sum_{n=0}^{\infty}\frac{1}{n!}A^n \tag{6.21}$$

of operators appear in many applications of quantum mechanics, because

- exponentials of Hamiltonians generate time evolution in quantum systems (time evolution operators);
- exponentials of operators generate continuous transformations (e.g. translations or rotations or phase rotations etc.) in quantum systems; and because
- the exponential $\exp(-\lambda A/\mathcal{C})$ shifts the eigenvalues of the complementary operator A_c (where $[A, A_c] = \mathcal{C} = const.$) by λ, i.e. $\exp(A)$ maps one eigenstate of A_c into another eigenstate. In this case the exponential operator is also denoted as a *shift operator*. We will see explicit examples for all these uses of exponentials of operators in later sections.

The definition (6.21) implies the property

$$\frac{d}{d\lambda}\exp(\lambda A) = A\exp(\lambda A) = \exp(\lambda A)A,$$

where λ is a complex variable. The property $\exp(A)\exp(-A) = 1$ is also easily proven, see Problem 6.5.

in addition, there are three other very useful theorems for products involving operator exponentials. The formulation of the first two of these theorems requires the notion of higher order commutators $\overset{n}{[}A, B]$, which can be recursively defined through

$$\overset{0}{[}A, B] \equiv B, \quad [A, \overset{0}{B}] \equiv A, \quad \overset{1}{[}A, B] = [A, \overset{1}{B}] = [A, B],$$

$$\overset{n+1}{[}A, B] = [A, \overset{n}{[}A, B]], \quad [A, \overset{n+1}{B}] = [[A, \overset{n}{B}], B].$$

With these definitions, we can state **Lemma 1:**

$$\exp(A)B\exp(-A) = \sum_{n\geq 0}\frac{1}{n!}\overset{n}{[}A, B]. \tag{6.22}$$

The proof simply proceeds by Taylor expansion of $\exp(\lambda A)B\exp(-\lambda A)$ with respect to λ. This uses the property

$$\frac{d^n}{d\lambda^n}\exp(\lambda A)B\exp(-\lambda A) = \exp(\lambda A)\overset{n}{[}A, B]\exp(-\lambda A),$$

which is proven by induction.

The second lemma is useful to combine certain products of three operator exponentials.

Lemma 2.

$$\exp(A)\exp(B)\exp(-A) = \exp[\exp(A)B\exp(-A)]. \tag{6.23}$$

The proof proceeds by applying the Taylor expansion $\exp(C) = \sum_{n=0}^{\infty} C^n/n!$ for $C = \exp(A)B\exp(-A)$.

The exponent on the right hand side of Lemma 2 can be evaluated with Lemma 1.

The third useful lemma concerns the combination of two exponentials of operators and requires that all higher order commutators of two operators A and B vanish:

$$\overset{2}{[}A, B] = 0, \quad \overset{2}{[}B, A] = 0. \tag{6.24}$$

Then the following equations hold[1],

[1]There is a generalization of equation (6.25) known as the Baker-Campbell-Hausdorff formula, which holds if the higher order commutators of A and B do not vanish. The recursive construction of higher order terms is outlined in Appendix E.

Lemma 3. *If equations (6.24) hold, then*

$$\exp(A)\exp(B) = \exp\left(A + B + \frac{1}{2}[A,B]\right) \tag{6.25}$$

$$= \exp(A+B)\exp\left(\frac{1}{2}[A,B]\right). \tag{6.26}$$

Proof. First we note that (6.26) is a direct consequence of (6.25) if we apply this equation to $\exp(A')\exp(B')$ with operators $A' = A+B$ and $B' = [A,B]/2$. Therefore it is enough to prove (6.25), which we will prove in the equivalent form

$$\exp(\lambda A)\exp(\lambda B) = \exp\left(\lambda A + \lambda B + \frac{\lambda^2}{2}[A,B]\right). \tag{6.27}$$

This equation is certainly correct for $\lambda = 0$. For the first order derivative of the left hand side of equation (6.27) one finds with (6.22) and (6.24)

$$\begin{aligned}\frac{d}{d\lambda}\exp(\lambda A)\exp(\lambda B) &= (A + B + \lambda[A,B])\exp(\lambda A)\exp(\lambda B)\\ &= \exp(\lambda A)\exp(\lambda B)(A + B + \lambda[A,B]),\end{aligned} \tag{6.28}$$

while the first order derivative of the right hand side of (6.27) is

$$\begin{aligned}&\frac{d}{d\lambda}\exp\left(\lambda A + \lambda B + \frac{\lambda^2}{2}[A,B]\right) = (A + B + \lambda[A,B])\\ &\times\exp\left(\lambda A + \lambda B + \frac{\lambda^2}{2}[A,B]\right) = \exp\left(\lambda A + \lambda B + \frac{\lambda^2}{2}[A,B]\right)\\ &\times(A + B + \lambda[A,B]).\end{aligned} \tag{6.29}$$

Therefore we also have

$$\begin{aligned}\left[\frac{d}{d\lambda}\exp(\lambda A)\exp(\lambda B)\right]_{\lambda=0} &= A + B\\ &= \left[\frac{d}{d\lambda}\exp\left(\lambda A + \lambda B + \frac{\lambda^2}{2}[A,B]\right)\right]_{\lambda=0}.\end{aligned} \tag{6.30}$$

Equations (6.28, 6.29) then also yield that in general

$$\begin{aligned}F_n &\equiv \left[\frac{d^n}{d\lambda^n}\exp(\lambda A)\exp(\lambda B)\right]_{\lambda=0}\\ &= \left[\frac{d^n}{d\lambda^n}\exp\left(\lambda A + \lambda B + \frac{\lambda^2}{2}[A,B]\right)\right]_{\lambda=0}\end{aligned}$$

by induction:

$$
\begin{aligned}
F_{n+1} &= \left[\frac{d^{n+1}}{d\lambda^{n+1}}\exp(\lambda A)\exp(\lambda B)\right]_{\lambda=0} \\
&= \left[\frac{d^{n}}{d\lambda^{n}}(A+B+\lambda[A,B])\exp(\lambda A)\exp(\lambda B)\right]_{\lambda=0} \\
&= (A+B)\left[\frac{d^{n}}{d\lambda^{n}}\exp(\lambda A)\exp(\lambda B)\right]_{\lambda=0} \\
&\quad + n[A,B]\left[\frac{d^{n-1}}{d\lambda^{n-1}}\exp(\lambda A)\exp(\lambda B)\right]_{\lambda=0} \\
&= (A+B)\left[\frac{d^{n}}{d\lambda^{n}}\exp\left(\lambda A+\lambda B+\frac{\lambda^2}{2}[A,B]\right)\right]_{\lambda=0} \\
&\quad + n[A,B]\left[\frac{d^{n-1}}{d\lambda^{n-1}}\exp\left(\lambda A+\lambda B+\frac{\lambda^2}{2}[A,B]\right)\right]_{\lambda=0} \\
&= \left[\frac{d^{n}}{d\lambda^{n}}(A+B+\lambda[A,B])\exp\left(\lambda A+\lambda B+\frac{\lambda^2}{2}[A,B]\right)\right]_{\lambda=0} \\
&= \left[\frac{d^{n+1}}{d\lambda^{n+1}}\exp\left(\lambda A+\lambda B+\frac{\lambda^2}{2}[A,B]\right)\right]_{\lambda=0}.
\end{aligned}
$$

Therefore the two operators have the same expansion in λ, and since they also agree for $\lambda = 0$ they must be the same.

We will often use these *lemmata* in quantum mechanical calculations.

6.5 Coherent states

Coherent states were introduced by Schrödinger in 1926 as quantum states which reproduce the classical oscillatory motion of a harmonic oscillator on the level of expectation values[2].

Equation (6.22) implies

$$\exp(-\lambda a^+)a\exp(\lambda a^+) = a+\lambda,$$

and therefore

$$a\exp(\lambda a^+)|0\rangle = \exp(\lambda a^+)(a+\lambda)|0\rangle = \lambda\exp(\lambda a^+)|0\rangle,$$

[2]E. Schrödinger, Naturwissenschaften 14, 664 (1926).

i.e. the state $\exp(\lambda a^+)|0\rangle$ is an eigenstate of the annihilation operator a with eigenvalue λ. It is not yet normalized, however. We can remedy this by replacing the shift operator $\exp(\lambda a^+)$ by a unitary shift operator with the same effect on a,

$$\exp(\lambda^+ a - \lambda a^+) a \exp(\lambda a^+ - \lambda^+ a) = a + \lambda,$$

and therefore the normalized eigenstate of a is

$$\begin{aligned} |\lambda\rangle &= \exp(\lambda a^+ - \lambda^+ a)|0\rangle = \exp\left(-\frac{1}{2}|\lambda|^2\right)\exp(\lambda a^+)|0\rangle \\ &= \exp\left(-\frac{1}{2}|\lambda|^2\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle. \end{aligned} \tag{6.31}$$

Here we used Lemma 3,

$$\exp(\lambda a^+)\exp(-\lambda^+ a) = \exp(\lambda a^+ - \lambda^+ a)\exp\left(\frac{1}{2}|\lambda|^2\right).$$

We also used an implicit convention that a state $|n\rangle$ labelled by an integer is an eigenstate of the Hamiltonian H of the harmonic oscillator, while a state $|\lambda\rangle$ labelled by a complex number is an eigenstate of a. Only the lowest energy state $|0\rangle$ is an eigenstate of both operators.

The states $|\lambda\rangle$ for $\lambda \neq 0$ are apparently superpositions of all energy eigenstates, and they are known as *coherent states*. They can be used to generate quantum states which move like a classical particle in the oscillator potential.

Classical motion with initial values $x(0) = X$ and $p(0) = P$ is described by

$$\begin{aligned} x_{cl}(t) &= X\cos(\omega t) + \frac{P}{m\omega}\sin(\omega t), \\ p_{cl}(t) &= P\cos(\omega t) - m\omega X\sin(\omega t), \end{aligned} \tag{6.32}$$

and what we want to construct is a state $|\lambda(t)\rangle$ with exactly these time dependences of its expectation values,

$$\begin{aligned} \langle\lambda(t)|\mathrm{x}|\lambda(t)\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle\lambda(t)|a + a^+|\lambda(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\lambda(t) + \lambda^+(t)\right] \\ &= x_{cl}(t) = X\cos(\omega t) + \frac{P}{m\omega}\sin(\omega t), \\ \langle\lambda(t)|\mathrm{p}|\lambda(t)\rangle &= -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\langle\lambda(t)|a - a^+|\lambda(t)\rangle = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\left[\lambda(t) - \lambda^+(t)\right] \\ &= p_{cl}(t) = P\cos(\omega t) - m\omega X\sin(\omega t). \end{aligned}$$

This yields

$$\lambda(t) = \sqrt{\frac{m\omega}{2\hbar}}x_{cl}(t) + \mathrm{i}\frac{p_{cl}(t)}{\sqrt{2m\omega\hbar}}$$
$$= \left(\sqrt{\frac{m\omega}{2\hbar}}X + \mathrm{i}\frac{P}{\sqrt{2m\omega\hbar}}\right)\exp(-\mathrm{i}\omega t). \tag{6.33}$$

We can also write the coherent state in terms of the operators x and p,

$$|\lambda(t)\rangle = \exp\left[\lambda(t)a^{+} - \lambda^{+}(t)a\right]|0\rangle = \exp\left(\frac{\mathrm{i}}{\hbar}\left[p_{cl}(t)\mathrm{x} - x_{cl}(t)\mathrm{p}\right]\right)|0\rangle. \tag{6.34}$$

We still have to show that the coherent states (6.34) satisfy the Schrödinger equation for the harmonic oscillator. We have with $|\lambda|^2(t) = |\lambda|^2$ time-independent,

$$\mathrm{i}\hbar\frac{d}{dt}|\lambda(t)\rangle = \mathrm{i}\hbar\dot{\lambda}(t)a^{+}|\lambda(t)\rangle = \hbar\omega\lambda(t)a^{+}|\lambda(t)\rangle = \hbar\omega a^{+}a|\lambda(t)\rangle$$
$$= H|\lambda(t)\rangle - \frac{1}{2}\hbar\omega|\lambda(t)\rangle.$$

Therefore the oscillating state which satisfies the Schrödinger equation of the harmonic oscillator including the zero point energy term is $|\lambda(t)\rangle\exp(-\mathrm{i}\omega t/2)$.

For the x representation of the coherent states, we notice

$$\langle x|\lambda(t)\rangle = \exp\left(-\frac{1}{2}|\lambda(t)|^2\right)\langle x|\exp\left(\lambda(t)a^{+}\right)|0\rangle$$
$$= \exp\left(-\frac{1}{2}|\lambda(t)|^2\right)\langle x|\exp\left[\frac{\lambda(t)}{\sqrt{2\hbar}}\left(\sqrt{m\omega}x - \mathrm{i}\frac{p}{\sqrt{m\omega}}\right)\right]|0\rangle$$
$$= \exp\left(\lambda(t)\sqrt{\frac{m\omega}{2\hbar}}x\right)\exp\left(-\frac{1}{2}|\lambda(t)|^2 - \frac{1}{4}\lambda^2(t)\right)$$
$$\times\langle x|\exp\left(-\mathrm{i}\lambda(t)\frac{p}{\sqrt{2m\omega\hbar}}\right)|0\rangle$$
$$= \exp\left(\lambda(t)\sqrt{\frac{m\omega}{2\hbar}}x\right)\exp\left(-\frac{1}{2}|\lambda(t)|^2 - \frac{1}{4}\lambda^2(t)\right)$$
$$\times\langle x - \lambda(t)\sqrt{\frac{\hbar}{2m\omega}}|0\rangle$$
$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(\lambda(t)\sqrt{\frac{m\omega}{2\hbar}}x\right)\exp\left(-\frac{1}{2}|\lambda(t)|^2 - \frac{1}{4}\lambda^2(t)\right)$$

$$\times \exp\left(-\frac{m\omega}{2\hbar}x^2 + \sqrt{\frac{m\omega}{2\hbar}}\lambda(t)x - \frac{1}{4}\lambda^2(t)\right)$$
$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2 + \sqrt{\frac{2m\omega}{\hbar}}\lambda(t)x - \frac{1}{2}|\lambda(t)|^2 - \frac{1}{2}\lambda^2(t)\right).$$

This yields with

$$\frac{1}{2}|\lambda(t)|^2 + \frac{1}{2}\lambda^2(t) = \frac{m\omega}{2\hbar}x_{cl}^2(t) + \mathrm{i}\frac{x_{cl}(t)p_{cl}(t)}{2\hbar}$$

the result

$$\langle x|\lambda(t)\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}\left(x - x_{cl}(t)\right)^2 + \mathrm{i}\frac{xp_{cl}(t)}{\hbar} - \mathrm{i}\frac{x_{cl}(t)p_{cl}(t)}{2\hbar}\right). \tag{6.35}$$

Comparison of equation (6.35) with equations (5.5) and (5.6) or direct evaluation yields the momentum representation of the coherent states,

$$\langle p|\lambda(t)\rangle = \frac{1}{(\pi m\omega\hbar)^{\frac{1}{4}}} \exp\left(-\frac{\left(p - p_{cl}(t)\right)^2}{2m\omega\hbar} - \mathrm{i}\frac{x_{cl}(t)p}{\hbar} + \mathrm{i}\frac{x_{cl}(t)p_{cl}(t)}{2\hbar}\right). \tag{6.36}$$

The variances of the expectation values $\langle x\rangle(t) = x_{cl}(t)$ and $\langle p\rangle(t) = p_{cl}(t)$ of the coherent state $|\lambda(t)\rangle$ (6.33, 6.34) are

$$\Delta x^2 = \frac{\hbar}{2m\omega}, \quad \Delta p^2 = \frac{m\omega\hbar}{2}. \tag{6.37}$$

In terms of the force constant $K = m\omega^2$ of the harmonic potential the width of the coherent states is $\Delta x^2 \propto 1/\sqrt{mK}$. This equation holds with an n-dependent factor also for the energy eigenstates $|n\rangle$, see Problem 6.2. We have seen in Section 3.4 that the kinetic term $p^2/2m$ drives wave packets apart. The attractive potential $V(x) = Kx^2/2$ on the other hand tries to collapse wave packets, and the balance of these terms yields the stable wave packets (6.17) and (6.33, 6.34). This is consistent with $\Delta x^2 \propto 1/\sqrt{mK}$, because $mK \to \infty$ would correspond to domination of the attractive potential while $mK \to 0$ would imply domination of the kinetic term. In the next chapter we will see that the same basic mechanism also stabilizes the bound states of atoms. Balance between kinetic terms driving wave packets apart and the attractive Coulomb potential trying to contract the wave function generates minimal possible sizes of wave functions for given kinetic parameters $1/m$ and force constants Ze^2, thus preventing electrons from the classically inevitable core collapse.

Scalar products and overcompleteness of coherent states

We use $\overline{\lambda}$ instead of λ^+ to denote the complex conjugate of λ in this section.

The decomposition (6.31) yields for the product of two coherent state the result

$$\begin{aligned}\langle\lambda|\mu\rangle &= \exp\left(-\frac{1}{2}\left(|\lambda|^2+|\mu|^2\right)\right)\sum_{n=0}^{\infty}\frac{\left(\overline{\lambda}\mu\right)^n}{n!}\\ &= \exp\left(\overline{\lambda}\mu-\frac{1}{2}\left(|\lambda|^2+|\mu|^2\right)\right),\end{aligned}\tag{6.38}$$

and therefore we also have

$$|\langle\lambda|\mu\rangle|^2 = \exp\left(-|\lambda-\mu|^2\right),$$

i.e. coherent states are never orthogonal. Therefore any completeness relation cannot be unique, and we will find indeed that coherent states are *overcomplete*, i.e. we can decompose every state in a series of coherent states in infinitely many different ways. However, we can still identify a kind of "canonical" completeness relation in which all coherent states contribute with the same weight. We use

$$dz = \exp(\mathrm{i}\varphi)dr + \mathrm{i}z d\varphi, \quad \frac{d\overline{z}\wedge dz}{2\mathrm{i}} = d\Re z\wedge d\Im z = dr\wedge d\varphi\, r$$

to find the particular completeness relation

$$\begin{aligned}\int\frac{d\overline{z}\wedge dz}{2\pi\mathrm{i}}|z\rangle\langle z| &= \int\frac{d\overline{z}\wedge dz}{2\pi\,\mathrm{i}}\exp\left(-|z|^2\right)\sum_{m,n}|m\rangle\frac{z^m\overline{z}^n}{\sqrt{m!n!}}\langle n|\\ &= \frac{1}{\pi}\int_0^{\infty}dr\int_0^{2\pi}d\varphi\,\exp\left(-r^2\right)\sum_{m,n}|m\rangle\frac{r^{m+n+1}}{\sqrt{m!n!}}\exp[\mathrm{i}(n-m)\varphi]\,\langle n|\\ &= 2\int_0^{\infty}dr\,r\exp\left(-r^2\right)\sum_n|n\rangle\frac{r^{2n}}{n!}\langle n|\\ &= \sum_n\int_0^{\infty}du\left(-\frac{d}{d\alpha}\right)^n\exp(-\alpha u)\Big|_{\alpha=1}\frac{1}{n!}|n\rangle\langle n|\\ &= \sum_n\left(-\frac{d}{d\alpha}\right)^n\frac{1}{\alpha}\Big|_{\alpha=1}\frac{1}{n!}|n\rangle\langle n| = \sum_n|n\rangle\langle n| = 1.\end{aligned}\tag{6.39}$$

For example, substitution of

$$\langle z|n\rangle = \frac{\overline{z}^n}{\sqrt{n!}}\exp\left(-\frac{1}{2}|z|^2\right)$$

yields the following decomposition of the energy eigenstates of the harmonic oscillator in terms of coherent states,

$$|n\rangle = \int \frac{d\bar{z} \wedge dz}{2\pi \mathrm{i}} |z\rangle \frac{\bar{z}^n}{\sqrt{n!}} \exp\left(-\frac{1}{2}|z|^2\right).$$

For another example of an expansion in terms of coherent states, substitution of equation (6.38) yields a decomposition of coherent states in terms of coherent states,

$$|\zeta\rangle = \int \frac{d\bar{z} \wedge dz}{2\pi \mathrm{i}} |z\rangle \exp\left(\bar{z}\zeta - \frac{1}{2}\left(|z|^2 + |\zeta|^2\right)\right). \tag{6.40}$$

This lack of orthogonality of the coherent states implies that we can shift contributions of different coherent states to the expansion of a state $|\psi\rangle$, see e.g. Problem 6.10.

Note that for the coherent state parameter $\lambda(t)$ (6.33)

$$\frac{d\overline{\lambda}(t) \wedge d\lambda(t)}{2\pi \mathrm{i}} = \frac{dx_{cl}(t) \wedge dp_{cl}(t)}{h} = \frac{dX \wedge dP}{h},$$

or if we denote the classical parameters simply with x and p,

$$\int \frac{dx \wedge dp}{h} |\lambda_{x,p}(t)\rangle\langle\lambda_{x,p}(t)| = 1.$$

Later we will encounter the measure $dx \wedge dp/h$ in phase space also in the density of states.

Squeezed states

We try to construct new oscillation operators b, b^+ from the oscillation operators (6.8). Substitution of the linear *ansatz*

$$b = Aa + Ba^+, \quad b^+ = \overline{A}a^+ + \overline{B}a$$

into the condition $[b, b^+] = 1$ yields

$$|A|^2 - |B|^2 = 1,$$

i.e. we find

$$b = \exp(\mathrm{i}\alpha)\cosh(u)a + \exp(\mathrm{i}\beta)\sinh(u)a^+,$$
$$b^+ = \exp(-\mathrm{i}\alpha)\cosh(u)a^+ + \exp(-\mathrm{i}\beta)\sinh(u)a.$$

The phase factors are irrelevant in the following. Therefore we study the ladder operators

$$a(u) = \cosh(u)a + \sinh(u)a^+ = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\mathrm{x}\exp(u) + \mathrm{i}\frac{\mathrm{p}\exp(-u)}{\sqrt{m\omega}}\right)$$

and

$$a^+(u) = \cosh(u)a^+ + \sinh(u)a = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\mathrm{x}\exp(u) - \mathrm{i}\frac{\mathrm{p}\exp(-u)}{\sqrt{m\omega}}\right).$$

The x and p operators of the original oscillator are

$$\mathrm{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^+\right) = \sqrt{\frac{\hbar}{2m\omega}}\exp(-u)\left(a(u) + a^+(u)\right)$$

and

$$\mathrm{p} = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^+\right) = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\exp(u)\left(a(u) - a^+(u)\right).$$

We can think of the new operators $a(u)$ and $a^+(u)$ as oscillation operators for a harmonic oscillator with a u-dependent product of mass and frequency,

$$(m\omega)(u) = (m\omega)\exp(2u). \tag{6.41}$$

The coherent state for the new oscillation operators $|\lambda\rangle(u) = \exp(\lambda a^+(u) - \lambda^+ a(u))|0\rangle_u$, $a(u)|0\rangle_u = 0$, has expectation values and variances

$$\langle x\rangle(u) = \sqrt{\frac{\hbar}{2m\omega}}\exp(-u)\left(\lambda + \lambda^+\right) = \langle x\rangle\Big|_{u=0} \times \exp(-u),$$

$$\langle p\rangle(u) = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\exp(u)\left(\lambda - \lambda^+\right) = \langle p\rangle\Big|_{u=0} \times \exp(u),$$

$$\Delta x^2(u) = \frac{\hbar}{2m\omega}\exp(-2u) = \Delta x^2\Big|_{u=0} \times \exp(-2u),$$

$$\Delta p^2(u) = \frac{m\omega\hbar}{2}\exp(2u) = \Delta p^2\Big|_{u=0} \times \exp(2u),$$

i.e. the uncertainty in x or p direction is squeezed at the expense of a corresponding increase of the uncertainty in the complementary direction.

We could formally write

$$\lambda a^+(u) - \lambda^+ a(u) = \lambda(u)a^+ - \lambda^+(u)a$$

with $\lambda(u) = \lambda\cosh(u) - \lambda^+\sinh(u)$. However, $|0\rangle_u \neq |0\rangle$ and therefore $|\lambda\rangle_u \neq |\lambda(u)\rangle$. Without the change in the vacuum, the variances could not change.

For the actual transformation, we note that

$$a(u) = \exp\Big(\frac{u}{2}\big[a^2 - (a^+)^2\big]\Big)\, a \exp\Big(-\frac{u}{2}\big[a^2 - (a^+)^2\big]\Big)$$

and therefore

$$|0\rangle_u = \exp\Big(\frac{u}{2}\big[a^2 - (a^+)^2\big]\Big)\,|0\rangle, \quad |\lambda\rangle_u = \exp\Big(\frac{u}{2}\big[a^2 - (a^+)^2\big]\Big)\,|\lambda\rangle.$$

6.6 Problems

6.1. Write down the p-representation of the Schrödinger equation for the one-dimensional harmonic oscillator. Which transformations between the parameters m and ω map the p-representation into the x-representation?

6.2. Calculate the widths Δx_n and Δp_n of the n-th energy eigenstate of the harmonic oscillator.

Remark. This is most conveniently done using the annihilation and creation operators.

6.3. A one-dimensional oscillator at time $t = 0$ is in a state

$$|\psi_\alpha\rangle = \cos\alpha|0\rangle + \exp(\mathrm{i}\varphi)\sin\alpha|1\rangle.$$

6.3a. Calculate the expectation values $\langle x\rangle(t)$, $\langle p\rangle(t)$ and $\langle E\rangle$ for the oscillator.

6.3b. Calculate the uncertainties $\Delta x(t)$, $\Delta p(t)$ and ΔE for the oscillator.

6.4. For the oscillator from Problem 6.3, how large is the probability density to find the oscillator in the location x at time t?

6.5. Show that

$$\exp(A)\exp(-A) = 1. \tag{6.42}$$

Hint: Define a corresponding λ-dependent operator $F(\lambda)$ by rescaling the operator A in (6.42) with the complex number λ. Show $dF(\lambda)/d\lambda = 0$. This implies $F(\lambda) = F(0)$.

6.6. Show that every coherent state $|\lambda\rangle = \exp(\lambda a^+ - \lambda^+ a)|0\rangle$ has the variances (6.37).

6.7. Calculate the energy expectation value $\langle E\rangle$ and the energy uncertainty ΔE for the coherent state $|\lambda\rangle$. Which values do you find in particular for the state $|\lambda(t)\rangle$ (6.34) which reproduces the classical trajectories (6.32)?

6.8. Show that

$$\begin{aligned}\exp\left(\frac{\mathrm{i}}{\hbar}\left[p_{cl}(t)\mathrm{x}-x_{cl}(t)\mathrm{p}\right]\right) &= \exp\left(\frac{\mathrm{i}}{\hbar}p_{cl}(t)\mathrm{x}\right)\exp\left(-\frac{\mathrm{i}}{\hbar}x_{cl}(t)\mathrm{p}\right)\\ &\quad\times\exp\left(-\frac{\mathrm{i}}{2\hbar}x_{cl}(t)p_{cl}(t)\right)\\ &= \exp\left(-\frac{\mathrm{i}}{\hbar}x_{cl}(t)\mathrm{p}\right)\exp\left(\frac{\mathrm{i}}{\hbar}p_{cl}(t)\mathrm{x}\right)\\ &\quad\times\exp\left(\frac{\mathrm{i}}{2\hbar}x_{cl}(t)p_{cl}(t)\right).\end{aligned}$$

6.9. Use the results from Problem 6.8 and equation (6.34) to re-derive the x and p representations (6.35, 6.36) of the coherent state $|\lambda(t)\rangle$.

6.10. Suppose $K\subset\mathbb{C}$ is a subset of the complex plane. Show that the relation (6.40) implies a completeness relation

$$\begin{aligned}1 &= \int_{\mathbb{C}\setminus K}\frac{d\bar{z}\wedge dz}{2\pi\mathrm{i}}|z\rangle\left(\langle z|+\int_K\frac{d\bar{\zeta}\wedge d\zeta}{2\pi\mathrm{i}}\exp\left(\bar{z}\zeta-\frac{1}{2}\left(|z|^2+|\zeta|^2\right)\right)\langle\zeta|\right)\\ &\quad+\int_K\frac{d\bar{z}\wedge dz}{2\pi\mathrm{i}}|z\rangle\int_K\frac{d\bar{\zeta}\wedge d\zeta}{2\pi\mathrm{i}}\exp\left(\bar{z}\zeta-\frac{1}{2}\left(|z|^2+|\zeta|^2\right)\right)\langle\zeta|.\end{aligned}$$

What we have done here is to use (6.40) to redistribute contributions from coherent states with eigenvalues in K and in $\mathbb{C}\setminus K$ to the completeness relation. Since there are non-enumerably many possibilities to choose the subset K, there are non-enumerably many completeness relations for coherent states.

6.11. Construct a coherent state $|\boldsymbol{\lambda}(t)\rangle$ that follows the orbit (6.1) in terms of its expectation values $\langle\boldsymbol{\lambda}(t)|\mathbf{x}|\boldsymbol{\lambda}(t)\rangle$, $\langle\boldsymbol{\lambda}(t)|\mathbf{p}|\boldsymbol{\lambda}(t)\rangle$.

How large are the uncertainties of the coordinate and momentum expectation values?

6.12. The classical solution (6.1) of the three-dimensional isotropic oscillator describes a curve on a five-dimensional ellipsoid in six-dimensional phase space.

How long are the main axes of the ellipsoid?

How can you then think geometrically of the evolution of the coherent state from Problem 6.11?