# Chapter 16
# Principles of Lagrangian Field Theory

The replacement of Newton's equation by quantum mechanical wave equations in the 1920s implied that by that time all known fundamental degrees of freedom in physics were described by fields like $\boldsymbol{A}(\boldsymbol{x},t)$ or $\Psi(\boldsymbol{x},t)$, and their dynamics was encoded in wave equations. However, all the known fundamental wave equations can be derived from a field theory version of Hamilton's principle[1], i.e. the concept of the Lagrange function $L(q(t),\dot{q}(t))$ and the related action $S=\int dt\,L$ generalizes to a Lagrange density $\mathcal{L}(\phi(\boldsymbol{x},t),\dot{\phi}(\boldsymbol{x},t),\nabla\phi(\boldsymbol{x},t))$ with related action $S=\int dt\int d^3\boldsymbol{x}\,\mathcal{L}$, such that all fundamental wave equations can be derived from the variation of an action,

$$\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}=0.$$

This formulation of dynamics is particularly useful for exploring the connection between symmetries and conservation laws of physical systems, and it also allows for a systematic approach to the quantization of fields, which allows us to describe creation and annihilation of particles.

## 16.1 Lagrangian field theory

Irrespective of whether we work with relativistic or non-relativistic field theories, it is convenient to use four-dimensional notation for coordinates and partial derivatives,

$$x^\mu=\{x^0,\boldsymbol{x}\}\equiv\{ct,\boldsymbol{x}\},\quad \partial_\mu=\frac{\partial}{\partial x^\mu}=\{\partial_0,\nabla\}.$$

[1]Please review Appendix A if you are not familiar with Lagrangian mechanics, or if you need a reminder.



R. Dick, *Advanced Quantum Mechanics*, Graduate Texts in Physics,
DOI 10.1007/978-3-319-25675-7_16

We proceed by first deriving the general field equations following from a Lagrangian $\mathcal{L}(\partial\phi_I, \phi_I)$ which depends on a set of fields $\phi_I(x) \equiv \phi_I(\boldsymbol{x}, t)$ and their first order derivatives $\partial_\mu \phi_I(x)$. These fields will be the Schrödinger field $\Psi(\boldsymbol{x}, t)$ and its complex conjugate field $\Psi^+(\boldsymbol{x}, t)$ in Chapter 17, but in Chapter 18 we will also deal with the wave function $\boldsymbol{A}(x)$ of the photon.

We know that the equations of motion for the variables $\boldsymbol{x}(t)$ of classical mechanics follow from action principles $\delta S = \delta \int dt L(\dot{\boldsymbol{x}}, \boldsymbol{x}) = 0$ in the form of the Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0.$$

The variation of a field dependent action functional

$$S[\phi] = \frac{1}{c}\int_{\mathcal{V}} d^4x\, \mathcal{L}(\partial\phi_I, \phi_I)$$

for fields $\phi_I(x)$ proceeds in the same way as in classical mechanics, the only difference being that we apply the Gauss theorem for the partial integrations.

To elucidate this, we require that arbitrary first order variation

$$\phi_I(x) \to \phi_I(x) + \delta\phi_I(x)$$

with fixed fields at initial and final times $t_0$ and $t_1$,

$$\delta\phi_I(\boldsymbol{x}, t_0) = 0, \quad \delta\phi_I(\boldsymbol{x}, t_1) = 0,$$

leaves the action $S[\phi]$ in first order invariant. We also assume that the fields and their variations vanish at spatial infinity.

The first order variation of the action between the times $t_0$ and $t_1$ is

$$\begin{aligned}
\delta S[\phi] &= S[\phi + \delta\phi] - S[\phi] \\
&= \int d^3\boldsymbol{x} \int_{t_0}^{t_1} dt\, [\mathcal{L}(\partial\phi_I + \partial\delta\phi_I, \phi_I + \delta\phi_I) - \mathcal{L}(\partial\phi_I, \phi_I)] \\
&= \int d^3\boldsymbol{x} \int_{t_0}^{t_1} dt \left(\delta\phi_I \frac{\partial\mathcal{L}}{\partial\phi_I} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I)}\partial_\mu\delta\phi_I\right).
\end{aligned}$$

Partial integration in the last term yields

$$\delta S[\phi] = \int d^3\boldsymbol{x} \int_{t_0}^{t_1} dt\, \delta\phi_I \left(\frac{\partial\mathcal{L}}{\partial\phi_I} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I)}\right), \tag{16.1}$$

where the boundary terms vanish because of the vanishing variations at spatial infinity and at $t_0$ and $t_1$.

Equation (16.1) implies that we can have $\delta S[\phi] = 0$ for arbitrary variations $\delta\phi_I(x)$ between $t_0$ and $t_1$ if and only if the equations

$$\frac{\partial\mathcal{L}}{\partial\phi_I} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I)} = 0 \tag{16.2}$$

hold for all the fields $\phi_I(x)$. These are the Euler-Lagrange equations for Lagrangian field theory.

The derivation of equation (16.2) does not depend on the number of four spacetime dimensions, $\mu \in \{0, 1, 2, 3\}$. It would just as well go through in any number $d$ of dimensions, where $d$ could be a number of spatial dimensions if we study equilibrium or static phenomena in field theory, or $d$ can be $d-1$ spatial and one time dimension. Relevant cases for observations include $d = 1$ (mechanics or equilibrium in one-dimensional systems), $d = 2$ (equilibrium phenomena on interfaces or surfaces, time-dependent phenomena in one-dimensional systems), $d = 3$ (equilibrium phenomena in three dimensions, time-dependent phenomena on interfaces or surfaces), and $d = 4$ (time-dependent phenomena in observable spacetime). In particular, classical particle mechanics can be considered as a field theory in one spacetime dimension.

## *The Lagrange density for the Schrödinger field*

An example is provided by the Lagrange density for the Schrödinger field,

$$\mathcal{L} = \frac{\mathrm{i}\hbar}{2}\left(\Psi^+\cdot\frac{\partial\Psi}{\partial t} - \frac{\partial\Psi^+}{\partial t}\cdot\Psi\right) - \frac{\hbar^2}{2m}\nabla\Psi^+\cdot\nabla\Psi - \Psi^+\cdot V\cdot\Psi. \tag{16.3}$$

In the notation of the previous paragraph, this corresponds to fields $\phi_1(x) = \Psi^+(x)$ and $\phi_2(x) = \Psi(x)$, or we could also denote the real and imaginary parts of $\Psi$ as the two fields.

We have the following partial derivatives of the Lagrange density,

$$\frac{\partial\mathcal{L}}{\partial\Psi^+} = \frac{\mathrm{i}\hbar}{2}\frac{\partial\Psi}{\partial t} - V\Psi, \quad \frac{\partial\mathcal{L}}{\partial(\partial_t\Psi^+)} = -\frac{\mathrm{i}\hbar}{2}\Psi, \quad \frac{\partial\mathcal{L}}{\partial(\partial_i\Psi^+)} = -\frac{\hbar^2}{2m}\partial_i\Psi,$$

and the corresponding adjoint equations. The Euler-Lagrange equation from variation of the action with respect to $\Psi^+$,

$$\frac{\partial\mathcal{L}}{\partial\Psi^+} - \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\Psi^+)} - \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\Psi^+)} = 0,$$

is the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi + \frac{\hbar^2}{2m}\Delta\Psi - V\Psi = 0.$$

The Euler-Lagrange equation from variation with respect to $\Psi$ in turn yields the complex conjugate Schrödinger equation for $\Psi^+$. This is of course required for consistency, and is a consequence of $\mathcal{L} = \mathcal{L}^+$.

The Schrödinger field is slightly unusual in that variation of the action with respect to $\phi_1(x) = \Psi^+(x)$ yields the equation for $\phi_2(x) = \Psi(x)$ and vice versa. Generically, variation of the action with respect to a field $\phi_I(x)$ yields the equation of motion for that field[2]. However, the important conclusion from this section is that Schrödinger's quantum mechanics is a Lagrangian field theory with a Lagrange density (16.3).

## 16.2 Symmetries and conservation laws

We consider an action with fields $\phi$ ($\phi_I$, $1 \leq I \leq N$) in a $d$-dimensional space or spacetime:

$$S = \frac{1}{c}\int d^dx\,\mathcal{L}(\phi, \partial\phi). \tag{16.4}$$

To reveal the connection between symmetries and conservation laws, we calculate the first order change of the action $S$ (16.4) if we perform transformations of the coordinates,

$$x'(x) = x - \epsilon(x). \tag{16.5}$$

This transforms the integration measure in the action as

$$d^dx' = d^dx\left(1 - \partial_\mu\epsilon^\mu\right),$$

and partial derivatives transform according to

$$\partial'_\mu = \partial_\mu + \left(\partial_\mu\epsilon^\nu\right)\partial_\nu. \tag{16.6}$$

We also include transformations of the fields,

$$\phi'(x') = \phi(x) + \delta\phi(x). \tag{16.7}$$

[2]The unconventional behavior for the Schrödinger field can be traced back to how it arises from the Klein-Gordon or Dirac fields in the non-relativistic limit, see Chapter 21.

Coordinate transformations often also imply transformations of the fields, e.g. if $\phi$ is a tensor field of $n$-th order with components $\phi_{\alpha\ldots\nu}(x)$, the transformation induced by the coordinate transformation $x \to x'(x) = x - \epsilon(x)$ is

$$\phi'_{\alpha'\ldots\nu'}(x') = \partial_{\alpha'}x^{\alpha} \cdot \partial_{\beta'}x^{\beta} \ldots \partial_{\nu'}x^{\nu} \cdot \phi_{\alpha\beta\ldots\nu}(x).$$

This yields is in first order

$$\begin{aligned}\delta\phi_{\alpha\beta\ldots\nu}(x) &= \phi'_{\alpha\ldots\nu}(x') - \phi_{\alpha\ldots\nu}(x)\\ &= \partial_{\alpha}\epsilon^{\sigma} \cdot \phi_{\sigma\beta\ldots\nu}(x) + \partial_{\beta}\epsilon^{\sigma} \cdot \phi_{\alpha\sigma\ldots\nu}(x) + \ldots + \partial_{\nu}\epsilon^{\sigma} \cdot \phi_{\alpha\beta\ldots\sigma}(x).\end{aligned}$$

Fields can also transform without a coordinate transformation, e.g. through a phase transformation.

We denote the transformations (16.5, 16.7) as a *symmetry* of the Lagrangian field theory (16.4) if they leave the volume form $d^d x\,\mathcal{L}$ invariant,

$$d^d x'\,\mathcal{L}(\phi', \partial'\phi'; x') = d^d x\,\mathcal{L}(\phi, \partial\phi; x). \tag{16.8}$$

Here we also allow for an explicit dependence of the Lagrange density on the coordinates $x$ besides the implicit coordinate dependence through the dependence on the fields $\phi(x)$. If we define a transformed Lagrange density from the requirement of invariance of the action $S$ under the transformations (16.5, 16.7),

$$\mathcal{L}'(\phi', \partial'\phi'; x') = \det(\partial' x)\mathcal{L}(\phi, \partial\phi; x), \tag{16.9}$$

the symmetry condition (16.8) amounts to form invariance of the Lagrange density.

The equations (16.6) and (16.7) imply the following first order change of partial derivative terms:

$$\delta\left(\partial_{\mu}\phi\right) = \partial_{\mu}\delta\phi + \left(\partial_{\mu}\epsilon^{\nu}\right)\partial_{\nu}\phi. \tag{16.10}$$

The resulting first order change of the volume form is (with the understanding that we sum over all fields in all multiplicative terms where the field $\phi$ appears twice):

$$\begin{aligned}\delta(d^d x\,\mathcal{L}) &= d^d x\left[\left(1 - \partial_{\mu}\epsilon^{\mu}\right)\left(\mathcal{L} + \delta\phi\frac{\partial\mathcal{L}}{\partial\phi} + \delta\left(\partial_{\rho}\phi\right)\frac{\partial\mathcal{L}}{\partial(\partial_{\rho}\phi)} - \epsilon^{\sigma}\delta_{\sigma}\mathcal{L}\right) - \mathcal{L}\right]\\ &= d^d x\left[(\partial_{\mu}\epsilon^{\nu})\left(\partial_{\nu}\phi \cdot \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} - \eta_{\nu}{}^{\mu}\mathcal{L}\right) + \partial_{\mu}\left(\delta\phi\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\right)\right.\\ &\quad + \delta\phi\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\right)\\ &\quad \left.- \epsilon^{\mu}\left(\partial_{\mu}\mathcal{L} - \partial_{\mu}\phi \cdot \frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\mu}\partial_{\nu}\phi \cdot \frac{\partial\mathcal{L}}{\partial(\partial_{\nu}\phi)}\right)\right]\end{aligned}$$

$$= d^d x \left\{ \partial_\mu \left[ \epsilon^\nu \left( \partial_\nu \phi \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \eta_\nu{}^\mu \mathcal{L} \right) + \delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \right.$$
$$\left. + (\delta\phi + \epsilon^\nu \partial_\nu \phi) \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right\} . \tag{16.11}$$

Here

$$\delta_\mu \mathcal{L} = \partial_\mu \mathcal{L} - \partial_\mu \phi \cdot \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \partial_\nu \phi \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}$$

is the partial derivative of $\mathcal{L}$ with respect to any *explicit* coordinate dependence.

If we have off-shell $\delta(d^d x\, \mathcal{L}) = 0$ for the proposed transformations $\epsilon$, $\delta\phi$, we find a local on-shell conservation law

$$\partial_\mu j^\mu = 0 \tag{16.12}$$

with the current density

$$j^\mu = \epsilon^\nu \left( \eta_\nu{}^\mu \mathcal{L} - \partial_\nu \phi \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} . \tag{16.13}$$

The corresponding charge in a $d$-dimensional spacetime

$$Q = \frac{1}{c} \int d^{d-1}\boldsymbol{x}\, j^0(\boldsymbol{x}, t) = \int d^{d-1}\boldsymbol{x}\, \varrho(\boldsymbol{x}, t) \tag{16.14}$$

is conserved if no charges are escaping or entering at $|\boldsymbol{x}| \to \infty$:

$$\lim_{|\boldsymbol{x}| \to \infty} \int d^{d-2}\Omega\, |\boldsymbol{x}|^{d-3} \boldsymbol{x} \cdot \boldsymbol{j}(\boldsymbol{x}, t) = 0.$$

Here $d^{d-2}\Omega = d\theta_1 \ldots d\theta_{d-2} \sin^{d-3}\theta_1 \ldots \sin\theta_{d-3}$ is the measure on the $(d-2)$-dimensional sphere in the $d-1$ spatial dimensions, see also (J.22) (note that in (J.22) the number of spatial dimensions is denoted as $d$).

If the off-shell variation of $d^d x \mathcal{L}$ satisfies $\delta(d^d x\, \mathcal{L}) \equiv d^d x\, \partial_\mu K^\mu$, the on-shell conserved current is $J^\mu = j^\mu + K^\mu$ and the charge is the spatial integral over $J^0/c$.

Symmetry transformations which only transform the fields, but leave the coordinates invariant ($\delta\phi \neq 0$, $\epsilon = 0$), are denoted as *internal symmetries*. Symmetry transformations involving coordinate transformations are denoted as *external symmetries*.

The connection between symmetries and conservation laws was developed by Emmy Noether[3] and is known as *Noether's theorem*.

[3] E. Noether, Nachr. König. Ges. Wiss. Göttingen, Math.-phys. Klasse, 235 (1918), see also arXiv:physics/0503066.

### *Energy-momentum tensors*

We now specialize to inertial (i.e. pseudo-Cartesian) coordinates in Minkowski spacetime. If the coordinate shift in (16.5) is a constant translation, $\partial_\mu \epsilon^\nu = 0$, all fields transform like scalars, $\delta\phi = 0$, and the conserved current becomes

$$j^\mu = \epsilon^\nu \left( \eta_\nu{}^\mu \mathcal{L} - \partial_\nu \phi \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \epsilon^\nu \Theta_\nu{}^\mu .$$

Omitting the $d$ irrelevant constants $\epsilon^\nu$ leaves us with $d$ conserved currents ($0 \le \nu \le d-1$)

$$\partial_\mu \Theta_\nu{}^\mu = 0, \qquad (16.15)$$

with components

$$\Theta_\nu{}^\mu = \eta_\nu{}^\mu \mathcal{L} - \partial_\nu \phi \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}. \qquad (16.16)$$

The corresponding conserved charges

$$p_\nu = \frac{1}{c} \int d^{d-1}\boldsymbol{x}\, \Theta_\nu{}^0 \qquad (16.17)$$

are the components of the four-dimensional energy-momentum vector of the physical system described by the Lagrange density $\mathcal{L}$, and the tensor with components $\Theta_\nu{}^\mu$ is therefore denoted as an *energy-momentum tensor*.

The spatial components $\Theta^{ij}$ of the energy-momentum tensor have dual interpretations in terms of momentum current densities and forces. To explain the meaning of $\Theta^{ij}$, we pick an arbitrary (but stationary) spatial volume $V$. Since we are talking about fields, part of the fields will reside in $V$. From equation (16.17), the fields in $V$ will carry a part of the total momentum $\boldsymbol{p}$ which is

$$\boldsymbol{p}_V = \boldsymbol{e}_i \frac{1}{c} \int_V d^{d-1}\boldsymbol{x}\, \Theta^{i0}.$$

The equations (16.15) and (16.17) imply that the change of $\boldsymbol{p}_V$ is given by

$$\frac{d}{dt}\boldsymbol{p}_V = \boldsymbol{e}_i \int_V d^{d-1}\boldsymbol{x}\, \partial_0 \Theta^{i0} = -\boldsymbol{e}_i \oint_{\partial V} d^{d-2}S_j\, \Theta^{ij}, \qquad (16.18)$$

where the Gauss theorem in $d-1$ spatial dimensions was employed and $d^{d-2}S_j$ is the outward bound surface element on the boundary $\partial V$ of the volume.

This equation tells us that the component $\Theta^{ij}$ describes the flow of the momentum component $p^i$ through the plane with normal vector $\boldsymbol{e}_j$, i.e. $\Theta^{ij}$ is the flow of momentum $p^i$ in the direction $\boldsymbol{e}_j$ and $\boldsymbol{j}^i = \Theta^{ij}\boldsymbol{e}_j$ is the corresponding current density. In the dual interpretation, we read equation (16.18) with the relation $\boldsymbol{F}_V = d\boldsymbol{p}_V/dt$ between force and momentum change in mind. In this interpretation, $\boldsymbol{F}_V$ *is the force exerted on the fields in the fixed volume* $V$, because it describes the rate of change of momentum of the fields in $V$. $-\boldsymbol{F}_V$ *is the force exerted by the fields in the fixed volume* $V$. The component $\Theta^{ij}$ is then the force in direction $\boldsymbol{e}_i$ per area with normal vector $\boldsymbol{e}_j$. This represents strain or pressure for $i = j$ and stress for $i \neq j$. The energy-momentum tensor is therefore also known as *stress-energy tensor*.

There is another equation for the energy-momentum tensor in general relativity, which agrees with equation (16.16) for scalar fields, but not for vector or relativistic spinor fields. Both definitions yield the same conserved energy and momentum of a system, but improvement terms have to be added to the tensor from equation (16.16) in relativistic field theories to get the correct expressions for local densities for energy and momentum. We will discuss the necessary modifications of $\Theta_\nu{}^\mu$ for the Maxwell field (photons) in Section 18.1 and for relativistic fermions in Section 21.4.

## 16.3 Applications to Schrödinger field theory

The energy-momentum tensor for the Schrödinger field is found by substituting (16.3) into equation (16.16). The corresponding energy density is usually written as a Hamiltonian density $\mathcal{H}$,

$$\mathcal{H} = c\mathcal{P}^0 = -\Theta_0{}^0 = \frac{\hbar^2}{2m}\nabla\Psi^+ \cdot \nabla\Psi + \Psi^+ \cdot V \cdot \Psi, \tag{16.19}$$

and the momentum density is

$$\mathcal{P} = \frac{1}{c}\boldsymbol{e}_i\Theta^{i0} = \frac{\hbar}{2\mathrm{i}}\left(\Psi^+ \cdot \nabla\Psi - \nabla\Psi^+ \cdot \Psi\right). \tag{16.20}$$

The energy current density for the Schrödinger field follows as

$$\boldsymbol{j}_{\mathcal{H}} = -c\Theta_0{}^i\boldsymbol{e}_i = -\frac{\hbar^2}{2m}\left(\nabla\Psi^+ \cdot \frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^+}{\partial t} \cdot \nabla\Psi\right). \tag{16.21}$$

The energy $E = \int d^3\boldsymbol{x}\,\mathcal{H}$ and momentum $\boldsymbol{p} = \int d^3\boldsymbol{x}\,\mathcal{P}$ agree with the corresponding expectation values of the Schrödinger wave function in quantum mechanics. The results of the previous section, or direct application of the Schrödinger equation, tell us that $E$ is conserved if the potential is time-independent, $V = V(\boldsymbol{x})$, and the momentum component e.g. in $x$-direction is conserved if the momentum does not depend on $x$, $V = V(y, z)$.

### *Probability and charge conservation from invariance under phase rotations*

The Lagrange density (16.3) is invariant under phase rotations of the Schrödinger field,

$$\delta\Psi(\boldsymbol{x},t)=\mathrm{i}\frac{q}{\hbar}\varphi\Psi(\boldsymbol{x},t),\quad \delta\Psi^{+}(\boldsymbol{x},t)=-\mathrm{i}\frac{q}{\hbar}\varphi\Psi^{+}(\boldsymbol{x},t).$$

We wrote the constant phase in the peculiar form $q\varphi/\hbar$ in anticipation of the connection to local gauge transformations (15.8, 15.9), which will play a recurring role later on. However, for now we note that substitution of the phase transformations into the equation (16.13) yields after division by the irrelevant constant $q\varphi$ the density

$$\varrho=\frac{j^0}{c}=-\frac{1}{q\varphi}\left(\delta\Psi\frac{\partial\mathcal{L}}{\partial(\partial_t\Psi)}+\delta\Psi^{+}\frac{\partial\mathcal{L}}{\partial(\partial_t\Psi^{+})}\right)=\Psi^{+}\Psi=\frac{1}{q}\varrho_q \tag{16.22}$$

and the related current density

$$\begin{aligned}\boldsymbol{j}&=-\frac{1}{q\varphi}\left(\delta\Psi\frac{\partial\mathcal{L}}{\partial(\nabla\Psi)}+\delta\Psi^{+}\frac{\partial\mathcal{L}}{\partial(\nabla\Psi^{+})}\right)=\frac{\hbar}{2\mathrm{i}m}\left(\Psi^{+}\cdot\nabla\Psi-\nabla\Psi^{+}\cdot\Psi\right)\\&=\frac{1}{q}\boldsymbol{j}_q.\end{aligned} \tag{16.23}$$

Comparison with equations (1.17) and (1.18) shows that probability conservation in Schrödinger theory can be considered as a consequence of invariance under global phase rotations.

Had we not divided out the charge $q$, we would have drawn the same conclusion for conservation of electric charge with $\varrho_q = q\Psi^{+}\Psi$ as the charge density and $\boldsymbol{j}_q = q\boldsymbol{j}$ as the electric current density. The coincidence of the conservation laws for probability and electric charge in Schrödinger theory arises because it is a theory for non-relativistic particles. Only charge conservation will survive in the relativistic limit, but probability conservation for particles will not hold any more, because $\varrho_q(\boldsymbol{x},t)/q$ will not be positive definite any more and therefore will not yield a quantity that could be considered as a probability density to find a particle in the location $\boldsymbol{x}$ at time $t$.

Comparison with equation (16.20) tells us that $\boldsymbol{j}$ is also proportional to the momentum density,

$$\boldsymbol{j}(\boldsymbol{x},t)=\frac{1}{m}\mathcal{P}(\boldsymbol{x},t), \tag{16.24}$$

which tells us that the probability current density of the Schrödinger field is also a velocity density.

## 16.4 Problems

**16.1.** Show that addition of any derivative term $\partial_\mu \mathcal{F}(\phi_I)$ to the Lagrange density $\mathcal{L}(\phi_I, \partial\phi_I)$ does not change the Euler-Lagrange equations.

**16.2.** We consider classical particle mechanics with a Lagrangian $L(q_I, \dot{q}_I)$.

**16.2a.** Suppose the action is invariant under constant shifts $\delta q_J$ of the coordinate $q_J(t)$. Which conserved quantity do you find from equation (16.13)? Which condition must $L$ fulfill to ensure that the action is not affected by the constant shift $\delta q_J$?

**16.2b.** Now we assume that the action is invariant under constant shifts $\delta t = -\epsilon$ of the internal coordinate $t$. Which conserved quantity do you find from equation (16.13) in this case?

**16.3.** Use the Schrödinger equation to confirm that the energy density (16.19) and the energy current density (16.21) indeed satisfy the local conservation law

$$\frac{\partial}{\partial t}\mathcal{H} = -\nabla \cdot \boldsymbol{j}_{\mathcal{H}}$$

if the potential is time-independent, $V = V(\boldsymbol{x})$.

How does $E$ change if $V = V(\boldsymbol{x}, t)$ is time-dependent?

**16.4.** We have only evaluated the components $\Theta_0{}^0$, $\Theta_i{}^0$ and $\Theta_0{}^i$ of the energy-momentum tensor of the Schrödinger field in equations (16.19)–(16.21). Which momentum current densities $\boldsymbol{j}^i_{\mathcal{P}}$ do you find from the energy-momentum tensor of the Schrödinger field?

**16.5.** Schrödinger fields can have different transformation properties under coordinate rotations $\delta\boldsymbol{x} = -\boldsymbol{\varphi} \times \boldsymbol{x}$, see Section 8.2. In this problem we analyze a Schrödinger field which transforms like a scalar under rotations,

$$\delta\Psi(\boldsymbol{x}, t) = \Psi'(\boldsymbol{x}', t) - \Psi(\boldsymbol{x}, t) = 0.$$

The Lagrange density (16.3) is invariant under rotations if $V = V(r, t)$. Which conserved quantity do you find from this observation?

**Solution.** Equation (16.13) yields with $\boldsymbol{\epsilon} = \boldsymbol{\varphi} \times \boldsymbol{x}$ a conserved charge density

$$\begin{aligned}\varrho = \frac{j^0}{c} &= -(\boldsymbol{\varphi} \times \boldsymbol{x}) \cdot \left( \nabla\Psi \frac{\partial\mathcal{L}}{\partial(\partial_t \Psi)} + \nabla\Psi^+ \frac{\partial\mathcal{L}}{\partial(\partial_t \Psi^+)} \right) \\ &= -\frac{\mathrm{i}\hbar}{2}\boldsymbol{\varphi} \cdot \left[\boldsymbol{x} \times \left(\Psi^+ \cdot \nabla\Psi - \nabla\Psi^+ \cdot \Psi\right)\right] = \boldsymbol{\varphi} \cdot \mathcal{M},\end{aligned}$$

with an angular momentum density

$$\mathcal{M} = \frac{\hbar}{2\mathrm{i}}\boldsymbol{x} \times \left(\Psi^+ \cdot \nabla\Psi - \nabla\Psi^+ \cdot \Psi\right) = \boldsymbol{x} \times \mathcal{P}. \tag{16.25}$$

Since the constant parameters $\boldsymbol{\varphi}$ are arbitrary, we find three linearly independent conserved quantities, *viz.* the angular momentum

$$\boldsymbol{M} = \int d^3\boldsymbol{x}\,\mathcal{M} = \langle \mathbf{x} \times \mathbf{p} \rangle$$

of the scalar Schrödinger field.

**16.6.** Now we assume that our Schrödinger field is a 2-spinor with the transformation property

$$\delta\Psi = \frac{\mathrm{i}}{2}(\boldsymbol{\varphi}\cdot\underline{\boldsymbol{\sigma}})\cdot\Psi, \quad \delta\Psi^{+} = -\frac{\mathrm{i}}{2}\Psi^{+}\cdot(\boldsymbol{\varphi}\cdot\underline{\boldsymbol{\sigma}}).$$

Show that the corresponding density of "total angular momentum" of the Schrödinger field in this case consists of an orbital and a spin part,

$$\begin{aligned}\mathcal{J} &= \frac{\hbar}{2\mathrm{i}}\boldsymbol{x}\times\left(\Psi^{+}\cdot\nabla\Psi - \nabla\Psi^{+}\cdot\Psi\right) + \frac{\hbar}{2}\Psi^{+}\cdot\underline{\boldsymbol{\sigma}}\cdot\Psi \\ &= \boldsymbol{x}\times\mathcal{P} + \Psi^{+}\cdot\underline{\boldsymbol{S}}\cdot\Psi = \mathcal{M} + \mathcal{S}.\end{aligned} \tag{16.26}$$

Rotational invariance implies only conservation of the total angular momentum $\boldsymbol{J} = \int d^3\boldsymbol{x}\,\mathcal{J}$. However, on the level of the Lagrange density (16.3), which does not contain spin-orbit interaction terms (8.20), the orbital and spin parts are preserved separately. We will see in Section 21.5 that spin-orbit coupling is a consequence of relativity.

**16.7.** Suppose the Hamiltonian has the spin-orbit coupling form $H = \alpha\boldsymbol{M}\cdot\boldsymbol{S}$, where $M_i$ and $S_i$ are angular momentum and spin operators. How do these operators evolve in the Heisenberg picture?

**16.7a.** Show that the Heisenberg evolution equations for the operators yield

$$\dot{\boldsymbol{M}} = \alpha\boldsymbol{S}\times\boldsymbol{M}, \quad \dot{\boldsymbol{S}} = \alpha\boldsymbol{M}\times\boldsymbol{S}. \tag{16.27}$$

**16.7b.** Show that $\boldsymbol{J} \equiv \boldsymbol{M} + \boldsymbol{S}$, $\boldsymbol{M}^2$, $\boldsymbol{S}^2$ and $\boldsymbol{M}\cdot\boldsymbol{S}$ are all constant.

**16.7c.** Show that the evolution equations (16.27) are solved by

$$\boldsymbol{M}(t) = \exp(-\alpha\boldsymbol{J}\cdot\underline{\boldsymbol{L}}t - \mathrm{i}\hbar\alpha t)\cdot\boldsymbol{M} \tag{16.28}$$

and

$$\boldsymbol{S}(t) = \exp(-\alpha\boldsymbol{J}\cdot\underline{\boldsymbol{L}}t - \mathrm{i}\hbar\alpha t)\cdot\boldsymbol{S}, \tag{16.29}$$

where $\boldsymbol{M} \equiv \boldsymbol{M}(0)$, $\boldsymbol{S} \equiv \boldsymbol{S}(0)$, and $\underline{\boldsymbol{L}} = (\underline{L}_1, \underline{L}_2, \underline{L}_3)$ is the vector of matrices with components $(\underline{L}_i)_{jk} = \epsilon_{ijk}$, see equation (7.18).

**16.7d.** Except for the phase rotations, the equations (16.28, 16.29) seem to suggest that $\boldsymbol{M}(t)$ and $\boldsymbol{S}(t)$ are rotating around the direction of the vector $\boldsymbol{J}$ with angular velocity $\boldsymbol{\omega} = \alpha\boldsymbol{J}$. This suggestive picture of coupled angular momentum type operators rotating around the total angular momentum vector is often denoted as the *vector model* of spin-orbit type couplings. However, note that the total angular momentum vector is acting on tensor products of eigenstates and in fully explicit notation has the form

$$\boldsymbol{J} = \boldsymbol{M} \otimes \underline{1} + \underline{1} \otimes \boldsymbol{S}.$$

That does not mean that the results (16.27–16.29) or the conservation laws expressed in 16.7b are incorrect, but we must beware of simple interpretations in terms of vectors living within one and the same vector space.

Repeat the previous problems 16.7a–c in terms of the explicit tensor product notation using the Hamiltonian

$$H = \alpha\boldsymbol{M} \dot{\otimes} \boldsymbol{S} = \alpha M_i \otimes S_i.$$

**16.8.** Show that the Lagrange density

$$\mathcal{L} = \frac{\mathrm{i}\hbar}{2}\left(\Psi^+ \cdot \frac{\partial\Psi}{\partial t} - \frac{\partial\Psi^+}{\partial t} \cdot \Psi\right) - q\Psi^+ \cdot \Phi \cdot \Psi$$
$$- \frac{\hbar^2}{2m}\left(\nabla\Psi^+ + \mathrm{i}\frac{q}{\hbar}\Psi^+\boldsymbol{A}\right) \cdot \left(\nabla\Psi - \mathrm{i}\frac{q}{\hbar}\boldsymbol{A}\Psi\right). \tag{16.30}$$

yields the equations of motion for the Schrödinger field in external electromagnetic fields

$$\boldsymbol{E}(\boldsymbol{x},t) = -\nabla\Phi(\boldsymbol{x},t) - \frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{x},t), \quad \boldsymbol{B}(\boldsymbol{x},t) = \nabla \times \boldsymbol{A}(\boldsymbol{x},t).$$

**16.9.** Derive the electric charge and current densities for the Schrödinger field in electromagnetic fields from the phase invariance of (16.30).

**Answers.** The charge density is

$$\varrho_q = q\Psi^+\Psi. \tag{16.31}$$

The current density is

$$\boldsymbol{j}_q = \frac{q\hbar}{2\mathrm{i}m}\left(\Psi^+ \cdot \nabla\Psi - \nabla\Psi^+ \cdot \Psi\right) - \frac{q^2}{m}\Psi^+\boldsymbol{A}\Psi. \tag{16.32}$$

Are the charge and current densities gauge invariant?