Chapter 2
Self-adjoint Operators and Eigenfunction Expansions

The relevance of waves in quantum mechanics naturally implies that the decomposition of arbitrary wave packets in terms of monochromatic waves, commonly known as Fourier decomposition after Jean-Baptiste Fourier's *Théorie analytique de la Chaleur* (1822), plays an important role in applications of the theory. Dirac's δ function, on the other hand, gained prominence primarily through its use in quantum mechanics, although today it is also commonly used in mechanics and electrodynamics to describe sudden impulses, mass points, or point charges. Both concepts are intimately connected to the completeness of eigenfunctions of self-adjoint operators. From the quantum mechanics perspective, the problem of completeness of sets of functions concerns the problem of enumeration of all possible states of a quantum system.

2.1 The δ function and Fourier transforms

Let $f(x)$ be a smooth function in the interval $[a, b]$. Dirichlet's equation [7]

$$\lim_{\kappa \to \infty} \int_a^b dx' \, \frac{\sin(\kappa(x - x'))}{\pi(x - x')} f(x') = \begin{cases} 0, & x \notin [a, b], \\ f(x), & x \in (a, b), \end{cases} \tag{2.1}$$

motivates the formal definition

$$\delta(x) = \lim_{\kappa \to \infty} \frac{\sin(\kappa x)}{\pi x} = \lim_{\kappa \to \infty} \frac{1}{2\pi} \int_{-\kappa}^{\kappa} dk \, \exp(ikx)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \exp(ikx), \tag{2.2}$$

such that equation (2.1) can (in)formally be written as

$$\int_a^b dx' \, \delta(x - x')f(x') = \begin{cases} 0, & x \notin [a, b], \\ f(x), & x \in (a, b). \end{cases}$$

A justification for Dirichlet's equation is given below in the derivation of equation (2.8).

The generalization to three dimensions follows immediately from Dirichlet's formula in a three-dimensional cube, and exhaustion of an arbitrary three-dimensional volume V by increasingly finer cubes. This yields

$$\delta(\boldsymbol{x}) = \prod_{i=1}^3 \lim_{\kappa_i \to \infty} \frac{\sin(\kappa_i x_i)}{\pi x_i} = \frac{1}{(2\pi)^3} \int d^3 k \, \exp(i\boldsymbol{k} \cdot \boldsymbol{x}), \tag{2.3}$$

$$\int_V d^3 x' \, \delta(\boldsymbol{x} - \boldsymbol{x}')f(\boldsymbol{x}') = \begin{cases} 0, & \boldsymbol{x} \notin V, \\ f(\boldsymbol{x}), & \boldsymbol{x} \text{ inside } V. \end{cases}$$

The case $\boldsymbol{x} \in \partial V$ ($\boldsymbol{x}$ on the boundary of V) must be analyzed on a case-by-case basis.

Equation (2.3) implies

$$\psi(\boldsymbol{x}, t) = \int d^3 x' \, \delta(\boldsymbol{x} - \boldsymbol{x}')\psi(\boldsymbol{x}', t)$$

$$= \frac{1}{(2\pi)^3} \int d^3 k \, \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \int d^3 x' \, \exp(-i\boldsymbol{k} \cdot \boldsymbol{x}')\psi(\boldsymbol{x}', t).$$

This can be used to introduce Fourier transforms by splitting the previous equation into two equations,

$$\psi(\boldsymbol{x}, t) = \frac{1}{\sqrt{2\pi}^3} \int d^3 k \, \exp(i\boldsymbol{k} \cdot \boldsymbol{x})\psi(\boldsymbol{k}, t), \tag{2.4}$$

with

$$\psi(\boldsymbol{k}, t) = \frac{1}{\sqrt{2\pi}^3} \int d^3 x \, \exp(-i\boldsymbol{k} \cdot \boldsymbol{x})\psi(\boldsymbol{x}, t). \tag{2.5}$$

Use of $\psi(\boldsymbol{x}, t)$ corresponds to the *x-representation* of quantum mechanics. Use of $\psi(\boldsymbol{k}, t)$ corresponds to the *k-representation* or *momentum-representation* of quantum mechanics.

The notation above for Fourier transforms is a little sloppy, but convenient and common in quantum mechanics. From a mathematical perspective, the Fourier transformed function $\psi(\boldsymbol{k}, t)$ should actually be denoted by $\tilde{\psi}(\boldsymbol{k}, t)$ to make it clear that it is *not* the same function as $\psi(\boldsymbol{x}, t)$ with different symbols for the first three variables. The physics notation is motivated by the observation that $\psi(\boldsymbol{x}, t)$ and $\psi(\boldsymbol{k}, t)$ are just different representations of the same quantum mechanical state ψ.

Another often used convention for Fourier transforms is to split the factor $(2\pi)^{-3}$ asymmetrically, or equivalently replace it with a factor 2π in the exponents,

$$\psi(x, t) = \frac{1}{(2\pi)^3} \int d^3k \, \exp(i k \cdot x) \psi(k, t),$$

$$\psi(k, t) = \int d^3x \, \exp(-i k \cdot x) \psi(x, t),$$

or equivalently

$$\psi(x, t) = \int d^3\tilde{v} \, \exp(2\pi i \tilde{v} \cdot x) \psi(\tilde{v}, t),$$

$$\psi(\tilde{v}, t) = \int d^3x \, \exp(-2\pi i \tilde{v} \cdot x) \psi(x, t),$$

with the vector of wave numbers

$$\tilde{v} = \frac{k}{2\pi}.$$

The conventions (2.4, 2.5) are used throughout this book.

The following is an argument for equation (2.1) and its generalizations to other representations of the δ function. The idea is to first construct a limit for the Heaviside step function or Θ function

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}$$

and go from there. The value of $\Theta(0)$ is often chosen to suite the needs of the problem at hands. The choice $\Theta(0) = 1/2$ seems intuitive and is also mathematically natural in the sense that any decomposition of a discontinuous functions in a complete set of functions (e.g. Fourier decomposition) will approximate the mean value between the left and right limit for a finite discontinuity, but in many applications other values of $\Theta(0)$ are preferred.

The Θ function helps us to explain Dirichlet's equation (2.1) through the following construction. Suppose $d(x)$ is a normalized function,

$$\int_{-\infty}^{\infty} dx \, d(x) = 1. \tag{2.6}$$

The integral

$$D(x) = \int_{-\infty}^{x} d\xi \, d(\xi)$$

satisfies

$$\lim_{\kappa \to \infty} D(\kappa \cdot x) = \Theta(x), \tag{2.7}$$

where we apparently defined $\Theta(0)$ as $\Theta(0) = \int_{-\infty}^{0} d\xi\, d(\xi)$, but this plays no role for the following reasoning.

Equation (2.7) yields for $f(x)$ differentiable in $[a, b]$

$$\int_{a}^{b} dx\, \kappa\, d(\kappa \cdot x) f(x) = D(\kappa \cdot x) f(x) \Big|_{a}^{b} - \int_{a}^{b} dx\, D(\kappa \cdot x) f'(x),$$

$$\lim_{\kappa \to \infty} \int_{a}^{b} dx\, \kappa\, d(\kappa \cdot x) f(x) = \Theta(b) f(b) - \Theta(a) f(a) - \int_{a}^{b} dx\, \Theta(x) f'(x)$$

$$= \Theta(b) f(b) - \Theta(a) f(a) - \Theta(b)[f(b) - f(0)] + \Theta(a)[f(a) - f(0)]$$

$$= [\Theta(b) - \Theta(a)] f(0), \tag{2.8}$$

where we simply split

$$\int_{a}^{b} dx\, \Theta(x) f'(x) = \int_{0}^{b} dx\, \Theta(x) f'(x) - \int_{0}^{a} dx\, \Theta(x) f'(x)$$

to arrive at the final result. Equation (2.8) confirms

$$\lim_{\kappa \to \infty} \kappa\, d(\kappa x) = \delta(x), \tag{2.9}$$

or after shifting the argument,

$$\lim_{\kappa \to \infty} \kappa\, d[\kappa(x - x_0)] = \delta(x - x_0).$$

From a mathematical perspective, equations like (2.9) mean that the action of the δ distribution on a smooth function corresponds to integration with a kernel $\kappa\, d(\kappa x)$ and then taking the limit $\kappa \to \infty$.

Equation (2.2) is an important particular realization of equation (2.9) with the normalized sinc function $d(x) = \mathrm{sinc}(x)/\pi = \sin(x)/\pi x$. Another important realization uses the function $d(x) = (\pi + \pi x^2)^{-1}$,

$$\delta(x) = \lim_{\kappa \to \infty} \frac{1}{\pi} \frac{\kappa}{1 + \kappa^2 x^2} = \lim_{a \to 0} \frac{1}{\pi} \frac{a}{a^2 + x^2}$$

$$= \lim_{a \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \exp(ikx - a|k|). \tag{2.10}$$

Note that we did not require $d(x)$ to have a maximum at $x = 0$ to derive (2.9), and indeed we do not need this requirement. Consider the following example,

$$d(x) = \frac{1}{2}\sqrt{\frac{\alpha}{\pi}}\exp[-\alpha(x-a)^2] + \frac{1}{2}\sqrt{\frac{\beta}{\pi}}\exp[-\beta(x-b)^2].$$

This function has two maxima if $\alpha \cdot \beta \neq 0$ and if a and b are sufficiently far apart, and it even has a minimum at $x = 0$ if $\alpha = \beta$ and $a = -b$. Yet we still have

$$\lim_{\kappa\to\infty}\kappa\,d(\kappa\cdot x) = \lim_{\kappa\to\infty}\left(\frac{\kappa}{2}\sqrt{\frac{\alpha}{\pi}}\exp[-\alpha(\kappa x - a)^2]\right.$$

$$\left. + \frac{\kappa}{2}\sqrt{\frac{\beta}{\pi}}\exp[-\beta(\kappa x - b)^2]\right) = \delta(x),$$

because the scaling with κ scales the initial maxima near a and b to $a/\kappa \to 0$ and $b/\kappa \to 0$.

Sokhotsky-Plemelj relations

The Sokhotsky-Plemelj relations are very useful relations involving a δ distribution[1],

$$\frac{1}{x - i\epsilon} = \mathcal{P}\frac{1}{x} + i\pi\delta(x), \quad \frac{1}{x + i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x). \tag{2.11}$$

Indeed, for the practical evaluation of integrals involving singular denominators, we virtually never use these relations but evaluate the integrals with the left hand sides directly using the Cauchy and residue theorems. The primary use of the Sokhotsky-Plemelj relations in physics and technology is to establish relations between different physical quantities. The relation between retarded Green's functions and local densities of states is an example for this and will be derived in Section 20.1.

I will give a brief justification for the Sokhotsky-Plemelj relations. The relations

$$\frac{1}{x + i\epsilon} = \frac{1}{i}\int_0^\infty dk\,\exp[ik(x + i\epsilon)] = \frac{1}{i}\int_{-\infty}^0 dk\,\exp[-ik(x + i\epsilon)]$$

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[1]Yu.V. Sokhotsky, Ph.D. thesis, University of St. Petersburg, 1873; J. Plemelj, Monatshefte Math. Phys. 19, 205 (1908). The "physics" version (2.11) of the Sokhotsky-Plemelj relations is of course more recent than the original references because the δ distribution was only introduced much later.

imply

$$\Im \frac{1}{x + i\epsilon} = -\frac{1}{2} \int_{-\infty}^{\infty} dk \, \cos(kx) = -\pi \delta(x).$$

On the other hand, the real part is

$$\Re \frac{1}{x + i\epsilon} = \frac{1}{2(x + i\epsilon)} + \frac{1}{2(x - i\epsilon)} = \frac{x}{x^2 + \epsilon^2}.$$

This implies for integration with a bounded function $f(x)$ in $[a, b]$

$$\int_a^b dx \, \frac{f(x)}{x + i\epsilon} = \int_a^b dx \, \frac{xf(x)}{x^2 + \epsilon^2} - i\pi[\Theta(b) - \Theta(a)]f(0).$$

However, the weight factor

$$K_\epsilon(x) = \frac{x}{x^2 + \epsilon^2}$$

essentially cuts the region $-3\epsilon < x < 3\epsilon$ symmetrically from the integral $\int_a^b dx f(x)/x$ (the value 3ϵ is chosen because $xK_\epsilon(x) = 0.9$ for $x = \pm 3\epsilon$), see Figure 2.1. Therefore we can use this factor as one possible definition of a principal value integral,

$$\mathcal{P} \int_a^b dx \, \frac{f(x)}{x} = \lim_{\epsilon \to 0} \int_a^b dx \, K_\epsilon(x) f(x).$$

2.2   Self-adjoint operators and completeness of eigenstates

The statistical interpretation of the wave function $\psi(\boldsymbol{x}, t)$ implies that the wave functions of single stable particles should be normalized,

$$\int d^3\boldsymbol{x} \, |\psi(\boldsymbol{x}, t)|^2 = 1. \tag{2.12}$$

Time-dependence plays no role and will be suppressed in the following investigations.

Indeed, we have to require a little more than just normalizability of the wave function $\psi(\boldsymbol{x})$ itself, because the functions $\nabla \psi(\boldsymbol{x})$, $\Delta \psi(\boldsymbol{x})$, and $V(\boldsymbol{x})\psi(\boldsymbol{x})$ for admissible potentials $V(\boldsymbol{x})$ should also be square integrable. We will therefore also encounter functions $f(\boldsymbol{x})$ which may not be normalized, although they are square integrable,
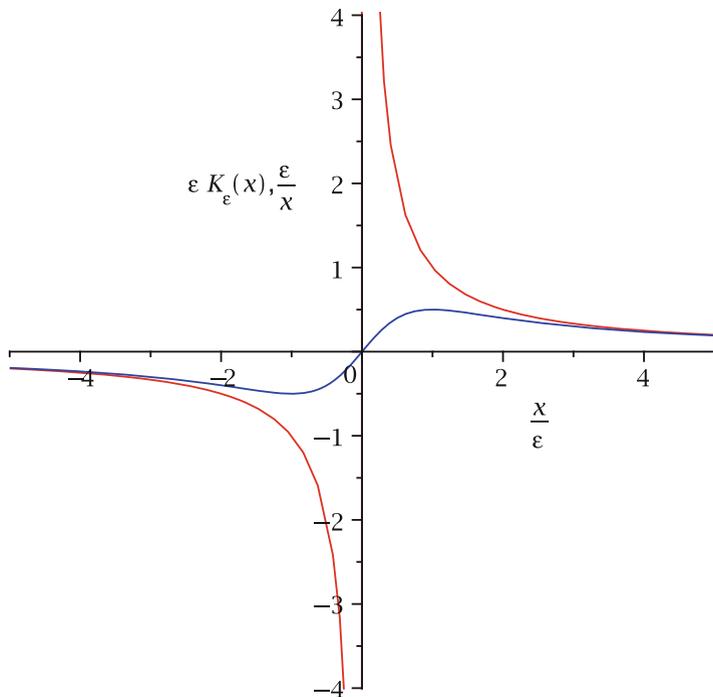
Fig. 2.1 Comparison of $1/x$ with the weight factor $K_\epsilon(x)$

$$\int d^3x \, |f(x)|^2 < \infty.$$

Let $\psi(x)$ and $\phi(x)$ be two square integrable functions. The identity

$$\int d^3x \, |\psi(x) - \lambda\phi(x)|^2 \geq 0$$

yields with the choice

$$\lambda = \frac{\int d^3x \, \phi^+(x)\psi(x)}{\int d^3x \, |\phi(x)|^2}$$

the Schwarz inequality

$$\left|\int d^3x \, \phi^+(x)\psi(x)\right|^2 \leq \int d^3x \, |\psi(x)|^2 \int d^3x' \, |\phi(x')|^2.$$

The differential operators $-i\hbar\nabla$ and $-(\hbar^2/2m)\Delta$, which we associated with momentum and kinetic energy, and the potential energy $V(\boldsymbol{x})$ all have the following properties,

$$\int d^3\boldsymbol{x}\,\phi^+(\boldsymbol{x})\frac{\hbar}{i}\nabla\psi(\boldsymbol{x}) = \left(\int d^3\boldsymbol{x}\,\psi^+(\boldsymbol{x})\frac{\hbar}{i}\nabla\phi(\boldsymbol{x})\right)^+, \tag{2.13}$$

$$\int d^3\boldsymbol{x}\,\phi^+(\boldsymbol{x})\Delta\psi(\boldsymbol{x}) = \left(\int d^3\boldsymbol{x}\,\psi^+(\boldsymbol{x})\Delta\phi(\boldsymbol{x})\right)^+, \tag{2.14}$$

and

$$\int d^3\boldsymbol{x}\,\phi^+(\boldsymbol{x})V(\boldsymbol{x})\psi(\boldsymbol{x}) = \left(\int d^3\boldsymbol{x}\,\psi^+(\boldsymbol{x})V(\boldsymbol{x})\phi(\boldsymbol{x})\right)^+. \tag{2.15}$$

Equation (2.15) is a consequence of the fact that $V(\boldsymbol{x})$ is a real function. Equations (2.13, 2.14) are a direct consequence of partial integrations and the fact that boundary terms at $|\boldsymbol{x}| \to \infty$ vanish under the assumptions that we had imposed on the wave functions.

If two operators $A_{\boldsymbol{x}}$ and $B_{\boldsymbol{x}}$ have the property

$$\int d^3\boldsymbol{x}\,\phi^+(\boldsymbol{x})A_{\boldsymbol{x}}\psi(\boldsymbol{x}) = \left(\int d^3\boldsymbol{x}\,\psi^+(\boldsymbol{x})B_{\boldsymbol{x}}\phi(\boldsymbol{x})\right)^+, \tag{2.16}$$

for *all* wave functions of interest, then $B_{\boldsymbol{x}}$ is denoted as *adjoint* to the operator $A_{\boldsymbol{x}}$. The mathematical notation for the adjoint operator to $A_{\boldsymbol{x}}$ is $A_{\boldsymbol{x}}^+$,

$$B_{\boldsymbol{x}} = A_{\boldsymbol{x}}^+.$$

Complex conjugation of (2.16) then immediately tells us $B_{\boldsymbol{x}}^+ = A_{\boldsymbol{x}}$.

An operator with the property $A_{\boldsymbol{x}}^+ = A_{\boldsymbol{x}}$ is denoted as a *self-adjoint* or *hermitian* operator[2]. Self-adjoint operators are important in quantum mechanics because they yield real expectation values,

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[2]We are not addressing matters of definition of domains of operators in function spaces, see e.g. [21] or Problem 2.6. If the operators $A_{\boldsymbol{x}}^+$ and $A_{\boldsymbol{x}}$ can be defined on different classes of functions, and $A_{\boldsymbol{x}}^+ = A_{\boldsymbol{x}}$ holds on the intersections of their domains, then $A_{\boldsymbol{x}}$ is usually denoted as a *symmetric operator*. The notion of self-adjoint operator requires identical domains for both $A_{\boldsymbol{x}}$ and $A_{\boldsymbol{x}}^+$ such that the domain of neither operator can be extended. If the conditions on the domains are violated, we can e.g. have a situation where $A_{\boldsymbol{x}}$ has no eigenfunctions at all, or where the eigenvalues of $A_{\boldsymbol{x}}$ are complex and the set of eigenfunctions is overcomplete. Hermiticity is sometimes defined as equivalent to symmetry or as equivalent to the more restrictive notion of self-adjointness of operators. We define Hermiticity as self-adjointness.

$$\left(\langle A\rangle_\psi\right)^+ = \left(\int d^3\boldsymbol{x}\,\psi^+(\boldsymbol{x})A_{\boldsymbol{x}}\psi(\boldsymbol{x})\right)^+ = \int d^3\boldsymbol{x}\,\psi^+(\boldsymbol{x})A_{\boldsymbol{x}}^+\psi(\boldsymbol{x})$$

$$= \int d^3\boldsymbol{x}\,\psi^+(\boldsymbol{x})A_{\boldsymbol{x}}\psi(\boldsymbol{x}) = \langle A\rangle_\psi.$$

Observable quantities like energy or momentum or location of a particle are therefore implemented through self-adjoint operators, e.g. momentum $\boldsymbol{p}$ is implemented through the self-adjoint differential operator $-i\hbar\nabla$. We have seen one method to figure this out in equation (1.21). We will see another method in equations (4.26, 4.27).

Self-adjoint operators have the further important property that their eigenfunctions yield *complete sets* of functions. Schematically this means the following: Suppose we can enumerate all constants a_n and functions $\psi_n(\boldsymbol{x})$ which satisfy the equation

$$A_{\boldsymbol{x}}\psi_n(\boldsymbol{x}) = a_n\psi_n(\boldsymbol{x}) \tag{2.17}$$

with the set of discrete indices $\boldsymbol{n}$. The constants a_n are *eigenvalues* and the functions $\psi_n(\boldsymbol{x})$ are *eigenfunctions* of the operator $A_{\boldsymbol{x}}$. Hermiticity of the operator $A_{\boldsymbol{x}}$ implies orthogonality of eigenfunctions for different eigenvalues,

$$a_n\int d^3\boldsymbol{x}\,\psi_m^+(\boldsymbol{x})\psi_n(\boldsymbol{x}) = \int d^3\boldsymbol{x}\,\psi_m^+(\boldsymbol{x})A_{\boldsymbol{x}}\psi_n(\boldsymbol{x})$$

$$= \left(\int d^3\boldsymbol{x}\,\psi_n^+(\boldsymbol{x})A_{\boldsymbol{x}}\psi_m(\boldsymbol{x})\right)^+$$

$$= a_m\int d^3\boldsymbol{x}\,\psi_m^+(\boldsymbol{x})\psi_n(\boldsymbol{x})$$

and therefore

$$\int d^3\boldsymbol{x}\,\psi_m^+(\boldsymbol{x})\psi_n(\boldsymbol{x}) = 0 \ \text{ if } \ a_n \neq a_m.$$

However, even if $a_n = a_m$ for different indices $\boldsymbol{n} \neq \boldsymbol{m}$ (i.e. if the eigenvalue a_n is *degenerate* because there exist at least two eigenfunctions with the same eigenvalue), one can always chose orthonormal sets of eigenfunctions for a degenerate eigenvalue. We therefore require

$$\int d^3\boldsymbol{x}\,\psi_m^+(\boldsymbol{x})\psi_n(\boldsymbol{x}) = \delta_{m,n}. \tag{2.18}$$

Completeness of the set of functions $\psi_n(\boldsymbol{x})$ means that an "arbitrary" function $f(\boldsymbol{x})$ can be expanded in terms of the eigenfunctions of the self-adjoint operator $A_{\boldsymbol{x}}$ in the form

$$f(\boldsymbol{x}) = \sum_n c_n\psi_n(\boldsymbol{x}) \tag{2.19}$$

with expansion coefficients

$$c_n = \int d^3x \, \psi_n^+(x) f(x). \tag{2.20}$$

If we substitute equation (2.20) into (2.19) and (in)formally exchange integration and summation, we can express the completeness property of the set of functions $\psi_n(x)$ in the *completeness relation*

$$\sum_n \psi_n(x)\psi_n^+(x') = \delta(x - x'). \tag{2.21}$$

Both the existence and the meaning of the series expansions (2.19, 2.20) depends on what large a class of "arbitrary" functions $f(x)$ one considers. Minimal constraints require boundedness of $f(x)$, and continuity if the series (2.19) is supposed to converge pointwise. The default constraints in non-relativistic quantum mechanics are continuity of wave functions $\psi(x)$ to ensure validity of the Schrödinger equation with at most finite discontinuities in potentials $V(x)$, and normalizability. Under these circumstances the expansion (2.19, 2.20) for a wave function $f(x) \equiv \psi(x)$ will converge pointwise to $\psi(x)$. However, it is convenient for many applications of quantum mechanics to use limiting forms of wave functions which are not normalizable in the sense of equation (2.12) any more, e.g. plane wave states $\psi_k(x) \propto \exp(ik \cdot x)$, and we will frequently also have to expand non-continuous functions, e.g. functions of the form $f(x) = V(x)\psi(x)$ with a discontinuous potential $V(x)$. However, finally we only have to use expansions of the form (2.19, 2.20) in the evaluation of integrals of the form $\int d^3x g^+(x)f(x)$, and here the concept of *convergence in the mean* comes to our rescue in the sense that substitution of the series expansion (2.19, 2.20) in the integral will converge to the same value of the integral, even if the expansion (2.19, 2.20) does not converge pointwise to the function $f(x)$.

A more thorough discussion of completeness of sets of eigenfunctions of self-adjoint operators in the relatively simple setting of wave functions confined to a finite one-dimensional interval is presented in Appendix C. However, for a first reading I would recommend to accept the series expansions (2.19, 2.20) with the assurance that substitutions of these series expansions is permissible in the calculation of observables in quantum mechanics.

2.3 Problems

2.1. Suppose the function $f(x)$ has only first order zeros, i.e. we have non-vanishing slope at all nodes x_i of the function,

$$f(x_i) = 0 \;\; \Rightarrow \;\; f'(x_i) \equiv \left.\frac{df(x)}{dx}\right|_{x=x_i} \neq 0.$$

Prove the following property of the δ function:

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \delta(x - x_i).$$

2.2. Calculate the Fourier transforms of the following functions, where in all cases $-\infty < x < \infty$. Do not use any electronic integration program.

2.2a. $\psi_1(x) = \exp(-ax^2)$, $\Re a > 0$,

2.2b. $\psi_2(x) = 1/(a^2 + x^2)$, $a > 0 \in \mathbb{R}$,

2.2c. $\psi_3(x) = x^n \exp(-a|x|)$, $a > 0 \in \mathbb{R}$, where n is a natural number.

2.3. The functions $f_1(x) = \exp(-x^2)$ and $f_2(x) = \exp(-|x|)$ are normalizable to functions $d(x)$ in the sense of equation (2.6). Use this to find other derivations of the Fourier representation of the δ function similar to equation (2.10).

2.4. We consider a finite interval $[a, b]$ together with the set $C^{(1,\alpha)}[a, b]$ of complex valued functions which are continuous in $[a, b]$ and differentiable in (a, b), and satisfy the pseudo-periodicity condition

$$\psi(b) = \exp(i\alpha)\psi(a), \quad \alpha \in \mathbb{R}.$$

Show that the differential operator $-id/dx$ is self-adjoint on $C^{(1,\alpha)}[a, b]$. Give a complete set of eigenstates of $-id/dx$ in $C^{(1,\alpha)}[a, b]$.

2.5. We consider the finite interval $[a, b]$ together with the set $C^{(2),0}[a, b]$ of complex valued functions which are continuous in $[a, b]$ and second order differentiable in (a, b), and satisfy the boundary conditions

$$\psi(a) = \psi(b) = 0.$$

Show that that the differential operator d^2/dx^2 is self-adjoint on $C^{(2),0}[a, b]$. Give a complete set of eigenstates of d^2/dx^2 in $C^{(2),0}[a, b]$.

2.6. We consider the finite interval $[a, b]$ together with the set $C^{(1),0}[a, b]$ of complex valued functions which are continuous in $[a, b]$ and differentiable in (a, b), and satisfy the boundary conditions

$$\psi(a) = \psi(b) = 0.$$

Show that the symmetric differential operator $h_1 = -id/dx$ with domain $C^{(1),0}[a, b]$ is not self-adjoint in the sense that h_1^+ can be defined on the larger set $L_2[a, b]$ of square integrable functions over $[a, b]$.

Show that h_1 has no eigenstates, while h_1^+ has complex eigenvalues and an overcomplete set of eigenstates.