Chapter 5
Curves in $\mathbb{R}^n$

Summary *We introduce and discuss the concept of directed curve in* $\mathbb{R}^n$. *We obtain a formula for the length of a curve, prove the existence of unit speed parametrizations and define piecewise smooth curves.*

Directed and parametrized curves play a role in many of the topics discussed in the remaining chapters of this book, e.g. line integrals, existence of a potential, Stokes' theorem and the geometry of surfaces in $\mathbb{R}^3$, and, furthermore, in a simple fashion introduce us to concepts such as parametrizations and orientations that are later developed and generalised in more involved settings.

We begin by giving a rigorous definition of directed curve. This may appear complicated and unnecessarily cumbersome at first glance and so we feel it proper to elaborate on why each condition is included. It is always important in mathematics to understand the basic *definition* and to refer to it until one appreciates each part separately and the totality of parts collectively. As progress is achieved there is usually less need to refer to the definition but in case of ambiguity the definition is the book of *rules*. The only requirement in a definition is that it be consistent, i.e. that the various conditions do not contradict one another or the rules of mathematics. Apart from this there is freedom in the choice of conditions and, indeed, many books would give a slightly different definition of directed curve. The differences depend on the degree of generality sought, the results aimed at and the methods used. However, all definitions of curve, directed curve and parametrized curve contain the same essential features.

Definition 5.1 A *directed* (or *oriented*) curve in $\mathbb{R}^n$ is a quadruple $\{\Gamma, A, B, \mathbf{v}\}$ where Γ is a set of points in $\mathbb{R}^n$; A and B are points in Γ, called respectively the *initial* and *final points* of Γ; $\mathbf{v}$ is a unit vector in $\mathbb{R}^n$ called the *initial direction*, for which there exists a mapping $P\colon [a, b] \to \mathbb{R}^n$, called a *parametrization* of Γ, such that the following conditions hold:

(a) there exists an *open* interval I, containing $[a, b]$ and a mapping from I into $\mathbb{R}^n$ which has derivatives of all orders and which coincides with P on $[a, b]$
(b) $P([a, b]) = \Gamma$, $P(a) = A$, $P(b) = B$ and $P'(a) = \alpha\mathbf{v}$ for some $\alpha > 0$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_5,

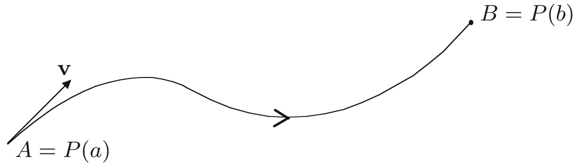

Fig. 5.1

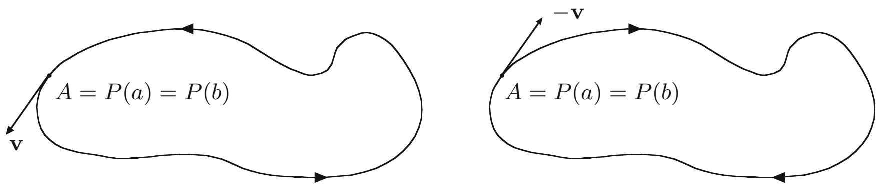

Fig. 5.2

(c) $P'(t) \neq 0$ for all $t \in [a, b]$
(d) P is injective (i.e. one to one) on $[a, b)$ and $(a, b]$.

Condition (a) is a rather strong regularity condition—we use the first derivative to define the tangent, the second to define curvature, the third to define torsion and the fourth to obtain the Frenet–Serret equations, and at this stage we felt it was just as easy to assume that we had derivatives of all orders. We also wished to have derivatives at the end points of the interval $[a, b]$ and for this reason we assumed that P has an extension as a *smooth* function (i.e. as a function with derivatives of all orders) to an open interval containing $[a, b]$. We could achieve precisely the same degree of smoothness by using one-sided derivatives at a and b but felt this *appears* even more complicated. All definitions of a curve will include, as an essential feature a *continuous* mapping P from an interval I in $\mathbb{R}$ *onto* Γ. The degree of differentiability, whether the interval I is open, closed, finite or infinite may be regarded as options that are available. We have chosen the options that suit our purposes.

The essential feature of condition (b), as we have just noted, is $P([a, b]) = \Gamma$. The remaining parts endow the set Γ with a sense of direction. If $A \neq B$ then the conditions $P(a) = A$ and $P(b) = B$ define a direction along Γ and in this case the condition $P'(a) = \alpha\mathbf{v}$ is redundant as $\mathbf{v}$ is determined by $\{\Gamma, A, B\}$ (Fig. 5.1).

If, however, $A = B$ then we have a *closed curve*, and it is necessary to distinguish between the two directions we may travel around Γ. In the case of curves in $\mathbb{R}^2$ we have clockwise and anticlockwise or counterclockwise directions. In $\mathbb{R}^n$ we do not have such a concept and instead specify the direction along the curve by giving an initial direction $\mathbf{v}$ (Fig. 5.2).

Note that when we know Γ and A then $\mathbf{v} = \pm P'(a) \big/ \|P'(a)\|$ and the condition $\alpha > 0$ distinguishes between the two signs and fixes the direction. Condition (c) is necessary to obtain a unit speed parametrization and we have already used $P'(a) \neq 0$ in (b). This condition excludes curves with corners but we get around this problem

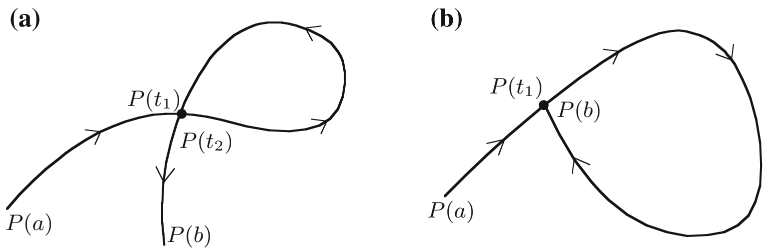

Fig. 5.3

by defining piecewise smooth curves. These are obtained by placing end to end a finite number of directed curves. The transition from directed curves to piecewise smooth curves is painless. Later we will require $P''(t) \neq 0$ in order to define a unit normal to a directed curve in $\mathbb{R}^3$.

Since we allow $A = B$ it follows that the mapping P may not be injective on $[a, b]$. However, we do not wish the curve to cross itself (Fig. 5.3a) or to half cross itself (Fig. 5.3b) as these lead to unnecessary complications and we have included condition (d) to exclude such possibilities.

A continuous mapping $P\colon [a, b] \to \mathbb{R}^n$ which satisfies (a), (c) and (d) is called a *parametrized curve*. A parametrized curve determines precisely one directed curve

$$\left\{P([a, b]),\ P(a),\ P(b),\ \frac{P'(a)}{\|P'(a)\|}\right\}$$

for which it is a parametrization.

Since the terminology "directed curve" and the notation $\{\Gamma, A, B, \mathbf{v}\}$ are rather cumbersome we will use the term *curve* and the notation Γ in all cases where there is little danger of confusion. If $A \neq B$ we sometimes write $\{\Gamma, A, B\}$. If we need to use coordinates we usually let $P(t) = \big(x_1(t), \dots x_n(t)\big)$ if $\Gamma \in \mathbb{R}^n$ and, when $n = 3$, we let $P(t) = \big(x(t), y(t), z(t)\big)$. It is helpful to think of $[a, b]$ as an interval of time and $P(t)$ as the position of a particle at time t as it travels along the route Γ from A to B.

We call $P'(t)$ the *velocity* and $\|P'(t)\|$ the *speed* at time t. Since *distance* = *speed* × *time* the formula

$$l(\Gamma) = \int_a^b \|P'(t)\| dt$$

where $l(\Gamma)$ is the length of Γ, is not surprising. We shall, however, pause to prove this formula in order to show the usefulness of vector notation. Since P is differentiable we have for all t and $t + \Delta t$ in $[a, b]$

$$P(t + \Delta t) = P(t) + P'(t)\Delta t + g(t, \Delta t) \cdot \Delta t$$

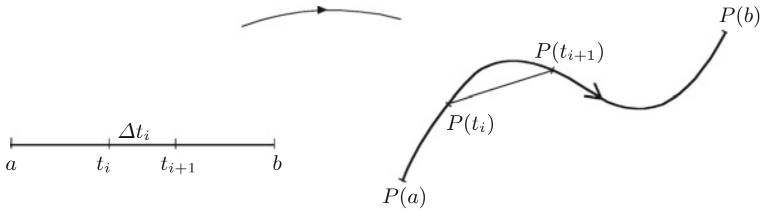

Fig. 5.4

where $g(t, \Delta t) \to 0$ as $\Delta t \to 0$ for any fixed t. Hence $P(t+\Delta t) - P(t) \approx P'(t)\Delta t$ for Δt close to zero. If we partition $[a, b]$ we get a corresponding partition of Γ and an approximation of the length of Γ (Fig. 5.4).

We have

$$\begin{aligned} l(\Gamma) &\approx \sum_i \|P(t_{i+1}) - P(t_i)\| \\ &\approx \sum_i \|P'(t_i)\|\Delta t_i \\ &\longrightarrow \int_a^b \|P'(t)\| dt \end{aligned}$$

as we take finer and finer partitions of $[a, b]$. In terms of coordinates we have $P(t) = \big(x_1(t), \ldots, x_n(t)\big)$, $P'(t) = \big(x_1'(t), \ldots, x_n'(t)\big)$ and

$$l(\Gamma) = \int_a^b \|P'(t)\| dt = \int_a^b \big(x_1'(t)^2 + \cdots + x_n'(t)^2\big)^{1/2} dt.$$

Example 5.2 Let $P(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $t \in [0, 2\pi]$, denote a parametrized curve Γ in $\mathbb{R}^3$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote the standard unit vector basis in $\mathbb{R}^3$, i.e. $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. The curve Γ is part of a *helix*—it *spirals* around a vertical *cylinder* (Fig. 5.5).

If we consider only the first two coordinates, this amounts to projecting onto the $\mathbb{R}^2$ plane in $\mathbb{R}^3$, we get the standard parametrization $t \to (\cos t, \sin t)$ of the unit circle. Hence, disregarding the final coordinate, which we take to be the height, the particle appears to move in a circle. We have

$$P'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}.$$

The rate of change of the height is given by the coefficient of the $\mathbf{k}$ term and so the particle is rising with constant speed. We have

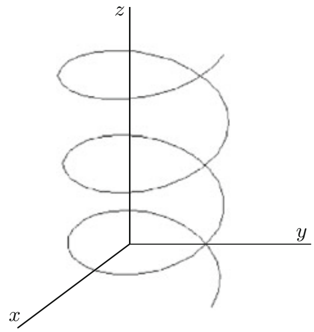

Fig. 5.5

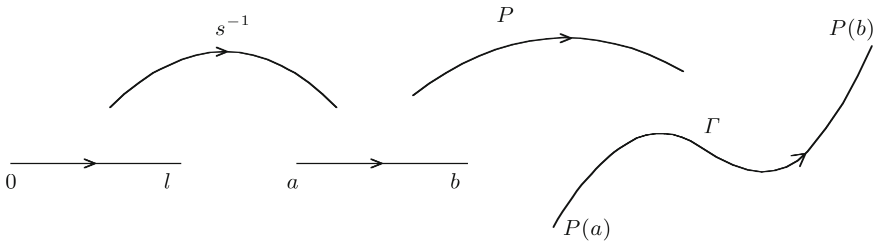

Fig. 5.6

$$\|P'(t)\| = \Big((-\sin t)^2 + (\cos t)^2 + 1^2\Big)^{1/2} = \sqrt{2}$$

and

$$l(\Gamma) = \int_0^{2\pi} \sqrt{2}dt = 2\sqrt{2}\pi.$$

If $P: [a, b] \to \mathbb{R}^n$ is a parametrization of the directed curve Γ we define the *length function* by the formula

$$s(t) = \int_a^t \|P'(x)\|dx$$

for all $t \in [a, b]$. If $l = l(\Gamma)$ then $s: [a, b] \to [0, l]$ and, by the one-variable fundamental theorem of calculus, $s'(t) = \|P'(t)\| > 0$. Hence s is strictly increasing, $s^{-1}: [0, l] \to [a, b]$ has derivatives of all orders on $[0, l]$ and $P \circ s^{-1}$ maps $[0, l]$ onto Γ (Fig. 5.6). For the inverse function s^{-1} we have

$$(s^{-1})'(t) = \frac{1}{s'\big(s^{-1}(t)\big)} = \frac{1}{\|P'\big(s^{-1}(t)\big)\|}$$

and hence

$$\|(P \circ s^{-1})'(t)\| = \frac{\|P'(s^{-1}(t))\|}{s'(s^{-1}(t))} = \frac{\|P'(s^{-1}(t))\|}{\|P'(s^{-1}(t))\|} = 1.$$

Since the remaining conditions (for a parametrization) are easily checked it follows that $P \circ s^{-1}$ is a parametrization of Γ and we have proved the following result.

Proposition 5.3 *Directed curves admit unit speed parametrizations.*

If two particles start at the point A at time zero and both proceed along Γ towards B at unit speed then their positions on Γ at time t will always agree. This shows that a directed curve of length l admits a *unique* unit speed parametrization on $[0, l]$. Using our construction of unit speed curve we may restate this as follows: if $P_1 : [a, b] \to \Gamma$ and $P_2 : [c, d] \to \Gamma$ are any two parametrizations of the directed curve Γ, of length l, and

$$s_1(t) = \int_a^t \|P_1'(x)\|dx, \qquad s_2(t) = \int_c^t \|P_2'(x)\|dx$$

are the associated length functions then

$$P_1 \circ s_1^{-1}(t) = P_2 \circ s_2^{-1}(t)$$

for all $t \in [0, l]$.

To include curves with corners we extend the concept of directed curve to that of piecewise smooth directed curve. A finite collection of directed curves $\{\Gamma_i, A_i, B_i, \mathbf{v_i}\}_{i=1}^n$ is called a *piecewise smooth directed curve* if

(a) each $\{\Gamma_i, A_i, B_i, \mathbf{v_i}\}$ is a directed curve,
(b) $B_i = A_{i+1}$ for $i = 1, \ldots, n-1$ (i.e. the *final* point of Γ_i coincides with the *initial* point of Γ_{i+1}).

If $B_n = A_1$ we say that the piecewise smooth directed curve is *closed*. We use the notation Γ for a piecewise smooth directed curve and A and B for its initial and final points respectively. This definition is rather general and apart from curves with corners it also includes curves which cross one another. In general such curves do not admit a unit speed parametrization but it can be shown that there exists a *continuous* parametrization

$$P : [a, b] \longrightarrow \Gamma = \bigcup_{i=1}^n \Gamma_i$$

and a partition $\{a_0 = a, a_1, a_2, \ldots, a_n = b\}$ of $[a, b]$ such that $P\big([a_{i-1}, a_i]\big) = \Gamma_i$, $P(a_{i-1}) = A_i$, $P(a_i) = B_i$, $i = 1, \ldots, n$, P has derivatives of all orders on $[a, b]$ and $P'(t) \neq 0$ for $t \neq a_0, \ldots, a_n$.

A piecewise smooth directed curve $\Gamma = \{\Gamma_i, A_i, B_i, \mathbf{v_i}\}_{i=1}^n$ is studied and applied by considering each of the component sections $\{\Gamma_i, A_i, B_i, \mathbf{v_i}\}$ in turn (see for instance Exercise 5.6 and our method of finding a potential in Chap. 6).

Exercises

5.1 Find the length of the curve parametrized by

$$P(t) = (2\cosh 3t, -2\sinh 3t, 6t), \quad 0 \leq t \leq 5.$$

5.2 Show that the following parametrizations are unit speed

(a) $P_1(s) = \frac{1}{2}\Big(s + \sqrt{s^2+1}, (s+\sqrt{s^2+1})^{-1}, \sqrt{2}\log(s+\sqrt{s^2+1})\Big)$, $s \in [0, 1]$

(b) $P_2(s) = \Big(\dfrac{(1+s)^{3/2}}{3}, \dfrac{(1-s)^{3/2}}{3}, \dfrac{s}{\sqrt{2}}\Big), \quad s \in [-1, +1]$

(c) $P_3(s) = \dfrac{1}{2}\big(\cos^{-1}(s) - s\sqrt{1-s^2}, 1-s^2, 0\big), \quad s \in [0, 1].$

5.3 Let r and h denote positive numbers. Find a unit speed parametrization of the helix $P(t) = (r\cos t, r\sin t, ht), \quad 0 \leq t \leq 6\pi$.

5.4 Obtain unit speed parametrizations of the curves defined by

(a) $t \longrightarrow (e^t \cos t, e^t \sin t, e^t), \quad t \in [0, 1]$

(b) $t \longrightarrow (\cosh t, \sinh t, t), \quad t \in [0, 1]$.

5.5 Parametrize the curve of intersection of the sphere $x^2 + y^2 + z^2 = 16$ and the cylinder $x^2 + (y-2)^2 = 4$ which lies in the first octant.

5.6 Parametrize the anticlockwise directed triangle in $\mathbb{R}^2$ with vertices $(1, 2)$, $(-1, -2)$ and $(4, 0)$ as a piecewise smooth curve.

5.7 Find the closest points on the curve $x^2 - y^2 = 1$ to $(a, 0)$ where (i) $a = 4$ (ii) $a = 2$ (iii) $a = \sqrt{2}$.

5.8 Let f denote a real-valued differentiable function defined on an open subset U of $\mathbb{R}^n$ and suppose $\nabla f(X) \neq \mathbf{0}$ for all X in U. Let P denote a parametrized curve in U. Use the chain rule to show that $\dfrac{d}{dt}(f \circ P)(t) = \langle \nabla f(P(t)), P'(t)\rangle$ and hence deduce, using the Cauchy–Schwarz inequality, that $\nabla f(X_0)$ gives the direction of maximum increase of f at X_0. Show that $\|\nabla f(X_0)\|$ is the maximum rate of increase.

5.9 If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping such that $\|TX\| = \|X\|$ for all $X \in \mathbb{R}^n$ show that T preserves the inner product, angles, area and the length of curves. When $n = 3$, show that T preserves the cross product.