Chapter 9
Double Integration

Summary *We define the double integral of a function over an open subset of* $\mathbb{R}^2$ *and use Fubini's theorem to evaluate such integrals. We discuss the fundamental theorem of calculus in* $\mathbb{R}^2$*—Green's theorem.*

We discuss *(double)* integration of a real-valued function of *two* variables $f(x, y)$ over an open set Ω in $\mathbb{R}^2$. Motivated by the one-dimensional theory we divide Ω into rectangles—the natural analogue of intervals—by first drawing horizontal and vertical lines and thus partitioning the x- and y-axes.

Let x_i denote a typical element of the partition of the the x-axis and let y_j be a typical element on the y-axis. The resulting grid of rectangles gives a partition of Ω (Fig. 9.1) and we form the Riemann sum

$$\sum_i \sum_j f(x_i, y_j) \cdot \Delta x_i \cdot \Delta y_j, \quad \Delta x_i = x_{i+1} - x_i, \quad \Delta y_j = y_{j+1} - y_j$$

where we sum over all rectangles which are strictly contained in Ω. If this sum tends to a limit as we take finer and finer partitions we say that f is *integrable over* Ω and denote the limit by

$$\iint_\Omega f(x, y)\, dxdy.$$

We call this the *integral* (or *double integral*) of f over Ω. If Ω is the inside of a closed curve Γ and f is *continuous* on $\overline{\Omega}$ it can be shown that f is integrable over Ω. When $f(x, y) = 1$ for all $(x, y) \in \Omega$ the Riemann sum is the area of the rectangles in the partition inside Γ and on taking a limit we obtain

$$\iint_\Omega dxdy = Area\, of\ \Omega.$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_9,

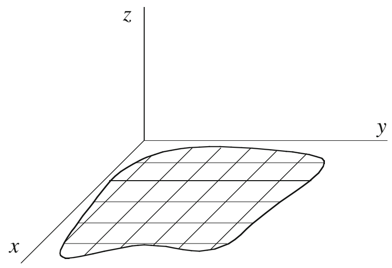

Fig. 9.1

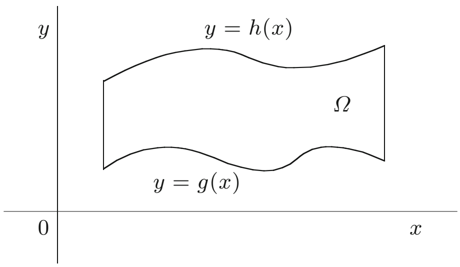

Fig. 9.2

If $f(x, y) \geq 0$ then the *volume* of the solid over Ω and beneath the graph of f is

$$\iint_{\Omega} f(x, y)dxdy.$$

We only evaluate double integrals over rather simple open sets. An open set is said to be of *type I* if it is bounded above by the graph of a continuous function $y = h(x)$, bounded below by the graph of a continuous function $y = g(x)$ and on the left and right by vertical lines of finite length (see Fig. 9.2).

Take a fixed interval in the partition of the x-axis, say (x_i, x_{i+1}), and consider the terms in the Riemann sum

$$\sum_i \sum_j f(x_i, y_j) \cdot \Delta x_i \cdot \Delta y_j$$

which only involve $\Delta x_i = x_{i+1} - x_i$. This gives the sum

$$\Big(\sum_j f(x_i, y_j) \cdot \Delta y_j\Big)\Delta x_i.$$

Taking limits—this can be justified for *continuous* functions—we get

$$\sum_{j} f(x_i, y_j) \cdot \Delta y_j \longrightarrow \int_{g(x_i)}^{h(x_i)} f(x_i, y)\, dy$$

as we take finer and finer partitions of the y-axis. Let

$$H(x) = \int_{g(x)}^{h(x)} f(x, y)\, dy.$$

Then

$$\sum_{i,j} f(x_i, y_j) \cdot \Delta x_i \cdot \Delta y_j \approx \sum_{i} H(x_i) \cdot \Delta x_i \to \int_{a}^{b} H(x)\, dx$$

and taking the limit on both sides we get

$$\iint_{\Omega} f(x, y)\, dydx = \int_{a}^{b} H(x)\, dx = \int_{a}^{b} \left\{ \int_{g(x)}^{h(x)} f(x, y)\, dy \right\} dx.$$

This method of integration, together with the similar method obtained by reversing the roles of x and y, is known as *Fubini's theorem.* We define an open set to be of *type II* if it is bounded on the left and right by the graphs of continuous functions of y, k and l, which are defined on the interval $[c, d]$ and above and below by horizontal lines of finite length. For domains of type II Fubini's theorem is

$$\iint_{\Omega} f(x, y)\, dxdy = \int_{c}^{d} \left\{ \int_{k(y)}^{l(y)} f(x, y)\, dx \right\} dy.$$

If we are given an open set we recognise it is of type I if each *vertical* line cuts the boundary at two points except possibly at the end points and type II if each *horizontal* line cuts the boundary at two points except, perhaps, at the end points.

Example 9.1 Evaluate

$$\iint_{\Omega} \frac{x}{\sqrt{16 + y^7}}\, dxdy$$

over the set Ω bounded above by the line $y = 2$, below by the graph of $y = x^{1/3}$ and on the left by the y-axis (Fig. 9.3). By inspection the domain Ω is of type I and type II and we have a choice of method, i.e. we can integrate first with respect to

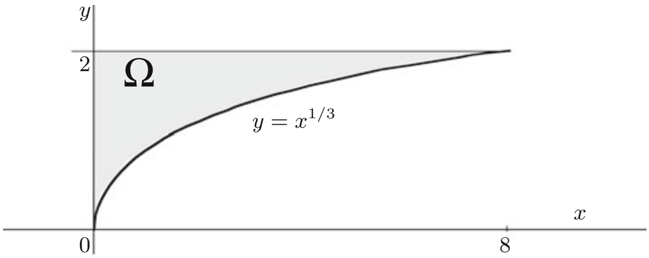

Fig. 9.3

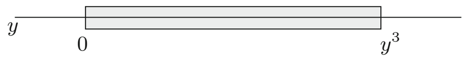

Fig. 9.4

either variable. Our choice may be important since one method may be very simple and the other quite difficult.

We have to evaluate two integrals of a single variable. In the first integral, the inner integral, one of the variables takes a fixed value and is really a constant. Thus we have to first evaluate either

$$\int x\,dx \qquad \text{or} \qquad \int \frac{dy}{\sqrt{16+y^7}}.$$

In these situations looks are usually not deceiving and we opt for the simpler looking integral. So, we choose to integrate first with respect to x. The limits of integration in the first integral will influence the degree of difficulty that arises in evaluating the second, or outer, integral. If you run into problems with the second integral you should consider starting again using a different order of integration. In our case we have decided to consider

$$\int \left\{ \int \frac{x}{\sqrt{16+y^7}}\,dx \right\} dy$$

and now need to determine the variation in x for fixed y. We draw a typical line through Ω on which y is constant—i.e. a horizontal line.

We must now express the end points in terms of y. Using once more Fig. 9.3 we see that the end points of the line of variation of x are $(0, y)$ and (x, y) where $y = x^{1/3}$. Hence $y^3 = x$ and we have the required variation of x (Fig. 9.4).

We see also that y varies from 0 to 2. Hence

$$\iint\limits_{\Omega} \frac{x}{\sqrt{16+y^7}}\,dx dy = \int_0^2 \left\{ \int_0^{y^3} \frac{x\,dx}{\sqrt{16+y^7}} \right\} dy$$

$$= \int_0^2 \frac{x^2\,dx}{2\sqrt{16+y^7}}\Bigg|_0^{y^3}\,dy$$

$$= \int_0^2 \frac{y^6}{2\sqrt{16+y^7}}\,dy$$

$$= \frac{1}{14}\int \frac{dw}{\sqrt{w}} \qquad \begin{aligned} w &= 16+y^7 \\ dw &= 7y^6\,dy \end{aligned}$$

$$= \frac{1}{14}\cdot\frac{w^{1/2}}{1/2} = \frac{1}{7}(16+y^7)^{1/2}\Big|_0^2$$

$$= \frac{1}{7}\big((144)^{1/2}-(16)^{1/2}\big) = \frac{8}{7}.$$

Example 9.2 In this example we reverse the order of integration and evaluate

$$\int_0^3 \left\{ \int_1^{\sqrt{4-y}} (x+y)\,dx \right\} dy.$$

From the limits of integration in the inner integral the left-hand side of the domain of integration, Ω, is bounded by the line $x = 1$ and the right-hand side by points satisfying $x = \sqrt{4-y}$, i.e. $x^2 = 4 - y$ or $y = 4 - x^2$. Hence the right-hand side is bounded by the graph of $y = 4 - x^2$ and we have the following diagram for our domain (Fig. 9.5).

Reversing the order of integration we get

$$\int_1^2 \left\{ \int_0^{4-x^2} (x+y)\,dy \right\} dx = \int_1^2 \left(xy + \frac{y^2}{2} \right)\Bigg|_0^{4-x^2} dx$$

$$= \int_1^2 \left(x(4-x^2) + \frac{(4-x^2)^2}{2} \right) dx = \frac{241}{60} \text{ (eventually).}$$

The *fundamental theorem of one-variable calculus*

$$f(b) - f(a) = \int_a^b f'(t)dt \tag{9.1}$$

is used to evaluate integrals of certain functions over intervals from their boundary values. In this theorem integration on the right-hand side is over a *directed interval*

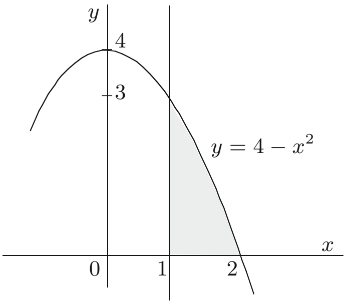

Fig. 9.5

while on the left positive and negative signs are assigned to the initial and final points of the interval respectively. Thus we see that a certain coherence has to be established between the orientations on the two sides of (9.1). The *fundamental theorem of two-variable calculus* is known as *Green's theorem*. To obtain this result by an immediate application of the one-variable theorem it is usual to begin with an open subset Ω in $\mathbb{R}^2$ which is of type I and type II. In applying (9.1) it is necessary to be careful with signs and this means that in Green's theorem the boundary of Ω, Γ, is oriented in an *anticlockwise* or *counterclockwise* direction.

Theorem 9.3 (Green's Theorem) *Let P and Q denote real-valued functions with continuous first-order partial derivatives on the open subset U of $\mathbb{R}^2$. If Γ is a closed curve directed in an anticlockwise direction such that the interior (inside) Ω of Γ is an open set of type I and type II and $\Gamma \cup \Omega \subset U$ then*

$$\int_\Gamma Pdx + Qdy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy. \tag{9.2}$$

Proof We show

$$\int_\Gamma Qdy = \iint_\Omega \frac{\partial Q}{\partial y}dxdy$$

and for this we use the type II property of Ω. From the representation in Fig. 9.6 we have

$$\iint_\Omega \frac{\partial Q}{\partial x}dxdy = \int_c^d \left\{\int_{k(y)}^{l(y)} \frac{\partial Q}{\partial x}dx\right\}dy.$$

By (9.1)

$$\int_{k(y)}^{l(y)} \frac{\partial Q}{\partial x}(x, y)dx = Q(x, y)\Big|_{k(y)}^{l(y)} = Q\big(l(y), y\big) - Q\big(k(y), y\big)$$

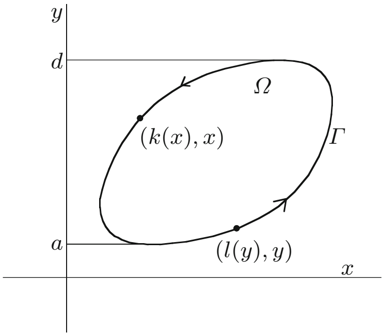

Fig. 9.6

and

$$\iint\limits_{\Omega} \frac{\partial Q}{\partial x} dxdy = \int_{c}^{d} \Big(Q\big(l(y), y\big) - Q\big(k(y), y\big)\Big) dy.$$

On the other hand, using the parametrizations $y \to \big(l(y), y\big)$ and $y \to \big(k(y), y\big)$, we obtain

$$\begin{aligned}\int_{\Gamma} Q dy &= \int_{(\text{graph of } l)} Q dy - \int_{(\text{graph of } k)} Q dy \\ &= \int_{c}^{d} Q\big(l(y), y\big) dy - \int_{c}^{d} Q\big(k(y), y\big) dy.\end{aligned}$$

This proves

$$\iint\limits_{\Omega} \frac{\partial Q}{\partial x} dxdy = \int_{\Gamma} Q dy.$$

The equality

$$\iint\limits_{\Omega} (-\frac{\partial P}{\partial y}) dxdy = \int_{\Gamma} P dx$$

is obtained in the same way and this completes the proof. □

Green's theorem is true for many other sets Ω and the proof usually proceeds by partitioning Ω into sets $(\Omega_i)_i$ and applying the simple case above to each Ω_i (Fig. 9.7). Note that each new curve created in partitioning Ω appears as part of the boundary of *two* Ω_i's and each direction along these new curves appears precisely once. Hence, when we apply (9.2) to each Ω_i and add them together the integrals

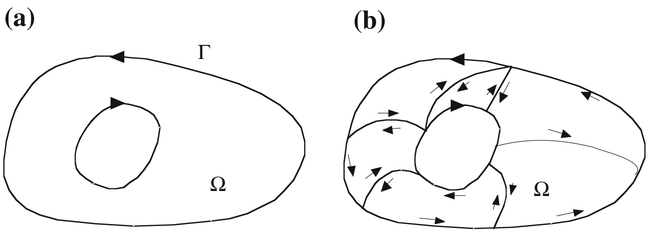

Fig. 9.7

along the newly created curves cancel and we are left with a line integral over the original curve Γ.

A glance at Fig. 9.7a shows that we no longer have an anticlockwise oriented boundary. In our new situation a *finite* number of piecewise smooth curves form the boundary of Ω and as we proceed along Γ in the given direction the open set Ω lies on the *left-hand side*. With these modifications Green's theorem is still true. We are, of course, always assuming that P and Q are nice smooth functions.

Example 9.4 We wish to calculate

$$I = \int_{\Gamma} (5 - xy - y^2)dx + (2xy - x^2)dy$$

where Γ is the boundary of the unit square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$. By Green's theorem

$$\begin{aligned}
I &= \int_0^1 \int_0^1 \Big(\frac{\partial}{\partial x}(2xy - x^2) - \frac{\partial}{\partial y}(5 - xy - y^2)\Big)dxdy \\
&= \int_0^1 \int_0^1 (2y - 2x + x + 2y)dxdy \\
&= \int_0^1 \int_0^1 (4y - x)dxdy \\
&= \Big(\int_0^1 4ydy\Big)\Big(\int_0^1 dx\Big) - \Big(\int_0^1 dy\Big)\Big(\int_0^1 xdx\Big) \\
&= \frac{4y^2}{2}\Big|_0^1 \; x\Big|_0^1 - y\Big|_0^1 \; \frac{x^2}{2}\Big|_0^1 \\
&= 2 - \frac{1}{2} = \frac{3}{2}.
\end{aligned}$$

The change of variables rule for double integrals can be considered as a special case of the same rule for triple integrals and this is discussed in Chap. 14.

Exercises

9.1 Find $\iint_U x^2 \sin^2 y dxdy$ where $A = \{(x, y) \in \mathbb{R}^2, \quad 0 < x < 1,\ 0 < y < \pi/4\}$.

9.2 Evaluate

(a) $\iint_U x\cos(x+y)dxdy$ where U is the subset of $\mathbb{R}^2$ bounded by the triangle with vertices $(0, 0)$, $(\pi, 0)$ and (π, π).

(b) $\iint_U (x^2+y^2)dxdy$ where $U = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \le 2y\}$.

(c) $\iint_U \frac{y^2}{x^2} dxdy$ where U is the region bounded by $y = x$, $y = 2$ and $xy = 1$.

9.3 Find the area bounded by the curves $x = y^2$ and $x = 4y - y^2$.

9.4 If Γ is a closed anticlockwise directed curve in $\mathbb{R}^2$ with interior Ω and $\mathbf{F}$ is a smooth vector field on an open set containing $\Omega \cup \Gamma$ show that Green's theorem is equivalent to

$$\iint_{\Omega} \mathrm{div}(\mathbf{F}) = \int (\mathbf{F}\cdot\mathbf{n})\mathrm{ds}$$

where $\mathbf{n}$ is the outward normal to Γ.

If f is harmonic on Ω, i.e.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

and continuous on $\Omega \cup \Gamma$ show that $\int_{\Gamma} (\nabla f \cdot \mathbf{n})ds = 0$.

9.5 By reversing the order of integration find

(a) $\int_0^6 \left\{ \int_{x/3}^{2} e^{y^2} dy \right\} dx$

(b) $\int_0^4 \left\{ \int_{y/2}^{\sqrt{y}} e^{y/x} dx \right\} dy$.

9.6 Find the volume of

$$\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 \le 1, \quad x \ge 0,\ y \ge 0,\ 0 \le z \le 1 - xy\}.$$

9.7 If Ω is an open set enclosed by the anticlockwise directed curve Γ show, using Green's theorem, that

$$\text{Area }(\Omega) = \int_{\Gamma} x\,dy = \int_{\Gamma} -y\,dx = \frac{1}{2}\int_{\Gamma} x\,dy - y\,dx\,.$$

Using one of these line integrals find the area of the interior of the ellipse $(x/4)^2 + (y/5)^2 = 1$.

9.8 Verify Green's theorem for the following integrals:

(a) $\int_{\Gamma} xy^2dx + 2x^2ydy$, Γ is the ellipse $4x^2 + 9y^2 = 36$

(b) $\int_{\Gamma} (x^2 + 2y^3)dy$, Γ is the circle $(x - 2)^2 + y^2 = 4$

(c) $\int_{\Gamma} 2x^2y^2dx - 3yxdy$, Γ is the square bounded by the lines $x = 3$, $x = 5$, $y = 1$, $y = 4$

where Γ is always directed in an anticlockwise direction.