Chapter 17
Gaussian Curvature

Summary *We define the Weingarten mapping or shape operator. This gives an intrinsic approach to Gaussian curvature.*

We first recall, from the previous chapter, the concepts introduced and the results obtained concerning the shape of a surface near a point p. All these results were obtained using plane curvature in $\mathbb{R}^2$. We defined or obtained the following:

(a) $k_p(\mathbf{v})$, the *normal curvature* at p in the direction of the unit tangent vector $\mathbf{v}$ at p

(b) *principal curvatures* at p, $k_1(p)$ and $k_2(p)$, where

$$k_1(p) = \max_{\|\mathbf{v}\|=1} k_p(\mathbf{v}) \quad \text{and} \quad k_2(p) = \min_{\|\mathbf{v}\|=1} k_p(\mathbf{v})$$

(c) *principal curvature directions*, i.e. tangent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that

$$k_p(\mathbf{v}_i) = k_i(p) \quad \text{for } i = 1, 2$$

(d) *umbilic points*, i.e. points where $k_1(p) = k_2(p)$, and *flat spots*, i.e. where $k_1(p) = k_2(p) = 0$

(e) at an *umbilic* point all (tangential) directions are principal curvature directions; at a *non-umbilic* point there are precisely two principal curvature directions $\mathbf{v}_1$ and $\mathbf{v}_2$ which are *perpendicular* to one another

(f) *Gaussian curvature* at p

$$K(p) = k_1(p)k_2(p) = \frac{ln - m^2}{EG - F^2}$$

where ϕ is any parametrization,

$$E = \phi_x \cdot \phi_x, \quad F = \phi_x \cdot \phi_y, \quad G = \phi_y \cdot \phi_y$$

$$l = \phi_{xx} \cdot \mathbf{n}, \quad m = \phi_{xy} \cdot \mathbf{n}, \quad n = \phi_{yy} \cdot \mathbf{n}$$

$$\mathbf{n} = \frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|}$$

(h) the coordinate curves are *lines of curvature* if and only if $F = m = 0$ at all non-umbilic points.

The Gaussian curvature contains less information than the principal curvatures, that is to say if we know the principal curvatures then we can calculate the Gaussian curvature but from the Gaussian curvature alone we cannot calculate the principal curvatures. Thus, at first glance, it appears that in using Gaussian curvature we may be neglecting important information.

However, experience and subsequent results show that the information lost is generously compensated by other gains. To begin with, Gaussian curvature is a single real number assigned to each point on a surface—the principal curvatures and directions involve two real numbers and two vectors. Thus Gaussian curvature has the advantage of *simplicity*. We have already seen that Gaussian curvature may be easily calculated from any parametrization whereas it may be difficult to calculate the principal curvatures. The principal curvatures depend, up to a factor ±1, on the choice of normal while Gaussian curvature has the same value for *any* choice of normal. In practice this means that any parametrization may be used to calculate Gaussian curvature while only parametrizations consistent with the choice of normal may be used to find principal curvatures. Indeed, along these lines, we have a celebrated theorem of Gauss—*theorema egregium*—which asserts that Gaussian curvature is an *intrinsic* property of the surface. Roughly speaking this says that Gaussian curvature *may be calculated* directly from functions defined internally on the surface and without using such external properties as the normal or the fact that the surface lies in $\mathbb{R}^3$. Our method of calculating K uses the normal so Gauss' theorem tells us that there is another way of calculating K which does not use the normal. At first glance this may appear a rather minor point but it was this result which paved the way for the development of a very powerful and a very general type of geometry—*Riemannian geometry*—in which the key concepts are *differentiation* and the *length of tangent vectors*. Thus for simplicity, for practical and intrinsic reasons, Gaussian curvature has many advantages. However, in studying surfaces *all* of the concepts we discussed play a useful role and none should be neglected.

So far we have studied the shape of a surface by examining curves (normal sections) of the surface but there are other intuitive approaches to the same problem. Almost invariably they lead back to Gaussian curvature. For example, following the successful approach to plane curvature obtained by taking limits of circles, it is natural to regard the reciprocal of the radius of the sphere that sits closest to the surface near p as a measure of the curvature at p. For surfaces the definition of the *sphere of closest fit* is not so obvious, especially at saddle points, and an indirect approach is taken. We use a normal $\mathbf{n}$ to project the surface near p onto the unit sphere in $\mathbb{R}^3$
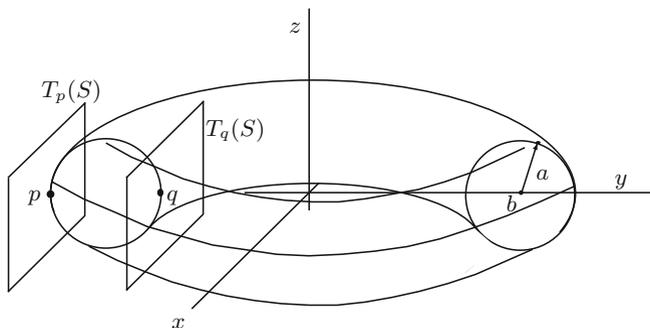
Fig. 17.1

and compare the area of the surface near p with the area of its image. When used in this way $\mathbf{n}$ is called the *Gauss map*. If B_ε is the ball with centre p and radius ε then

$$|K(p)| = \lim_{\varepsilon \to 0} \left| \frac{\text{Area}(\mathbf{n}(S \cap B_\varepsilon))}{\text{Area}(S \cap B_\varepsilon)} \right|.$$

Note that if S is a plane then $\mathbf{n}(S)$ consists of a single point and $K(p) = 0$ while if S is a sphere of radius r then $\mathbf{n}(x) = \pm x/r$ for all $x \in S$ and $K(p) = 1/r^2$. Further geometric interpretations appear in the final chapter.

We list now a number of results on Gaussian curvature which give some idea of its uses; more advanced and deeper results are given later. Think about these results, ask yourself if they are geometrically plausible, how they fit in with your intuition, what they say about the surfaces with which you are already familiar, and how you might go about proving them.

(i)   If S is a connected surface in $\mathbb{R}^3$ consisting entirely of umbilics then S is either an open subset of a sphere or a plane.

(ii) Every compact surface in $\mathbb{R}^3$ contains a point p with $K(p) > 0$.

(iii) A compact connected surface of *constant Gaussian curvature* is a sphere.

(iv) *Hilbert's Lemma:* If p is a non-umbilic point in S, k_1 has a local maximum at p and k_2 has a local minimum at p, then $K(p) \le 0$.

Example 17.1   In this example we discuss the torus. From Fig. 17.1 we see that at the point p the surface lies on one side of the tangent plane and $K(p) > 0$ while at q it lies on both sides and $K(q) < 0$. From Figs. 17.1 and 17.2 it is clear that $1/a$ will always be a principal curvature and that $1/b - a$ (respectively $1/b + a$) is a principal curvature at p (respectively q). We confirm this in our analysis.

We use toroidal polar coordinates for our parametrization,

$$P : (\theta, \phi) \to ((b + a \cos\theta) \cos\phi, \ (b + a \cos\theta) \sin\phi, \ a \sin\theta)$$

$(\theta, \phi) \in (0, 2\pi) \times (0, 2\pi)$. From Examples 11.1 and 14.5, $E = a^2$, $F = 0$, $G = (b + a \cos\theta)^2$, and
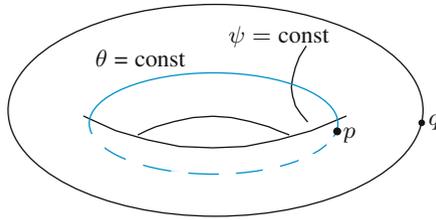
Fig. 17.2

$$\mathbf{n} = -(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ \sin\theta)$$
$$P_\theta = (-a\sin\theta\cos\phi,\ -a\sin\theta\sin\phi,\ a\cos\theta)$$
$$P_\phi = (-(b+a\cos\theta)\sin\phi,\ (b+a\cos\theta)\cos\phi,\ 0)$$
$$P_{\theta\theta} = (-a\cos\theta\cos\phi,\ -a\cos\theta\sin\phi,\ -a\sin\theta)$$
$$P_{\theta\phi} = (a\sin\theta\sin\phi,\ -a\sin\theta\cos\phi,\ 0)$$
$$P_{\phi\phi} = (-(b+a\cos\theta)\cos\phi,\ -(b+a\cos\theta)\sin\phi,\ 0)\,.$$

Hence

$$l = \langle P_{\theta\theta}, \mathbf{n}\rangle = a\cos^2\theta\cos^2\phi + a\cos^2\theta\sin^2\phi + a\sin^2\theta = a$$
$$m = \langle P_{\theta\phi}, \mathbf{n}\rangle = -a\cos\theta\sin\theta\cos\phi\sin\phi + a\cos\theta\sin\theta\cos\phi\sin\phi = 0$$
$$n = \langle P_{\phi\phi}, \mathbf{n}\rangle = (b+a\cos\theta)(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi)$$
$$= (b+a\cos\theta)\cos\theta.$$

By Proposition 16.5 the coordinate curves are lines of curvature, $\frac{l}{E} = \frac{1}{a}$ and $\frac{n}{G} = \frac{\cos\theta}{b+a\cos\theta}$ are the principal curvatures and

$$K = \frac{ln - m^2}{EG - F^2} = \frac{a(b+a\cos\theta)\cos\theta}{a^2(b+a\cos\theta)^2} = \frac{\cos\theta}{a(b+a\cos\theta)}.$$

We now have a formula for Gaussian curvature and also a diagram (Fig. 17.2) and we may compare and combine them. By differentiating $K(\theta,\ \phi)$, which only depends on θ, we can locate the points of maximum and minimum Gaussian curvature, p and q, and this confirms what the sketch tells us. For the torus $F = m = 0$. Hence $(-a\sin\theta\cos\phi, -a\sin\theta\sin\psi, a\cos\theta)$ and $(-(b+a\cos\theta)\sin\psi, (b+a\cos\theta)\cos\psi, 0)$ are principal curvature directions and the principal curvatures, associated with toroidal polar coordinates, are $1/a$ and $\cos\theta/(b+a\cos\theta)$.

We now consider a more intrinsic approach to Gaussian curvature. If $F : S \longrightarrow \mathbb{R}^n$, where $(S, \mathbf{n})$ is an oriented surface in $\mathbb{R}^3$, $p \in S$ and $\mathbf{v} \in T_p(S)$, let

$$D_{\mathbf{v}} F(p) = \frac{d}{dt} \Big(F\left(P(t) \right) \Big) \Big|_{t=0} \tag{17.1}$$

where $P : [-a, a] \longrightarrow \Gamma$ is any parametrized curve in S with $P(0) = p$ and $P'(0) = \mathbf{v}$. It can be shown that this definition does not depend on Γ or P. The mapping $\mathbf{v} \longrightarrow D_{\mathbf{v}} F(p)$ is a linear mapping.

Let $\phi : U \longrightarrow S$ denote a parametrization of a part of S. Here we use ϕ_x to denote the mapping $\phi(U) \cap S \longrightarrow \mathbb{R}^3$ given by

$$\phi_x(\phi(x, y)) = \frac{d}{dt} \left(\phi(x + t, y) \right) \Big|_{t=0}. \tag{17.2}$$

Using (17.1) and (17.2) we define $D_{\phi_x}\phi_x$, $D_{\phi_x}\phi_y$, $D_{\phi_y}\phi_x$ and $D_{\phi_y}\phi_y$. The expressions for these derivatives may initially appear cumbersome, e.g. for instance

$$\left(D_{\phi_y}\phi_x \right)(\phi(x, y)) = \frac{d}{dt} \left(\frac{d}{ds} \phi(x + s, y + t) \Big|_{s=0} \right) \Big|_{t=0}.$$

However, since ϕ is a parametrization all its partial derivatives exist and hence, using a two variables Taylor series expansion, we see that

$$\left(D_{\phi_y}\phi_x \right)(\phi(x, y)) = \frac{\partial^2 \phi}{\partial y \partial x}(x, y). \tag{17.3}$$

We now use the notation ϕ_{xx}, ϕ_{xy}, ϕ_{yx} and ϕ_{yy}, respectively, in place of $D_{\phi_x}\phi_x$, $D_{\phi_x}\phi_y$, $D_{\phi_y}\phi_x$ and $D_{\phi_y}\phi_y$. By (17.3), $\phi_{xy} = \phi_{yx}$ (see also the introduction to Chap. 4).

The product rule for differentiation and $\langle \mathbf{n}(p), \mathbf{n}(p) \rangle = 1$ imply

$$\langle D_{\mathbf{v}}\mathbf{n}(p), \mathbf{n}(p) \rangle + \langle \mathbf{n}(p), D_{\mathbf{v}}\mathbf{n}(p) \rangle = 0$$

i.e. $\langle D_{\mathbf{v}}\mathbf{n}(p), \mathbf{n}(p) \rangle = 0$ at any point $p \in S$ and any $\mathbf{v} \in T_p(S)$. Hence $D_{\mathbf{v}}\mathbf{n}(p) \perp \mathbf{n}(p)$ and $D_{\mathbf{v}}\mathbf{n}(p)$ belongs to the tangent space at p. This allows us to define a *linear* mapping from the *tangent space* at p into itself by letting

$$L_p : \mathbf{v} \in T_p(S) \longrightarrow -D_{\mathbf{v}}\mathbf{n}(p) \in T_p(S).$$

This important mapping is called the *Weingarten mapping* or *shape operator*. We now suppose that ϕ is consistent with the orientation and write $\mathbf{n}$ in place of $\mathbf{n}(\phi(x, y))$. Since

$$\frac{d}{dt} \mathbf{n}(\phi(x + t, y))|_{t=0} = D_{\phi_x}\mathbf{n} = -L_p\phi_x.$$

and $\langle \phi_x, \mathbf{n} \rangle = 0$ we have

$$0 = \frac{d}{dt} \langle \phi_x(x + t, y), \mathbf{n}(\phi(x + t, y)) \rangle|_{t=0}$$

$$= \langle \phi_{xx}, \mathbf{n} \rangle + \langle \phi_x, \ D_{\phi_x} \mathbf{n} \rangle$$
$$= \langle \phi_{xx}, \mathbf{n} \rangle - \langle \phi_x, \ L_p \phi_x \rangle$$

and

$$\langle \phi_x, \ L_p \phi_x \rangle = \langle \phi_{xx}, \mathbf{n} \rangle = l.$$

Similarly $m = \langle \phi_y, L_p \phi_x \rangle = \langle \phi_{xy}, \mathbf{n} \rangle = \langle \phi_{yx}, \mathbf{n} \rangle = \langle \phi_x, L_p \phi_y \rangle$ and $\langle \phi_y, L_p \phi_y \rangle = \langle \phi_{yy}, \mathbf{n} \rangle = n$.

Since $\{\phi_x, \phi_y\}$ is a basis for the tangent space at p we can find scalars $\{a, b, c, d\}$ so that $L_p(\phi_x) = a\phi_x + b\phi_y$ and $L_p(\phi_y) = c\phi_x + d\phi_y$. On solving the system of linear equations,

$$L_p(\phi_x) \cdot \phi_x = aE + bF = l$$
$$L_p(\phi_x) \cdot \phi_y = aF + bG = m$$
$$L_p(\phi_y) \cdot \phi_x = cE + dF = m$$
$$L_p(\phi_y) \cdot \phi_y = cF + dG = n$$

we see that

$$A_p = \frac{1}{EG - F^2} \begin{pmatrix} lG - mF & mE - lF \\ mG - nF & nE - mF \end{pmatrix}$$

is the matrix for L_p with respect to the basis $\{\phi_x, \phi_y\}$ for $T_p(S)$. Since

$$(EG - F^2)^2 \det(A_p) = (lG - mF)(nE - mF) - (mE - lF)(mG - nF)$$
$$= (EG - F^2)(ln - m^2)$$

we have proved, in view of Proposition 16.3, the following result.

Proposition 17.2 *If ϕ parametrizes a surface S in $\mathbb{R}^3$ then the determinant of the matrix of the Weingarten mapping at p with respect to the basis for the tangent space induced by ϕ is the Gaussian curvature of S at p.*

If $\mathbf{v} = v_1 \phi_x + v_2 \phi_y$ and $\mathbf{w} = w_1 \phi_x + w_2 \phi_y$ then, by the above,

$$\langle L_p(\mathbf{v}), \mathbf{w} \rangle = \langle L_p(v_1 \phi_x + v_2 \phi_y), w_1 \phi_x + w_2 \phi_y \rangle$$
$$= v_1 w_1 \langle L_p \phi_x, \phi_x \rangle + (v_1 w_2 + v_2 w_1) \langle L_p(\phi_x), \phi_y \rangle + v_2 w_2 \langle L_p(\phi_y), \phi_y \rangle$$
$$= \langle \mathbf{v}, L_p(\mathbf{w}) \rangle$$

and $L_p : T_p(S) \longrightarrow T_p(S)$ is a symmetric linear operator. Moreover,

$$\langle L_p(\mathbf{v}), \mathbf{v} \rangle = l v_1^2 + 2m v_1 v_2 + n v_2^2$$

and if $\mathbf{v}$ is a unit tangent vector at p then

$$\langle L_p(\mathbf{v}), \mathbf{v} \rangle = k_p(\mathbf{v}). \tag{17.4}$$

Since L_p is symmetric we can choose an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for $T_p(S)$ consisting of eigenvectors for L_p. Let β_1 and β_2 denote the corresponding eigenvalues. If $x, y \in \mathbb{R}$, $x^2 + y^2 = 1$ then $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2$ is a unit tangent vector at p and, by (17.4),

$$
\begin{aligned}
k_p(\mathbf{v}) &= \langle L_p(x\mathbf{e}_1 + y\mathbf{e}_2), x\mathbf{e}_1 + y\mathbf{e}_2 \rangle \\
&= \langle x\beta_1\mathbf{e}_1 + y\beta_2\mathbf{e}_2, x\mathbf{e}_1 + y\mathbf{e}_2 \rangle \\
&= x^2\beta_1\langle \mathbf{e}_1, \mathbf{e}_1 \rangle + 2xy\langle \mathbf{e}_1, \mathbf{e}_2 \rangle + y^2\langle \mathbf{e}_2, \mathbf{e}_2 \rangle \\
&= x^2\beta_1 + y^2\beta_2.
\end{aligned}
$$

This shows that the principal curvatures are the eigenvalues of L_p and the eigenvectors are the principal curvature directions. Since the matrix for L_p with respect to this basis is a diagonal matrix with the eigenvalues as entries we also have $K(p) = \beta_1\beta_2 = \det(L_p)$. We summarise what we have proved in the following proposition.

Proposition 17.3 *At a point p in a surface S the principal curvatures are the eigenvalues of L_p, the principal curvature directions are the eigenvectors of L_p, and the Gaussian curvature, $K(p)$, is the determinant of L_p with respect to any orthonormal basis for $T_p(S)$.*

Example 17.4 If p is umbilic then all normal curvatures are equal. If p is *non-umbilic* and k_1, k_2, $\mathbf{v}_1$, and $\mathbf{v}_2$ are the principal curvatures and the corresponding principal curvature directions then any *unit tangent vector* $\mathbf{v}$ at p has the form

$$\mathbf{v} = \cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2$$

for some real number θ. By Proposition 17.3

$$L_p(\mathbf{v}_1) = k_1(p)\mathbf{v}_1 \quad \text{and} \quad L_p(\mathbf{v}_2) = k_2(p)\mathbf{v}_2$$

and, as p is non-umbilic, $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0$. By (17.2),

$$
\begin{aligned}
k_p(\mathbf{v}) = \langle L_p(\mathbf{v}), \mathbf{v} \rangle &= \langle L_p(\cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2), \cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2 \rangle \\
&= \langle \cos\theta L_p(\mathbf{v}_1) + \sin\theta L_p(\mathbf{v}_2), \cos\theta\mathbf{v}_1 + \sin\theta\mathbf{v}_2 \rangle \\
&= k_1(p)\cos^2\theta + k_2(p)\sin^2\theta.
\end{aligned}
$$

This is known as *Euler's formula* and shows that normal curvature in any direction can be recovered from the principal curvatures and the principal curvature directions.
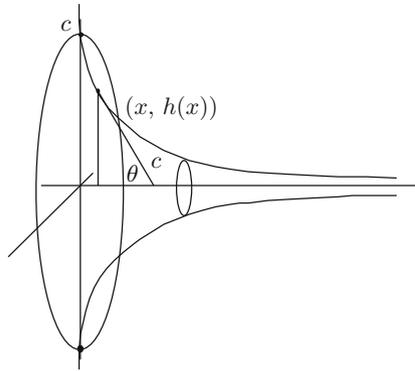
Fig. 17.3

Exercises

17.1 Find the Weingarten mapping for the torus.

17.2 Let **S** denote the graph of the function $f(u, v) = u^2 - 2v^2$ oriented by the usual parametrization. Find a curve Γ in **S** such that the normal curvature in the tangent direction to Γ is always zero. Hence find a straight line that lies in **S**.

17.3 Suppose the two oriented surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$ intersect in a curve Γ. Let κ denote the curvature of $\Gamma \subset \mathbb{R}^3$ and let λ_i denote the normal curvature of Γ in $\mathbf{S_i}$, $i = 1, 2$. If θ is the angle between the normals to $\mathbf{S}_1$ and $\mathbf{S}_2$ show that

$$\kappa^2 \sin^2 \theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1 \lambda_2 \cos \theta.$$

17.4 Use $-D_{\mathbf{v}}(\mathbf{n})$ to give another solution to Exercise 16.3.

17.5 Show that the average of the normal curvature over all directions is the mean curvature.

17.6 Consider the plane curve h described by the following geometric condition: start at the point $(0, c)$ and move so that the tangent line always reaches the x-axis after traveling a distance c. From Fig. 17.3 deduce that

$$h' = -\frac{h}{\sqrt{c^2 - h^2}}.$$

(Note that $\sin \theta = -h/c$, $\tan \theta = h' = \dfrac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$.) Show that

$$h'' = \frac{c^2 h}{(c^2 - h^2)^2}.$$

Orient the surface of revolution, B, of this curve so that the principal curvatures are h'/c and $-1/ch'$. Hence deduce that B has constant negative Gaussian curvature. Sketch the surface B—it may help you to know that it is called the *bugle surface*.