Chapter 18
Geodesic Curvature

Summary *We define geodesic curvature and geodesics. For a curve on a surface we derive a formula connecting intrinsic curvature, normal curvature and geodesic curvature. We discuss paths of shortest distance, further interpretations of Gaussian curvature and introduce, informally and geometrically, a number of important results in differential geometry*.

Our study of normal curvature was based on identifying the normal of a curve with the normal of the surface. It is also possible to arrange things so that the normal to a curve in a surface is a tangent vector to the surface. This leads to a new type of curvature, *geodesic curvature*, that we discuss and interpret in this chapter.

Let $\mathbf{S}$ denote an oriented surface in $\mathbb{R}^3$ with smooth unit normal $\mathbf{n}$ and let $P: [a, b] \to \Gamma \subset \mathbf{S}$ denote a unit speed parametrized curve on the surface. Since the tangent space at each point on the surface is two-dimensional and $P'(t)$ is a tangent vector at $P(t)$ it follows that there are precisely two unit tangent vectors at $P(t)$ which are perpendicular to $P'(t)$. We distinguish between them by using the normal to the surface at $P(t)$, $\mathbf{n}\big(P(t)\big)$, and define the *surface normal* to Γ at $P(t)$, $\mathbf{n_S}(P(t))$ to be $\mathbf{n}(P(t)) \times T(t)$. To simplify our notation we sometimes write $\mathbf{n}(t)$ and $\mathbf{n_S}(t)$ in place of $\mathbf{n}\big(P(t)\big)$ and $\mathbf{n_S}(P(t))$, respectively. By construction

$$\{P'(t) = T(t), \mathbf{n}_S(t), \mathbf{n}(t)\}$$

is a right-handed orthogonal system, and in particular, an orthonormal basis for $\mathbb{R}^3$. In Chap. 7 we encountered a similar situation when we obtained the orthonormal basis $\{T, N, B\}$ at a point on a curve Γ. In that case we proceeded to obtain the Frenet–Serret equations by differentiation and using properties of the orthonormal basis. We follow *precisely* the same path to a similar end here and obtain real-valued functions $a(t)$, $b(t)$ and $c(t)$ such that

$$\begin{pmatrix} T(t) \\ \mathbf{n_S}(t) \\ \mathbf{n}(t) \end{pmatrix}' = \begin{pmatrix} 0 & a(t) & c(t) \\ -a(t) & 0 & b(t) \\ -c(t) & -b(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ \mathbf{n_S}(t) \\ \mathbf{n}(t) \end{pmatrix}. \tag{18.1}$$

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In particular we see that

$$\mathbf{n}'_{\mathbf{S}}(t) = -a(t)T(t) + b(t)\mathbf{n}(t) \tag{18.2}$$

and the "part" of $\mathbf{n}'_{\mathbf{S}}(t)$ which "lies" in the tangent space at $P(t)$ is parallel to $T(t)$. This equation is similar to Eq. 7.11 in Chap. 7. We define the *geodesic curvature* of Γ at $P(t)$, $\kappa_g(t)$, to be $a(t)$. By (18.2)

$$\kappa_g(t) = -\langle \mathbf{n}_{\mathbf{S}}'(t), T(t)\rangle = a(t)\,.$$

The entry $b(t)$ is called the *geodesic torsion* of Γ at $P(t)$ and written $\tau_g(t)$. Rewriting Eq. (18.2) we obtain

$$\mathbf{n}'_{\mathbf{S}}(t) = -\kappa_g(t)T(t) + \tau_g(t)\mathbf{n}(t)\,.$$

From (18.1) and the definition of normal curvature we have

$$c(t) = -\langle \frac{d}{dt}\mathbf{n}\big(P(t)\big), T(t)\rangle = k_{P(t)}\big(T(t)\big)$$

and $c(t)$ is the *normal curvature* at $P(t)$ in the direction $T(t)$. Hence, rewriting (18.1) we obtain

$$\begin{pmatrix} T(t) \\ \mathbf{n}_{\mathbf{S}}(t) \\ \mathbf{n}(t) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_g(t) & \kappa_n(t) \\ -\kappa_g(t) & 0 & \tau_g(t) \\ -\kappa_n(t) & -\tau_g(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ \mathbf{n}_{\mathbf{S}}(t) \\ \mathbf{n}(t) \end{pmatrix} \tag{18.3}$$

where we have written $\kappa_n(t)$ in place of $k_{P(t)}\big(T(t)\big)$.

By (18.3)

$$T'(t) = \kappa_g(t)\mathbf{n}_{\mathbf{S}}(t) + \kappa_n(t)\mathbf{n}(t)\,. \tag{18.4}$$

but if Γ, as a parametrized curve in $\mathbb{R}^3$ has strictly positive curvature, we can add to this, by using (7.1′) and (18.4), and obtain

$$T'(t) = \kappa(t)N(t) = \kappa_g(t)\mathbf{n}_{\mathbf{S}}(t) + \kappa_n(t)\mathbf{n}(t)\,. \tag{18.5}$$

To avoid confusion between the different types of curvature we call κ the *intrinsic curvature* of Γ and use N to denote the normal to Γ in $\mathbb{R}^3$ whenever it exists. By (18.5) and Pythagoras' Theorem,

$$\kappa^2(t) = \kappa_g^2(t) + \kappa_n^2(t)\,.$$

Equation (18.5) is a decomposition of the intrinsic curvature into its normal and tangential components and establishes a relationship between the three different kinds of curvature. If we consider curvature as a measure of "bending" towards the normal then, since we have chosen our normal $\mathbf{n_S}(t)$ to lie in the tangent space, this and (18.5) suggest that we consider geodesic curvature as the surface curvature of the parametrized curve. So far we have only considered a unit speed parametrized curve. We define the geodesic curvature of a parametrized curve in an oriented surface as the geodesic curvature of a unit speed reparametrization of the curve which preserves the original sense of direction. Note that our definition of geodesic curvature is based, as was normal curvature, on curvature in $\mathbb{R}^2$—see the introduction to Chap. 7. Since the two unit normals at any point on a surface are parallel, Eq. (18.4) shows that $|\kappa_g|$—the *absolute geodesic curvature*—does not depend on either the choice of normal or parametrization. As surfaces can be covered by simple, and hence orientable, surfaces it follows that absolute geodesic curvature is *well defined* for *any* curve on *any* orientable surface.

Curves with $\kappa_g = 0$ are said to have *zero geodesic curvature*. We give a geometrical interpretation of this phenomena and afterwards discuss a practical method for identifying such curves. If we are dealing with a curve in $\mathbb{R}^2$ then zero curvature implies that the curve is a straight line. If we identify $\mathbb{R}^2$ with the oriented surface $\mathbb{R}^2_{(x,y)}$ in $\mathbb{R}^3$ and consider a curve Γ in $\mathbb{R}^2$ as a curve in $\mathbb{R}^3$ then we see easily that its plane curvature in $\mathbb{R}^2$ coincides with its geodesic curvature in $\mathbb{R}^2_{(x,y)}$. In $\mathbb{R}^2$ we also note that a curve is a straight line if and only if it follows the shortest route between any pair of its points. Now the *tangent plane* is the *closest plane* to the surface near a given point p and since geodesic curvature is essentially curvature on the tangent plane it is at least plausible that zero geodesic curvature implies that Γ follows the *shortest path on the surface* between points on Γ close to p. This is indeed the case.

Formally we have the following definitions and results. A surface S is *connected* if between any two points p and q on S there exists at least one path (or directed curve) with initial point p and final point q. We define the *distance* between p and q, $d(p, q)$, as

$$\inf \{\text{length}(\gamma),\ \gamma \text{ is a path with initial point } p \text{ and final point } q\}.$$

A path γ joining p and q is a shortest path if

$$d(p, q) = \text{length}(\gamma).$$

Shortest paths may or may not exist and if they exist they may not be unique. For instance it is easily seen that there is no shortest path on the surface $S = \{(x, y, 0) : 0 < x^2 + y^2 < 2\}$ in $\mathbb{R}^3$ joining the points $(-1, 0, 0)$ and $(1, 0, 0)$ although it is easy to see that the distance between them is 2. On the other hand there exist an infinite number of shortest paths on a sphere joining the North and South poles (any line of longitude is a shortest path). Equation 18.5 leads to a simple practical criterion for identifying unit speed curves of zero geodesic curvature since it is easily seen, using (18.5), that $P''(t)$ (or $N(t)$) is parallel to $\mathbf{n}(t)$, for any choice of normal, if and only if $\kappa_g(t) = 0$. This motivates the following definition.

Definition 18.1 A parametrized curve $P: [a, b] \to \Gamma \subset S$, where S is a surface in $\mathbb{R}^3$, is a geodesic in S if $P''(t)$ (the acceleration of the parametrization) is parallel to the normal to the surface at $P(t)$ for all t.

Our remarks above show that *a unit speed parametrized curve is a geodesic if and only if it has zero geodesic curvature.*

Proposition 18.2 *A parametrized curve $P: [a, b] \to \Gamma \in S$ is a geodesic if and only if it has constant speed and zero geodesic curvature.*

Proof We may assume, without loss of generality, that $[a, b] = [0, \alpha]$. We first suppose that the parametrized curve is a geodesic. We have

$$\frac{d}{dt}\langle P'(t), P'(t)\rangle = 2\langle P''(t), P'(t)\rangle.$$

Since the curve lies in S, $P'(t)$ is a tangent vector at $P(t)$, and as P is a geodesic $P''(t)$ is perpendicular to every tangent vector. This implies $\langle P'(t), P''(t)\rangle = 0$ and $\frac{d}{dt}\left(\|P'(t)\|^2\right) = 0$. Hence $\|P'(t)\|$ is a constant function of t and the parametrization has constant speed c. This implies that the parametrization $Q : [0, c\alpha] \to \Gamma \in S$, where $Q(t) = P(t/c)$, is unit speed. Since $Q''(t) = P''(t/c)/c^2$, we see that $Q''(t)$ is parallel to $P''(t/c)$ and hence to $\mathbf{n}(P(t/c)) = \mathbf{n}(Q(t))$.

Conversely suppose $P : [0, \alpha] \longrightarrow S$ has constant speed, c, and zero geodesic curvature. Then $Q : [0, c\alpha] \longrightarrow S$ where $Q(t) = P(t/c)$ is unit speed and has zero geodesic curvature. By (18.4), $Q''(t)$ is parallel to $\mathbf{n}(t/c)$ and since $P''(t/c)$ and $Q''(t)$ are parallel this shows that the curve parametrized by P is a geodesic. This completes the proof. □

From the above considerations it is not difficult to show that a parametrized curve is a geodesic if and only if it satisfies a certain ordinary differential equation. Existence theorems for ordinary differential equations show that any surface admits an abundance of geodesics. The following is true.

Proposition 18.3 *If S is a surface in $\mathbb{R}^3$, $p \in S$ and $\mathbf{v} \in T_p(S)$ is non-zero then there exists $\varepsilon > 0$ and a unique geodesic $P: [-\varepsilon, \varepsilon] \to S$ such that $P(0) = p$ and $P'(0) = \mathbf{v}$.*

Example 18.4 Let S_2 denote the unit sphere with centre at the origin in $\mathbb{R}^3$. Let $\mathbf{v}$ and $\mathbf{w}$ denote perpendicular unit vectors in $\mathbb{R}^3$ and let

$$P(t) = \cos(at)\mathbf{v} + \sin(at)\mathbf{w}, \quad t \in \mathbb{R}$$

where a is a fixed real number. By Pythagoras' Theorem P defines a parametrized curve in S_2. We have

$$P''(t) = -a^2\cos(at)\mathbf{v} - a^2\sin(at)\mathbf{w} = -a^2 P(t)\,.$$

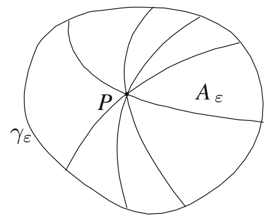

Fig. 18.1

Since $\mathbf{n}(t) = \pm P(t)$ for the unit sphere with centre at the origin this implies $P''(t) \parallel \mathbf{n}(t)$ for all t and P is a geodesic. Proposition 18.3 shows that a geodesic is completely determined by its position and velocity at a single point. Since $P(0) = \mathbf{v}$ and $P'(0) = a\mathbf{w}$ it follows that we have found *all* geodesics on the unit sphere.

We return to the problem we started with—the existence of a shortest path. Propositions 18.2 and 18.3 show that there are many curves on a surface with zero geodesic curvature. From this it is possible to prove the following result which shows, at least locally, that there are always shortest paths.

Proposition 18.5 *If p is a point on a surface S in $\mathbb{R}^3$ then there exists $\varepsilon > 0$ such that for any q in S, $d(p, q) < \varepsilon$, there is a unique shortest path in S joining p and q. This path has zero geodesic curvature and may be parametrized as a geodesic.*

Geodesics also lead to a new derivation of Gaussian curvature. Take a point p on the surface S. An extension of Proposition 18.3 shows that there exists $\varepsilon > 0$ such that for every unit tangent vector $\mathbf{v}$ at p the geodesic with initial point p and initial velocity $\mathbf{v}$ is defined on $[0, \varepsilon]$. If we consider the set of positions taken at time ε by *all* unit speed geodesics starting at p we obtain a curve γ_ε in S. We denote the inside of this curve by A_ε (Fig. 18.1).

Thus γ_ε consists of those points in S whose distance to p is ε and A_ε are the points whose distance to p is less than ε. It can be shown that γ_ε is a closed subset of S and A_ε is open. If S is flat then γ_ε is a circle of radius ε and length $2\pi\varepsilon$ and A_ε is a disc of area $\pi\varepsilon^2$. Hence the quantities $2\pi\varepsilon - l(\gamma_\varepsilon)$ and $\pi\varepsilon^2 - \mathrm{A}(A_\varepsilon)$ where l = length and A = Area are some measure of the curvature of S. In fact, taking limits we obtain the following

$$K(p) = \lim_{\varepsilon \to 0} \frac{3}{\pi\varepsilon^3}\left(2\pi\varepsilon - l(\gamma_\varepsilon)\right) = \lim_{\varepsilon \to 0} \frac{12}{\pi\varepsilon^4}\left(\pi\varepsilon^2 - A(A_\varepsilon)\right).$$

Thus both normal and geodesic curvature lead quite naturally to Gaussian curvature. It is also worth noting that the two most important geometrical concepts that we associated with a surface—Gaussian curvature and geodesics—both turned out to be local properties independent of any orientations used for calculations or motivation.

To complete our introduction to the geometry of surfaces we present without proof some rather remarkable results involving the concepts we have introduced.

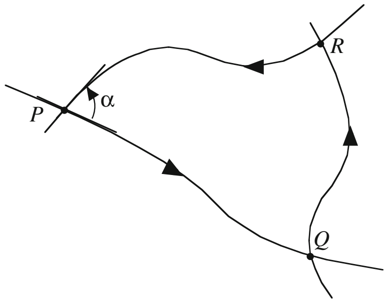

Fig. 18.2

These results are easily stated while the proofs are rather involved. Even in the absence of any ideas regarding the proofs it is well worth thinking about these results and their geometric significance. You may consider verifying them on some of the classical surfaces we have studied, e.g. the sphere, ellipsoid or torus. Trying to prove them will take some time but if you make the effort and are patient you will learn a lot regardless of how successful you are in completing the proofs.

The first result we discuss is a generalization of the well-known result in Euclidean geometry which says that the sum of the interior angles in a (plane) triangle is equal to π. This corresponds to the case where the surface is a plane and the Gaussian curvature is zero. A *triangle* in a surface is a simple closed oriented curve formed by three smooth directed curves (Fig. 18.2).

Each of these smooth curves is called an *edge*, the edges meet at a *vertex* and the interior of the triangle is called its *face*. A triangle is called a *geodesic triangle* if each edge is a geodesic. In the plane, geodesics are straight lines so the usual triangle in the plane is a geodesic triangle. At the vertices P, Q and R the tangents of the two curves which meet are in the same tangent space on the surface and we can define the *angle* between them. It is also possible to define what we mean by an *interior angle* (e.g. the angle α at the vertex P). Let A denote the face of the triangle.

Theorem 18.6 (Local Gauss–Bonnet Theorem) *In a geodesic triangle on a simple surface*

$$\iint_A K = \sum \textit{interior angles} - \pi.$$

Stokes' Theorem plays an important role in the proof. For example consider the sphere of radius r. By Example 18.4 the lines of longitude and the equator are geodesics. Hence the triangle formed by the lines of longitude corresponding to $\psi = 0$ and $\psi = \pi/2$ and the equator are a geodesic triangle (Fig. 18.3).

The three interior angles are all $\pi/2$ and the area of the triangle is $\frac{1}{8} \times$ (total area of the sphere) and hence equal to $4\pi r^2/8$. Since the Gaussian curvature is $1/r^2$ this implies

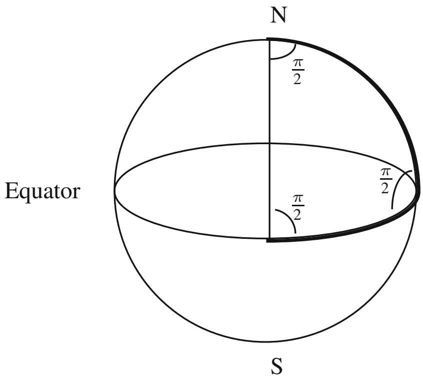

Fig. 18.3

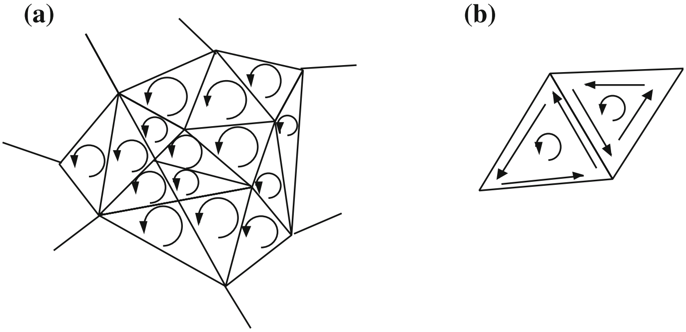

Fig. 18.4

$$\iint_A K = \frac{1}{r^2} \cdot \frac{4\pi r^2}{8} = \frac{\pi}{2}$$

while

$$\sum \text{interior angles} - \pi = \frac{3\pi}{2} - \pi = \frac{\pi}{2}$$

and we have verified the local Gauss–Bonnet Theorem for this triangle.

Any compact (i.e. closed and bounded) oriented surface **S** in $\mathbb{R}^3$ can be partitioned into a finite number of triangles each of which is oriented in an anticlockwise direction about the normal (Fig. 18.4a). This means that an edge which is in two triangles has opposite orientations in each (Fig. 18.4b). Let V denote the total number of vertices, E the total number of edges and F the total number of faces.

A remarkable result of Euler says that no matter how we partition the surface into (not necessarily geodesic) triangles the quantity $V - E + F$ remains unchanged. We call this number the *Euler–Poincaré characteristic* of S and denote it by $\chi(S)$. As

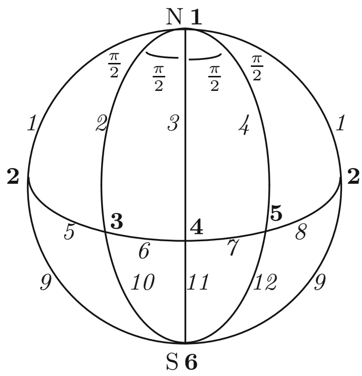

Fig. 18.5

a consequence of the local Gauss–Bonnet Theorem we have the following global result:

Theorem 18.7 (Global Gauss–Bonnet Formula) *If S is a compact oriented surface then*

$$\iint_S K = 2\pi\chi(S).$$

We call $\iint_S K$ the *total curvature* of the surface and it is remarkable that this quantity is always an integer multiple of 2π. Applying this to a sphere of radius r we find

$$\iint_S K = \frac{1}{r^2}\text{ Surface Area (sphere)} = \frac{4\pi r^2}{r^2} = 2\pi\chi(S)$$

and see that $\chi(\text{sphere}) = 2$.

On the other hand we may partition the sphere by the lines of longitude corresponding to $0,\ \pm\pi/2,\ \pi$ and the equator (Fig. 18.5). The outer edges are on the back of the sphere and coincide with 180° East or 180° West (i.e. the international date line). By counting we get $V = 6$, $E = 12$ and $F = 8$ and again

$$\chi\ (\text{sphere}) = 6 - 12 + 8 = 2\,.$$

Triangles can be replaced by rectangles in calculating the Euler-Poincaré characteristic since each rectangle can be partitioned into triangles using diagonals. This doubles the number of faces F and adds F new edges. Overall the sum $V - E + F$ is unchanged. On a box (Fig. 3.2) we have $V = 8$, $E = 12$, $F = 6$ and hence $V - E + F = 2$. Since a box can be inflated into a sphere this shows once more that the Euler-Poincaré characteristic of the sphere equals 2.

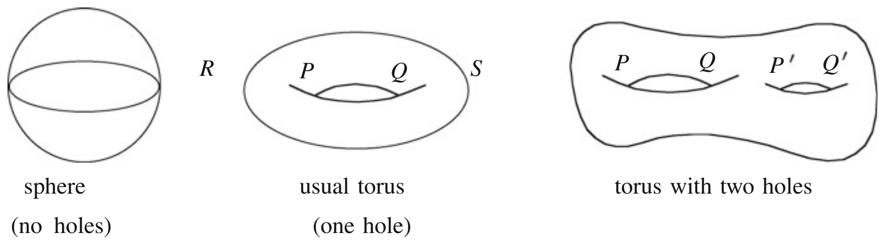

Fig. 18.6

Any compact oriented surface can be smoothly deformed into a surface with a finite number of holes (Fig. 18.6). We call the number of holes the *genus* of the surface. This smooth deformation does not change the number of faces, edges or vertices of any partition and hence the Euler–Poincaré characteristic is unchanged. It does, however, change the Gaussian curvature in many places, e.g. the sphere can be changed into an ellipsoid and we know that these do not have the same Gaussian curvature. The global Gauss–Bonnet Theorem says that the total curvature is unchanged.

For any compact surface S we have

$$\chi(S) = 2 - 2g$$

where g is the genus. This implies that the Euler–Poincaré characteristic is always an even integer and the total curvature is an integer multiple of 4π. Thus we have a remarkable set of relationships between total curvature, the number $V - E + F$ and the number of holes on a compact oriented surface **S**. This is not the end of the story as they are all equal to the index of any smooth vector field on **S**. The *index* of a vector field X, $i(X)$, is obtained by assigning an integer to each zero following a prescribed formula. Since χ (sphere) $= 2$ it follows that every smooth tangent vector field on a sphere has at least one zero and explains the existence of the bald spot which most people usually have at the point of maximal curvature on the midline of the calva.

It is interesting to speculate in a purely geometric way why these things are the way they are. For instance, we have seen that the total curvature of a sphere does not depend on the radius. Think of blowing up a balloon. The bigger the radius the larger the surface area. On the other hand the sphere is becoming less curved, i.e. the Gaussian curvature is decreasing, and over the whole surface the increase in surface area is counterbalanced by the decrease in curvature. Another simple observation: from the formula $\chi(S) = 2 - 2g$ we see that adding holes decreases the Euler–Poincaré characteristic and hence adds *negative* Gaussian curvature to the surface. Why should this be so? If we recall our study of the Gaussian curvature of the torus (Example 17.1) we noted that at the points P and Q in Fig. 18.6 we had negative Gaussian curvature while at R and S we had positive Gaussian curvature. Adding holes creates points like P' and Q' while the outside, where the Gaussian curvature is positive, is relatively unchanged and it is at least plausible that we are increasing the overall negative Gaussian curvature by adding holes.

Exercises

18.1 Show that every geodesic on the cylinder $\{(x, y, z) : x^2 + y^2 = 1\}$ has the form
$$\phi(t) = \big(\cos(at + b), \sin(at + b), ct\big), \quad t \in \mathbb{R}.$$

18.2 Let $\phi(t) = \big(x(t), y(t)\big)$, $a < t < b$ denote a unit speed parametrized curve in $\mathbb{R}^2$ and suppose $y(t) > 0$. Let S denote the surface obtained by rotating this curve about the x-axis. If
$$P(t, \theta) = \big(x(t), y(t)\cos\theta, y(t)\sin\theta\big)$$
show that the mapping
$$\varphi : t \longrightarrow P(t, \theta_0)$$
is unit speed and a geodesic for every fixed θ_0. Find the normal curvature in the direction $\varphi'(t)$ at the point $P(t, \theta_0)$.

18.3 Prove that a straight line which lies in a surface is a geodesic. By using this result find for each point P on the surface $z = x^2 - y^2$ two geodesics passing through P.

18.4 Let $P : [a, b] \to \Gamma$ denote a unit speed parametrized curve on the surface S. Suppose Γ has positive curvature in $\mathbb{R}^3$. Show that Γ is a line of curvature if and only if the geodesic torsion $\tau_g = 0$. If Γ is a geodesic show that $\tau_g = \tau$ where τ is the torsion of Γ as a curve in $\mathbb{R}^3$ (Chap. 7).

18.5 If Γ is a directed curve in a sphere with zero torsion show that Γ is part of a circle.

18.6 Let $P : [a, b] \to \Gamma$ denote a unit speed curve with strictly positive intrinsic curvature in an oriented surface **S**. Show that the normal curvature at $P(t)$ in the direction $P'(t)$ is zero if and only if the osculating plane to the curve coincides with the tangent plane to the surface. (A curve with this property at all points is called an *asymptotic curve* on the surface.)

18.7 Show that a curve in a sphere with constant geodesic curvature is part of a circle.

18.8 Find the Euler–Poincaré characteristic of the torus by partitioning it into triangles and calculating $V - E + F$. Verify your result by calculating the total curvature (see Example 17.1).

18.9 If T_1 and T_2 are two triangulations of a compact oriented surface **S** then $T_1 \subset T_2$ (or T_2 is a refinement of T_1) if every triangle in T_1 is a union of triangles from T_2. Let V_i, E_i and F_i denote, respectively, the number of vertices, edges and faces in T_i, $i = 1, 2$. Show, by induction on $V_2 - V_1$ that $V_1 - E_1 + F_1 = V_2 - E_2 + F_2$. Hence show that the Euler-Poincaré characteristic is well defined– that is that $V - E + F$ has the same value for any triangulation of $\mathbf{S}$.

18.10 Let T_i denote a triangulation of the compact oriented surface $\mathbf{S}_i$ of genus g_i, $i = 1, 2$. Let t_i denote a fixed triangle in T_i, $i = 1, 2$ and let $\mathbf{S}_3$ denote the surface obtained by taking the union of $\mathbf{S}_1$ and $\mathbf{S}_2$, identifying the edges of t_1 and t_2 (preserving orientation) and removing the face $t_1 \simeq t_2$. Show that $\mathbf{S}_3$ is a compact oriented surface of genus $g_1 + g_2$. Use this result, $\chi(Torus) = 0$ and induction to show

$$\chi(\mathbf{S}) = 2 - 2g$$

for any oriented surface $\mathbf{S}$.