Chapter 8
Geometry of Curves in $\mathbb{R}^3$

Summary *We apply the Frenet–Serret equations to study the geometric significance of torsion, to analyse curves in spheres and to characterise generalised helices.*

We first provide a geometrical interpretation of *zero torsion*.

Proposition 8.1 *If Γ is a directed curve in $\mathbb{R}^3$ with positive curvature at all points then the following are equivalent*

(a) *Γ is a plane curve.*
(b) *the function $t \rightarrow B(t)$ is constant.*
(c) *$\tau(t) = 0$ for all t.*

Proof Since $\kappa(t) > 0$, $N(t)$ is defined. By the Frenet–Serret equations $B'(t) = -\tau(t)N(t)$ and since $\|N(t)\| = 1$ we have:

$$B(t) \text{ is independent of } t \iff B'(t) = 0 \iff \tau(t) = 0.$$

Hence (b) and (c) are equivalent. Now suppose Γ is a plane curve, i.e. there exists a plane in $\mathbb{R}^3$ which contains Γ. Then there exists a unit vector A in $\mathbb{R}^3$ and a real number c such that $\Gamma \subset \{X \in \mathbb{R}^3 : X \cdot A = c\}$. Let P denote a unit speed parametrization of Γ with domain $[a, b]$. For all $t \in [a, b]$, $P(t) \cdot A = c$. We have

$$\frac{d}{dt}(P(t) \cdot A) = T(t) \cdot A = 0$$

and

$$\frac{d^2}{dt^2} P(t) \cdot A = T'(t) \cdot A = \kappa(t)N(t) \cdot A = 0.$$

Hence A is perpendicular to both $T(t)$ and $N(t)$ and $B(t) = \pm A$ for all t. Suppose $B(t_1) = A$ and $B(t_2) = -A$. The mapping $g : t \longrightarrow B(t) \cdot A$ is a continuous real-valued function on $[a, b]$, $g(t) = \pm 1$ for all t, $g(t_1) = 1$ and $g(t_2) = -1$. This is impossible, since it would imply, by the Intermediate Value Theorem, that there exists $t_0 \in (a, b)$ such that $g(t_0) = 0$. Hence $g(t) = +1$ for all t or $g(t) = -1$ for

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all t and this implies $B(t) = A$ for all t or $B(t) = -A$ for all t. In either case B is a constant function and (a) $\Longrightarrow$ (b).

Now suppose (b) holds. Let P denote a unit speed parametrization of Γ with domain $[a, b]$. If $t_0 \in (a, b)$, $X_0 = P(t_0)$ and $B(t) = B$ for all t, then

$$\begin{aligned}\frac{d}{dt}\langle P(t) - X_0, B\rangle &= \langle P'(t), B(t)\rangle + \langle P(t) - X_0, 0\rangle \\ &= \langle T(t), B(t)\rangle + 0 \\ &= 0 \ \ (\text{since } T(t) \perp B(t)).\end{aligned}$$

Hence there exists a constant c such that

$$\langle P(t) - X_0, B\rangle = c$$

and P lies in the plane through X_0 perpendicular to B. This shows that (b) $\Longrightarrow$ (a) and completes the proof. □

Proposition 8.1 gives a precise geometric interpretation of *zero torsion*. To interpret *non-zero torsion* we look at an expansion of the parametrization P about a fixed point relative to the basis $\{T(t_0), N(t_0), B(t_0)\}$. For convenience we may suppose $t_0 = 0$.

In Chap. 7 we obtained, using orthogonality, the Taylor series expansion and the Frenet–Serret equations, the following three expansions of $P(t)$:

$$P(t) = \langle P(t), T(0)\rangle T(0) + \langle P(t), N(0)\rangle N(0) + \langle P(t), B(0)\rangle B(0) \tag{8.1}$$

$$= P(0) + P'(0)t + P''(0)\frac{t^2}{2} + g(t)t^2 \tag{8.2}$$

$$= P(0) + T(0)t + \frac{\kappa(0)N(0)}{2}t^2 + g(t)t^2 \tag{8.3}$$

where $g(t) \to 0$ in $\mathbb{R}^3$ as $t \to 0$. From (8.3) we can identify the main influence—the first non-constant term in the Taylor series expansion—on the shape of the curve in the $T(0)$ and $N(0)$ directions. Comparing (8.1–8.3) we see also that the first possible non-zero term in the $B(0)$ direction will be $\frac{d^3}{dt^3}\langle P(t), B(0)\rangle \Big|_{t=0}$. By repeated differentiation and use of the Frenet–Serret equation for T' we obtain

$$\begin{aligned}\langle P'(t), B(0)\rangle &= \langle T(t), B(0)\rangle \\ \langle P''(t), B(0)\rangle &= \langle T'(t), B(0)\rangle = \langle \kappa(t)N(t), B(0)\rangle\end{aligned}$$

and, using the Frenet–Serret equation for N',

$$\begin{aligned}\langle P'''(t), B(0)\rangle &= \langle \kappa'(t)N(t) + \kappa(t)N'(t), B(0)\rangle \\ &= \langle \kappa'(t)N(t) + \kappa(t)\big(-\kappa(t)T(t) + \tau(t)B(t)\big), B(0)\rangle.\end{aligned}$$

Letting $t = 0$ we get

$$\begin{aligned}\langle P'''(0), B(0)\rangle &= \kappa'(0)\langle N(0), B(0)\rangle - \kappa^2(0)\langle T(0), B(0)\rangle \\ &\quad + \kappa(0)\tau(0)\langle B(0), B(0)\rangle \\ &= \kappa(0)\tau(0)\end{aligned}$$

since $\{T, N, B\}$ are mutually perpendicular unit vectors. This gives us the approximation

$$Q(t) = P(0) + T(0)t + \kappa(0)N(0)\frac{t^2}{2} + \kappa(0)\tau(0)B(0)\frac{t^3}{6}$$

called the *Frenet approximation* to the curve Γ at 0. The Frenet approximation is clearly a refinement of (8.3) which takes account of torsion. The function $t \to Q(t)$ defines a parametrized curve which has the *same* Frenet–Serret apparatus as the original curve at $P(0)$. From the Frenet approximation we see the influence of non-zero torsion on the shape of the curve. Torsion controls the motion of the curve *orthogonal to the osculating plane*. If $\tau(0) > 0$ then the curve twists towards the side of the osculating plane which contains $B(0)$ and the greater $\tau(0)$ the more dramatic the twist. If $\tau(0) < 0$ the curve twists towards $-B(0)$.

An everyday example of non-zero torsion is given by the curve on the edge of a screw. In tightening a screw one usually uses the right-hand and follows the right-hand rule while in loosening a screw one follows the left-hand rule. If you have any doubts about the difference change hands. This also illustrates the two different orientations of $\mathbb{R}^3$.

Example 8.2 In this example we study a directed curve Γ which lies in a sphere with centre c and radius r. Let P denote a unit speed parametrization of Γ. Our hypothesis states that

$$\|P(t) - c\|^2 = \langle P(t) - c, P(t) - c\rangle = r^2.$$

Consider the expansion of $P(t) - c$ relative to the orthonormal basis $\{T(t), N(t), B(t)\}$, i.e.

$$\begin{aligned}P(t) - c &= \langle P(t) - c, T(t)\rangle T(t) + \langle P(t) - c, N(t)\rangle N(t) \\ &\quad + \langle P(t) - c, B(t)\rangle B(t).\end{aligned} \tag{8.4}$$

Differentiating we get

$$\frac{d}{dt}\langle P(t) - c, P(t) - c\rangle = 0 = 2\langle P(t) - c, P'(t)\rangle\,.$$

Since P is unit speed, $P'(t) = T(t)$, and we may restate this as follows:

$$\langle P(t) - c, T(t)\rangle = 0. \tag{8.5}$$

Differentiating again and using the Frenet–Serret equation for T' gives us

$$\begin{aligned}0 = \frac{d}{dt}\langle P(t) - c, T(t)\rangle &= \langle T(t), T(t)\rangle + \langle P(t) - c, T'(t)\rangle \\ &= 1 + \kappa(t)\langle P(t) - c, N(t)\rangle.\end{aligned}$$

Hence $\kappa(t) \neq 0$ for all t, $N(t)$ is defined and

$$\langle P(t) - c, N(t)\rangle = -\frac{1}{\kappa(t)}. \tag{8.6}$$

Differentiating (8.6) and applying the Frenet–Serret equation for N' we obtain

$$\begin{aligned}-\left(\frac{1}{\kappa(t)}\right)' &= \frac{d}{dt}\langle P(t) - c, N(t)\rangle \\ &= \langle T(t), N(t)\rangle + \langle P(t) - c, N'(t)\rangle \\ &= \langle P(t) - c, -\kappa(t)T(t) + \tau(t)B(t)\rangle \text{ (since } N \perp T) \\ &= \tau(t)\langle P(t) - c, B(t)\rangle \text{ (by (8.5)).}\end{aligned}$$

If $\tau(t_0) \neq 0$ for some t_0 then, by continuity, $\tau(t) \neq 0$ for all t near t_0 and

$$\langle P(t) - c, B(t)\rangle = -\frac{1}{\tau(t)}\left(\frac{1}{\kappa(t)}\right)'. \tag{8.7}$$

Substituting (8.5–8.7) into (8.4) we get

$$P(t) - c = -\frac{1}{\kappa(t)}N(t) - \frac{1}{\tau(t)}\left(\frac{1}{\kappa(t)}\right)' B(t)$$

and we have found $P(t)$ in terms of the Frenet–Serret apparatus of Γ. By Pythagoras' theorem

$$r^2 = \|P(t) - c\|^2 = \frac{1}{\kappa(t)^2} + \left(\left(\frac{1}{\kappa(t)}\right)' \cdot \frac{1}{\tau(t)}\right)^2.$$

Hence we have recovered the radius of the sphere from the curvature and torsion *when we know* that the curve lies in a sphere. In particular, we see that $r^2 \geq 1/\kappa(t)^2$, or $\kappa(t) \geq 1/r$, which we may loosely rephrase as saying that a curve in a sphere is at least as curved as the sphere in which it lies.

If $\tau(t) = 0$ for all t in an open interval I then the above implies that $\kappa(t)$ is constant on I. Since a plane circle has constant curvature, Proposition 8.1 and Example 8.5 imply that the part of Γ parametrized by restricting P to I is part of a circle contained in a plane in $\mathbb{R}^3$.

Example 8.3 The helix in Example 7.2 satisfies $\langle T_p, (0, 0, 1)\rangle = h\omega$ for all p where T_p is the unit tangent at the point p. We generalise this by defining a *generalised* *helix* in $\mathbb{R}^3$, as a curve Γ of positive curvature (at all points) for which there exists a unit vector $\mathbf{u}$ in $\mathbb{R}^3$ such that

$$\langle T_p, \mathbf{u} \rangle = c \text{ (constant)}$$

for all $p \in \Gamma$. Let P denote a unit speed parametrization of the generalised helix Γ and let $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$ denote the Frenet–Serret apparatus at $P(t)$. We prove the following characterisation:

$$\Gamma \text{ is a generalised helix} \iff \frac{\tau(t)}{\kappa(t)} \text{ is constant (i.e. independent of } t).$$

Since $\langle T(t), \mathbf{u} \rangle$ is constant it follows, by the Cauchy–Schwarz inequality (Example 3.4), that

$$|\langle T(t), \mathbf{u} \rangle| \leq \|T(t)\| \cdot \|\mathbf{u}\| \leq 1$$

and $\langle T(t), \mathbf{u} \rangle = \cos\theta$ for some θ. If $\theta = n\pi$ then, by the equality case in the Cauchy–Schwarz inequality, $T(t) = \pm\mathbf{u}$. By the Intermediate Value Theorem (see the proof of Proposition 8.1) this implies that $T(t)$ is a constant function of t. Hence $T'(t) = 0$ and this contradicts our hypothesis. We thus have $\langle T(t), \mathbf{u} \rangle = \cos\theta$ for some $\theta \neq n\pi$. Again using the orthonormal basis $\{T(t), N(t), B(t)\}$ we have

$$\mathbf{u} = \langle \mathbf{u}, T(t) \rangle T(t) + \langle \mathbf{u}, N(t) \rangle N(t) + \langle \mathbf{u}, B(t) \rangle B(t).$$

Since $\langle T(t), \mathbf{u} \rangle = \cos\theta$

$$\frac{d}{dt}\big(\langle T(t), \mathbf{u} \rangle\big) = 0 = \langle T'(t), \mathbf{u} \rangle + \langle T(t), (\mathbf{u})' \rangle = \langle \kappa(t) N(t), \mathbf{u} \rangle$$

(since $\mathbf{u}$ is a constant, $(\mathbf{u})' = 0$). By our hypothesis $\kappa(t) \neq 0$ and this implies $\langle \mathbf{u}, N(t) \rangle = 0$. Hence

$$\mathbf{u} = \cos\theta T(t) + \sin\theta B(t).$$

A further application of the Frenet–Serret equations implies

$$\begin{aligned} 0 = \frac{d}{dt}\mathbf{u} &= \cos\theta T'(t) + \sin\theta B'(t) \\ &= \cos\theta \kappa(t) N(t) - \sin\theta \tau(t) N(t) \\ &= \big(\cos\theta \kappa(t) - \sin\theta \tau(t)\big) N(t). \end{aligned}$$

Since $\|N(t)\| = 1$ this implies $\cos\theta\kappa(t) = \sin\theta\tau(t)$ and

$$\frac{\tau(t)}{\kappa(t)} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

We have shown $\tau(t)/\kappa(t)$ is constant for any generalised helix. In obtaining this result we obtained a formula for $\mathbf{u}$ and now use this to prove the converse. Let Γ denote a directed curve with non-zero curvature in $\mathbb{R}^3$ such that $\tau(t)/\kappa(t)$ is constant (i.e. independent of t). This hypothesis implies that there exists a real number θ, $0 < \theta < \pi$, such that $\cot\theta = \tau(t)/\kappa(t)$ for all t (note that θ does not depend on t). Let

$$\mathbf{u}(t) = \cos\theta\, T(t) + \sin\theta\, B(t).$$

By Pythagoras' theorem $\|\mathbf{u}\|^2 = \cos^2\theta + \sin^2\theta = 1$ and $\mathbf{u}$ is a unit vector. To show that $\mathbf{u}$ does not depend on t we prove $\dfrac{d}{dt}\big(\mathbf{u}(t)\big) = 0$. By the Frenet–Serret equations

$$\begin{aligned}\frac{d}{dt}\big(\mathbf{u}(t)\big) &= \cos\theta\, T'(t) + \sin\theta\, B'(t)\\ &= \big(\kappa(t)\cos\theta - \tau(t)\sin\theta\big)N(t) = 0.\end{aligned}$$

Hence $\mathbf{u}$ does not depend on t and so is a constant. Moreover, since $T \perp B$,

$$\langle T(t), \mathbf{u}\rangle = \langle T(t), \cos\theta\, T(t) + \sin\theta\, B(t)\rangle = \cos\theta.$$

This shows that Γ is a generalised helix and justifies our claim.

Our analysis so far applies to unit speed parametrizations of a directed curve. Unfortunately, many natural parametrizations of curves are not unit speed. It is thus useful to be able to calculate the Frenet–Serret apparatus directly from an arbitrary parametrization.

Let $P\colon [a,b] \to \mathbb{R}^3$ denote an *arbitrary* parametrization of the directed curve Γ. We suppose $P'(t)$ and $P''(t)$ are both non-zero for all t. Let $s\colon [a,b] \to [0,l]$ denote the length function associated with P (see Chap. 5). Then l is the length of Γ, $\|P'(t)\| = s'(t)$ and $Q := P \circ s^{-1}$ is a unit speed parametrization of Γ. Let $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$ denote the Frenet–Serret apparatus *at the point* $P(t)$ on Γ. We have

$$Q \circ s(t) = P \circ s^{-1} \circ s(t) = P(t).$$

Hence

$$\frac{d}{dt}\Big(Q\big(s(t)\big)\Big) = Q'\big(s(t)\big)s'(t) = P'(t) = \|P'(t)\|T(t)$$

and

$$Q'\big(s(t)\big) = T(t) = \frac{P'(t)}{\|P'(t)\|}. \tag{8.8}$$

Differentiating again

$$\frac{d^2}{dt^2}\Big(Q\big(s(t)\big)\Big) = \frac{d}{dt}\Big(Q'\big(s(t)\big)\cdot s'(t)\Big)$$
$$= Q''\big(s(t)\big)\big(s'(t)\big)^2 + Q'\big(s(t)\big)s''(t) = P''(t).$$

Since Q has unit speed the Frenet–Serret equations imply

$$Q''\big(s(t)\big) = \kappa(t)N(t)$$

and

$$P''(t) = \big(s'(t)\big)^2\kappa(t)N(t) + s''(t)T(t).$$

Hence

$$P'(t)\times P''(t) = s'(t)T(t)\times\Big(s'(t)^2\kappa(t)N(t) + s''(t)T(t)\Big)$$
$$= s'(t)^3\kappa(t)B(t)$$

since $T\times N = B$ and $T\times T = 0$. Since $\|B(t)\| = 1$ and $s'(t) = \|P'(t)\|$ this implies

$$\kappa(t) = \frac{\|P'(t)\times P''(t)\|}{\|P'(t)\|^3} \tag{8.9}$$

and

$$B(t) = \frac{P'(t)\times P''(t)}{\|P'(t)\times P''(t)\|}. \tag{8.10}$$

The simplest way to obtain N is to use the formula

$$N = B\times T. \tag{8.11}$$

The Frenet–Serret equation for N' implies

$$Q'''\big(s(t)\big) = (\kappa N)'(t) = \kappa'(t)N(t) + \kappa(t)N'(t)$$
$$= \kappa'(t)N(t) + \kappa(t)\big(-\kappa(t)T(t) + \tau(t)B(t)\big)$$

and hence

$$\langle Q'''\big(s(t)\big), B(t)\rangle = \kappa(t)\tau(t).$$

On the other hand

$$\begin{aligned}\frac{d^3}{dt^3}\Big(Q\big(s(t)\big)\Big) &= P'''(t)\\ &= Q'''\big(s(t)\big)s'(t)^3 + 3Q''\big(s(t)\big)s'(t)s''(t) + Q'\big(s(t)\big)s'''(t)\\ &= Q'''\big(s(t)\big)s'(t)^3 + 3\kappa(t)s'(t)s''(t)N(t) + s'''(t)T(t).\end{aligned}$$

By orthogonality

$$\begin{aligned}\left\langle P'''(t),\ \frac{P'(t)\times P''(t)}{\|P'(t)\times P''(t)\|}\right\rangle &= \langle P'''(t), B(t)\rangle = s'(t)^3\langle Q'''\big(s(t)\big), B(t)\rangle\\ &= \|P'(t)\|^3\kappa(t)\tau(t).\end{aligned}$$

Finally

$$\tau(t) = \frac{\langle P'''(t), P'(t)\times P''(t)\rangle}{\|P'(t)\times P''(t)\|\cdot\|P'(t)\|^3\kappa(t)} = \frac{\langle P'''(t), P'(t)\times P''(t)\rangle}{\|P'(t)\times P''(t)\|^2}. \tag{8.12}$$

Equations (8.8–8.12) are the Frenet–Serret apparatus at $P(t)$ for Γ in terms of the parametrization P.

Example 8.4 Calculate the Frenet–Serret apparatus of the curve parametrized by

$$P(t) = (t - \cos t, \sin t, t).$$

We first calculate the required derivatives of P; P', P'' and P'''. We have $P'(t) = (1 + \sin t, \cos t, 1)$, $P''(t) = (\cos t, -\sin t, 0)$ and $P'''(t) = (-\sin t, -\cos t, 0)$. Next, we obtain the cross product

$$P'(t)\times P''(t) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1+\sin t & \cos t & 1\\ \cos t & -\sin t & 0\end{vmatrix} = (\sin t, \cos t, -\sin t - 1)$$

and finally the norms or lengths

$$\begin{aligned}\|P'(t)\| &= (1 + 2\sin t + \sin^2 t + \cos^2 t + 1)^{1/2} = (3 + 2\sin t)^{1/2}\\ \|P'(t)\times P''(t)\| &= (\sin^2 t + \cos^2 t + \sin^2 t + 2\sin t + 1)^{1/2}\\ &= (2 + 2\sin t + \sin^2 t)^{1/2}.\end{aligned}$$

Hence

$$T(t) = \frac{P'(t)}{\|P'(t)\|} = \frac{(1+\sin t, \cos t, 1)}{(3 + 2\sin t)^{1/2}}$$

$$\kappa(t) = \frac{\|P'(t) \times P''(t)\|}{\|P'(t)\|^3} = \frac{(2 + 2\sin t + \sin^2 t)^{1/2}}{(3 + 2\sin t)^{3/2}}$$
$$B(t) = \frac{P'(t) \times P''(t)}{\|P'(t) \times P''(t)\|} = \frac{(\sin t, \cos t, -\sin t - 1)}{(2 + 2\sin t + \sin^2 t)^{1/2}}$$
$$\begin{aligned}\tau(t) &= \frac{\langle P'''(t), P'(t) \times P''(t)\rangle}{\|P'(t) \times P''(t)\|^2}\\ &= \frac{(-\sin t, -\cos t, 0)\cdot(\sin t, \cos t, -\sin t - 1)}{2 + 2\sin t + \sin^2 t}\\ &= \frac{-1}{2 + 2\sin t + \sin^2 t}\end{aligned}$$
$$\begin{aligned}N(t) &= B(t) \times T(t)\\ &= \frac{1}{\sqrt{3 + 2\sin t}\sqrt{2 + 2\sin t + \sin^2 t}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sin t & \cos t & -\sin t - 1\\ 1 + \sin t & \cos t & 1\end{vmatrix}\\ &= \frac{(2\cos t + \sin t\cos t, -1 - 3\sin t - \sin^2 t, -\cos t)}{(6 + 10\sin t + 7\sin^2 t + 2\sin^3 t)^{1/2}}\end{aligned}$$

Example 8.5 In Example 8.2 we showed that curvature and torsion together allowed us to deduce properties of spherical curves. In this example we show that curvature and torsion completely determine the shape of a curve in $\mathbb{R}^3$. Let Γ and Γ_1 denote two directed curves having the same length l in $\mathbb{R}^3$. We suppose that both have positive curvature. Now transfer Γ_1 so that its initial point coincides with the initial point of Γ and rotate it so that the tangents, normals and binormals of Γ and Γ_1 coincide at the initial point. These operations do not affect the shape of Γ_1. Let $P\colon [0, l] \to \Gamma$ and $P_1\colon [0, l] \to \Gamma_1$ denote unit speed parametrizations. We now suppose that the curvature and torsion of Γ and Γ_1 at $P(t)$ and $P_1(t)$ coincide for all t and thus we have the Frenet–Serret apparatus $\{T, N, B, \kappa, \tau\}$ and $\{T_1, N_1, B_1, \kappa, \tau\}$ for Γ and Γ_1 respectively. Using the dot product in $\mathbb{R}^3$ we define g: $[0, l] \to \mathbb{R}$ by

$$g(t) = T(t)\cdot T_1(t) + N(t)\cdot N_1(t) + B(t)\cdot B_1(t).$$

By our hypothesis $g(0) = 3$ and by the Cauchy–Schwarz inequality (Example 3.4) $-3 \le g(t) \le 3$ for all t and $g(t) = 3$ if and only if $T(t) = T_1(t)$, $N(t) = N_1(t)$ and $B(t) = B_1(t)$. From the Frenet–Serret equations

$$g' = \kappa N\cdot T_1 + \kappa T\cdot N_1 + (-\kappa T + \tau B)\cdot N_1 + N\cdot(-\kappa T_1 + \tau B_1) - \tau N\cdot B_1 - \tau B\cdot N_1 = 0.$$

Hence g is a constant mapping and, since $g(0) = 3$, we have $g(t) = 3$ for all t and the Frenet–Serret apparatus is the same for both curves. In particular

$$(P - P_1)'(t) = P'(t) - P_1'(t) = T(t) - T_1(t) = 0$$

and $P(t) = P_1(t) + C$ for all t. Since $P(0) = P_1(0)$ this implies $P(t) = P_1(t)$ for all t and one curve lies on top of the other. We conclude that Γ and Γ_1 have the same shape.

Exercises

8.1 Show that each of the directed curves in Exercise 7.5 is a generalised helix. In each case find a unit vector $\mathbf{u}$ such that $\langle T(t), \mathbf{u}\rangle$ is independent of t.

8.2 Let $P(t) = (t, 1 + t^{-1}, t^{-1} - t)$, $1 \le t \le 2$, denote a parametrization of the curve Γ in $\mathbb{R}^3$. Show that $B(t) = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ for all t and hence deduce that Γ lies in the plane $x - y + z = -1$. Find the Frenet–Serret apparatus for Γ.

8.3 If Γ is parametrized by

$$P(\theta) = (\log\cos\theta, \log\sin\theta, \sqrt{2}\theta), \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{3}$$

show that Γ has curvature $\sin 2\theta/\sqrt{2}$ at $P(\theta)$.

8.4 For the curve parametrized by $P(t) = (3t^2, 3t - t^3, 3t + t^3)$, $-1 \le t \le 1$, show that

$$\kappa(t) = -\tau(t) = \frac{1}{3(1+t^2)^2}.$$

Find a unit vector $\mathbf{u}$ such that $\langle T(t), \mathbf{u}\rangle$ is independent of t.

8.5 Find the curvature and torsion of the curve parametrized by

$$P(t) = (e^t \cos t, e^t \sin t, e^t), \quad t \in \mathbb{R}.$$

8.6 The plane through a point on a curve perpendicular to the tangent line is called the *normal plane* to the curve at the point. Show that a curve lies on a sphere if the intersection of all normal planes is non-empty. Hence show that the curve parametrized by

$$P(\theta) = (-\cos 2\theta, -2\cos\theta, \sin 2\theta), \quad \theta \in [0, 2\pi]$$

lies in a sphere. Find the centre and radius of the sphere.

8.7 Show that the curve parametrized by $P(t) = (at, bt^2, t^3)$, $ab \neq 0$, is a generalised helix if and only if $4b^4 = 9a^2$.