Chapter 16
Geometry of Surfaces in $\mathbb{R}^3$

Summary *Using normal sections we define normal curvature, principal curvatures and Gaussian curvature. Geometric interpretations and a method of calculating the Gaussian curvature using parametrization are given.*

Our geometric study of surfaces in $\mathbb{R}^3$ is motivated by some very simple basic questions such as; what is curvature and how does one measure it? Is there any relationship between surface area and curvature? What is the shortest distance between two points on a surface? We start by taking an intuitive and non-rigorous look at an apparently very special case and this leads us to mathematical concepts which are both useful and natural. The surface we study is one with which we are already familiar and this simple example gives us *everything*. We have already used *all* the techniques that we require, and *all* the facts that we need are known to us—we just have to look at things in a slightly different way. The surface S we consider is the graph of the smooth function f where $f(x, y)$ is the height above sea level. We suppose f has a local maximum at (x_0, y_0) and we study S near $p = \big(x_0, y_0, f(x_0, y_0)\big)$. Since the point (x_0, y_0) is a critical point of f the tangent plane to S at p is the plane through the point $(x_0, y_0, f(x_0, y_0))$ *parallel* to the xy-plane in $\mathbb{R}^3$. The unit normals to the surface at p are $\pm(0, 0, 1)$ and, for convenience, we *choose* $(0, 0, 1)$ as our unit normal $\mathbf{n}(p)$. The tangent plane of S at p consists of all vectors of the form $(v_1, v_2, 0)$.

If our notion of curvature is meaningful it should say something when we take cross-sections of a surface. For instance, if we keep getting circles when we take cross-sections we should not be surprised if the surface is a sphere and if each cross-section is either a line or a plane we expect the surface to be a plane. We consider cross-sections of $\mathbb{R}^3$ through the point p which contain the unit normal at p, $\mathbf{n}(p)$. Since cross-sections of $\mathbb{R}^3$ are two-dimensional this will cut the tangent plane and we can find a unit tangent vector at p, $\mathbf{v}$, such that our cross-section has the form

$$p + \{x\mathbf{v} + y\mathbf{n}(p) : x, y \in \mathbb{R}^2\}.$$

We identify this cross-section of $\mathbb{R}^3$ with $\mathbb{R}^2_{(x,y)}$ by the correspondence

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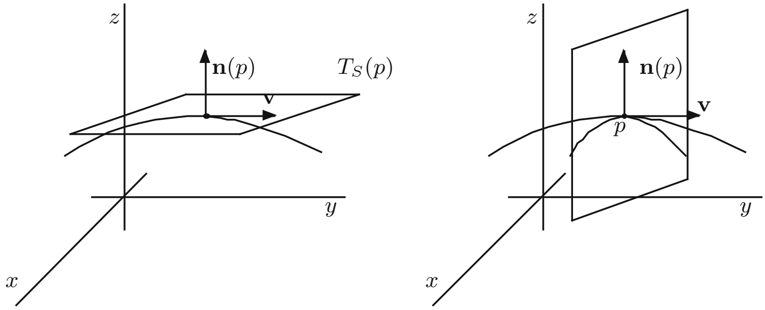

Fig. 16.1

$$p + x\mathbf{v} + y\mathbf{n}(p) \in \mathbb{R}^3 \longleftrightarrow (x, y) \in \mathbb{R}^2.$$

The intersection of this cross-section of $\mathbb{R}^3$ and the surface S is a *curve* on the surface called a *normal section* of the surface at p (Fig. 16.1).

Using this correspondence we may identify the normal section of the surface with a curve Γ in $\mathbb{R}^2$ passing through the origin. We direct Γ by requiring $(0, 1)$ to be the unit normal at the origin and calculate its curvature using the concept of plane curvature in Chap. 7. The curvature of Γ at the origin in $\mathbb{R}^2$ will depend on the unit vector $\mathbf{v}$ and the choice of normal and we denote it by $k_p(\mathbf{v})$. We call $k_p(\mathbf{v})$ the *normal curvature* of S at p in the direction $\mathbf{v}$. By examining $k_p(\mathbf{v})$ as $\mathbf{v}$ ranges over all unit tangent vectors to S at p we hope to draw conclusions on the shape of the surface near p. For example, if $k_p(\mathbf{v}) = 1/r$ for all $\mathbf{v}$ then we might expect the surface to approximate a sphere of radius r *near* p. All normal sections are paths on the surface leading to the point p (i.e. to the top of the mountain) and the different normal curvatures distinguish between the steep and the not so steep paths. Visualising circles of different radii going through p, with p as their highest point, gives us an idea of the shape of the surface near p. Of course it is very laborious to examine all these curves and to calculate all their curvatures so instead we examine them collectively to see if any particular features of the set of *all* normal curvatures captures the essence of the shape near p.

Since f has a local maximum at (x_0, y_0) and $\mathbf{n}(p) = (0, 0, 1)$ *all* normal circles of curvature will lie on the *same* side of the tangent plane and on the opposite side to the normal. This means that all normal curvatures will be negative. Choosing $(0, 0, -1)$ as unit normal at p changes the signs of all normal curvatures. If we were examining a local minimum at (x_0, y_0) and $\mathbf{n}(p) = (0, 0, 1)$ we would have found that all normal curvatures were positive.

But changing the normal or turning the surface upside down—this is equivalent to changing local maxima into local minima and conversely—does not affect the shape of the surface in any way and we conclude: if all normal curvatures are strictly positive or all are strictly negative then the shape of S near p is similar to an ellipsoid (Fig. 16.2). These points are called *elliptic* points of the surface. We can extend

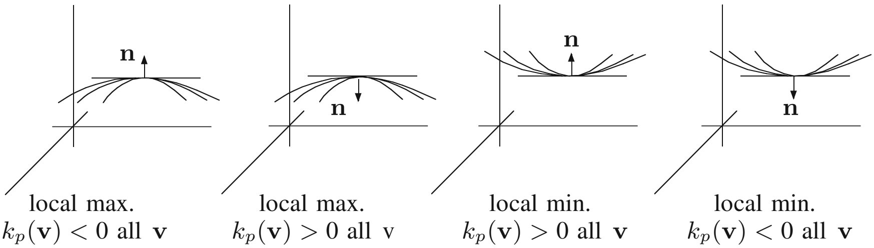

Fig. 16.2

this analysis to arbitrary critical points and clearly if, at a point p, normal curvature takes both positive and negative values then the surface near p is similar in shape to a saddle point. Such points are called *hyperbolic* points of the surface. Our preliminary investigation has shown that information on the shape of the surface can be obtained from the range of values taken by $k_p(\mathbf{v})$ as $\mathbf{v}$ varies over the set of unit tangent vectors at p. Since the range is determined by the extremal values we let $k_1(p) = \max\limits_{\|\mathbf{v}\|=1} k_p(\mathbf{v})$ and $k_2(p) = \min\limits_{\|\mathbf{v}\|=1} k_p(\mathbf{v})$ and call $k_1(p)$ and $k_2(p)$ the *principal curvatures* at p. If the maximum or minimum occurs at a vector $\mathbf{v}$ we call $\mathbf{v}$ a *principal curvature direction*.

We consider the various possibilities that may arise for $k_1(p)$ and $k_2(p)$. If $k_1(p) = k_2(p)$ then we call p an *umbilic point* and if $k_1(p) = k_2(p) = 0$ we call p a *flat spot.* If p is umbilic then $k_p(\mathbf{v}) = k_1(p)$ for all $\mathbf{v}$, $\|\mathbf{v}\| = 1$, and if $k_1(p) \neq 0$ then the surface near p is similar to a portion of a sphere of radius $1/|k_1(p)|$. If p is a flat spot then all normal curvatures are zero and the surface near p is very flat and almost like a part of a plane. At an umbilic point all directions are principal curvature directions.

We now consider non-umbilic points, i.e. $k_1(p) > k_2(p)$. If $k_2(p) > 0$ then for any $\mathbf{v}$, $\|\mathbf{v}\| = 1$, we have $0 < k_2(p) \leq k_p(\mathbf{v})$ and if $k_1(p) < 0$ then $k_p(\mathbf{v}) \leq k_1(p) < 0$ for all $\mathbf{v}$, $\|\mathbf{v}\| = 1$. Hence, in both cases, all normal curvatures have the same sign and near p the surface is shaped like an ellipsoid at one of its extreme points (i.e. like the surface $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ at the point $(0, 0, c)$). If $k_2(p) < 0 < k_1(p)$ then the surface near p is shaped like a saddle point.

To summarise our conclusions more concisely we introduce *Gaussian curvature*.

Definition 16.1 The Gaussian curvature, $K(p)$, at a point p on a surface S, is the product of the principal curvatures, $k_1(p)k_2(p)$.

We have noted already that $k_p(\mathbf{v})$ depends on the choice of normal—changing the normal changes the sign of $k_p(\mathbf{v})$—but since two changes of sign cancel one another it follows that $K(p)$ *does not depend* on the choice of normal.

Although we have only considered critical points on the graph in our analysis we will see shortly that our analysis applies to *all* points on a surface and for this reason we state the following result in its full generality.

Proposition 16.2 *At a non-umbilic point on a surface S in $\mathbb{R}^3$ we have:*

$$\begin{aligned} K(p) > 0 &\iff \textit{near } p, S \textit{ is shaped like an ellipsoid},\\ K(p) < 0 &\iff \textit{near } p, S \textit{ is shaped like a saddle point},\\ K(p) = 0 &\iff \textit{near } p, S \textit{ is shaped like a cylinder or cone}. \end{aligned}$$

At an umbilic point $K(p) \geq 0$ and

$$\begin{aligned} K(p) > 0 &\iff \textit{near } p, S \textit{ is shaped like a sphere},\\ K(p) = 0 &\iff \textit{near } p, S \textit{ is very flat}. \end{aligned}$$

Consider again the graph of a function f near a critical point (x_0, y_0). We suppose for simplicity that $(x_0, y_0) = (0, 0)$ and $f(0, 0) = 0$. The tangent plane at $p = (x_0, y_0, f(0, 0)) = (0, 0, 0)$ is the xy-plane and consists of the points $\{(v_1, v_2, 0) : v_1, v_2 \in \mathbb{R}\}$, and $\mathbf{n} = \mathbf{n}(p) = (0, 0, 1)$ is our choice of unit normal. Let $\mathbf{v} = (v_1, v_2, 0)$ be a fixed unit vector. The section of $\mathbb{R}^3$ determined by $\mathbf{n}$ and $\mathbf{v}$ is the plane

$$\{(tv_1, tv_2, s) : t, s \in \mathbb{R}\} \tag{16.1}$$

and the graph of f is the set

$$\{(x, y, f(x, y)) : (x, y) \in domain(f)\}. \tag{16.2}$$

The points which satisfy both (16.1) and (16.2) form the normal section of the surface through p defined by $\mathbf{v}$ and are easily seen near p to have the form

$$\{(tv_1, tv_2, f(tv_1, tv_2)),\ t \in [-a, a]\} = \{t\mathbf{v} + f(tv_1, tv_2)\mathbf{n} : t \in [-a, a]\} \tag{16.3}$$

for some positive a.

On identifying the plane in $\mathbb{R}^3$ spanned by $\mathbf{v}$ and $\mathbf{n}$ with $\mathbb{R}^2_{(x,y)}$ we see that the normal section can be parameterised by

$$P(t) = (t, f(tv_1, tv_2)), t \in [-a, a],$$

and may be identified with the graph of the function $h\colon [-a, a] \to \mathbb{R}, h(t) = f(tv_1, tv_2)$. If we parametrize the graph of h by $t \to \big(t, h(t)\big)$ we obtain $(1, 0)$ as unit tangent at $(0, 0)$ and $(0, 1)$ as unit normal (see Chap. 7). Hence the curvature of the graph at the origin, directed by this parametrization, is the normal curvature of S at p in the direction $\mathbf{v}$. By Example 7.1 the curvature at $(t, h(t))$ is

$$\frac{h''(t)}{(1 + h'(t)^2)^{3/2}}.$$

By the chain rule

$$h'(t) = v_1 f_x(tv_1, tv_2) + v_2 f_y(tv_1, tv_2)$$

and

$$h''(t) = f_{xx}(tv_1, tv_2)v_1^2 + 2f_{xy}(tv_1, tv_2)v_1v_2 + f_{yy}(tv_1, tv_2)v_2^2.$$

On letting $t = 0$ we obtain

$$k_p(\mathbf{v}) = f_{xx}(0,0)v_1^2 + 2f_{xy}(0,0)v_1v_2 + f_{y,y}(0,0)v_2^2 = \mathbf{v}H_{f(0,0)}{}^t\mathbf{v}$$

where

$$H_{f(0,0)} = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{pmatrix}$$

is the *Hessian* of f at $(0,0)$. (We have taken the liberty of using $\mathbf{v}$ to denote both (v_1, v_2) and $(v_1, v_2, 0)$ and hope that this does not cause any confusion—it is a practice that we do not recommend except in special circumstances.)

This is familiar territory and we use our knowledge of the function $\mathbf{v}H_{f(0,0)}{}^t\mathbf{v}$ as $\mathbf{v}$ ranges over all unit vectors. In Chap. 4 we showed that the maximum and minimum values of $\mathbf{v}H_{f(0,0)}{}^t\mathbf{v}$ over $\|\mathbf{v}\| = 1$ are the *eigenvalues* of $H_{f(0,0)}$ and are achieved at the corresponding eigenvectors. Hence the eigenvectors of $H_f(0,0)$ are the principal curvature directions and p is non-umbilic if and only if $H_f(0,0)$ has two *distinct* eigenvalues. Since eigenvectors corresponding to different eigenvalues are perpendicular (Exercise 1.21) it follows that at a non-umbilic point there exist *precisely two* principal curvature directions which are perpendicular to one another—hence if we know one we can easily find the other. Note that we do not distinguish between the directions $\mathbf{v}$ and $-\mathbf{v}$.

We now discuss the case previously omitted, i.e. $K(p) = 0$, which corresponds to a degenerate critical point of the function f. If $K(p) = 0$ and p is umbilic then since $K(p) = k_1(p)k_2(p)$ and $k_1(p) = k_2(p)$ it follows that $k_1(p) = k_2(p) = 0$ and p is a *flat spot* and we have already considered this case. If $K(p) = 0$ and p is not umbilic then we have two possibilities, $k_2(p) < 0 = k_1(p)$ and $k_2(p) = 0 < k_1(p)$, and which occurs depends on the choice of normal. Since both have the same geometrical interpretation we just consider the first one. If $\mathbf{v}$ is the principal curvature direction associated with $k_1(p)$, i.e. $k_{\mathbf{v}}(p) = k_1(p)$, then the normal section near p in the direction $\mathbf{v}$ is approximately a straight line. If $\mathbf{w} \perp \mathbf{v} = 0$ then $\mathbf{w}$ is the other principal curvature direction and the normal section in this direction is approximately a circle of radius $1/|k_2(p)|$ on the opposite side of the surface to the normal (Fig. 16.3a, b).

Figure 16.3b suggests a *cylinder* as an example and indeed a cylinder has Gaussian curvature zero at *all* points. For the cylinder in Fig. 16.3c we have $k_1(p) = 0$ and $k_2(p) = -1/r$. In general, if $K(p) = 0$ then all normal circles of curvature will lie on the same side of the tangent plane and all normal curvatures will have the same sign, that is, either all non-negative or all non-positive. The cylinder and cone (see Exercise 16.2) are typical examples of surfaces without umbilics and with $K(p) = 0$. One should, however, not assume that every surface with $K(p) = 0$ is of this type as

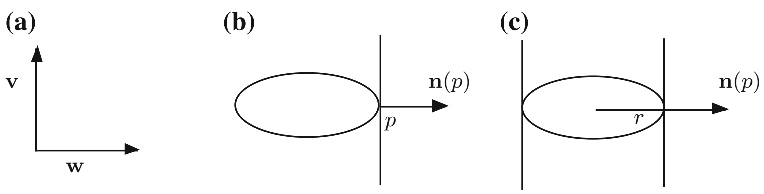

Fig. 16.3

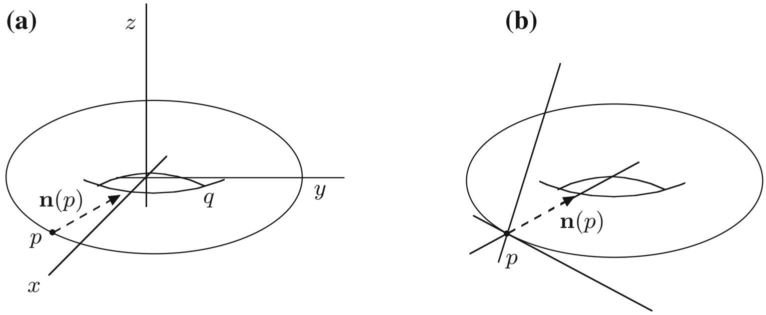

Fig. 16.4

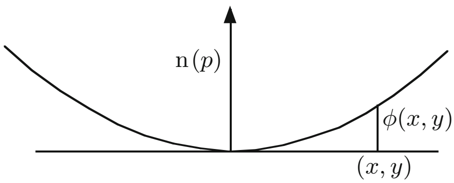

Fig. 16.5

one can find quite strange surfaces with $K(p) = 0$ at isolated points. Our geometric interpretations are meant as a rough guide and as such are reasonably useful in visualising the shape of the surface but do not, of course, explain the full subtlety of many situations—this requires further analysis.

So far we have considered a rather special situation—a critical point on a surface which is the graph of a function—and it is time to see how representative this is of the general situation.

What about non-critical points? Well, if looked at the right way, *every point is a critical point*. For example, consider the torus in $\mathbb{R}^3$ and take p as a typical point on the surface (Fig. 16.4a). To turn p into a critical point we must choose a new coordinate system for $\mathbb{R}^3$. We choose our first two coordinates so that the tangent plane to the surface at p corresponds to the (x, y)-plane and then take the z-direction as one of the normal directions (Fig. 16.4b).

We define a function ϕ on the tangent plane near p by letting $\phi(x, y)$ denote the distance squared from the tangent plane in the $\mathbf{n}(p)$ direction to the surface (Fig. 16.5). This means that the surface S near p is the graph of ϕ and that ϕ, defined on the tangent space near p, has a local minimum at p. One can carry out a similar analysis at any point on the surface, although it is not so easy to sketch at points which become saddle points, e.g. the point q in Fig. 16.4a.

We conclude that our analysis applies *everywhere* on a *graph*. Our examination of graphs, level sets, simple surfaces and surfaces at the end of Chap. 10 shows that these objects are locally equivalent and since normal curvature, principal curvatures and Gaussian curvature are locally defined our results obtained using graphs are valid at all points on all surfaces.

Gaussian curvature is the most important number that we can geometrically associate with a point on a surface. To use it effectively we need to be able to calculate it directly from any parametrization, without first finding normal curvatures or principal curvatures. To find a way to do this we turn again to normal curvatures and results from Chaps. 7 and 8 on the curvature of plane curves. If P is a unit speed parametrized curve in $\mathbb{R}^2$ then

$$P'(t) = T(t), \qquad T'(t) = \kappa(t)N(t), \qquad \kappa(t) = \langle T'(t), N(t)\rangle. \tag{16.4}$$

Let ϕ denote a parametrization of the simple surface S. Let $p \in S, \mathbf{v} \in T_p(S)$, $\|\mathbf{v}\| = 1$, and let $\mathbf{n}(p)$ denote a choice of normal to S at p. Let $P : [-a, a] \longrightarrow \Gamma$ be a parameterized curve in S with $P(0) = p$ and $P'(0) = \mathbf{v}$. If s denotes the length function on Γ defined by P and $Q = P \circ s^{-1}$ then (see Chap. 5 and (8.9)),

$$P''(t) = (s'(t))^2 Q''(s(t)) + s''(t)Q'(s(t)).$$

Since $s'(0) = 1$ and $Q'(s(0)) \perp \mathbf{n}(p)$, (16.4) implies

$$\langle P''(0), \mathbf{n}(p)\rangle = \langle Q''(s(0)), \mathbf{n}(p)\rangle = k_p(\mathbf{v}). \tag{16.5}$$

Since $\Gamma \subset S$ we may suppose

$$P : t \in [-a, a] \mapsto \phi(x(t), y(t))$$

where $t \mapsto x(t)$ and $y \mapsto y(t)$ are real-valued smooth functions on $[-a, a]$. By the chain rule

$$P'(t) = x'(t)\phi_x(x(t), y(t)) + y'(t)\phi_y(x(t), y(t))$$

and

$$\begin{aligned} P''(t) = &\, x''(t) \cdot \phi_x(x(t), y(t)) + y''(t) \cdot \phi_y(x(t), y(t)) \\ &+ (x'(t))^2 \cdot \phi_{xx}(x(t), y(t)) + 2x'(t) \cdot y'(t) \cdot \phi_{xy}(x(t), y(t)) \\ &+ (y'(t))^2 \cdot \phi_{yy}(x(t), y(t)). \end{aligned}$$

If $\overline{p} = (x(0), y(0))$, $v_1 = x'(0)$, $v_2 = y'(0)$ and we write, for simplicity, ϕ_x in place of $\phi_x(\overline{p})$, ϕ_{xx} in place of $\phi_{xx}(\overline{p})$, $\mathbf{n}$ in place of $\mathbf{n}(p)$ etc., and let $l := \langle \phi_{xx}, \mathbf{n} \rangle$, $m := \langle \phi_{xy}, \mathbf{n} \rangle = \langle \phi_{yx}, \mathbf{n} \rangle$ and $n := \langle \phi_{yy}, \mathbf{n} \rangle$. Then $P'(0) = \mathbf{v} = v_1\phi_x + v_2\phi_y$ and, since ϕ_x and ϕ_y are tangent vectors, $\langle \phi_x, \mathbf{n} \rangle = \langle \phi_y, \mathbf{n} \rangle = 0$. Hence

$$\begin{aligned}\langle P''(0), \mathbf{n}(p) \rangle = k_p(\mathbf{v}) &= v_1^2 \langle \phi_{xx}, \mathbf{n} \rangle + 2v_1v_2 \langle \phi_{xy}, \mathbf{n} \rangle + v_2^2 \langle \phi_{yy}, \mathbf{n} \rangle \\ &= lv_1^2 + 2mv_1v_2 + nv_2^2\end{aligned}$$

for any unit tangent vector $\mathbf{v} \in T_p(S)$. If $\mathbf{v} = v_1\phi_x + v_2\phi_y$ is any non-zero vector in $T_p(S)$ then $\|\mathbf{v}\|^2 = Ev_1^2 + 2Fv_1v_2 + Gv_2^2$ and

$$k_p\Big(\frac{v_1\phi_x + v_2\phi_y}{\|v_1\phi_x + v_2\phi_y\|}\Big) = \frac{lv_1^2 + 2mv_1v_2 + nv_2^2}{\|\mathbf{v}\|^2} = \frac{lv_1^2 + 2mv_1v_2 + nv_2^2}{Ev_1^2 + 2Fv_1v_2 + Gv_2^2}. \quad (16.6)$$

This is an extremely useful formula, see Exercises 16.4, 16.9 and 17.2.

The principal curvatures are the maximum and minimum values of $lv_1^2 + 2mv_1v_2 + nv_2^2$ on the set $v_1^2E + 2v_1v_2F + v_2^2G = 1$. To simplify the notation we let $v_1 = x$ and $v_2 = y$. Then

$$f(x, y) = lx^2 + 2mxy + ny^2 = (x, y)\begin{pmatrix} l & m \\ m & n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$g(x, y) = Ex^2 + 2Fxy + Gy^2 = (x, y)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Since $EG - F^2 > 0$ it is easily verified that $\{(x, y) : g(x, y) = 1\}$ is a compact subset of $\mathbb{R}^2$ and $\nabla g \neq 0$ on this level set. Hence f achieves its maximum and minimum on $\{(x, y) : g(x, y) = 1\}$ and we may apply the method of Lagrange multipliers. On writing the equation $\nabla f(x, y) = \lambda \nabla g(x, y)$ in matrix form we obtain

$$\begin{pmatrix} l & m \\ m & n \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (16.7)$$

and we may rewrite this as follows

$$\begin{pmatrix} l - \lambda E & m - \lambda F \\ m - \lambda F & n - \lambda G \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (16.8)$$

By elementary linear algebra the λ for which (16.8) has non-zero solutions (x, y) satisfy the quadratic equation

$$(l - \lambda E)(n - \lambda G) - (m - \lambda F)^2 = 0$$

or equivalently

$$\lambda^2(EG - F^2) - \lambda(En - 2Fm + Gl) + ln - m^2 = 0. \qquad (16.9)$$

If λ_1 and λ_2 solve (16.9) then, since $K(p) = \lambda_1\lambda_2$, we have proved the following result.

Proposition 16.3 *The Gaussian curvature at a point p on a surface S is*

$$K(p) = \frac{ln - m^2}{EG - F^2}. \qquad (16.10)$$

By (16.7) the principle curvatures satisfy

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} l & m \\ m & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

and we see that the principle curvatures are eigenvalues of a symmetric matrix and we recover, using Exercise 1.21, a result already observed using graphs: *at a non-umbilic point the principal curvature directions are perpendicular to one another.*

The appearance of $EG - F^2$, previously encountered while calculating surface area in Proposition 16.3, suggests a relationship between Gaussian curvature and surface area. This comes to light in the Gauss–Bonnet theorem (Chap. 18).

Example 16.4 We calculate the Gaussian curvature of the surface $z = xy$. This surface is the graph of the function $f(x, y) = xy$ and we obtain a parametrization ϕ by letting $\phi(x, y) = (x, y, xy)$. We have

$$\begin{gathered} \phi_x = (1, 0, y), \qquad \phi_y = (0, 1, x) \\ E = \phi_x.\phi_x = 1 + y^2, \quad F = \phi_x.\phi_y = xy, \quad G = \phi_y.\phi_y = 1 + x^2 \\ EG - F^2 = (1 + y^2)(1 + x^2) - x^2y^2 = 1 + x^2 + y^2 \end{gathered}$$

$$\phi_x \times \phi_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = (-y, -x, 1)$$

$$\begin{gathered} \mathbf{n} = \frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|} = \frac{(-y, -x, 1)}{(1 + y^2 + x^2)^{1/2}} \\ \phi_{xx} = (0, 0, 0), \quad \phi_{xy} = (0, 0, 1), \quad \phi_{yy} = (0, 0, 0) \\ l = \langle \phi_{xx}, \mathbf{n} \rangle = 0, \quad m = \langle \phi_{xy}, \mathbf{n} \rangle = \frac{1}{(1 + x^2 + y^2)^{1/2}}, \quad n = \langle \phi_{yy}, \mathbf{n} \rangle = 0. \end{gathered}$$

Hence, if $p = (x, y, xy)$, then

$$K(p) = \frac{-m^2}{EG - F^2} = \frac{-1}{(1 + x^2 + y^2)^2}.$$

Since the Gaussian curvature is always strictly negative the surface $z = xy$ consists entirely of non-umbilic points which, looked at critically, are saddle points.

Although, in general, it may be difficult to find principal curvatures and principal curvature directions it is possible in special cases and this is the content of the following proposition. We say that a curve Γ on a surface is a *line of curvature* if its tangents are principal curvature directions at each point.

Proposition 16.5 *If $\phi\colon U \subset \mathbb{R}^2 \to S$ is a parametrized surface then the coordinate curves of ϕ are lines of curvature if and only if $F = m = 0$ at all non-umbilic points. In this case, at non-umbilic points, the principal curvatures are l/E and n/G.*

Proof Fix $p \in S$ and suppose the principles curvatures at p are λ_1 and λ_2, $\lambda_1 \neq \lambda_1$. Since the lines of curvature passing through p cross one another at right angles it follows, if the coordinate curves are lines of curvature, that $F = \phi_x \cdot \phi_y = 0$. Since the vectors $(1, 0)$ and $(0, 1)$ solve (16.8) we obtain after making these substitutions $l - \lambda E = m = n - \lambda G = 0$. In particular, we see that $m = 0$ and note also $\{\lambda_1, \lambda_2\} = \{\frac{l}{E}, \frac{n}{G}\}$.

Conversely, suppose $F = m = 0$ at all non-umbilic points. By (16.6)

$$k_p\left(\frac{v_1\phi_x + v_2\phi_y}{\|v_1\phi_x + v_2\phi_y\|}\right) = \frac{lv_1^2 + nv_2^2}{Ev_1^2 + Gv_2^2}.$$

when $\mathbf{v} = v_1\phi_x + v_2\phi_y \neq 0$. The maximum and minimum of $f(v_1, v_2) = lv_1^2 + nv_2^2$ on the set $Ev_2 + Gv_2^2 = 1$ are easily found using Lagrange multipliers and we see that $\{\lambda_1, \lambda_2\} = \left\{\frac{l}{E}, \frac{n}{G}\right\}$. Since $\|\phi_x\| = \sqrt{E}$ and $\|\phi_y\| = \sqrt{G}$ this implies

$$k_p\left(\frac{\phi_x}{\sqrt{E}}\right) = \frac{l}{E} \text{ and } k_p\left(\frac{\phi_y}{\sqrt{G}}\right) = \frac{n}{G}$$

and the coordinate curves are lines of curvature. This completes the proof. □

Example 16.6 The surface obtained by rotating the graph of $h\colon (a, b) \to \mathbb{R}$ (Example 10.4) is parametrized by

$$P(t, \theta) = (t, h(t)\cos\theta, h(t)\sin\theta).$$

For this parametrization $P_t = (1, h'\cos\theta, h'\sin\theta)$ and $P_\theta = (0, -h\sin\theta, h\cos\theta)$. Hence $E = 1 + (h')^2$, $F = 0$, $G = h^2$,

$$P_{tt} = (0, h''\cos\theta, h''\sin\theta)$$

$$P_{t\theta} = (0,\ -h' \sin\theta,\ h' \cos\theta)$$

and

$$P_{\theta\theta} = (0,\ -h\cos\theta,\ -h\sin\theta).$$

Moreover,

$$P_t \times P_\theta = (h'h,\ -h\cos\theta,\ -h\sin\theta)$$

and

$$\mathbf{n} = \frac{(h',\ -\cos\theta,\ -\sin\theta)}{\big(1+(h')^2\big)^{1/2}}.$$

Hence

$$\begin{aligned} l &= \langle P_{tt}, \mathbf{n}\rangle = \frac{-h''}{\big(1+(h')^2\big)^{1/2}} \\ m &= \langle P_{t\theta}, \mathbf{n}\rangle = 0 \\ n &= \langle P_{\theta\theta}, \mathbf{n}\rangle = \frac{h}{\big(1+(h')^2\big)^{1/2}}. \end{aligned}$$

By Proposition 16.5 the coordinate curves are lines of curvature and the principal curvatures and Gaussian curvature are

$$\frac{l}{E} = \frac{-h''}{\big(1+(h')^2\big)^{3/2}}, \quad \frac{n}{G} = \frac{1}{h\big(1+(h')^2\big)^{1/2}} \quad \text{and} \quad K = \frac{-h''}{h\big(1+(h')^2\big)^2}.$$

An interesting case occurs when $h(t) = c\cosh(t/c)$. This is the shape assumed by a *hanging chain* and is called a *catenary*. The surface of revolution is called a *catenoid*. Using the identities

$$\frac{d}{dt}(\cosh t) = \sinh t, \quad \frac{d}{dt}(\sinh t) = \cosh t, \quad \cosh^2 t - \sinh^2 t = 1,$$

we see that the principal curvatures for the catenoid are $\pm 1/c\cosh^2(t/c)$. Hence $k_1 + k_2 = 0$. A surface with this property is called a *minimal surface*. This terminology arose in the following way. Take a closed curve in $\mathbb{R}^3$—shaped, for instance, from a piece of wire—and place a bubble over it. This will assume a certain shape in order to *minimise* a physical quantity on the boundary called surface tension. The shape assumed by the bubble is a minimal surface. The catenoid is the only surface of revolution which is a minimal surface. On minimal surfaces $k_2(p) = -k_1(p)$ and hence

$$K(p) = k_1(p)k_2(p) = -k_1(p)^2 \le 0.$$

The quantity $\dfrac{k_1(p)+k_2(p)}{2}$ is called the *mean curvature*.

In this chapter we have covered a lot of theoretical and practical material in identifying and calculating Gaussian curvature. In the next chapter we will summarise the information we already have on this important concept and discuss further geometric implications.

Exercises

16.1 Calculate the Gaussian curvature at an arbitrary point of the helicoloid parametrized by

$$P(t, \theta) = (t\cos\theta, t\sin\theta, b\theta)$$

$t \in \mathbb{R}, \theta \in (0, 2\pi)$ and b a non-zero real number.

16.2 Find the principal curvatures and the Gaussian curvature at an arbitrary point on the cone $z^2 = x^2 + y^2$. Show that the cone has no umbilics.

16.3 Let S denote the surface parametrized by $\phi(u, v) = (u, v, u^2+v^2)$, $(u, v) \in \mathbb{R}^2$. Show that the curve Γ parametrized by

$$P(t) = \phi(t^2, t), \quad -\frac{1}{2} < t < 2,$$

lies in S. Find the unit tangent to Γ at $P(1)$ and find the (absolute) normal curvature to S at $P(1)$ in the direction $P'(1)$.

16.4 If $P\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a parametrization of a simple surface S show that a point p is an umbilic point if and only if there exists a real number α such that $(l, m, n) = \alpha(E, F, G)$ where each term is evaluated at p. Hence find all umbilics on the surface $z = xy$.

16.5 Find the elliptic and hyperbolic points on the surface parametrized by $\phi(u, v) = (u, v, u^3 + v^3)$.

16.6 Prove that the surface parametrized by

$$f(s, t) = (\cos s, 2\sin s, t), \quad 0 < s < 2\pi, \ t > 0$$

has constant Gaussian curvature. Describe this surface.

16.7 If the surface S is defined by the equations

$$x = a(u + v), \quad y = b(u - v), \quad z = uv$$

show that the coordinate curves are straight lines.

16.8 Find the Gaussian curvature at an arbitrary point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

16.9 Show that a surface S is a minimal surface if and only if

$$En + Gl = 2Fm.$$

Show that the level set $e^z \cos x = \cos y$ is a minimal surface.