# Chapter 6
# Line Integrals

**Summary** *We integrate vector-valued and scalar-valued functions along a directed curve in* $\mathbb{R}^n$. *We discuss scalar and vector potentials and define the curl of a vector field in* $\mathbb{R}^3$.

The differential calculus was developed to study extremal (i.e. maximal and minimal) values of functions. Since it is only possible to discuss the maximum and minimum of a real-valued function it is not surprising that such functions occupy a prominent role in several-variable differential calculus. However, in moving to integration theory it is more natural (and more natural in mathematics usually means more useful, more efficient and more elegant) to consider vector-valued functions where the domain and the range share, in perhaps a loose way, a *common dimension*. Formally, we have the following definition of a *vector field*.

**Definition 6.1** *A function F which maps a subset U of* $\mathbb{R}^n$ *into* $\mathbb{R}^n$ *is called a vector field on U.*

If $U$ is an open subset of $\mathbb{R}^n$ and the vector field has derivatives of all orders we call it a *smooth vector field* and if $U$ is arbitrary and the vector field is continuous we use the term *continuous vector field*. We shall also use the notation $\mathbf{F}$ to denote a vector field.

The gradient is an important example of a vector field, i.e. if $U$ is open and $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable then $\nabla f : U \to \mathbb{R}^n$ is a vector field on $U$.

Another useful example occurs when $\Gamma$ is a directed curve in $\mathbb{R}^n$ and $F$ is a function which assigns a vector in $\mathbb{R}^n$ to each point on $\Gamma$—in this case we say that $F$ is a vector field *along* $\Gamma$. For example, if $P$ is a parametrization of a directed curve $\Gamma$ in $\mathbb{R}^n$ then the mapping

$$P(t) \in \Gamma \to P'(t) \in \mathbb{R}^3$$

is a smooth vector field along $\Gamma$.

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_6,

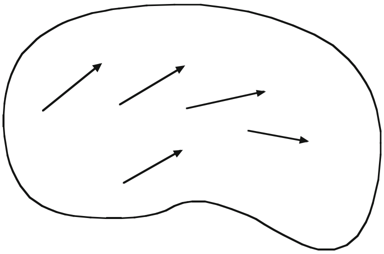

**Fig. 6.1**

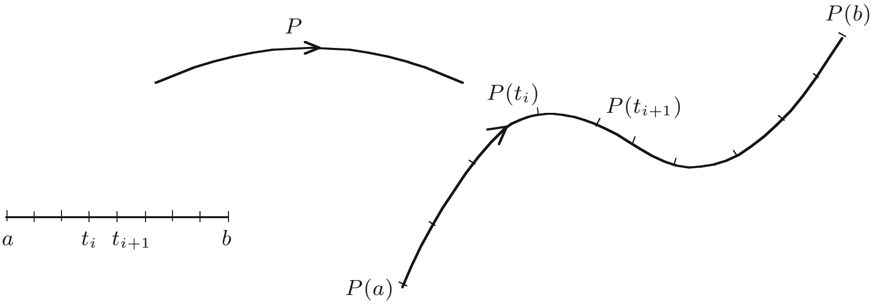


**Fig. 6.2**

Vector fields, which assign vectors to points in the domain of definition, are often represented as in Fig. 6.1.

This representation is useful in locating zeros and suggesting properties such as continuity and smoothness. Furthermore, it allows various physical interpretations of vector fields, e.g. as the velocity of a moving fluid and the flow of an electric current which lead, in turn, to important physical and engineering applications.

We begin our study of integration theory by defining the integral of a vector field $F$ along a directed curve $\Gamma$. Let $P\colon [a, b] \to \Gamma$ denote a parametrization of $\Gamma$. To each partition of $[a, b]$ we obtain a partition of $\Gamma$ (Fig. 6.2) and the Riemann sum

$$\sum_i F\big(P(t_i)\big) \cdot \big(P(t_{i+1}) - P(t_i)\big) \approx \sum_i F\big(P(t_i)\big) \cdot P'(t_i)\Delta t_i$$

where $\Delta t_i = t_{i+1} - t_i$ and $\cdot$ denotes the inner product in $\mathbb{R}^n$. Note that we are using, as usual, the linear approximation to $P(t+\Delta t)$, $P(t) + P'(t)\Delta t$. If $F$ is a continuous vector field along $\Gamma$, i.e. if the mapping $t \in [a, b] \to F(P(t))$ is continuous, then as we take finer and finer partitions of $[a, b]$ the Riemann sums converge to the limit

$$\int_a^b F\big(P(t)\big) \cdot P'(t)\, dt.$$

We denote this integral by $\int_\Gamma F$, since we shall shortly prove that it is independent of the parametrization $P$, and call it the *line integral* of $F$ over $\Gamma$. In terms of coordinates, for instance in the case $n = 3$, we have

$$F = (f, g, h), \quad P(t) = \big(x(t), y(t), z(t)\big), \quad P'(t) = \big(x'(t), y'(t), z'(t)\big)$$

and

$$\int_\Gamma F = \int_a^b \Big[f\big(x(t), y(t), z(t)\big) \cdot x'(t) \,+\, g\big(x(t), y(t), z(t)\big) \cdot y'(t) \\ + \, h\big(x(t), y(t), z(t)\big) \cdot z'(t)\Big]\, dt.$$

This is frequently written in the form

$$\int_\Gamma f\, dx + g\, dy + h\, dz.$$

If $(\Gamma, A, B, \mathbf{v})$ is a directed curve and $\tilde{\Gamma}$ is obtained by changing the direction along the curve $\Gamma$, i.e. $\tilde{\Gamma} = (\Gamma, B, A, -\mathbf{v})$, then clearly $\int_{\tilde{\Gamma}} F = -\int_\Gamma F$ for any continuous vector field $F$.

*Example 6.2* We evaluate

$$\int_\Gamma xy\, dx + xz^2\, dy + xyz\, dz$$

where the curve $\Gamma$ is parametrized by

$$P(t) = (t, t^2, t^3), \quad 0 \le t \le 1.$$

In coordinates we have

$$\begin{cases} x = x(t) = t \\ y = y(t) = t^2 \\ z = z(t) \ = t^3 \end{cases} \Longrightarrow \begin{cases} \dfrac{dx}{dt} = x'(t) = 1 \\ \dfrac{dy}{dt} = y'(t) = 2t \\ \dfrac{dz}{dt} = z'(t) = 3t^2 \end{cases} \Longrightarrow \begin{cases} dx = dt \\ dy = 2t\, dt \\ dz = 3t^2\, dt \end{cases}$$

and

$$\int_\Gamma xy\, dx + xz^2\, dy + xyz\, dz = \int_0^1 t^3\, dt + t^7 \cdot 2t\, dt + t^6 \cdot 3t^2\, dt \\ = \left[\frac{t^4}{4} + \frac{2t^9}{9} + \frac{3t^9}{9}\right]_0^1 = \frac{29}{36}.$$

Alternatively, changing to vector notation, we let $F(x, y, z) = (xy, xz^2, xyz)$. Then $F\big(P(t)\big) = (t^3, t^7, t^6)$ and $P'(t) = (1, 2t, 3t^2)$. Hence

$$\int_\Gamma F = \int_0^1 F\big(P(t)\big) \cdot P'(t)\,dt = \int_0^1 (t^3, t^7, t^6) \cdot (1, 2t, 3t^2)\,dt$$
$$= \int_0^1 (t^3 + 2t^8 + 3t^8)\,dt = \frac{29}{36}.$$

We return to the general situation. If $P\colon [a,b] \to \Gamma$ and $Q\colon [c,d] \to \Gamma$ are two parametrizations of $\Gamma$ with length functions $s$ and $s_1$ respectively, then, as we saw in Chap. 5,

$$P \circ s^{-1} = Q \circ s_1^{-1} \text{ on } [0,l], \quad l = \text{length of } \Gamma.$$

Using the one-variable change of variables, $y = s^{-1}(t)$ and $x = s_1^{-1}(t)$, we obtain

$$\int_0^l F\big(P \circ s^{-1}(t)\big) \cdot (P \circ s^{-1})'(t)\,dt$$
$$= \int_0^l F\big(P \circ s^{-1}(t)\big) \cdot P'\big(s^{-1}(t)\big)(s^{-1})'(t)\,dt$$
$$= \int_a^b F\big(P(y)\big) \cdot P'(y)\,dy.$$

Similarly

$$\int_0^l F\big(Q \circ s_1^{-1}(t)\big) \cdot (Q \circ s_1^{-1})'(t)\,dt = \int_c^d F\big(Q(x)\big) \cdot Q'(x)\,dx.$$

Since $P \circ s^{-1} = Q \circ s_1^{-1}$ this implies

$$\int_a^b F\big(P(y)\big) \cdot P'(y)\,dy = \int_c^d F\big(Q(x)\big) \cdot Q'(x)\,dx$$

and we get the same value no matter which parametrization is used. This justifies the notation $\int_\Gamma F$.

If $P\colon [a,b] \to \Gamma$ is a parametrization of the directed curve $\Gamma$ in $\mathbb{R}^n$ and $t \in [a,b]$ let $T(t) = P'(t) \,\big/\, \|P'(t)\|$. We call $T(t)$ the *unit tangent* to the (directed) curve at $P(t)$. It is easily seen that any two parametrizations define the same unit tangent vector at each point of $\Gamma$. This leads to another way of writing line integrals. If $F$ is a vector field along $\Gamma$ (always of course assumed to be continuous) then

$$\int_\Gamma F = \int_a^b F\big(P(t)\big) \cdot P'(t)\,dt$$
$$= \int_a^b F\big(P(t)\big) \cdot \frac{P'(t)}{\|P'(t)\|}\,\|P'(t)\|\,dt$$

$$= \int_a^b F\big(P(t)\big) \cdot T(t)\, \|P'(t)\|\, dt$$
$$= \int_a^b (F \cdot T)\, ds$$

and when written in this form one should remember, in applying a parametrization $P$, that $F$ and $T$ are both evaluated at $P(t)$ and $ds = \|P'(t)\|\, dt$.

Real-valued functions or *scalar fields*, such as the speed of a parametrization, can also be defined along a directed curve $\Gamma$. Since these are not endowed with a sense of direction we cannot apply directly our definition of integral. Fortunately, we have just observed a *special* or *privileged* direction associated with each point on a curve, the tangent direction, and by associating the continuous real-valued function $f : \Gamma \to \mathbb{R}$ with the vector field $fT : \Gamma \to \mathbb{R}^n$ we can define $\int_\Gamma f$. The privileged direction on an oriented surface in $\mathbb{R}^3$ is the *normal* direction and in this case it is also possible to consider scalar-valued integration as a special case of vector-valued integration.

If $P : [a, b] \to \Gamma$ is a parametrization of the directed curve $\Gamma$ we define

$$\int_\Gamma f = \int_\Gamma fT = \int_a^b f\big(P(t)\big)T(t) \cdot T(t)\|P'(t)\|\, dt$$
$$= \int_a^b f\big(P(t)\big)\|P'(t)\|\, dt \tag{6.1}$$

Because of (6.1) we sometimes write $\int_\Gamma f ds$ in place of $\int_\Gamma f$.

We now seek to identify those vector fields in $\mathbb{R}^n$ which are the gradient of a scalar-valued function. Vector fields of this kind are called *conservative* and are said to have a scalar potential. If $\nabla f = F$ we call $f$ a *scalar potential* of $F$. Our investigation of this problem leads to a generalisation of the *fundamental theorem of one-variable calculus*

$$\int_a^b g'(t)\, dt = g(b) - g(a)$$

where $g$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

We begin by considering properties of the gradient of $f : U \subset \mathbb{R}^n \to \mathbb{R}$. If $\Gamma$ is a directed curve in $U$ parametrized by $P : [a, b] \to \mathbb{R}^n$ then

$$\int_\Gamma \nabla f = \int_a^b \nabla f\big(P(t)\big) \cdot P'(t)\, dt$$
$$= \int_a^b \frac{d}{dt}(f \circ P)(t)\, dt \quad \text{(chain rule)}$$
$$= \Big[ f \circ P(t) \Big]_a^b \quad \text{(fundamental theorem of calculus in } \mathbb{R}\text{)}$$
$$= f\big(P(b)\big) - f\big(P(a)\big).$$

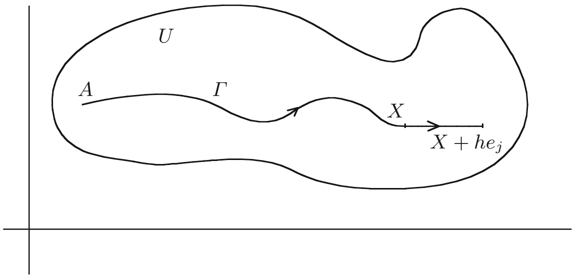


**Fig. 6.3**

Thus the line integral of $\nabla f$ along $\Gamma$ depends only on the values of $f$ at the initial and final points of the curve. This, as we shall see in Proposition 6.3, characterises vector fields which have a scalar potential. If $\Gamma$ is a piecewise smooth directed curve in $\mathbb{R}^n$ which is the union of directed curves $(\Gamma_i)_{i=1}^k$ and $F$ is either a vector field along $\Gamma$ or a real-valued function on $\Gamma$ we let

$$\int_\Gamma F = \sum_{i=1}^k \int_{\Gamma_i} F.$$

In the next proposition it is necessary to assume that any pair of points in the open set $U$ can be joined by a piecewise smooth directed curve which lies in $U$; an open set of this kind is said to be *connected*.

**Proposition 6.3** *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ denote a continuous vector field on the connected open subset $U$ of $\mathbb{R}^n$. If for any two points $A$ and $B$ in $\mathbb{R}^n$ and any two piecewise smooth directed curves $\Gamma_1$ and $\Gamma_2$ joining $A$ and $B$ we have*

$$\int_{\Gamma_1} F = \int_{\Gamma_2} F$$

*then $F$ has a potential.*

*Proof* Let $A$ denote a fixed point in $U$. For any $X$ in $U$ let $f(X) = \int_\Gamma F$ where $\Gamma$ is any piecewise smooth directed curve in $U$ joining $A$ to $X$. By our hypothesis $f$ is well defined, i.e. there is no ambiguity in the definition. Let $F = (f_1, \ldots, f_n)$ and let $\Gamma$ denote a curve joining $A$ to $X$. Fix $j$, $1 \leq j \leq n$, and let $Y = he_j$ where $h$ denotes a real number close to 0. Let $\Gamma_1$ denote the directed curve parametrized by $P(t) = X + tY$, $0 \leq t \leq 1$. Then $\Gamma_1$ joins $X$ to $X + Y$ and $\Gamma \cup \Gamma_1$ is a piecewise smooth directed curve in $U$ which joins $A$ to $X + Y$ (Fig. 6.3).

Hence

$$f(X+Y)-f(X)=\int_{\Gamma\cup\Gamma_1}F-\int_{\Gamma}F$$
$$=\int_{\Gamma_1}F=\int_0^1 F\big(P(t)\big)\cdot P'(t)\,dt.$$

Since $P'(t)=Y=he_j$

$$f(X+Y)-f(X)=\int_0^1 F\big(P(t)\big)\cdot he_j\,dt$$
$$=h\int_0^1 f_j(X+tY)\,dt$$
$$=hf_j(X)+h\int_0^1\Big(f_j(X+tY)-f_j(X)\Big)\,dt.$$

As $F$ is continuous, each component in $F$ and, in particular, $f_j$ is also continuous. Hence

$$\max_{0\le t\le 1}\big|f_j(X+tY)-f_j(X)\big|\longrightarrow 0 \text{ as } h\to 0$$

and

$$\Big|\int_0^1\Big(f_j(X+tY)-f_j(X)\Big)\,dt\Big|\le\max_{0\le t\le 1}\big|f_j(X+tY)-f_j(X)\big|.$$

We have shown

$$f(X+he_j)=f(X)+hf_j(X)+g(X,h)h$$

where $g(X,h)\to 0$ as $h\to 0$. Hence $\dfrac{\partial f}{\partial x_j}(X)=f_j(X)$ and this completes the proof. □

Proposition 6.3 can be used to *find* potentials but is not very practical for *showing* existence. We need a simpler method which follows from the observation:

if $F=(f_1,\dots,f_n)$ is a continuously differentiable vector field with potential $f$, i.e. $f_j=\dfrac{\partial f}{\partial x_j}$ for all $j$, then for all $i$ and $j$

$$\frac{\partial f_i}{\partial x_j}=\frac{\partial}{\partial x_j}\Big(\frac{\partial f}{\partial x_i}\Big)=\frac{\partial^2 f}{\partial x_j\partial x_i}=\frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial x_j}\Big)=\frac{\partial f_j}{\partial x_i}.$$

The converse is true for *suitable* open sets—the whole space $\mathbb{R}^n$ is always suitable and so also is any convex set in $\mathbb{R}^n$. In $\mathbb{R}^2$ an open set $U$ is suitable if and only if the "interior" of any closed curve in $U$ also lies in $U$; roughly speaking this means that $U$ contains no holes. In particular the open set $\mathbb{R}^2\backslash\{(0,0)\}$ *is not* suitable.

However, in $\mathbb{R}^3$ the whole space with a finite number of points removed *is* suitable. These examples show that the concept of suitability is rather subtle. The proof of the following result involves Green's theorem (Chap. 9) and Proposition 6.3.

**Proposition 6.4** *If $F = (f_1, \dots, f_n)$ is a continuously differentiable vector field on a suitable open set in $\mathbb{R}^n$ then $F$ has a potential if and only if*

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \tag{6.2}$$

*for all $i$ and $j$.*

*Example 6.5* We wish to show that $F(x, y, z) = (y\mathrm{e}^z, x\mathrm{e}^z, xy\mathrm{e}^z)$ has a potential. Let $F = (f_1, f_2, f_3)$. To apply Proposition 6.4 we must verify (6.2), that is we must show

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}.$$

We have

$$\frac{\partial f_1}{\partial y} = \mathrm{e}^z = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = y\mathrm{e}^z = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = x\mathrm{e}^z = \frac{\partial f_3}{\partial y}$$

and hence $F$ has a potential on $\mathbb{R}^3$.

The following is probably the simplest way to find a potential. If $f$ is a potential for $F$ then

$$\frac{\partial f}{\partial x} = y\mathrm{e}^z, \quad \frac{\partial f}{\partial y} = x\mathrm{e}^z, \quad \frac{\partial f}{\partial z} = xy\mathrm{e}^z. \tag{6.3}$$

Hence

$$f = \int \frac{\partial f}{\partial x}\, dx = \int y\mathrm{e}^z\, dx = xy\mathrm{e}^z + \phi(y, z)$$

where $\phi$ is the constant of integration *with respect to* $x$ which may, however, depend on $y$ and $z$. Differentiating with respect to $y$ we get

$$\frac{\partial f}{\partial y} = x\mathrm{e}^z + \frac{\partial \phi}{\partial y}$$

and comparing this with the formula for $\dfrac{\partial f}{\partial y}$ in (6.3) we have

$$x\mathrm{e}^z + \frac{\partial \phi}{\partial y} = x\mathrm{e}^z$$

and

$$\frac{\partial \phi}{\partial y} = 0.$$

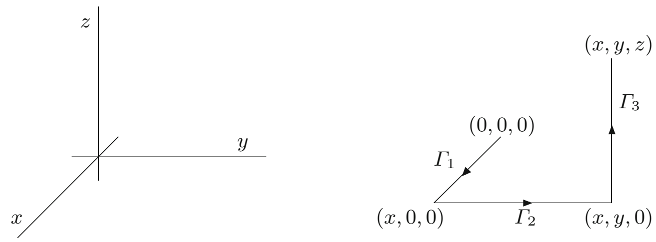


**Fig. 6.4**

Hence $\phi$ does not depend on $y$ and we let $\phi(y, z) = \psi(z)$. Now differentiating

$$f(x, y, z) = xye^z + \psi(z)$$

and comparing this with (6.3) gives

$$\frac{\partial f}{\partial z} = xye^z + \psi'(z) = xye^z$$

and $\psi'(z) = 0$. This implies that $\psi$ is a constant and we have shown

$$f(x, y, z) = xye^z + c$$

for some constant $c$.

We can also use Proposition 6.3 to find a potential $f$. This proposition tells us that

$$f(x, y, z) = \int_\Gamma F$$

where $\Gamma$ is any piecewise smooth directed curve in $\mathbb{R}^3$, joining a fixed point to $(x, y, z)$, is a potential of $F$. We take the fixed point to be the origin in $\mathbb{R}^3$ and a piecewise smooth curve $\Gamma$ consisting of three straight lines parallel to the axis $\Gamma_1$, $\Gamma_2$, $\Gamma_3$. Specifically we use the following:

$$\begin{array}{llll} \Gamma_1 \text{ joins } (0,0,0) \text{ to } (x,0,0), & P_1(t) = (t,0,0), & P_1'(t) = (1,0,0), & 0 \le t \le x \\ \Gamma_2 \text{ joins } (x,0,0) \text{ to } (x,y,0), & P_2(t) = (x,t,0), & P_2'(t) = (0,1,0), & 0 \le t \le y \\ \Gamma_3 \text{ joins } (x,y,0) \text{ to } (x,y,z), & P_3(t) = (x,y,t), & P_3'(t) = (0,0,1), & 0 \le t \le z \end{array}$$

(see Fig. 6.4).

Then

$$\begin{aligned} f(x,y,z) &= \int_{\Gamma_1} F + \int_{\Gamma_2} F + \int_{\Gamma_3} F \\ &= \int_0^x f_1\big(P_1(t)\big)\,dt + \int_0^y f_2\big(P_2(t)\big)\,dt + \int_0^z f_3\big(P_3(t)\big)\,dt \\ &= \int_0^x f_1(t,0,0)\,dt + \int_0^y f_2(x,t,0)\,dt + \int_0^z f_3(x,y,t)\,dt \\ &= \int_0^x 0\cdot dt + \int_0^y x\mathrm{e}^0\,dt + \int_0^z xy\mathrm{e}^t\,dt \\ &= 0 + xy + \big[xy\cdot \mathrm{e}^t\big]_0^z \\ &= xy + xy\mathrm{e}^z - xy \\ &= xy\mathrm{e}^z. \end{aligned}$$

We verify this result by noting

$$\frac{\partial}{\partial x}(xy\mathrm{e}^z) = y\mathrm{e}^z = f_1, \quad \frac{\partial}{\partial y}(xy\mathrm{e}^z) = x\mathrm{e}^z = f_2, \quad \frac{\partial}{\partial z}(xy\mathrm{e}^z) = xy\mathrm{e}^z = f_3.$$

We now define the *cross-product* of two vectors in $\mathbb{R}^3$. This allows us to present the classical notation for what are essentially vector derivatives. We shall also need this product when we discuss the Frenet-Serret equations in Chap. 7. If $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are two vectors in $\mathbb{R}^3$ then the *cross product* of $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v}\times\mathbf{w}$, is defined as

$$\begin{aligned} \mathbf{v}\times\mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= (v_2w_3 - v_3w_2)\mathbf{i} - (v_1w_3 - v_3w_1)\mathbf{j} + (v_1w_2 - v_2w_1)\mathbf{k} \\ &= (v_2w_3 - v_3w_2, -v_1w_3 + v_3w_1, v_1w_2 - v_2w_1). \end{aligned}$$

Since $\nabla f = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(f)$ we define $\mathrm{curl}(F)$ or $\nabla \times F$, $F = (f_1, f_2, f_3)$ a vector field, by the formula

$$\nabla\times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z})\mathbf{i} - (\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z})\mathbf{j} + (\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})\mathbf{k}.$$

One easily sees that $\mathrm{curl}(\nabla f) = 0$ for any real-valued function with continuous first- and second-order partial derivatives while Proposition 6.4 says the converse is true on suitable domains, that is

$$F = \nabla f \Longleftrightarrow \operatorname{curl}(F) = 0.$$

If $F$ is the velocity of a fluid then curl $(F)$ measures the tendency of the fluid to curl or rotate about an axis. Curl$(F)$ gives the direction of the axis of rotation and $\|\operatorname{curl}(F)\|$ measures the speed of rotation.

Since the symbolism $\nabla \times F$ has proved useful we consider the analogous symbol $\nabla \cdot F$ where the dot replaces the cross product. This makes sense for a vector field on $\mathbb{R}^n$. If $F = (f_1, f_2, \ldots, f_n)$, we let

$$\nabla \cdot F = \Big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big) \cdot (f_1, \ldots, f_n) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} := \operatorname{div}(\mathbf{F}).$$

This is called the *divergence* of $F$ and written div$(F)$ (see Chap. 15). A vector field $G$ is a *vector potential* for the vector field $F$ on the open subset $U$ of $\mathbb{R}^3$ if on $U$ we have

$$\operatorname{curl}(G) = \nabla \times G = F.$$

On "suitable" open sets a vector field $F$ has a vector potential if and only if $\nabla \cdot F = \operatorname{div}(F) = 0$.

*Example 6.6* We show that $\mathbf{F}(X) = X\big/\|X\|^3$ has a vector potential on $\mathbb{R}^3\backslash\{$ the $z$-axis$\}$. Let

$$\mathbf{G}(x, y, z) = \frac{(yz, -xz, 0)}{(x^2 + y^2)(x^2 + y^2 + z^2)^{1/2}} = \frac{z}{\|X\|^2 - z^2} \cdot \frac{1}{\|X\|}(y, -x, 0)$$

on $\mathbb{R}^3\backslash\{$ the $z$-axis$\} = \mathbb{R}^3\backslash\{(x, y, z) : x = y = 0\}$. We calculate curl$(\mathbf{G})$. To simplify our calculations we use symmetry and the following result which follows immediately from Exercise 1.10:

$$\frac{\partial}{\partial x}\big(\|X\|\big) = \frac{x}{\|X\|}.$$

We have

$$\operatorname{curl}(\mathbf{G}) = (g_1, g_2, g_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{yz}{(x^2 + y^2)\|X\|} & \dfrac{-xz}{(x^2 + y^2)\|X\|} & 0 \end{vmatrix}$$

which implies

$$\begin{aligned} g_1 = \frac{\partial}{\partial z}\Big(\frac{xz}{(x^2+y^2)\|X\|}\Big) &= \frac{x}{x^2+y^2}\frac{\partial}{\partial z}\Big(\frac{z}{\|X\|}\Big) \\ &= \frac{x}{x^2+y^2}\cdot\frac{\Big(\|X\| - \frac{z^2}{\|X\|}\Big)}{\|X\|^2} \\ &= \frac{x}{x^2+y^2}\cdot\frac{\big(\|X\|^2 - z^2\big)}{\|X\|^3} \\ &= \frac{x}{\|X\|^3}, \end{aligned}$$

since $x^2 + y^2 + z^2 = \|X\|^2$. By symmetry $g_2 = y \big/ \|X\|^3$. Finally

$$\begin{aligned} g_3 &= \frac{\partial}{\partial x}\Big(\frac{-xz}{(x^2+y^2)\|X\|}\Big) - \frac{\partial}{\partial y}\Big(\frac{yz}{(x^2+y^2)\|X\|}\Big) \\ &= -z\Bigg[\frac{(x^2+y^2)\|X\| - 2x^2\|X\| - \frac{x^2(x^2+y^2)}{\|X\|}}{(x^2+y^2)^2\|X\|^2} \\ &\qquad + \frac{(x^2+y^2)\|X\| - 2y^2\|X\| - \frac{y^2(x^2+y^2)}{\|X\|}}{(x^2+y^2)^2\|X\|^2}\Bigg] \\ &= \frac{-z(-1)}{\|X\|^3} = \frac{z}{\|X\|^3} \end{aligned}$$

and we have shown

$$\mathrm{curl}(\mathbf{G}) = \frac{(x, y, z)}{(x^2+y^2+z^2)^{3/2}} = \frac{X}{\|X\|^3} = \mathbf{F}.$$

## Exercises

6.1 Evaluate $\int_\Gamma F$ where $F$ is a vector field and $P$ is a parametrization of the directed curve $\Gamma$

(a) $F(x, y, z) = (x, y, z)$, $P(t) = (\sin t, \cos t, t)$, $0 \le t \le 2\pi$,
(b) $F(x, y, z) = x^2\,dx + xyz\,dy + xz\,dz$, $P(t) = (t, t^2, t^3)$, $0 \le t \le 1$,
(c) $F(x, y, z) = \cos z\,\mathbf{i} + e^x\mathbf{j} + e^y\mathbf{k}$, $P(t) = (1, t, e^t)$, $0 \le t \le 4$.

6.2 Find, if they exist, scalar potentials for the following vector fields

(a) $F(x, y, z) = (2x + ze^{xz},\ z,\ y + xe^{zx})$,
(b) $G(x, y, z) = (y,\ z\cos yz,\ y\cos yz)$,
(c) $H(x, y, z) = \big(y + yz\cos(xyz),\ x + xz\cos(xyz),\ 2z + xy\cos(xyz)\big)$,
(d) $K(x, y, z) = \big(2x\cos(x^2 + yz),\ z\cos(x^2 + yz),\ y\cos(x^2 + yz)\big)$.

6.3 Let $f(x, y, z) = x^2y^2 + y^2z^2$. Verify directly that $\nabla \times \nabla f = 0$.

6.4 Compute the curl of each of the following vector fields:

(a) $F_1(x, y, z) = \dfrac{(3, 1, 2)}{\|X\|}$
(b) $F_2(x, y, z) = \dfrac{(yz,\ zx,\ xy)}{\|X\|}$
(c) $F_3(X) = \dfrac{\langle X, X\rangle X}{\|X\|^4}$
where $X = (x, y, z) \in \mathbb{R}^3 \setminus \{0\}$.

6.5 If **a**, **b**, **c** and **d** are vectors in $\mathbb{R}^3$ show that

(a) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.
(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$.

6.6 Let $f,\ g \colon \mathbb{R}^3 \to \mathbb{R}$ and $\mathbf{F},\ \mathbf{G} \colon \mathbb{R}^3 \to \mathbb{R}^3$ denote smooth functions. Prove

(a) $\mathrm{div}(f\mathbf{F}) = f\,\mathrm{div}(\mathbf{F}) + \nabla f \cdot \mathbf{F}$
(b) $\mathrm{div}(\mathbf{F} \times \mathbf{G}) = \mathrm{curl}(\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot \mathrm{curl}(\mathbf{G})$
(c) $\mathrm{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f\,\mathrm{curl}(\mathbf{F})$.

6.7 If $f \colon U \text{ (open)} \subset \mathbb{R}^3 \to \mathbb{R}$ has continuous second-order derivatives show that

$$\mathrm{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Symbolically the left-hand side has the form $\nabla \cdot \nabla f$ and is written (for this reason) $\nabla^2 f$. ($\nabla^2 f$ is called the *Laplacian* of $f$ and if $\nabla^2 f = 0$ then $f$ is called *harmonic*).

6.8 Show that $\dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}$ is harmonic on $\mathbb{R}^3 \setminus \{0, 0, 0\}$.

6.9 If $f \colon \mathbb{R}^+ \to \mathbb{R}$ and $g(X) = f(\|X\|)$ for $X \in \mathbb{R}^3 \setminus \{0, 0, 0\}$ show that

$$\nabla g(X) = f'(\|X\|)\frac{X}{\|X\|} \quad \text{and} \quad \nabla^2 g(X) = f''(\|X\|) + \frac{2}{\|X\|} f'(\|X\|).$$

Show that $g$ is harmonic on $\mathbb{R}^3 \setminus \{0, 0, 0\}$ if and only if

$$f(r) = \frac{A}{r} + B$$

for all $r \neq 0$ in $\mathbb{R}$.