Chapter 10
Parametrized Surfaces in $\mathbb{R}^3$

Summary *We discuss theoretical and practical approaches to parametrizing a surface in* $\mathbb{R}^3$.

In this chapter we begin a systematic study of surfaces in $\mathbb{R}^3$, a topic which will occupy the remaining chapters of this book. We begin with an informal discussion of the background we bring to this investigation and reveal our general intentions.

Daily we encounter most of the classical Euclidean surfaces in $\mathbb{R}^3$ such as a *sphere* (football, globe), *cone* (ice-cream cone), *ellipsoid* (egg, American football), *cylinder* (jar, can), *plane* (floor, wall, ceiling) and the non-Euclidean *torus* (doughnut, tube). Many of our examples involve these surfaces and a solid geometric understanding of these should be deliberately cultivated. Initially these surfaces are merely subsets of $\mathbb{R}^3$ which have a certain recognisable shape. We have already established mathematical contact with them as level sets and graphs of functions. Our studies have been confined to the classical problem in differential calculus of finding the maximum and minimum of a scalar-valued function over a surface and for this we introduced the geometric concepts of *tangent plane* and *normal line*. This essentially summarises the formal knowledge we have acquired but we do have at our disposal a range of mathematical ideas and techniques which, if not directly applicable, will often hint at the way forward.

Our plan is to develop integration theory on surfaces and to investigate the geometry of surfaces. Keep in mind the following topics as particularly relevant to the theories we develop:

(a) integration over open subsets of $\mathbb{R}^2$ (Chap. 9)
(b) integration along directed curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Chap. 6)
(c) the geometry of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ (Chaps. 7 and 8).

For instance an open subset U in $\mathbb{R}^2$ can be considered, or identified with, a surface $\widetilde{U}$ in $\mathbb{R}^3$ by means of the mapping

$$(x, y) \in U \longrightarrow (x, y, 0) \in \widetilde{U}.$$

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_10,

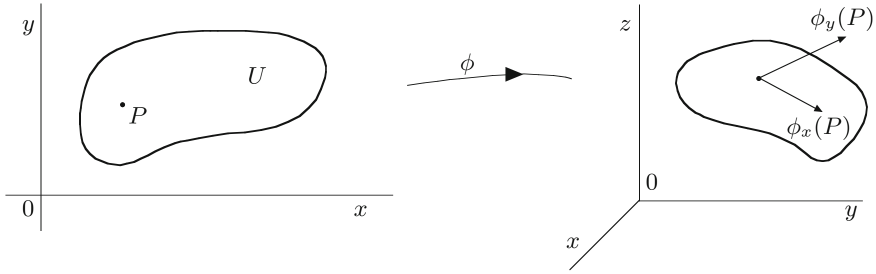

Fig. 10.1

Thus any theory of integration over a surface will extend integration theory over open subsets of $\mathbb{R}^2$. We can say more since parametrizations (Definition 10.2) are a means of identifying a simple surface with an open subset of $\mathbb{R}^2$ and this eventually allows us to reduce integration over a surface to integration over an open subset of $\mathbb{R}^2$ in much the same way that we reduced line integrals in $\mathbb{R}^n$ to integration over an interval in $\mathbb{R}$ (the key word in this sentence is unfortunately "eventually"). Thus our theory of integration over surfaces is obtained by combining and developing techniques already used in (a) and (b). We shall see later how (b) and (c) and, in particular, the methods used to derive the Frenet–Serret equations, can be extended to investigate the geometry of surfaces in $\mathbb{R}^3$.

We begin our formal study of surfaces by defining the concept of a parametrized surface in $\mathbb{R}^3$. Our definition is mathematically simple, a good starting point, and carries us a long way but does have certain inadequacies that we discuss as we proceed.

Definition 10.1 A parametrized surface in $\mathbb{R}^3$ consists of a pair (S, ϕ) where S is a subset of $\mathbb{R}^3$ and ϕ is a bijective mapping from an open subset of $\mathbb{R}^2$ onto S such that the following conditions hold:

(i) ϕ has derivatives of all orders (we say that ϕ is smooth or $\mathscr{C}^\infty$)
(ii) $\phi_x \times \phi_y \neq 0$ at all points.

Condition (ii) is the analogue of $P' \neq 0$ for a parametrized curve in $\mathbb{R}^3$ and is *equivalent* to the requirement that ϕ_x and ϕ_y are linearly independent vectors at all points (see Fig. 10.1).

Definition 10.2 A simple surface in $\mathbb{R}^3$ is a subset S of $\mathbb{R}^3$ for which there exists a mapping ϕ such that (S, ϕ) is a parametrized surface. We call ϕ a parametrization of S.

One notable difference between the above definitions and the corresponding definition of directed curve (Definition 5.1) is the change from a *closed* interval in $\mathbb{R}$ to an *open* subset of $\mathbb{R}^2$ for the domain of parametrization. For a directed curve the inclusion of end points in the domain of definition leads to a sense of *direction* along the curve and we will need an analogous concept, a sense of *orientation*, to develop vector-valued integration theory over a surface. However, a sense of direction along a curve can also be obtained by using tangent vectors at interior points of the curve and we develop the concept we require for surfaces by using the interior of the surface.

Unfortunately, many of the classical Euclidean surfaces, e.g. the sphere, are not simple surfaces but, fortunately, for many practical purposes, e.g. the calculation of surface area and the evaluation of surface integrals, they may be considered as simple surfaces. We will make this precise later and also define a general surface in $\mathbb{R}^3$.

We now examine three specific examples—*graphs*, *surfaces of revolution* and the classical *ellipsoid*. These, although apparently rather limited, appear in many different contexts and we allow them to divert us to essential ideas associated with any parametrized surface.

Example 10.3 (see Example 7.1) Let $f\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ denote a smooth function defined on the open subset U of $\mathbb{R}^2$ and let S denote the *graph* of f, i.e.

$$S = \big\{\big(x, y, f(x, y)\big) : (x, y) \in U\big\}.$$

The form of S immediately gives a parametrization ϕ defined by

$$\phi(x, y) = \big(x, y, f(x, y)\big)$$

with domain U. Since $\phi_x = (1, 0, f_x)$ and $\phi_y = (0, 1, f_y)$ we have

$$\phi_x \times \phi_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1).$$

As the final coordinate of $\phi_x \times \phi_y$ is always 1 we have $\phi_x \times \phi_y \neq 0$. If $\phi(x_1, y_1) = \phi(x_2, y_2)$ then

$$\big(x_1, y_1, f(x_1, y_1)\big) = \big(x_2, y_2, f(x_2, y_2)\big).$$

Hence $x_1 = x_2$ and $y_1 = y_2$ and so ϕ is injective. Clearly $\phi(U) = S$.

As a particular example consider the unit sphere $S : x^2 + y^2 + z^2 = 1$. We have $z^2 = 1 - x^2 - y^2$ and $z = \pm(1 - x^2 - y^2)^{1/2}$. Hence the *upper hemisphere* of S is the graph of the function

$$g(x, y) = (1 - x^2 - y^2)^{1/2}$$

and the mapping $\widetilde{g}\colon (x, y) \to \big(x, y, (1-x^2-y^2)^{1/2}\big)$ is a parametrization of the upper hemisphere. The domain of $\widetilde{g}$ is the disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. To visualise this physically imagine the hemisphere as a dome S and take U to be the floor of the dome. If you can now accept yourself as a point p on the floor and look directly upwards your eyes will focus on a unique point q in the dome S (Fig. 10.2). Moving around you will be *identifying* the points of U with points of the surface. The floor U is the domain of the parametrization while the parametrization itself is the mapping which takes p to q. The fact that you are always looking in the same direction means

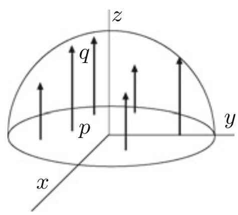

Fig. 10.2

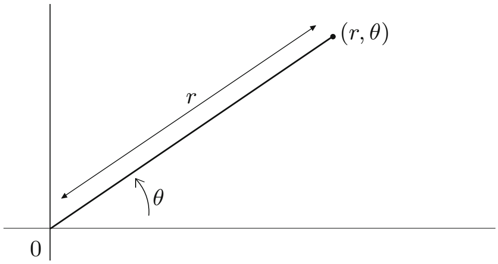

Fig. 10.3

that the identification or parametrization is given by a set of *parallel* lines (Fig. 10.2) and since parallel lines never meet the mapping is injective or one-to-one. Since each point on the dome is hit by an arrow we also have an onto or surjective mapping and thus a bijective mapping. In general any natural identification of a flat set with a surface by means of parallel lines will lead to a parametrization (see Example 11.2).

A point P in $\mathbb{R}^2$ can be identified by means of its distance r from the origin and the angle θ between the positive x-axis and the vector OP (Fig. 10.3).

We call (r, θ) the *polar coordinates* of P. Polar coordinates are particularly useful when dealing with circles and discs and with functions involving the expression $x^2 + y^2$ in Cartesian coordinates. Unfortunately, the correspondence

$$(r, \theta) \longrightarrow (x, y) = (r\cos\theta, r\sin\theta)$$

is not bijective but we get around this difficulty by removing a small portion of the domain. The domain of the parametrization of the hemisphere S given above is the disc $x^2 + y^2 < 1$ and suggests the use of polar coordinates. Since

$$(1 - x^2 - y^2)^{1/2} = (1 - r^2\cos^2\theta - r^2\sin^2\theta)^{1/2} = (1 - r^2)^{1/2}$$

the mapping

$$F\colon (r, \theta) \in (0, 1) \times (-\pi, \pi) \to \big(r\cos\theta, r\sin\theta, (1 - r^2)^{1/2}\big)$$

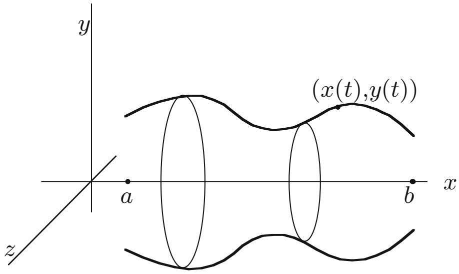

Fig. 10.4

is a bijective mapping onto $S \setminus \Gamma$ where Γ is the set

$$\Big\{(-r, 0, (1-r^2)^{1/2}),\ 0 \le r < 1\Big\}.$$

On $(0,1) \times (-\pi, \pi)$

$$F_r \times F_\theta = \left(\frac{r^2\cos\theta}{(1-r^2)^{1/2}}, \frac{r^2\sin\theta}{(1-r^2)^{1/2}}, r\right)$$

is non-zero and we have found a second parametrization of (almost) the whole hemisphere.

Example 10.4 Let $P(t) = \big(x(t), y(t)\big)$, $t \in [a,b]$ denote a directed curve Γ in $\mathbb{R}^2$. We suppose $y(t) > 0$ for all t. The surface obtained by revolving this curve about the x-axis is called the *surface of revolution* of P about the x-axis (Fig. 10.4).

Consider a typical point on the circle obtained by rotating the point $\big(x(t), y(t)\big)$. The first coordinate remains unchanged. The second coordinate will generate a circle in the (y,z)-plane with centre $(0,0)$ and radius $y(t)$. Using the standard parametrization of the circle we see that a typical point on this circle has coordinates $\big(y(t)\cos\theta, y(t)\sin\theta\big)$. Putting these together we obtain a parametrization

$$(t,\theta) \longrightarrow \big(x(t), y(t)\cos\theta, y(t)\sin\theta\big)$$

where $t \in (a,b)$ and $\theta \in (0, 2\pi)$. Now

$$\phi_t = \big(x'(t), y'(t)\cos\theta, y'(t)\sin\theta\big)$$

and

$$\phi_\theta = \big(0, -y(t)\sin\theta, y(t)\cos\theta\big).$$

Hence

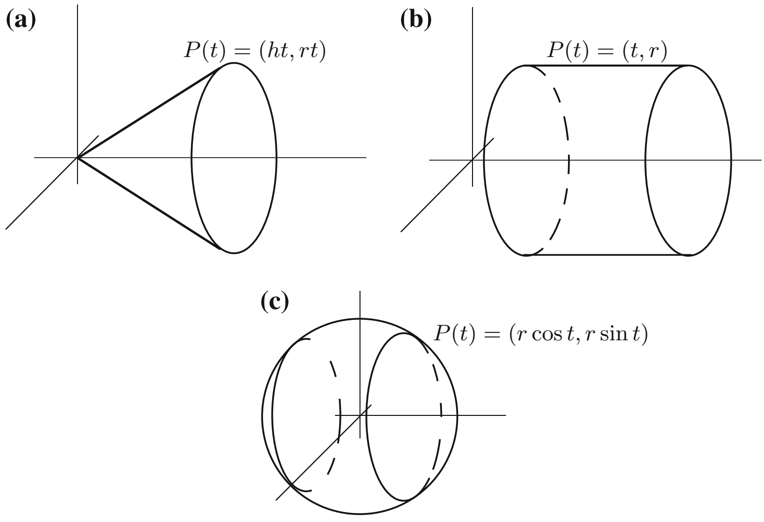

Fig. 10.5

$$\begin{aligned}
\phi_t \times \phi_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'(t) & y'(t)\cos\theta & y'(t)\sin\theta \\ 0 & -y(t)\sin\theta & y(t)\cos\theta \end{vmatrix} \\
&= \big(y'(t)y(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta\big).
\end{aligned}$$

Since

$$\begin{aligned}
\|\phi_t \times \phi_\theta\| &= \big((y')^2y^2 + (x')^2y^2\cos^2\theta + (x')^2y^2\sin^2\theta\big)^{1/2} \\
&= y(t)\big(y'(t)^2 + x'(t)^2\big)^{1/2} = y(t)\|P'(t)\| \neq 0
\end{aligned}$$

we have $\phi_t \times \phi_\theta \neq 0$. Since the mapping ϕ is bijective as long as we do not include θ and $\theta + 2\pi$ in the domain we let $U = (a, b) \times (0, 2\pi)$ and obtain a bijective mapping onto the surface of revolution with one curve, the original or *profile* curve, removed. Many classical surfaces may be realised as surfaces of revolution; e.g. the cone (Fig. 10.5a), cylinder (Fig. 10.5b) and sphere (Fig. 10.5c).

These induce the following parametrizations:

$$\begin{aligned}
&\textit{Cone} && (t, \theta) \longrightarrow (ht, rt\cos\theta, rt\sin\theta), \\
&\textit{Cylinder} && (t, \theta) \longrightarrow (t, r\cos\theta, r\sin\theta), \\
&\textit{Sphere} && (t, \theta) \longrightarrow (r\cos t, r\sin t\cos\theta, r\sin t\sin\theta).
\end{aligned}$$

We parametrize, in Example 11.1 the *torus* as a surface of revolution.

We now examine in considerable detail a particular surface—the standard ellipsoid. This includes the sphere as a special case. We use this example as an excuse to explore and comment on many of the practical and theoretical considerations that arise in studying any surface and to shed some light on the necessity and significance of later developments. In studying this example the reader should keep in mind that parametrizations are nothing more than coordinate systems, that we are always trying to interpret mathematical facts geometrically and constantly attempting to articulate mathematically our geometric observations.

Example 10.5 We consider the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

This specialises to the sphere of radius a centred at the origin when $a = b = c$. Two of the most frequently used coordinate systems on the sphere are *spherical polar coordinates* and *geographical coordinates*. These, as we shall see, are closely related. Spherical polar coordinates are more popular in the mathematical literature but since geographical coordinates are in everyone's normal experience we devote more time to them here.

The representation of the ellipsoid as a level set of a sum of squares suggests we use the elementary identity $\sin^2\theta + \cos^2\theta = 1$ to develop our parametrization. Rewriting the formula for the ellipsoid as a sum of *two* squares we get

$$\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

Now let

$$\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} = \cos\theta \quad \text{and} \quad \frac{z}{c} = \sin\theta.$$

Hence $z = c\sin\theta$ and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta, \quad \text{i.e.} \quad \left(\frac{x}{a\cos\theta}\right)^2 + \left(\frac{y}{b\cos\theta}\right)^2 = 1.$$

A further similar substitution gives

$$\frac{x}{a\cos\theta} = \cos\psi \quad \text{and} \quad \frac{y}{b\cos\theta} = \sin\psi.$$

This implies $x = a\cos\theta\cos\psi$ and $y = b\cos\theta\sin\psi$. Our parametrization, denoted by F, has the form

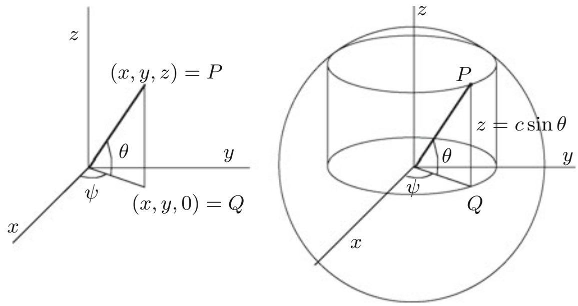

Fig. 10.6

$$\begin{aligned} F\colon U &\longrightarrow S \text{ (ellipsoid)} \\ (\theta, \psi) &\longrightarrow (a\cos\theta\cos\psi, b\cos\theta\sin\psi, c\sin\theta) \end{aligned}$$

but we must specify U, check that F is bijective and see that conditions (i) and (ii) are satisfied. Geometrically we have the following diagram (Fig. 10.6) in the case of the sphere $a = b = c$.

Thus we project the point $P = (x, y, z)$ onto the xy-plane to get the point $Q = (x, y, 0)$. The angle θ is the angle between the vectors OQ and OP and hence $\theta = \tan^{-1}\big(z \;/\; (x^2 + y^2)^{1/2}\big)$. The level set of the surface through P parallel to the xy-plane is a circle (remember we are just considering a sphere here) and this is also projected onto the xy-plane and the angle ψ is obtained by using *polar coordinates* on this circle. Hence $\psi = \tan^{-1}(y/x)$. From Fig. 10.6 it is clear that θ ranges over the interval $(-\pi/2, \pi/2)$ and ψ over the interval $(0, 2\pi)$. Clearly the smoothness condition (i) is satisfied no matter what domain U we choose for F. In many examples condition (ii) is easily checked and formal identification of the range of F is obtained using a diagram. In this example, however, we adopt a more analytic approach in verifying that F is a parametrization.

We have

$$F_\theta = (-a\sin\theta\cos\psi, -b\sin\theta\sin\psi, c\cos\theta)$$

and

$$F_\psi = (-a\cos\theta\sin\psi, b\cos\theta\cos\psi, 0).$$

Hence,

$$\begin{aligned} F_\theta \times F_\psi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta\cos\psi & -b\sin\theta\sin\psi & c\cos\theta \\ -a\cos\theta\sin\psi & b\cos\theta\cos\psi & 0 \end{vmatrix} \\ &= (-bc\cos^2\theta\cos\psi, -ac\cos^2\theta\sin\psi, -ab\cos\theta\sin\theta) \\ &= -abc\cos\theta\left(\frac{\cos\theta\cos\psi}{a}, \frac{\cos\theta\sin\psi}{b}, \frac{\sin\theta}{c}\right). \end{aligned}$$

If $a = b = c$ then

$$\begin{aligned}\|F_\theta \times F_\psi\| &= a^2 \cos\theta (\cos^2\theta \cos^2\psi + \cos^2\theta \sin^2\psi + \sin^2\theta)^{1/2} \\ &= a^2 \cos\theta. \end{aligned} \tag{10.1}$$

Before proceeding we make a brief observation which we develop in more detail in Chaps. 16–18. If we fix ψ and let θ vary we obtain a mapping

$$\theta \longrightarrow F(\theta, \psi)$$

where θ ranges over an interval in $\mathbb{R}$. This defines a directed curve which lies in S with tangent F_θ. Similarly F_ψ is tangent to the curve $\psi \to F(\theta, \psi)$ in S. These curves are called *coordinate curves* (of the parametrization). The vectors F_θ and F_ψ lie in the tangent space of S at P and are called *tangent vectors*. Tangent vectors enable us to define directional derivatives of functions defined on the surface. If $G : S \to \mathbb{R}^n$ and $P \in S$ then the *directional derivative* of G at P in the direction of the tangent vector $\mathbf{v}$ at P is given by

$$D_{\mathbf{v}}G(p) = \frac{d}{dt}(G \circ \phi)(t)|_{t=0}$$

where ϕ is *any* differentiable mapping from $(-a, a)$ into S such that $\phi(0) = P$ and $\phi'(0) = \mathbf{v}$.

Condition (ii) says that F_ϕ and F_θ span the tangent space and the following relationship holds in *any* surface S.

$$\begin{aligned}\{\text{The tangent space of } S \text{ at } P\} &= \left\{\begin{array}{c}\text{set of tangents at } P \text{ of all curves} \\ \text{in } S \text{ which pass through } P\end{array}\right\} \\ &= \left\{\begin{array}{c}\text{all linear combinations spanned} \\ \text{by the partial derivatives at } P \\ \text{of any parametrization}\end{array}\right\} \\ &= \left\{\begin{array}{c}\text{set of directional derivatives} \\ \text{at } P \text{ which operate on} \\ \text{functions defined on } S.\end{array}\right\}\end{aligned}$$

Since $F_\theta \times F_\psi \neq 0$, the vector $F_\theta \times F_\psi$ is perpendicular to the tangent space and hence parallel to the normal. In our case the surface S is the level set $g^{-1}(0)$ where

$$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

We now confine ourselves to the special case of the unit sphere, i.e. the ellipsoid with $a = b = c = 1$. The adjustments necessary, when this case has been considered, in order to examine the general situation are fairly minor. We have seen in Chap. 2 that the normal is parallel to ∇g and by inspecting (10.1) we see that $\nabla g(P)$ is parallel to $F_\theta(P) \times F_\psi(P)$. This observation can be used to partially check calculations and, in some cases, may even be used to avoid calculating $F_\phi \times F_\psi$. We have thus related our earlier concepts of *tangent space* and *normal* with terms which can be calculated from any parametrization.

We next turn to find a domain U for F and to verify that all requirements necessary for a parametrization are satisfied by (F, U). Initially, it is usually better to choose the domain of parametrization to be as large and as simple as possible. These twin aims may not always be compatible with the requirements for a parametrization or, indeed, with one another so some judgement is necessary and this usually comes with experience.

By (10.1), $F_\theta \times F_\psi \neq 0$ if and only if $\cos\theta \neq 0$ and a natural choice for the domain of θ is the interval $-\pi/2 < \theta < \pi/2$. This is also suggested by Fig. 10.6. Since both the sine and cosine functions have period 2π the length of the interval of definition of ψ *cannot* be greater than 2π. The interval $(0, 2\pi)$ is a natural choice and leads to

$$U = \{(\theta, \psi) : -\pi/2 < \theta < \pi/2,\ 0 < \psi < 2\pi\}$$

as our domain for F. Clearly the set U is open. If $F(\theta_1, \psi_1) = F(\theta_2, \psi_2)$ then, comparing final coordinates in F, we get $\sin\theta_1 = \sin\theta_2$. Since the sine function is injective on $(-\pi/2, \pi/2)$ this implies $\theta_1 = \theta_2$. Moreover, $\cos\theta \neq 0$ on $(-\pi/2, \pi/2)$ and comparing the first two coordinates of F we see that $\cos\psi_1 = \cos\psi_2$ and $\sin\psi_1 = \sin\psi_2$. Since $(\cos(\psi), \sin(\psi)), 0 < \psi < 2\pi$, are the coordinates of a unique point on the unit circle we have $\psi_1 = \psi_2$ and F is injective on U.

We have thus shown that F is a parametrization of its range or image, $F(U)$. By our construction the image of F lies in the ellipsoid. Does it cover the full ellipsoid? If $F(\theta, \psi) = (0, 0, c)$ then $\sin\theta = 1$. However, $-\pi/2 < \theta < \pi/2$, and so $(0, 0, c) \notin F(U)$ and F does *not* cover the ellipsoid. To find the range of F we proceed as follows. The set U is an *open* rectangle in $\mathbb{R}^2$ and its boundary is the perimeter of this rectangle. The function F has a nice smooth extension from U to its boundary (Fig. 10.7) and the image of this boundary is the boundary of the image of F in the surface we are examining.

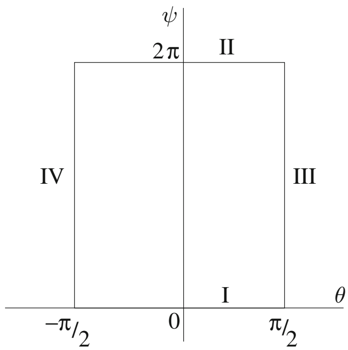

Fig. 10.7

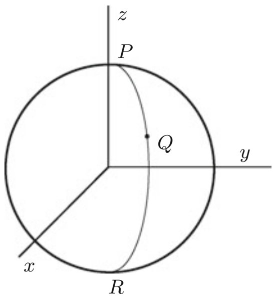

Fig. 10.8

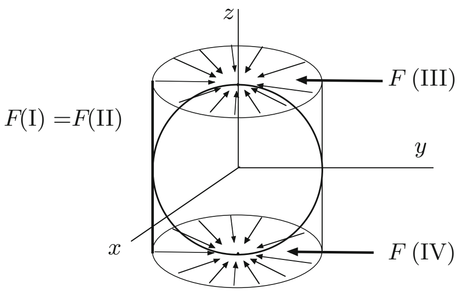

Fig. 10.9

Since $F(\theta, 0) = F(\theta, 2\pi) = (-a\cos\theta, 0, c\sin\theta)$, $F(\pi/2, \psi) = (0, 0, c)$, $F(-\pi/2, \psi) = (0, 0, -c)$, and $-\pi/2 < \theta < \pi/2$ the boundary of U is mapped onto one half of the ellipse $(x/a)^2 + (z/c)^2 = 1, y = 0$. Hence F is a parametrization of the ellipsoid with one curve removed, the half ellipse PQR in Fig. 10.8. As we develop geometric insight we will find that a good sketch often leads to a rapid identification of the domain of definition of the parametrization and it will not be necessary to go through the protracted investigation we have just completed.

A different choice of domain would have led to a parametrization which almost certainly would have covered a different part of the ellipsoid and perhaps covered the curve that we missed. However, *no parametrization* will cover the full ellipsoid. This, although intuitively clear, is a highly non-trivial mathematical result.

A parametrization identifies, with F as the mode of identification, a flat set of points, the open subset U of $\mathbb{R}^2$, with a subset of the surface, $F(U)$, in a one-to-one fashion. So essentially we are taking a sheet of paper U and using F to wrap it around a sphere. The first stage of wrapping turns the sheet of paper into a cylinder with the sphere inside it (Fig. 10.9). This identifies the boundary lines I and II with one another (Fig. 10.7) and is equivalent mathematically to $F(\theta, 0) = F(\theta, 2\pi)$ for all θ.

The final steps in the wrapping collapses III onto $(0, 0, c)$ and IV onto $(0, 0, -c)$. Mathematically this says $F(\pi/2, \psi) = (0, 0, c)$ and $F(-\pi/2, \psi) = (0, 0, -c)$ for

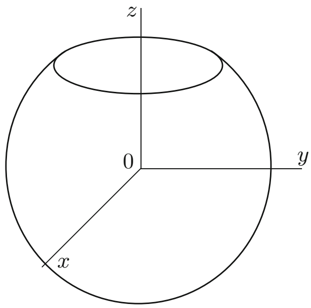

Fig. 10.10

all ψ. We thus see geometrically that if we *include* the boundary of U in the domain of F then the curve, which we previously missed, is covered twice and F is not one-to-one. If we do *not include* the boundary then this curve is not in the image of F.

Next we discuss parametrizations as *coordinate systems*. A useful initial approach to any such system is to sketch and examine the coordinate curves on the surface. In our case, since we are still looking at the unit sphere, we consider the curves obtained by fixing one of the variables of the function

$$F(\theta, \psi) = (\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta), \quad -\pi/2 < \theta < \pi/2,\ 0 < \psi < 2\pi.$$

Since F is bijective, a point on the surface corresponds to a unique pair (θ_1, ψ_1) and $F(\theta_1, \psi_1)$ is the point of intersection of the *coordinate curves* $\theta \to F(\theta, \psi_1)$, $\psi \to F(\theta_1, \psi)$. For this reason, the pair (θ, ψ) is often referred to as the *curvilinear coordinates* of the point $F(\theta, \psi)$. Fixing θ is clearly equivalent to taking a fixed value of z and amounts to taking a cross-section of the unit sphere parallel to the xy-plane. Geometrically we get a circle (Fig. 10.10).

This may also be seen analytically since the mapping $F(\theta, \psi)$ can be written as

$$(0, 0, \sin\theta) + \cos\theta(\cos\psi, \sin\psi, 0).$$

As ψ varies over $(0, 2\pi)$ we get a circle (with one point removed) in a plane parallel to the xy-plane, with centre $(0, 0, \sin\theta)$ on the z-axis, and radius $\cos\theta$. A number of level sets of θ are given in Fig. 10.11.

Now fix ψ and let θ range over the interval $-\pi/2 < \theta < \pi/2$. We have

$$F(\theta, \psi) = \cos\theta(\cos\psi, \sin\psi, 0) + \sin\theta(0, 0, 1).$$

Since $(\cos\psi, \sin\psi, 0)$ and $(0, 0, 1)$ are perpendicular unit vectors we get, by Pythagoras' theorem, a semicircle of radius 1. We do not get the full circle because of the restricted range of θ. This is easily seen, either geometrically or from the above formula, to be a semicircle running from $(-1, 0, 0)$ to $(+1, 0, 0)$, i.e.

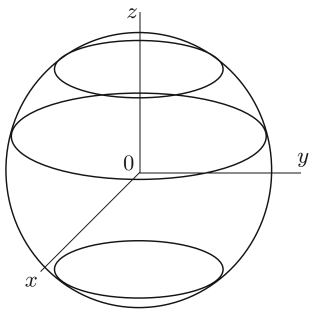

Fig. 10.11

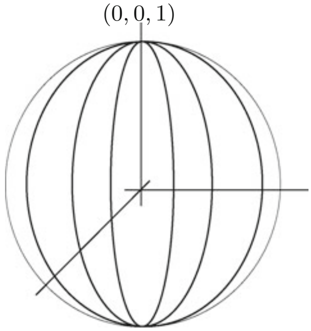

Fig. 10.12

from the bottom to the top of the sphere. In Fig. 10.12 we sketch a number of these level sets of ψ.

One can find a ready-made example of this by examining a *globe* representing the Earth. The level sets of θ are the *lines* or *parallels of latitude* with the equator corresponding to $\theta = 0$. Instead of $+$ and $-$ the terms North and South are used. The level sets of ψ are called *lines of longitude* or *meridians* with the level set $\psi = 0$ going through Greenwich, England. In place of $+$ and $-$ we use East and West (of Greenwich). The range here is from 0 to 180° while for θ we use 0 to 90°. If we represent the Earth on a map we are unwrapping the sphere, or mathematically, taking the inverse of a parametrization. The most common map (Fig. 10.13) is obtained by using *Mercator's projection*. This first unwraps the sphere onto a cylinder in such a way that when the cylinder is unrolled shapes are preserved (but area is distorted—see Exercise 11.8). What happens if we replace the cylinder by a cone?

Note that East meets West on the International Date Line in the middle of the Pacific Ocean. There is a certain ambiguity about this line—is it 180° East or 180° West of Greenwich? This, once more, reflects the fact that we do not obtain a bijective mapping if we include the boundary in the domain of definition of the parametrization.

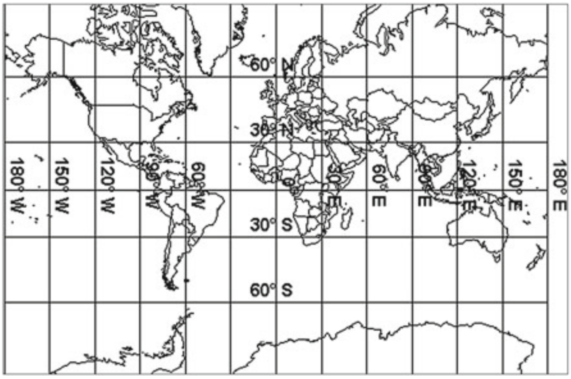

Fig. 10.13

Different maps result from using different parametrizations or coordinate systems. The mapping F that we have been discussing is called the *geographical coordinate system*. It is as useful and as intuitive a system as the more popular spherical polar coordinate system that we now discuss.

A small change in our method of introducing geographical coordinates leads to the definition of spherical polar coordinates. Recall that, for the ellipsoid, we have

$$\left(\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{1/2}\right)^2+\left(\frac{z}{c}\right)^2=1.$$

Now we interchange the role of sine and cosine and let

$$\frac{z}{c}=\cos\theta \quad \text{and} \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{1/2}=\sin\theta.$$

This implies

$$\left(\frac{x}{a\sin\theta}\right)^2+\left(\frac{y}{b\sin\theta}\right)^2=1\,.$$

Let $x/a\sin\theta=\cos\psi$ and $y/b\sin\theta=\sin\psi$. This defines the mapping

$$G(\theta,\psi)=(a\sin\theta\cos\psi,\, b\sin\theta\sin\psi,\, c\cos\theta)$$

which are the *spherical polar coordinates* of the point (x, y, z).

From Fig. 10.14 we see that $0<\theta<\pi$ and $0<\psi<2\pi$. The angle θ is called the *colatitude* and ψ is sometimes called the *azimuth*. Note that we obtain the same two sets of coordinate curves for geographical and spherical polar coordinates—the labelling is, however, different.

For a sphere of radius r with centre at the origin this gives

$$G(\theta,\psi)=(r\sin\theta\cos\psi,\, r\sin\theta\sin\psi,\, r\cos\theta).$$

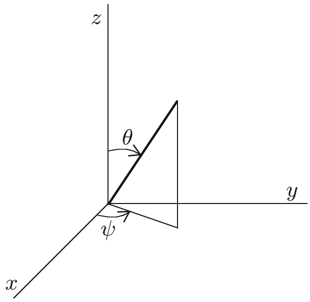

Fig. 10.14

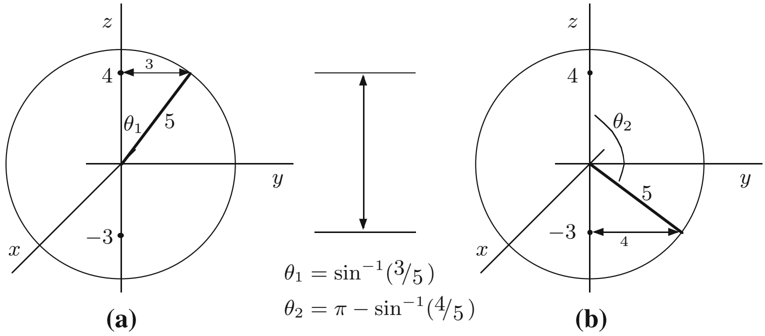

Fig. 10.15

Note that

$$G_\theta \times G_\psi = r^2 \sin\theta(\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$$

and

$$\|G_\theta \times G_\psi\| = r^2 \sin\theta.$$

If we wish to parametrize a part of the sphere we can still use G—and hence all calculations involving G and its derivatives—but it is necessary to identify the restricted domain on which we are working. Fig. 10.14 is useful for this purpose. Suppose, for instance, we wish to parametrize that portion of the sphere of radius 5 with centre at the origin which lies between the planes $z = -3$ and $z = 4$.

Figure 10.15a and b gives the minimum and maximum values of θ, θ_1 and θ_2, and there is no restriction on ψ. Hence our parametrization G has domain

$$U = \{(\theta, \psi) : \sin^{-1}\left(\frac{3}{5}\right) < \theta < \pi - \sin^{-1}\left(\frac{4}{5}\right),\ 0 < \psi < 2\pi\}.$$

From Fig. 10.14 we get the spherical polar coordinates in terms of the Cartesian coordinates (x, y, z):

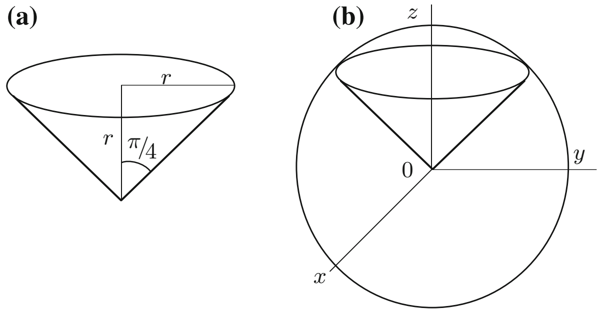

Fig. 10.16

$$r = (x^2 + y^2 + z^2)^{1/2}, \quad \theta = \tan^{-1}\left(\frac{(x^2 + y^2)^{1/2}}{z}\right) \quad \text{and} \quad \psi = \tan^{-1}\left(\frac{y}{x}\right).$$

The parametrization of the sphere can also be used to parametrize the *inverted vertical cone* of height a and base radius a (Fig. 10.16a).

Note that the angle between the z-axis and the curved surface of the cone is $\pi/4$. If we take the cross-section of the cone of height r, $0 < r < a$, then we obtain a circle of colatitude $\pi/4$ on the sphere of radius $r\sqrt{2}$ (Fig. 10.16b). The spherical polar coordinates of this circle are

$$\begin{aligned}\big(r\sqrt{2}\sin\frac{\pi}{4}\cos\psi,\ r\sqrt{2}\sin\frac{\pi}{4}\sin\psi,\ r\sqrt{2}\cos\frac{\pi}{4}\big), \quad 0 < \psi < 2\pi,\\ = (r\cos\psi, r\sin\psi, r)\end{aligned}$$

and we have the parametrization

$$(r, \psi) \longrightarrow (r\cos\psi, r\sin\psi, r), \quad 0 < \psi < 2\pi,\ 0 < r < a.$$

This completes our analysis of the ellipsoid and sphere. We remark that many of our comments apply to a wide variety of surfaces.

A simple surface may have *different parametrizations* and depending on the circumstances one may be more useful than another. In Examples 10.3 and 10.4 we derived other parametrizations for the "sphere". An important technique that we discuss in Chap. 14 is how to go from one parametrization to another. Since parametrizations are coordinate systems involving variables this process is known as a *change of variable*.

In the above discussion we have seen that ellipsoids and spheres are not simple surfaces, although they may be written as a union of a simple surface and a parametrized curve and for practical purposes, such as integration, they may be regarded as simple surfaces. Nevertheless, it is desirable to have a definition of surface in $\mathbb{R}^3$ which implies, for instance, that a sphere is a surface. Such a definition, as we have just noted, requires more than one parametrization.

Definition 10.6 A subset of $\mathbb{R}^3$ is a surface if it can be covered by a collection of (generally overlapping) simple surfaces.

With the aid of the following important and highly non-trivial result we outline a relationship between surfaces, simple surfaces, level sets and graphs. This also shows that an ellipsoid is a surface.

Theorem 10.7 (Inverse Mapping Theorem.) *If $F : U(open) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuously differentiable and $F'(p)$ is invertible at $p \in U$ then there exist V and W open in $\mathbb{R}^n$, $p \in V$, such that the restriction of F to V defines a bijective mapping from V onto W with continuously differentiable inverse.*

Theorem 10.8 *If S is a subset of $\mathbb{R}^3$ then the following are equivalent:*

1. *S is a surface,*
2. *for each $P \in S$ there exists $\varepsilon > 0$ such that $S \cap \{X : \|X - P\| < \varepsilon\}$ is a simple surface,*
3. *for each $P \in S$ there exists $\varepsilon > 0$ such that $S \cap \{X : \|X - P\| < \varepsilon\}$ is the graph of a smooth function,*
4. *for each $P \in S$ there exists $\varepsilon > 0$ and a smooth function $f : \{X : \|X - P\| < \varepsilon\} \longrightarrow \mathbb{R}$ such that $\nabla f(X) \neq 0$ for $\|X - P\| < \varepsilon$ and*

$$S \cap \{X : \|X - P\| < \varepsilon\} = \{X : \|X - P\| < \varepsilon, f(X) = 0\}.$$

Proof The first two conditions are equivalent by Definition 10.6. By Example 1.7, graphs are level sets and, by the Implicit Function Theorem (Theorem 2.1), level sets are locally graphs and the final two conditions are also equivalent. By Example 10.3 every graph is a simple surface and thus it suffices to show that every simple surface is locally a level set.

Suppose $\phi : U(\text{open in } \mathbb{R}^2) \longrightarrow S$ is a parameterized surface. Define $F : U \times \mathbb{R} \longrightarrow \mathbb{R}^3$ by letting

$$F(x, y, z) = \phi(x, y) + zG(x, y)$$

where $G(x, y, z) = \phi_x(x, y) \times \phi_y(x, y)$. Then F is continuously differentiable and

$$F'(x, y, 0) = [\phi_x(x, y), \phi_y(x, y), G(x, y)]$$

is invertible. If $P \in S$ then, by the Inverse Mapping Theorem, we can find open subsets of $\mathbb{R}^3$, V and W, such that $P \in W$, $F(V) = W$ and F restricted to V has a continuously differentiable inverse. Let $g = \pi_3 \circ F^{-1}$ where $\pi_3(x, y, z) = z$. If $(x, y, z) \in W$, then for a unique $(x', y', z') \in V$ we have

$$(x, y, z) = F(x', y', z') = \phi(x', y') + G(x', y')z'.$$

This implies

$$\begin{aligned} g(x, y, z) = \pi_3 \circ F^{-1}(x, y, z) = 0 &\iff \pi_3 \circ F^{-1} \circ F(x', y', z') = z' = 0 \\ &\iff (x, y, z) = F(x', y', 0) = \phi(x', y') \\ &\iff (x, y, z) \in S \end{aligned}$$

and hence $g^{-1}(0) \cap W = S \cap W$. Since $\nabla g(X) \neq 0$ for all $X \in W$ this completes the proof. $\square$

The above shows that graphs are much more than simple examples. This will become very evident when we examine curvature in the final three chapters.

Exercises

10.1 By using $\cosh^2 - \sinh^2 = 1$ and $\cos^2 + \sin^2 = 1$ parametrize the surfaces

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1.$

10.2 Show that the following are parametrizations of simple surfaces

(a) $P_1(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \quad 0 < \theta < 2\pi,\ 0 < r < 1$
(b) $P_2(x, \theta) = (x\cos\theta, y\sin\theta, x + y), \quad 0 < x < 1,\ 0 < \theta < 2\pi, y \neq 0$
(c) $P_3(u, v) = (u + v, u - v, uv), \quad 0 < u < 1,\ 0 < v < 1.$

10.3 By using a parametrization of an ellipsoid parametrize an inverted cone of height h and base radius r.

10.4 Use geographical and spherical polar coordinates to parametrize that portion of the sphere $x^2 + y^2 + z^2 = r^2$ which lies in the first octant $\{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$.

10.5 Let $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ denote the unit sphere in $\mathbb{R}^3$. If $(u, v) \in \mathbb{R}^2$ then the line determined by $(u, v, 0)$ and $(0, 0, 1)$ intersects S in a point other than $(0, 0, 1)$. Let $\phi(u, v)$ denote this point. Find the coordinates of $\phi(u, v)$ in terms of u and v. Show that $\phi(\mathbb{R}^2)$ is a simple surface parametrized by ϕ. Identify $\phi(\mathbb{R}^2)$.

10.6 Find the length of the portion of the curve $u = v$ lying in the surface parametrized by

$$\phi(u, v) = (u\cos v, u\sin v, u\sqrt{3})$$

which lies between the planes $z = 0$ and $z = 2\sqrt{3}$.

10.7 Let $f(x, y, z) = x + xy + yz$ and let $S = f^{-1}(1)$. Find two parametrized surfaces which lie in S and cover S.

10.8 Consider a line L initially lying along the positive y-axis and attached orthogonally to the z-axis. If L moves along the z-axis with constant speed a and at the same time rotates about the z-axis with constant speed b show that it sweeps out a surface parametrized by

$$\phi(u, t) = (u\sin bt, u\cos bt, at), \quad u > 0,\ t > 0.$$