Chapter 13
Stokes' Theorem

Summary *We discuss Stokes' theorem for oriented surfaces in* $\mathbb{R}^3$.

Stokes' theorem, the *fundamental theorem of calculus for surfaces*, generalises Green's theorem to oriented surfaces $\mathbf{S} = (S, \mathbf{n})$ with *edge* or *boundary* Γ (the term edge avoids confusion with our other use of the word boundary) consisting of a finite number of piecewise smooth directed curves. We suppose that the *positive side* of $\mathbf{S}$ lies on the *left-hand side* as we move along Γ in the *positive direction*. In practice this consistency between the orientations of the surface and its edge may be verified by sketching. In certain cases the normal $\mathbf{n}$ admits a *continuous extension* to the boundary and a parametrization P of the boundary has the correct orientation if at one point, say $P(t_0)$, we have $P'(t_0) \cdot \mathbf{n}\big(P(t_0)\big) > 0$ where $\mathbf{n}\big(P(t_0)\big)$ is the value of the extension of $\mathbf{n}$ at $P(t_0)$. If a consistent parametrization of the surface extends to give a parametrization of the boundary then the boundary is also correctly directed (see Example 13.5).

Theorem 13.1 (Stokes' Theorem) *Let* $\mathbf{S} = (S, \mathbf{n})$ *denote an oriented surface in* $\mathbb{R}^3$ *with boundary* Γ *consisting of a finite number of piecewise smooth directed curves. We suppose that the positive side of* $\mathbf{S}$ *lies on the left of the positive side of* Γ. *If* $\mathbf{F}$ *is a smooth vector field on* $S \cup \Gamma$ *then*

$$\int_\Gamma \mathbf{F} = \iint_{\mathbf{S}} \mathrm{curl}(\mathbf{F}) \tag{13.1}$$

i.e.

$$\int_\Gamma \langle \mathbf{F}, T\rangle ds = \iint_S \langle \mathrm{curl}(\mathbf{F}), \mathbf{n}\rangle dA$$

where T *is the unit tangent to the directed curves* Γ.

The proof, which we omit, is obtained by applying Green's theorem to the projections onto the coordinate planes. In Chaps. 6 and 12 we developed techniques to evaluate the left- and right-hand sides of (13.1), respectively. Thus the only new

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_13,

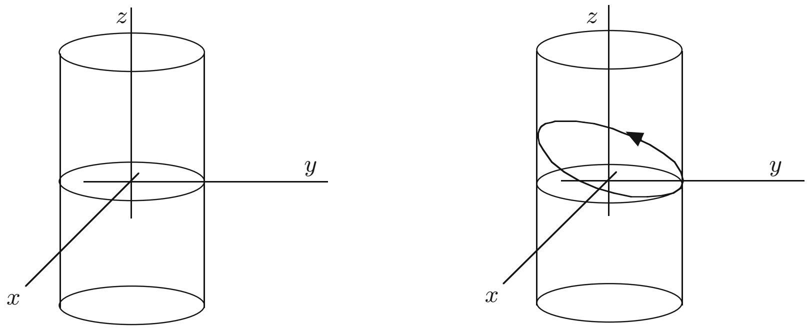

Fig. 13.1

factor in Stokes' theorem is the correlation between the orientations of the surface and its boundary.

Example 13.2 We use Stokes' theorem to evaluate the line integral

$$\int_C -y^3 dx + x^3 dy - z^3 dz$$

where C is the intersection of the cylinder $x^2 + y^2 = 3$ and the plane $x + y + z = 1$ and the orientation on C is anticlockwise when viewed from a point sufficiently high up on the z-axis. Let $\mathbf{S}$ denote the portion of the plane inside the cylinder oriented so that the normal lies above the surface (Fig. 13.1).

Let $\mathbf{F} = (-y^3, x^3, -z^3)$ then

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (0, 0, 3x^2 + 3y^2).$$

By Stokes' theorem

$$\int_C \mathbf{F} = \iint_{\mathbf{S}} \text{curl}(\mathbf{F}) = \iint_{\mathbf{S}} (0, 0, 3x^2 + 3y^2).$$

Since only the final coordinate of curl($\mathbf{F}$) is non-zero our analysis in the previous chapter implies that we only need consider the projection of $\mathbf{S}$ onto $\mathbb{R}^2_{(x,y)}$. As C projects onto an anticlockwise oriented curve Γ_1 (Fig. 13.1) our projection is onto the positive side of $\mathbb{R}^2_{(x,y)}$. Hence

$$\int_C \mathbf{F} = \iint_{x^2+y^2 \leq 3} (3x^2 + 3y^2) dx dy.$$

We parametrize the surface $x^2 + y^2 < 3$ in $\mathbb{R}^3$ by

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta, 0), \quad 0 < r < \sqrt{3},\ 0 < \theta < 2\pi.$$

We have $\phi_r = (\cos\theta, \sin\theta, 0)$ and $\phi_\theta = (-r\sin\theta, r\cos\theta, 0)$. Hence

$$\phi_r \times \phi_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0, 0, r).$$

Since

$$\frac{\phi_r \times \phi_\theta}{\|\phi_r \times \phi_\theta\|} = (0, 0, 1)$$

our parametrization is consistent with the positive orientation of $\mathbb{R}^2_{(x,y)}$. We have

$$\int_C \mathbf{F} = \int_0^{\sqrt{3}} \int_0^{2\pi} 3r^2 \cdot \|\phi_r \times \phi_\theta\| dr d\theta = 6\pi \int_0^{\sqrt{3}} r^3 dr = \frac{27\pi}{2}.$$

Moral A reasonable sketch is not just optional but necessary. The form of **F**, i.e. the fact that the first two coordinates were zero, combined with information on how surface integrals can be projected onto the coordinate planes greatly simplified the calculations required.

Example 13.3 We evaluate $\iint_{\mathbf{S}} \operatorname{curl}(\mathbf{F})$ where

$$\mathbf{F}(x, y, z) = (y^2 \cos xz, x^3 e^{yz}, -e^{-xyz})$$

and **S** is the portion of the sphere $x^2 + y^2 + (z-2)^2 = 8$ which lies above the xy-plane oriented outwards. The edge or boundary of **S**, Γ, is where the sphere cuts the xy-plane, i.e. where $z = 0$. We have $x^2 + y^2 + 4 = 8$, i.e. $x^2 + y^2 = 4$ (Fig. 13.2).

Since the positive side of the sphere is the outside we see from Fig. 13.2 that the surface **S** is on the left as we move along Γ in an anticlockwise direction in the xy-plane. This gives us our direction along Γ. By Stokes' theorem

$$\int_\Gamma \mathbf{F} = \iint_{\mathbf{S}} \operatorname{curl}(\mathbf{F}).$$

But Γ with anticlockwise direction is also the boundary or edge of the surface

$$\mathbf{S}_1 = \{(x, y, z) : x^2 + y^2 < 4, z = 0\}$$

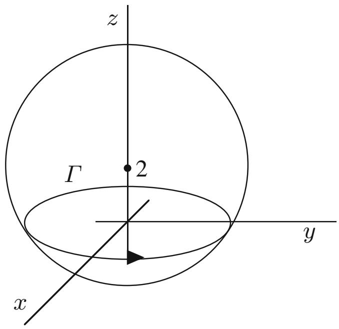

Fig. 13.2

oriented by the normal $(0, 0, 1)$. Hence a further application of Stokes' theorem implies

$$\int_{\Gamma} \mathbf{F} = \iint_{\mathbf{S}_1} \operatorname{curl}(\mathbf{F}).$$

Now

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos xz & x^3 e^{yz} & -e^{-xyz} \end{vmatrix}.$$

Since, however, $\mathbf{S}_1$ projects onto smooth curves, which have zero surface area in $\mathbb{R}^2_{(y,z)}$ and $\mathbb{R}^2_{(x,z)}$, it suffices to consider the final coordinate of curl $(\mathbf{F})$. This is

$$\frac{\partial}{\partial x}\left(x^3 e^{yz}\right) - \frac{\partial}{\partial y}\left(y^2 \cos xz\right) = 3x^2 e^{yz} - 2y \cos xz$$

and, since $z = 0$ on $\mathbf{S}_1$, we need only evaluate

$$\iint_{x^2+y^2\le 4} (3x^2 - 2y)\,dxdy.$$

By symmetry

$$\iint_{x^2+y^2\le 4} (-2y)\,dxdy = 0.$$

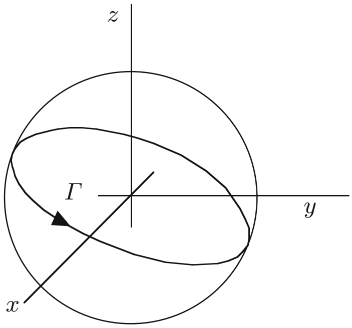

Fig. 13.3

If we use the parametrization

$$(r,\theta) \longrightarrow (r\cos\theta, r\sin\theta, 0), \quad 0 < r < 2,\ 0 < \theta < 2\pi$$

then

$$\iint_{\mathbf{S}_1} \mathbf{F} = \int_0^2 \int_0^{2\pi} 3r^2\cos^2\theta r dr d\theta = \int_0^{2\pi} \cos^2\theta d\theta \int_0^2 3r^3 dr = 12\pi.$$

Moral A closed curve may be the edge or boundary of more than one surface and a suitable choice (of surface) may simplify calculations. Projections and symmetry are helpful.

Example 13.4 We wish to use Stokes' theorem to find a suitable orientation of the curve of intersection of $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$, Γ, so that

$$\int_\Gamma y dx + z dy + x dz = \sqrt{3}\pi a^2.$$

The curve Γ is the intersection of a sphere and a plane through the centre of the sphere and hence is a "great circle" or "equator" on a sphere (see Fig. 13.3).

The curve Γ is the edge or boundary of the two hemispheres on either side of it and also of a portion of the plane $x + y + z = 0$. Which we use will depend on the function being integrated. Let $\mathbf{F}(x, y, z) = (y, z, x)$. We have

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1, -1, -1).$$

The plane $x + y + z = 0$ has unit normals $\pm(1, 1, 1)/\sqrt{3} = \mathbf{n}_1(x, y, z)$ while the sphere has unit normals $\pm(x, y, z)/a = \mathbf{n}_2(x, y, z)$. By Stokes' theorem

$$\int_\Gamma y dx + z dy + x dz = \iint_{(S,\mathbf{n})} (-1, -1, -1)$$

where S and $\mathbf{n}$ have to be chosen and Γ directed. If we take $\mathbf{S}$ as part of the plane with normal $(1, 1, 1)/\sqrt{3}$ then

$$\iint_{\mathbf{S}} (-1, -1, -1) = \iint_S (-1, -1, -1) \cdot \frac{(1, 1, 1)}{\sqrt{3}} dA = -\frac{3}{\sqrt{3}} \iint_S dA$$
$$= -\sqrt{3}\pi a^2$$

since S is a disc of radius a.

Since we obtained a negative answer we have been using the incorrect orientation on S and so take $(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$ as the normal which describes the orientation. Hence Γ is oriented as in Fig. 13.3, i.e. it appears clockwise when looked at from, say, the point $(1, 1, 1)$ or from any point sufficiently far out in the first octant.

Moral The curl of a vector field is a form of derivative. If the entries are linear, as in this example, the curl is constant. If the entries are of degree 2 then the curl has linear entries.

Example 13.5 In this example we verify Stokes' theorem for the portion S of the surface $z = \tan^{-1}(y/x)$ which lies inside the cone $x^2 + y^2 = z^2$ and between the planes $z = 0$ and $z = 2\pi$ by using the vector field

$$\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} - (x^2 + y^2)\mathbf{k}.$$

We first examine the surface $z = \tan^{-1}(y/x)$. This can be parametrized as a graph using Cartesian coordinates (see Example 10.3) but it is preferable to use polar coordinates.

Let $x = r\cos\theta$, $y = r\sin\theta$ then

$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta$$

and we obtain the parametrization

$$(r, \theta) \longrightarrow (r\cos\theta, r\sin\theta, \theta) \tag{13.2}$$

where $r > 0$ and $0 < \theta < 2\pi$.

What sort of surface is this? Well, if we fix different values of r and let θ vary we obtain a *helix* (see Example 5.2). In Fig. 13.4 we have sketched the surface for $1/2 \le r \le 1$. We are considering the portion of the surface in Fig. 13.4, extended

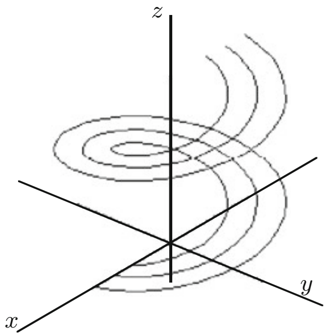

Fig. 13.4

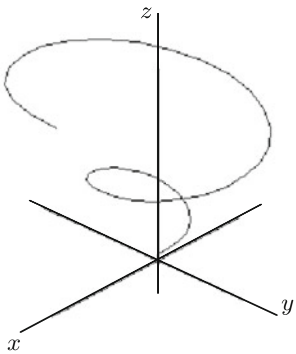

Fig. 13.5

over all r, which lies inside the cone and between the two planes. It is thus a *screw-shaped* surface or *spiral staircase* with the spirals or steps getting wider as we rise. The fact that the surface lies between the planes $z = 0$ and $z = \pi$ means that we have just one full twist of the screw or one turn of the staircase (Fig. 13.5).

We can use the parametrization (13.2) but the restriction caused by lying inside the other surfaces means that we must restrict the range. Translating the boundaries into polar coordinates, we get

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta = r^2 = z^2$$

and since $0 \leq z \leq 2\pi$ this implies $r = z$, $0 \leq r \leq 2\pi$. Hence our parametrization of the surface is

$$f : (r, \theta) \longrightarrow (r \cos \theta, r \sin \theta, \theta), \quad 0 < r < \theta, \; 0 < \theta < 2\pi.$$

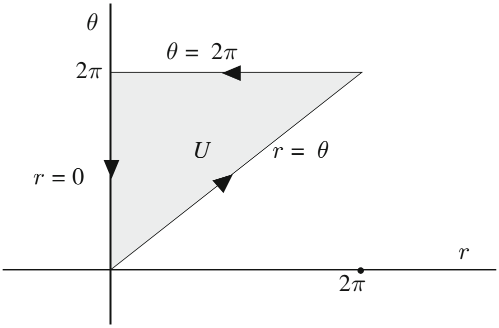

Fig. 13.6

We have $\mathbf{F}(f(r,\theta)) = (r\theta\cos\theta, r\theta\sin\theta, -r^2)$ and

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathrm{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & yz & -(x^2+y^2) \end{vmatrix} = -3y\mathbf{i} + 3x\mathbf{j} = (-3r\sin\theta, 3r\cos\theta, 0).$$

We define our orientation on **S** by

$$f_r \times f_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = (\sin\theta, -\cos\theta, r).$$

Hence

$$\iint_{\mathbf{S}} \text{curl}(\mathbf{F}) = \int_{\substack{0 \\ (\theta)}}^{2\pi} \int_{\substack{0 \\ (r)}}^{\theta} \langle \text{curl}(\mathbf{F}), f_r \times f_\theta \rangle dr d\theta$$
$$= \int_0^{2\pi} \left(\int_0^{\theta} -3r dr \right) d\theta = \int_0^{2\pi} -\frac{3r^2}{2} \Big|_0^{\theta} d\theta$$
$$= -\frac{3}{2} \int_0^{2\pi} \theta^2 d\theta = -\frac{3}{2}\frac{\theta^3}{3} \Big|_0^{2\pi}$$
$$= -\frac{(2\pi)^3}{2} = -4\pi^3.$$

The parametrization of S is over the set U in $\mathbb{R}^2$ given in Fig. 13.6 and the boundary or edge of the surface can be found by examining

$$f(r,\theta) = (r\cos\theta, r\sin\theta, \theta) \tag{13.3}$$

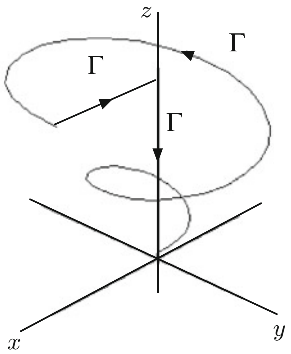

Fig. 13.7

on the boundary of U. We can also look at it geometrically by examining where the boundary curves intersect S. We first look at the curve of intersection of the cone and the screw. The cone is parametrized (Examples 10.4 and 10.5) by

$$(r, \theta) \longrightarrow (r\cos\theta, r\sin\theta, \theta)$$

and comparing this with (13.3) we get a curve of intersection Γ_1 when $r = \theta$. This curve is parametrized by

$$\theta \longrightarrow (\theta\cos\theta, \theta\sin\theta, \theta), \quad 0 \leq \theta \leq 2\pi.$$

The curve Γ_1 joins the origin to $P_1 = (2\pi, 0, 2\pi)$. The second curve Γ_2 is obtained by putting $\theta = 2\pi$ and from (13.3) this is parametrized by

$$r \longrightarrow (r\cos 2\pi, r\sin 2\pi, 2\pi) = (r, 0, 2\pi)$$

where $0 < r < 2\pi$. This is the straight line joining P_1 to $(0, 0, 2\pi)$. From Fig. 13.6 it runs in the *negative direction* and so we must reverse the orientation.

The third curve Γ_3 is obtained by letting $r = 0$ in (13.3) and we have a parametrization

$$\theta \longrightarrow (0, 0, \theta), \quad 0 \leq \theta \leq 2\pi.$$

Γ_3 joins $(0, 0, 2\pi)$ to the origin and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is a closed piecewise smooth directed curve (Fig. 13.7).

We have oriented the surface and directed its boundary. Are these consistent in order to apply Stokes' theorem? Yes, because the parametrization f of Γ is obtained from a *continuous extension* of a consistent parametrization of the surface. You have already seen two other ways in which it is possible to check this consistency. We now have to evaluate

$$\int_{\Gamma} \mathbf{F} = \int_{\Gamma_1} \mathbf{F} + \int_{\Gamma_2} \mathbf{F} + \int_{\Gamma_3} \mathbf{F}.$$

We have

$$\begin{aligned}
\int_{\Gamma_1} \mathbf{F} &= \int_0^{2\pi} \Big(\frac{d}{d\theta}(\theta\cos\theta, \theta\sin\theta, \theta)\Big)\cdot(\theta^2\cos\theta, \theta^2\sin\theta, -\theta^2)\,d\theta \\
&= \int_0^{2\pi} (\theta^2\cos^2\theta - \theta^3\sin\theta\cos\theta + \theta^2\sin^2\theta + \theta^3\sin\theta\cos\theta - \theta^2)\,d\theta \\
&= \int_0^{2\pi} 0\,d\theta = 0 \\
\int_{\Gamma_2} \mathbf{F} &= \int_{2\pi}^{0} \Big(\frac{d}{dr}(r, 0, 2\pi)\Big)\cdot(2\pi r, 0, -r^2)dr \\
&= \int_{2\pi}^{0} 2\pi r\,dr = \frac{2\pi r^2}{2}\Big|_{2\pi}^{0} = -\pi(2\pi)^2 = -4\pi^3 \\
\int_{\Gamma_3} \mathbf{F} &= \int_{2\pi}^{0} \Big(\frac{d}{d\theta}(0, 0, \theta)\Big)\cdot(0, 0, 0)d\theta = 0.
\end{aligned}$$

Hence

$$\int_{\Gamma} \mathbf{F} = -4\pi^3 = \iint_{\mathbf{S}} \mathrm{curl}(\mathbf{F})$$

and we have verified Stokes' theorem.

Moral It is possible to verify Stokes' theorem.

Exercises

13.1 Let **S** denote the portion of the sphere $x^2 + y^2 + z^2 = 4a^2$ in the first octant which lies inside the cylinder $x^2 + y^2 = 2ax$ oriented outwards. Let Γ denote the boundary or edge of **S** directed in accordance with Stokes' theorem. Sketch **S** and Γ. Using Stokes' theorem evaluate

(a) $\int_\Gamma zdx - xdz$,

(b) $\int_\Gamma xdy - ydx$,

(c) $\int_\Gamma ydz - zdy$.

13.2 Sketch the surfaces $az = xy$ and $x^2 + y^2 = b^2$ in $\mathbb{R}^3$. Show that

$$\theta \longrightarrow (b\cos\theta, b\sin\theta, \frac{b^2\sin 2\theta}{2a}), \quad 0 \le \theta \le 2\pi$$

is a parametrization of the intersection Γ of the two surfaces oriented clockwise when viewed from a high point on the z-axis. Show that the surface parametrized by

$$P\colon (r,\theta) \longrightarrow (r\cos\theta, r\sin\theta, \frac{r^2\sin 2\theta}{2a}), \quad 0 < r < b,\ 0 < \theta < 2\pi$$

has Γ as its edge or boundary. Using Stokes' theorem find

$$\int_\Gamma y\,dx + z\,dy + x\,dz.$$

13.3 Let Γ denote the curve of intersection of $x + y = 2b$ and $x^2 + y^2 + z^2 = 2b(x + y)$ oriented in a clockwise sense when viewed from the origin. Sketch the appropriate diagram. Using Stokes' theorem evaluate

$$\int_\Gamma y\,dx + z\,dy + x\,dz.$$

13.4 Let $0 < a < b$ and let S denote the torus obtained by rotating the circle $(x-b)^2 + z^2 = a^2$ in the xz-plane about the z-axis. Sketch S. Let Γ denote the directed curve parametrized by

$$t \longrightarrow \big((b + a\cos nt)\cos t, (b + a\cos nt)\sin t, a\sin nt\big).$$

Show that Γ is a closed curve in S. Describe and sketch S. Let $\mathbf{S}$ denote the surface parametrized and oriented by

$$P\colon (r,t) \longrightarrow \big(r(b + a\cos nt)\cos t, r(b + a\cos nt)\sin t, a\sin nt\big)$$

where $0 \le r < 1,\ 0 < t < 2\pi$. Show that Γ is the boundary of $\mathbf{S}$. Let $\mathbf{F}(x,y,z) = (-y, x, 0)$. By computing both $\int_\Gamma \mathbf{F}$ and $\int_{\mathbf{S}} \mathrm{curl}(\mathbf{F})$ verify Stokes' theorem. Show that the area of the projection of $\mathbf{S}$ onto $\mathbb{R}^2_{(x,y)}$ is $\pi(a^2+2b^2)/2$.

13.5 Let $\mathbf{S}$ denote the unit sphere oriented outwards. For $0 < b < c < 1$ let $\mathbf{S}_{b,c}$ denote the part of the sphere between the planes $z = b$ and $z = c$. Let $\mathbf{F}(X) = X\|X\|^{-3}$ for all $X \neq 0$ in $\mathbb{R}^3$. By using the result in Exercise 6.3 and Stokes Theorem find $\iint_{\mathbf{S}_{b,c}} \mathbf{F}$.

13.6 Let Γ denote the curve of intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x/a + z/b = 1$, $a > 0$, $b > 0$. Use Stokes' theorem to find a direction along Γ so that $\int_\Gamma (y-z, z-x, x-y)$ is positive. Find the value of this positive number.

13.7 Use Stokes' theorem to find a suitable orientation of the curve of intersection Γ of the hemisphere $x^2+y^2+z^2 = 2ax$, $z > 0$, and the cylinder $x^2+y^2 = 2bx$, $0 < b < a$, so that

$$\int_\Gamma (y^2 + z^2)dx + (x^2 + z^2)dy + (x^2 + y^2)dz = 2\pi ab^2.$$