Chapter 11
Surface Area

Summary *We define and calculate surface area.*

We follow the method used in Chap. 5 to calculate the length of a curve in order to define the (surface) area of a simple surface S in $\mathbb{R}^3$. Let $\phi\colon U \to S$ denote a parametrization of S where U is an open subset of $\mathbb{R}^2$. We take a rectangular partition of U (Fig. 11.1), find the approximate area of the image of each rectangle in the partition, form a Riemann sum and obtain the surface area as the limit of the Riemann sums.

If R denotes a typical rectangle in U (Fig. 11.2) then

$$\phi(x + \Delta x, y) - \phi(x, y) \approx \phi_x(x, y)\Delta x$$

and

$$\phi(x, y + \Delta y) - \phi(x, y) \approx \phi_y(x, y)\Delta y.$$

If $\theta = \theta(x, y)$ is the *angle* between $\phi_x(x, y)$ and $\phi_y(x, y)$ then using the well-known formula for the *area* of a triangle, $\frac{1}{2}ab\sin C$, we get

$$\begin{aligned}\text{Area}\big(\phi(R)\big) &\approx 2 \cdot \frac{1}{2}\|\phi_x(x, y)\| \cdot \|\phi_y(x, y)\| \cdot |\sin\theta(x, y)|\Delta x\Delta y\\ &= \|\phi_x \times \phi_y(x, y)\|\Delta x\Delta y.\end{aligned}$$

If ϕ is integrable over U, and this will be the case if, for instance U is bounded, with smooth boundary, and ϕ has a continuous extension to $\overline{U}$, then

$$\begin{aligned}\sum_{i,j}\text{Area}\big(\phi(R_{ij})\big) &\cong \sum_{i,j}\|\phi_x \times \phi_y(x_i, y_j)\|\Delta x_i\Delta y_j\\ &\longrightarrow \iint_U \|\phi_x \times \phi_y\|\, dxdy\end{aligned}$$

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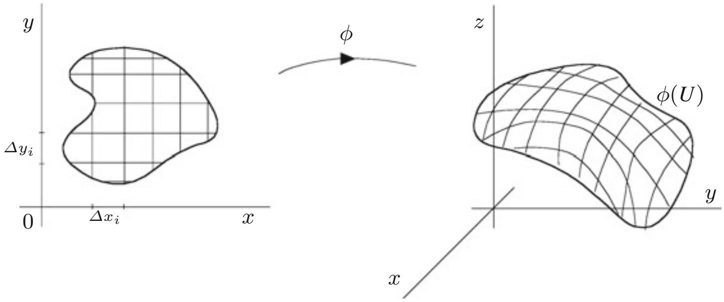

Fig. 11.1

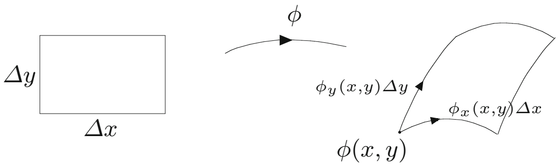

Fig. 11.2

as we take finer and finer partitions of U. Hence the *surface area* of S, $A(S)$, has the form

$$A(S) = \iint_U \|\phi_x \times \phi_y\| \, dxdy$$

and is calculated using a parametrization. In general, a surface will admit many different parametrizations but, as we will see later, they all give the same value for surface area. In this chapter we are using the usual *physical* idea of area and angle. These *non-negative absolute quantities* do not require a sense of direction or orientation on the surface and lead, as we have just seen, to a relatively straightforward form of integration. In the next chapter we require more sophisticated concepts of area and angle to integrate *vector fields* over a surface.

We now obtain another formula for surface area which avoids the cross product. We maintain the notation $\phi\colon U \to S$ for our parametrization and introduce, in their traditional form, three quantities that make regular and important appearances in the remaining chapters of this book. Let

$$E \text{ (or } E(x,y)) = \phi_x \cdot \phi_x = \|\phi_x\|^2$$
$$F \text{ (or } F(x,y)) = \phi_x \cdot \phi_y$$
$$G \text{ (or } G(x,y)) = \phi_y \cdot \phi_y = \|\phi_y\|^2.$$

We have

$$\|\phi_x \times \phi_y\|^2 = \|\phi_x\|^2 \|\phi_y\|^2 \sin^2 \delta$$

where δ is the angle between ϕ_x and ϕ_y. Hence

$$\cos\delta = \frac{\phi_x \cdot \phi_y}{\|\phi_x\| \cdot \|\phi_y\|}$$

and

$$\begin{aligned}\|\phi_x \times \phi_y\|^2 &= \|\phi_x\|^2\|\phi_y\|^2(1-\cos^2\delta)\\ &= \|\phi_x\|^2\|\phi_y\|^2\Big(1-\frac{(\phi_x\cdot\phi_y)^2}{\|\phi_x\|^2\|\phi_y\|^2}\Big)\\ &= \|\phi_x\|^2\|\phi_y\|^2-(\phi_x\cdot\phi_y)^2 = EG-F^2.\end{aligned}$$

This gives the following useful formula for surface area

$$A(S) = \iint_U \sqrt{EG-F^2}\,dxdy.$$

Figure 11.2 shows how E, F and G quantify the *distortion* of a rectangle by the parametrization. The stretching or contraction of the sides is measured by E and G while F measures the change in angle. Thus we see that *shape* is preserved if $E = G$ and $F = 0$ while (relative) *area* is preserved if $EG - F^2$ is constant. For many important parametrizations, including geographical and spherical polar coordinates, $F = 0$. This implies that angles between curves are preserved and, in particular, parallels (of latitude) and meridians (of longitude) cross one another at right angles. For geographical coordinates on a sphere of radius r, $E = r^2$ and $G = r^2\cos^2\theta$ and hence neither shape nor area are preserved. On the Equator, where $\theta = 0$, we have $E = G = r^2$ but as one moves towards the North and South Poles, $\theta \to \pm\pi/2$ and $G \to 0$ while $E = r^2$. Consequently, near the Equator shape is fairly well preserved but as one moves towards the polar regions it becomes more and more distorted. Mercator's projection is a *modification* of geographical coordinates—notice how the lines of latitude in Fig. 10.13 are not equally spaced in order to preserve shape. If the cylinder in Mercator's construction is replaced by a cone we obtain *Lambert's equal area projection* which preserves (relative) area but distorts shape and distance.

In Chap. 10 we noted that many well-known surfaces S could be written in the form $S = S_1 \cup \Gamma$ where S_1 is a simple surface and Γ is a curve parametrized by a *smooth* function on a closed interval. We see now that the surface area of Γ is zero and from this conclude that $A(S) = A(S_1)$ and thus we may use a parametrization of S_1 to calculate the surface area of S. To show $A(\Gamma) = 0$ enclose Γ in a finite union of rectangles of small width ε (Fig. 11.3).

The sum of the lengths of the rectangles is approximately the length of Γ, $l(\Gamma)$. Hence

$$A(\Gamma) \approx \varepsilon \times l(\Gamma).$$

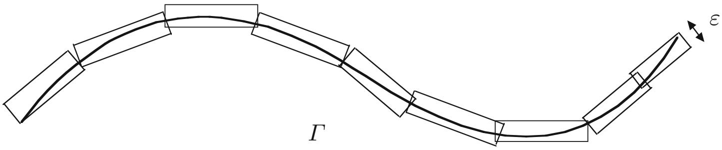

Fig. 11.3

Since ε is arbitrarily small this implies $A(\Gamma) = 0$. This intuitive "proof" can be developed into a rigorous proof using the *Mean Value Theorem* for differentiable functions of one variable. A differentiable parametrization of the curve is *necessary*, although our intuition might suggest otherwise, since there is a famous example of a *square filling curve* parametrized by a continuous function.

Example 11.1 In this example we consider the surface of revolution S parametrized as in Example 10.4. Let $P\colon [a, b] \to \mathbb{R}^2$ denote a parametrization of the curve to be revolved. Our parametrization ϕ of S is defined on $U = \{(t, \theta) : a < t < b,\ 0 < \theta < 2\pi\}$ and has the form

$$\phi(t, \theta) = \big(x(t), y(t)\cos\theta, y(t)\sin\theta\big)$$

where $P(t) = \big(x(t), y(t)\big)$. We have already shown in Example 10.4 that

$$\|\phi_t \times \phi_\theta\| = y(t)\|P'(t)\|$$

and so the surface area of S is

$$\int_0^{2\pi}\int_a^b y(t)\|P'(t)\|dtd\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_a^b y(t)\|P'(t)\|dt\right)$$
$$= 2\pi\int_a^b y(t)\|P'(t)\|dt.$$

For the *cone* of height h and (base) radius r, $P(t) = (ht, rt)$, $P'(t) = (h, r)$, $\|P'(t)\| = (h^2 + r^2)^{1/2}$ and the curved surface area is

$$2\pi\int_0^1 rt(r^2 + h^2)^{1/2}dt = 2\pi r(h^2 + r^2)^{1/2}\int_0^1 tdt = \pi r(h^2 + r^2)^{1/2}.$$

For the *cylinder* of radius r and height h we have $P(t) = (t, r)$, $P'(t) = (1, 0)$ and $\|P'(t)\| = 1$. Hence the curved surface area of the cylinder is

$$2\pi \int_0^h r \cdot 1 dt = 2\pi rh.$$

For the *sphere* of radius r, $P(t) = (r\cos t, r\sin t)$, $P'(t) = (-r\sin t, r\cos t)$ and $\|P'(t)\| = r$ and the surface area of the sphere of radius r is

$$2\pi \int_0^\pi r\sin t \cdot r dt = 2\pi r^2(-\cos t)\Big|_0^\pi = 4\pi r^2.$$

We now parametrize the *torus* by realising it as a surface of revolution. We adopt a slightly different approach to that in Example 10.4 in order to obtain what is regarded as the standard parametrization. Place a circle of radius r in the (y, z)-plane with centre on the y-axis at a distance b, $r > b$, from the origin and revolve this circle about the z-axis (Fig. 11.4).

The coordinates of a typical point on the original curve in the (x, z)-plane are $(b + r\cos\theta, r\sin\theta)$. When rotated about the z-axis the third coordinate remains unchanged while the first two coordinates describe a circle of radius $b + r\cos\theta$ about the origin. This gives a parametrization f with formula

$$f(\theta, \psi) = \big((b + r\cos\theta)\cos\psi, (b + r\cos\theta)\sin\psi, r\sin\theta\big)$$

and domain $(0, 2\pi) \times (0, 2\pi)$. Hence

$$\begin{aligned} f_\theta &= (-r\sin\theta\cos\psi, -r\sin\theta\sin\psi, r\cos\theta) \\ f_\psi &= \Big(-(b + r\cos\theta)\sin\psi, (b + r\cos\theta)\cos\psi, 0\Big). \end{aligned}$$

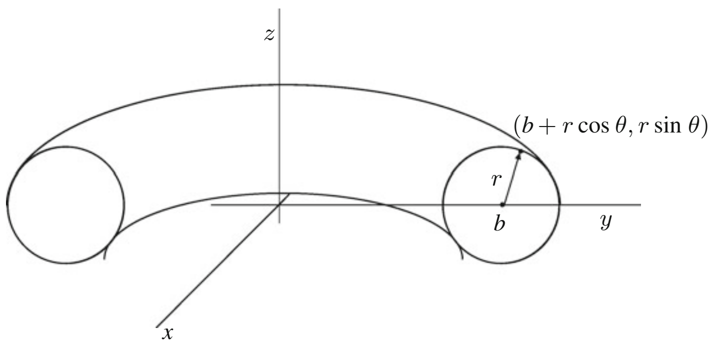

Fig. 11.4

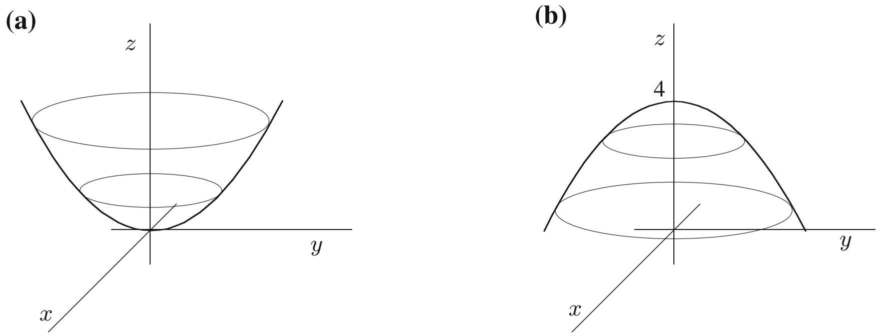

Fig. 11.5

By inspection we see that f_θ and f_ψ are perpendicular and hence $F = 0$. We have

$$\begin{aligned} E = \|f_\theta\|^2 &= r^2 \sin^2\theta\cos^2\psi + r^2\sin^2\theta\sin^2\psi + r^2\cos^2\theta \\ &= r^2\sin^2\theta(\cos^2\psi + \sin^2\psi) + r^2\cos^2\theta \\ &= r^2\sin^2\theta + r^2\cos^2\theta = r^2 \end{aligned}$$

and

$$\begin{aligned} G = \|f_\psi\|^2 &= (b + r\cos\theta)^2\sin^2\psi + (b + r\cos\theta)^2\cos^2\psi \\ &= (b + r\cos\theta)^2. \end{aligned}$$

Hence, $\sqrt{EG - F^2} = r(b + r\cos\theta)$, and

$$\begin{aligned} A(\text{Torus}) &= \int_0^{2\pi}\int_0^{2\pi} \sqrt{EG - F^2}\,d\theta\, d\psi \\ &= \int_0^{2\pi}\int_0^{2\pi} r(b + r\cos\theta)\,d\theta\, d\psi \\ &= \left(\int_0^{2\pi} d\psi\right)\left(\int_0^{2\pi} r(b + r\cos\theta)\,d\theta\right) \\ &= 4\pi^2 rb. \end{aligned}$$

Some geometric insight might have led you to this answer without *any* calculations or integration (see Example 14.5). The coordinates (θ, ψ) for the torus defined above are called *toroidal polar coordinates*.

Example 11.2 We wish to find the surface area of the portion of the *paraboloid* $z = 4 - x^2 - y^2$ that lies above the xy-plane. This is the graph of the function $f(x, y) = 4 - x^2 - y^2$ and we could find a parametrization using the method in Example 10.3. We prefer, however, to take a more geometric approach.

The standard paraboloid $z = x^2 + y^2$ can be sketched by noting that cross-sections parallel to the xy-plane are circles (Fig. 11.5a). This surface is turned upside down by taking $z = -x^2 - y^2$ and then moved up 4 units in the direction of the z-axis to give us the original surface $z = 4 - x^2 - y^2$ (Fig. 11.5b).

The surface cuts the xy-plane when $0 = z = 4 - x^2 - y^2$, i.e. on the circle $x^2 + y^2 = 4$. Clearly the geometry shows that we can project the surface in a one-to-one fashion onto the disc $x^2 + y^2 \leq 4$ and if we reverse this we obtain the parametrization

$$(x, y) \in \{x^2 + y^2 < 4\} \longrightarrow (x, y, 4 - x^2 - y^2) \in \text{ Paraboloid.}$$

We could proceed directly to use this parametrization to compute the surface area. This would require a change of variable in working out the double integral. It is just as easy to initially use a more appropriate parametrization which avoids a later change of variable.

We note that the domain of the above parametrization is a *disc* and the formula for the parametrization function involves $x^2 + y^2$. Either of these on their own suggest polar or spherical coordinates. Since, however, the disc shape and $x^2 + y^2$ only involve the first two coordinates this inclines us towards polar coordinates for the pair (x, y). We use the parametrization

$$f\colon (r, \theta) \longrightarrow (r\cos\theta, r\sin\theta, 4 - r^2), \quad 0 < r < 2 \text{ and } 0 < \theta < 2\pi.$$

We have $f_r = (\cos\theta, \sin\theta, -2r)$, $f_\theta = (-r\sin\theta, r\cos\theta, 0)$. Hence $E = 1 + 4r^2$, $G = r^2$ and since $f_r \cdot f_\theta = 0$ we have $F = 0$. The surface area is

$$\begin{aligned}
\int_0^2 \int_0^{2\pi} \sqrt{EG - F^2}\,drd\theta &= \int_0^2 \int_0^{2\pi} r\sqrt{1 + 4r^2}\,drd\theta \\
&= 2\pi \int_0^2 r\sqrt{1 + 4r^2}\,dr \\
&= \frac{2\pi}{8} \int u^{1/2}du, \qquad 1 + 4r^2 = u,\ 8rdr = du \\
&= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} = \frac{\pi}{6}(1 + 4r^2)^{3/2} \Big|_0^2 \\
&= \frac{\pi}{6}(17^{3/2} - 1).
\end{aligned}$$

Example 11.3 We wish to find in this example the surface area of that portion of the sphere $x^2 + y^2 + z^2 = 4z$ which lies outside the paraboloid $x^2 + y^2 = 3z$.

For this problem we first get some idea of the geometry and see, once this is reasonably clear, that the parametrization, the domain of the parametrization and even the integration are relatively straightforward.

We have two surfaces, a sphere and a paraboloid. Rewriting the equation for the sphere by completing the square we get

$$x^2 + y^2 + z^2 - 4z + 4 = 4$$

i.e.

$$x^2 + y^2 + (z - 2)^2 = 2^2$$

and this is the sphere of radius 2 with centre (0, 0, 2) (Fig. 11.6).

If we take the cross-section of the paraboloid corresponding to the plane $z = c$ we get a circle of radius $\sqrt{3c}$ and so we can sketch the paraboloid (Fig. 11.7). The points of intersection of the two surfaces are identified by letting

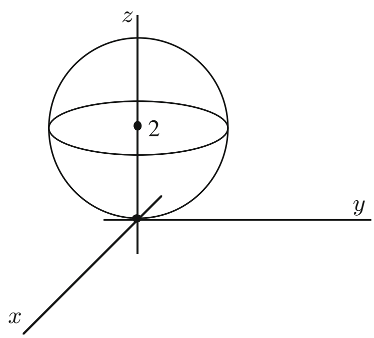

Fig. 11.6

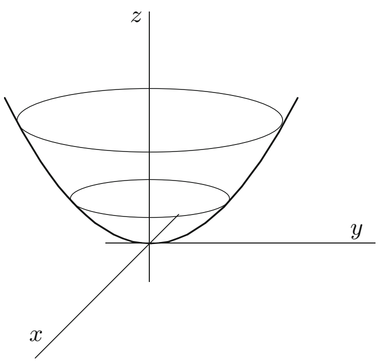

Fig. 11.7

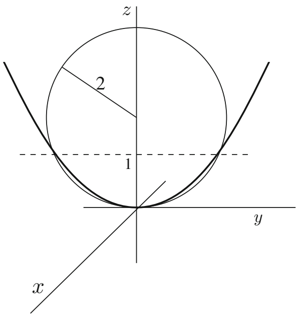

Fig. 11.8

$$x^2 + y^2 + z^2 - 4z = 0 = x^2 + y^2 - 3z.$$

This implies $z^2 - 4z = -3z$, i.e. $z^2 - z = 0$ and $z = 1$ or $z = 0$. When $z = 0$ we get the point $(0, 0, 0)$ on both surfaces and when $z = 1$ we get the curve $x^2 + y^2 = 3$, i.e. a circle of radius $\sqrt{3}$. This usually indicates where one surface crosses over from the inside to the outside of the other and gives us the working diagram (Fig. 11.8).

To find when the sphere is outside the paraboloid it suffices to compare the points (x, y, z_p) and (x, y, z_s) on the paraboloid and sphere, respectively, which project onto the *same* point $(x, y, 0)$ in the (x, y)-plane. From the defining equations of the surfaces we have

$$x^2 + y^2 = 4z_s - z_s^2 = 3z_p.$$

Hence

$$z_s - z_s^2 = 3z_p - 3z_s, \quad \text{i.e.} \quad z_s(1 - z_s) = 3(z_p - z_s).$$

Since $z_s \geq 0$ we have $z_p \geq z_s$ if and only if $1 - z_s \geq 0$, i.e. when $0 \leq z_s \leq 1$. Thus the part of the sphere which lies outside the paraboloid is precisely that portion which lies between the parallel planes $z = 0$ and $z = 1$.

We thus have to find the surface area of the sphere $x^2 + y^2 + (z - 2)^2 = 2^2$ between the planes $z = 0$ and $z = 1$. We use the formula for spherical polar coordinates obtained in Example 10.5 translated so that the origin is at the point $(0, 0, 2)$, i.e. let

$$x = 2\sin\theta\cos\psi, \quad y = 2\sin\theta\sin\psi, \quad z - 2 = 2\cos\theta$$

and it is now only necessary to find the domain of the parametrization. Since $0 \leq z \leq 1$ we consider $-2 \leq z - 2 \leq -1$, i.e. $-2 \leq 2\cos\theta \leq -1$. This implies (see Fig. 10.14) $2\pi/3 \leq \theta \leq \pi$ and, since there is no restriction on ψ, $0 < \psi < 2\pi$.

Denote this parametrization by f. Then

$$f_\theta = (2\cos\theta\cos\psi, 2\cos\theta\sin\psi, -2\sin\theta)$$

and

$$f_\psi = (-2\sin\theta\sin\psi, 2\sin\theta\cos\psi, 0).$$

Hence

$$\begin{aligned} E &= 4\cos^2\theta\cos^2\psi + 4\cos^2\theta\sin^2\psi + 4\sin^2\theta \\ &= 4(\cos^2\psi + \sin^2\psi)\cos^2\theta + 4\sin^2\theta \\ &= 4\cos^2\theta + 4\sin^2\theta = 4 \end{aligned}$$

and

$$G = 4\sin^2\theta\sin^2\psi + 4\sin^2\theta\cos^2\psi = 4\sin^2\theta.$$

Since $f_\theta \cdot f_\psi = 0$, $F = 0$ and the surface area is

$$\begin{aligned} \int_0^{2\pi}\int_{2\pi/3}^{\pi} 2\cdot 2\sin\theta d\theta d\psi &= \Big(4\int_0^{2\pi} d\psi\Big)\Big(\int_{2\pi/3}^{\pi} \sin\theta d\theta\Big) \\ &= 8\pi(-\cos\theta)\Big|_{2\pi/3}^{\pi} = 8\pi(1-1/2) = 4\pi. \end{aligned}$$

Example 11.4 We calculate the surface area of the level set $x^2 - 2y - 2z = 0$ which projects onto the region U of the xy-plane bounded by the lines $x = 2$, $y = 0$ and $y = 4x$. On the surface S we have $z = \frac{1}{2}x^2 - y$ and the surface is the graph of the function $g(x, y) = \frac{1}{2}x^2 - y$. Using Example 10.3 we obtain the parametrization

$$\phi\colon (x, y) \in U \longrightarrow (x, y, \frac{1}{2}x^2 - y)$$

and, moreover,

$$\|\phi_x \times \phi_y\| = \|(-g_x, -g_y, 1)\| = \|(x, -1, 1)\| = (x^2 + 2)^{1/2}.$$

Note that only part of the surface lies above the xy-plane. Our surface area is

$$\iint_U (x^2 + 2)^{1/2}\, dxdy.$$

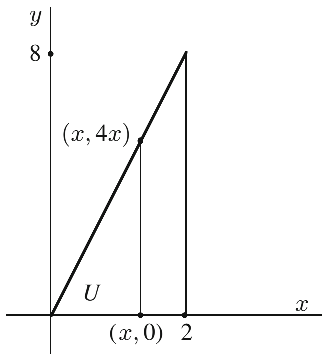

Fig. 11.9

We integrate first with respect to y and note, from Fig. 11.9, that y varies from 0 to $4x$ and afterwards integrate with respect to x which varies from 0 to 2. We have

$$\iint_U (x^2+2)^{1/2}\,dxdy = \int_0^2 \left\{ \int_0^{4x} (x^2+2)^{1/2} dy \right\} dx$$

$$= \int_0^2 (x^2+2)^{1/2} y \,\Big|_0^{4x}\, dx = \int_0^2 4x(x^2+2)^{1/2} dx$$

$$= 2\int u^{1/2} du = 2\frac{u^{3/2}}{3/2} = \frac{4}{3}(x^2+2)^{3/2}\,\Big|_0^2$$

$$= \frac{4}{3}\left(6^{3/2} - 2^{3/2}\right) = \frac{8}{3}\left(3\sqrt{6} - \sqrt{2}\right)$$

where $u = x^2 + 2$, $du = 2xdx$.

Exercises

11.1 Calculate the surface area of the paraboloid $z = x^2 + y^2$ which lies between the planes $z = 0$ and $z = 4$.

11.2 Find the area of the surface generated by revolving about the x-axis the curves

(a) $y = x^3, 0 \le x \le 1$,
(b) $y = x^2, 0 \le x \le 1$.

11.3 Show that the area of the *helicoid* defined by the parametrization

$$P(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \quad 0 < \theta < 2\pi, \ 0 < r < 1$$

is $\pi\left(\sqrt{2} + \log(1 + \sqrt{2})\right)$.

11.4 If U is open in $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ is a smooth function show

$$\text{Area}\left(\text{graph}(f)\right) = \iint_U \left(1 + \|\nabla f\|^2\right)^{1/2} dxdy.$$

11.5 Find the surface area of the paraboloid in Example 11.3 that lies inside the sphere.

11.6 Find the area of the portion of the surface $z = xy$ which lies inside the cylinder $x^2 + y^2 = a^2$.

11.7 Use the parametrization $(r, \theta) \to (r^2/16, r\cos\theta, r\sin\theta)$ to find the area of the part of $y^2 + z^2 = 16x$ which lies in the first octant ($x \geq 0, y \geq 0, z \geq 0$) between the planes $x = 0$ and $x = 12$ and inside the cylinder $y^2 = 4x$.

11.8 Let $\phi : U \to S$ denote a parametrization of the simple surface S in $\mathbb{R}^3$. Give some intuitive reasons why ϕ preserves

(a) angles if $F = 0$
(b) relative area if $EF - G^2$ is constant
(c) shape if $E = G$ and $F = 0$.

11.9 Use the Cauchy–Schwarz inequality (Example 3.4) to show that $EG - F^2 \geq 0$ where $E,\ F$ and G are calculated from any parametrization of a surface.

11.10 Let Γ denote a directed curve in a surface parametrized by a mapping ϕ. Show that Γ has a parametrization of the form $P : [a, b] \longrightarrow \Gamma$, $P(t) = \phi(\alpha(t), \beta(t))$ where α and β are real valued functions. Show that

$$l(\Gamma) = \int_a^b \sqrt{E(P(t))\alpha'(t)^2 + 2F(P(t))\alpha'(t)\beta'(t) + G(P(t))\beta'(t)^2}\ dt.$$

(see Table 11.1)

Table 11.1 Useful parametrizations

1. Sphere $x^2+y^2+z^2=r^2$; **geographical coordinates**

$$f(\theta,\psi)=(r\cos\theta\cos\psi, r\cos\theta\sin\psi, r\sin\theta),\ -\pi/2<\theta<\pi/2,\ 0<\psi<2\pi$$

$$f_\theta=(-r\sin\theta\cos\psi, -r\sin\theta\sin\psi, r\cos\theta),\quad f_\psi=(-r\cos\theta\sin\psi, r\cos\theta\cos\psi, 0)$$

$$f_\theta\times f_\psi=-r^2\cos\theta(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)$$

$$E=r^2,\ F=0,\ G=r^2\cos^2\theta,\ \|f_\theta\times f_\psi\|=\sqrt{EG-F^2}=r^2\cos\theta$$

$$\mathbf{n}=-(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)$$

2. Sphere $x^2+y^2+z^2=r^2$; **spherical polar coordinates**

$$f(\theta,\psi)=(r\sin\theta\cos\psi, r\sin\theta\sin\psi, r\cos\theta),\quad 0<\theta<\pi,\ 0<\psi<2\pi$$

$$f_\theta=(r\cos\theta\cos\psi, r\cos\theta\sin\psi, -r\sin\theta),\quad f_\psi=(-r\sin\theta\sin\psi, r\sin\theta\cos\psi, 0)$$

$$f_\theta\times f_\psi=r^2\sin\theta(\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$$

$$E=r^2,\ F=0,\ G=r^2\sin^2\theta,\ \|f_\theta\times f_\psi\|=\sqrt{EG-F^2}=r^2\sin\theta$$

$$\mathbf{n}=(\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$$

3. Cylinder $x^2+y^2=r^2,\ 0<z<h$; **cylindrical polar coordinates**

$$f(\theta,z)=(r\cos\theta, r\sin\theta, z),\quad 0<\theta<2\pi,\ 0<z<h$$

$$f_\theta=(-r\sin\theta, r\cos\theta, 0),\quad f_z=(0,0,1)$$

$$f_\theta\times f_z=(r\cos\theta, r\sin\theta, 0)$$

$$E=r^2,\ F=0,\ G=1,\ \|f_\theta\times f_z\|=\sqrt{EG-F^2}=r$$

$$\mathbf{n}=(\cos\theta, \sin\theta, 0)$$

4. Inverted cone $x^2+y^2=z^2,\ 0<z<a$

$$f(r,\theta)=(r\cos\theta, r\sin\theta, r),\quad 0<r<a,\ 0<\theta<2\pi$$

$$f_r=(\cos\theta, \sin\theta, 1),\quad f_\theta=(-r\sin\theta, r\cos\theta, 0)$$

$$f_r\times f_\theta=(-r\cos\theta, -r\sin\theta, r)$$

$$E=2,\ F=0,\ G=r^2,\ \|f_r\times f_\theta\|=\sqrt{EG-F^2}=\sqrt{2}r$$

$$\mathbf{n}=-\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, -1)$$

(Continued)

Table 11.1 (Continued)

5. Torus—rotate circle of radius r in yz-plane, centre $(b, 0)$, $b > r$, about z-axis; **toroidal polar coordinates**

$$f(\theta, \psi) = \big((b + r\cos\theta)\cos\psi, (b + r\cos\theta)\sin\psi, r\sin\theta\big), \quad 0 < \theta, \psi < 2\pi$$

$$f_\theta = (-r\sin\theta\cos\psi, -r\sin\theta\sin\psi, r\cos\theta),$$

$$f_\psi = \big(-(b + r\cos\theta)\sin\psi, (b + r\cos\theta)\cos\psi, 0\big)$$

$$f_\theta \times f_\psi = -r(b + r\cos\theta)(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)$$

$$E = r^2, \; F = 0, \; G = (b + r\cos\theta)^2, \|f_\theta \times f_\psi\| = \sqrt{EG - F^2} = r(b + r\cos\theta)$$

$$\mathbf{n} = -(\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta)$$

6. Graph of $f : U \subset \mathbb{R}^2 \to \mathbb{R}$

$$F(x, y) = (x, y, f(x, y)), \; (x, y) \in U$$

$$F_x = (1, 0, f_x), \; F_y = (0, 1, f_y), \; F_x \times F_y = (-f_x, -f_y, 1)$$

$$E = 1 + f_x^2, \; F = f_x f_y, \; G = 1 + f_y^2, \; \|F_x \times F_y\| = \sqrt{EG - F^2} = \sqrt{1 + f_x^2 + f_y^2}$$

$$\mathbf{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$$

7. Surface of revolution of Γ parametrized by $P(t) = ((x(t), y(t)), \; a \le t \le b$, $y(t) > 0$ for all t, $\|P'(t)\| \ne 0$

$$f(t, \theta) = \big(x(t), y(t)\cos\theta, y(t)\sin\theta\big), \quad a < t < b, \; 0 < \theta < 2\pi$$

$$f_t = \big(x'(t), y'(t)\cos\theta, y'(t)\sin\theta\big), \quad f_\theta = \big(0, -y(t)\sin\theta, y(t)\cos\theta\big)$$

$$f_t \times f_\theta = \big(y'(t)y(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta\big)$$

$$E = \|P'(t)\|^2, \; F = 0, \; G = y(t)^2, \; \|f_t \times f_\theta\| = \sqrt{EG - F^2} = y(t)\|P'(t)\|$$

$$\mathbf{n} = \tfrac{1}{\|P'(t)\|}\big(y'(t), -x'(t)\cos\theta, -x'(t)\sin\theta\big)$$