Chapter 12
Surface Integrals

Summary *We define the integral of a vector field over an oriented surface. Geometrical interpretations are discussed.*

Integrals are used to measure quantities such as length, area, expected value, etc., and as with all measurements, there has to be a unit of measurement. Our basic unit of measurement in integration theory over $\mathbb{R}$ is obtained by assigning the value 1 to the rectangle of height 1 over an interval of length 1 measured from *left to right*. From this we are able to define Riemann sums and afterwards the Riemann integral of a continuous function over a closed interval. The inclusion of "left to right" is crucial for without it we would have an ambiguous definition—and to a mathematician an ambiguous definition is no definition. To emphasis this point we call an interval $[a, b]$ directed from left to right a *positive* interval in $\mathbb{R}$.

Now suppose Γ is a curve in $\mathbb{R}^2$ and, for the sake of simplicity, we suppose that Γ is not closed and that we are interested in defining the integral of a function over Γ. We have seen in Chap. 7 that we can define *two* integrals over Γ since Γ can be directed in two different ways and we have to decide, before doing any calculations, which interests us. We may think of a curve as having *two sides*—one for each direction of motion—like two escalators side by side, one going up and the other coming down (Fig. 12.1). We call one side positive and the other negative. If we are interested in evaluating integrals over $\{\Gamma, A, B\}$ we call Γ directed from A to B the *positive side*.

We use a parametrization to transfer the integral over $\{\Gamma, A, B\}$ to an integral over an interval in $\mathbb{R}$ which we subsequently evaluate. However, the parametrization must be directed correctly. This means that it must map a *positive* interval in $\mathbb{R}$ onto the *positive* side of Γ. Surface integrals are defined in the same way—just step up a dimension.

Consider the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. It has two *sides*—an inside and an outside. More generally, the surface S which is the level set $\{X \in U : f(X) = 0 \text{ and } \nabla f(X) \neq 0\}$ has $\pm\nabla f(X)\big/\|\nabla f(X)\|$ as unit normals at each point and we may consider $\nabla f(X)\big/\|\nabla f(X)\|$ and $-\nabla f(X)\big/\|\nabla f(X)\|$ as lying on different sides. We *distinguish* between the sides of a surface by *using normals*.

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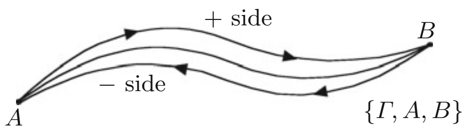

Fig. 12.1

Definition 12.1 An oriented surface in $\mathbb{R}^3$ is a pair $(S, \mathbf{n})$ where S is a surface and $\mathbf{n}$ is a continuous mapping from S into $\mathbb{R}^3$ such that $\mathbf{n}(P)$ is a unit normal to the surface S at P.

We will also use $\mathbf{S}$ to denote an oriented surface. The appearance of $\mathbf{S}$ signals that we have fixed an orientation on the underlying surface S. The notation $\mathbf{S}$ is simpler but if we need to specify the normal or if there is any possibility of confusion we write $(S, \mathbf{n})$.

Clearly if $(S, \mathbf{n})$ is an oriented surface, then $(S, -\mathbf{n})$ is also an oriented surface and the continuity requirement in the choice of normal implies that a *connected surface* can have at most *two orientations*. There exist, however, surfaces which do not admit *any* orientation and integration theory *cannot* be defined over such surfaces. Fortunately, every simple surface admits an orientation. If $\phi\colon U \subset \mathbb{R}^2 \to S$ is a parametrization of S then $\big(S, \phi_x \times \phi_y / \|\phi_x \times \phi_y\|\big)$ and $\big(S, -\phi_x \times \phi_y / \|\phi_x \times \phi_y\|\big)$ are two oriented surfaces associated with S and if S is connected these are the only two oriented surfaces associated with S. Given an oriented surface $(S, \mathbf{n})$ we call the side of S containing $\mathbf{n}$ the *positive side* and the side which contains $-\mathbf{n}$ the negative side. In the case of a simple oriented surface $(S, \mathbf{n})$ a parametrization ϕ is said to be *consistent* with the orientation if

$$\frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|} = \mathbf{n}.$$

If $\mathbf{F}$ is a continuous vector field on a simple oriented surface $\mathbf{S}$ we define

$$\iint_{\mathbf{S}} \mathbf{F}$$

by using a parametrization (ϕ, U) consistent with the orientation and a process involving Riemann sums similar to that used to define surface area in the previous chapter (see Figs. 11.2 and 11.3 for reference). We find it convenient to use (u, v) for the variables in the domain of ϕ and (x, y, z) for the range, i.e. $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$. We begin by partitioning the domain U into small rectangles and consider a typical rectangle R in this partition. Now, however, instead of taking the *absolute area* of $\phi(R)$ we take the *(signed) volume* of the *parallelepiped* with base $\phi(R)$ and height determined by $\mathbf{F}$ (see Fig. 12.2).

We use $\mathbf{n}$ as our *positive unit of measurement* perpendicular to the surface. This is reasonable since we are just considering a small portion of the surface $\phi(R)$, which

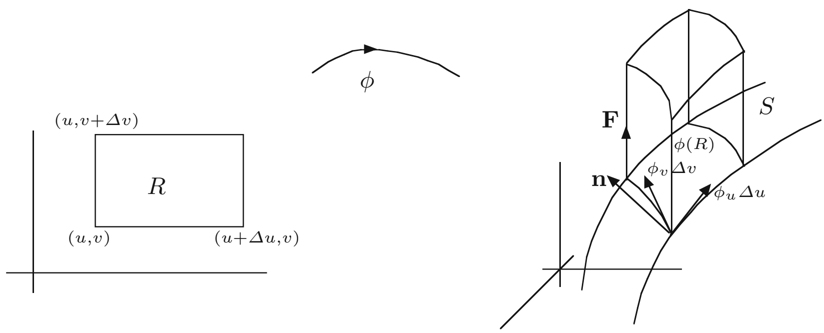

Fig. 12.2

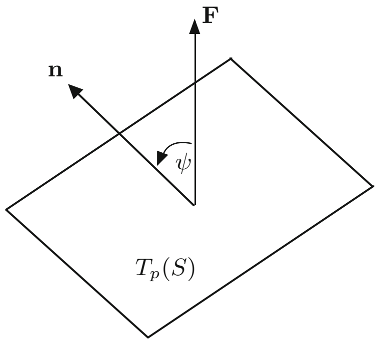

Fig. 12.3

lies approximately in the tangent plane, and $\mathbf{n}\big(\phi(u, v)\big)$ is perpendicular to the tangent plane at $\phi(u, v)$. The height of the parallelepiped is $\|\mathbf{F}\| \cos\psi = \mathbf{F} \cdot \mathbf{n}$ where ψ is the angle between $\mathbf{F}$ and $\mathbf{n}$ at $\phi(u, v)$ (see Fig. 12.3).

We have already seen, in calculating surface area, that the area of $\phi(R)$, the base of the parallelepiped, is $\|\phi_x \times \phi_y\|\Delta x \Delta y$ and hence the (signed) volume is

$$(\mathbf{F} \cdot \mathbf{n})\|\phi_u \times \phi_v\|\Delta u \Delta v = (\mathbf{F} \cdot \phi_u \times \phi_v)\Delta u \Delta v.$$

Now taking the limit of Riemann sums in the usual way we define

$$\begin{aligned}\iint\limits_{\mathbf{S}} \mathbf{F} &= \iint\limits_{U} (\mathbf{F} \cdot \mathbf{n})\|\phi_u \times \phi_v\| du dv \\ &= \iint\limits_{U} (\mathbf{F} \cdot \phi_u \times \phi_v) du dv.\end{aligned}$$

From this analysis we expect the volume of the solid one unit high on the positive side of the surface to equal the total surface area and to be obtained by integrating the vector field **n** over **S**. Indeed, we have

$$\iint_{(S,\mathbf{n})} \mathbf{n} = \iint_U (\mathbf{n}\cdot\mathbf{n})\|\phi_u \times \phi_v\| du dv = \text{Surface area } (S).$$

It is now natural to define the surface integral of a *scalar-valued function* f on $(S, \mathbf{n})$ by identifying f with $f\mathbf{n}$, i.e.

$$\iint_{(S,\mathbf{n})} f \stackrel{\text{def}}{=} \iint_{(S,\mathbf{n})} f\mathbf{n} = \iint_S f\big(\phi(u,v)\big)\|\phi_u \times \phi_v\| du dv$$

where ϕ is a parametrization consistent with the orientation. This is analogous to the way we defined, in Chap. 7, the integral of a scalar-valued function f along a curve by identifying f with fT. Although we needed an orientation to define this integral it is easily seen that

$$\iint_{(S,\mathbf{n})} f = \iint_{(S,-\mathbf{n})} f$$

and the value of the integral is independent of the orientation. This might suggest that it is possible to integrate a scalar-valued function over an *arbitrary* surface (see Definition 10.6). This is not true. What is required is an *orientable surface*, i.e. a surface which can be oriented. Simple surfaces are orientable. This means that we can use *any* parametrization to integrate a scalar-valued function over a simple surface.

If the vector field **F** is represented by vectors emanating from the surface and if these all lie on the *same* side (of the surface) as **n** then the integral of **F** over $(S, \mathbf{n})$ is non-negative. On the right-hand diagram in Fig. 12.2 we see that $\{\phi_u, \phi_v, \mathbf{n}\}$ follows the *right-hand rule* and note that **F** and **n** lie on the same side of $(S, \mathbf{n})$ if and only if the angle between them lies in $[-\pi/2, \pi/2]$.

Example 12.2 We evaluate $\iint_{(S,\mathbf{n})} \mathbf{F}$ where **S** is the sphere of radius r with centre at the origin oriented outwards and $\mathbf{F}(x, y, z) = (x, y, z)$. Although a sphere is not a simple surface we can treat it as one for integration theory (see our remarks in the previous chapter). Here, however, we do not need any parametrization. The unit normals at a point P on S are $\pm P/\|P\|$ and since S is oriented outwards

$$\mathbf{n}(P) = \frac{P}{\|P\|} = \frac{P}{r}.$$

Hence

$$\mathbf{F}(P)\cdot\mathbf{n}(P) = P\cdot\frac{P}{r} = \frac{\|P\|^2}{r} = \frac{r^2}{r} = r$$

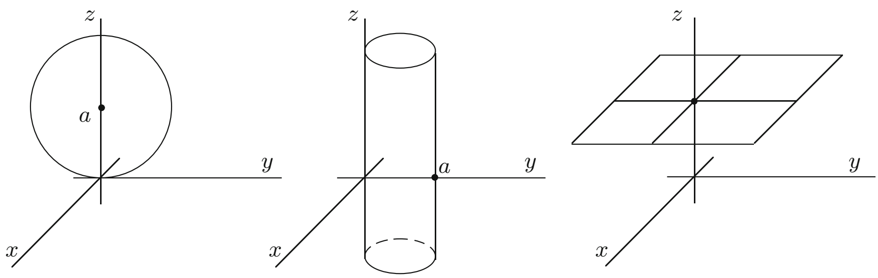

Fig. 12.4

and

$$\iint_{\mathbf{S}} \mathbf{F} = \iint_{S} r\,dA = r(\text{Surface Area of } S) = r \cdot 4\pi r^2 = 4\pi r^3.$$

Example 12.3 We compute

$$\iint_{\mathbf{S}} \frac{(x, y, z-a)}{\sqrt{2az - z^2}}$$

where **S** is the part of the surface $x^2 + y^2 + (z-a)^2 = a^2$ which lies inside the cylinder $x^2 + y^2 = ay$ and underneath the plane $z = a$ oriented with outward pointing normal.

We first sketch the surface over which we are integrating. Initially this appears as a rather formidable task but by sketching each part separately (Fig. 12.4) and then combining (Fig. 12.5) them it becomes relatively simple. The surface $x^2 + y^2 + (z-a)^2 = a^2$ is a sphere of radius a with centre $(0, 0, a)$. The equation $x^2 + y^2 = ay$ can be rewritten as

$$x^2 + y^2 - ay + \frac{a^2}{4} = \frac{a^2}{4}$$

i.e.

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$$

and this surface is the cylinder parallel to the z-axis above the circle with centre $(0, a/2)$ and radius $a/2$ in the xy-plane. The plane $z = a$ is parallel to the xy-plane and a units above it. The outward normal to the sphere of radius a with centre $(0, 0, a)$ at (x, y, z) is $(x, y, z-a)/a$. Hence

$$\iint_{\mathbf{S}} \frac{(x, y, z-a)}{\sqrt{2az - z^2}} = \iint_{S} \frac{(x, y, z-a)}{\sqrt{2az - z^2}} \cdot \frac{(x, y, z-a)}{a} dA$$

$$= \iint_{S} \frac{a}{\sqrt{2az - z^2}} dA.$$

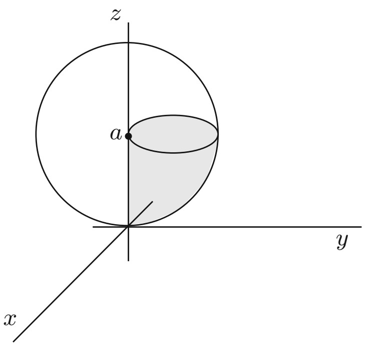

Fig. 12.5

From Fig. 12.5 we observe that our integral is over part of the lower right-hand quarter of the sphere. We use geographical coordinates about the point $(0, 0, a)$

$$F: (\theta, \psi) \longrightarrow (a\cos\theta\cos\psi, a\cos\theta\sin\psi, a + a\sin\theta).$$

Since θ refers to latitude and we are considering the lower portion we have $-\pi/2 < \theta < 0$. From the sketch $y > 0$ and hence $\sin\psi > 0$, i.e. $0 < \psi < \pi$, and since the surface lies inside the cylinder we also have $x^2 + y^2 < ay$, i.e. $a^2\cos^2\theta < a^2\cos\theta\sin\psi$. Hence $\cos\theta < \sin\psi = \cos(\psi - \frac{\pi}{2})$. By considering separately the cases $0 < \psi < \frac{\pi}{2}$ and $\frac{\pi}{2} < \psi < \pi$ we obtain $\frac{\pi}{2} + \theta < \psi < \frac{\pi}{2} - \theta$. Now

$$2az - z^2 = -a^2 + 2az - z^2 + a^2 = a^2 - (a - z)^2$$

and changing this into our new coordinates we get

$$\sqrt{2az - z^2} = (a^2 - a^2\sin^2\theta)^{1/2} = (a^2\cos^2\theta)^{1/2} = a\cos\theta.$$

From Example 10.5 we recall that

$$dA = \sqrt{EG - F^2}d\theta d\psi = a^2\cos\theta d\theta d\psi.$$

Hence

$$\iint\limits_{\mathbf{S}} \frac{(x, y, z - a)}{\sqrt{2az - z^2}} = \int_{-\pi/2}^{0} \left\{ \int_{\pi/2+\theta}^{\pi/2-\theta} \frac{a^2\cos\theta \cdot a}{a\cos\theta} d\psi \right\} d\theta = \pi^2 a^2/4.$$

An alternative approach to vector-valued integration is possible by using oriented (coordinate) planes in $\mathbb{R}^3$. In oriented planes we can define *positive* and *negative area* and an anticlockwise sense of direction. The idea is to use consistent

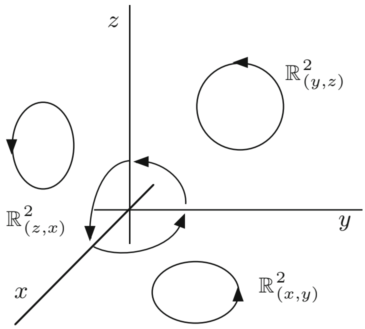

Fig. 12.6

parameterizations to transfer integration from an oriented surface in $\mathbb{R}^3$ to integration over subsets of the *coordinate planes* in $\mathbb{R}^3$, i.e. the xy-plane, the yz-plane and the xz-plane. Each of these planes is a surface in $\mathbb{R}^3$ and to define our integral we must assign a positive side to each of these planes for the same reason that we required positive intervals in $\mathbb{R}$.

We first define the *positive unit vectors* in the x, y and z directions in $\mathbb{R}^3$ as $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. These choices are natural in view of the way we sketch graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$. Consider now the xy-plane in $\mathbb{R}^3$ as the surface defined by $f^{-1}(0)$ where $f(x, y, z) = z$. The unit normals to this surface are $\pm(0, 0, 1)$ and we must choose between them to define a positive side to the xy-plane. We use the positive unit vectors in the x and y directions *in that order* and the *right-hand rule* or *cross product* to define the *positive side* of the xy-plane as that oriented by the normal

$$(1, 0, 0) \times (0, 1, 0) = (0, 0, 1).$$

We denote the positively oriented xy-plane in $\mathbb{R}^3$ by $\mathbb{R}^2_{(x,y)}$. Similarly the positive side of the yz-plane is defined by the normal $(0, 1, 0) \times (0, 0, 1) = (1, 0, 0)$ and denoted by $\mathbb{R}^2_{(y,z)}$ and the positive side of the zx-plane is defined by $(0, 0, 1) \times (1, 0, 0) = (0, 1, 0)$ and denoted by $\mathbb{R}^2_{(z,x)}$.

In Fig. 12.6 we see the direction of rotation (i.e. rotation to the *left*) used to measure angles in each of the coordinate planes and also the direction of a closed anticlockwise oriented curve.

If we look at each individual coordinate plane we get, using Fig. 12.6, the diagrams in Fig. 12.7 showing the anticlockwise directions.

In Chap. 9 we discussed Green's theorem

$$\oint_\Gamma Pdx + Qdy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy \tag{12.1}$$

where the boundary of Ω, Γ is oriented in an *anticlockwise* direction. From (12.1) it does not appear that Ω has been assigned any orientation but, in fact, if we identify

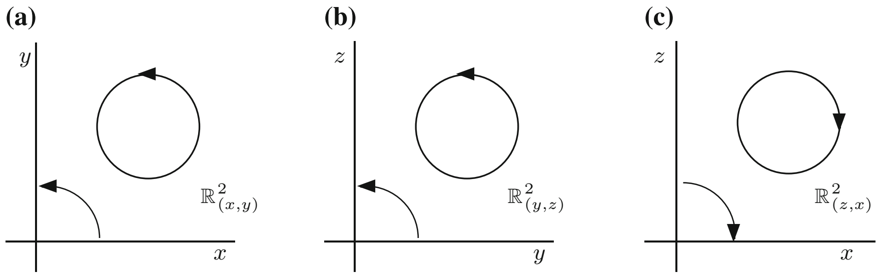

Fig. 12.7

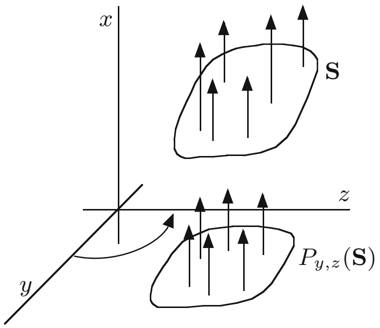

Fig. 12.8

$\mathbb{R}^2$ with $\mathbb{R}^2_{(x,y)} \subset \mathbb{R}^3$ then the integration on $\mathbb{R}^2$ is really over the *positive side* of $\mathbb{R}^2_{(x,y)}$. This is the matching up of orientations in Green's theorem.

Let $\mathbf{F} = (f, g, h)$ denote a continuous vector field on the simple oriented surface **S**. Let

$$\phi\colon (u, v) \in U \longrightarrow \big(x(u, v), y(u, v), z(u, v)\big)$$

denote a parametrization consistent with the orientation. We now take an independent approach to defining separately $\iint\limits_{\mathbf{S}} (f, 0, 0)$, $\iint\limits_{\mathbf{S}} (0, g, 0)$ and $\iint\limits_{\mathbf{S}} (0, 0, h)$.

First observe that $(f, 0, 0)$ is parallel to the x-axis and may be represented as in Fig. 12.8, where we have rotated the axes but preserved the orientation.

Since f points in the direction of the normal $(1, 0, 0)$ to $\mathbb{R}^2_{(y,z)}$ it is natural to project onto $\mathbb{R}^2_{(y,z)}$ and to define $\iint\limits_{\mathbf{S}} (f, 0, 0)$ as $\iint\limits_{P_{y,z}(\mathbf{S})\subset\mathbb{R}^2_{(y,z)}} f$ where $P_{y,z}$ is the projection $(x, y, z) \to (y, z)$ and $P_{y,z}(\mathbf{S})$ is given the induced orientation from $\mathbb{R}^2_{(y,z)}$. Although the set $P_{y,z}(\mathbf{S})$ may not be open and the mapping

$$P_{y,z}\colon (u, v) \longrightarrow \big(y(u, v), z(u, v)\big)$$

is not necessarily a parametrization of $P_{y,z}(\mathbf{S})$ we *may* proceed as if they were. Partition U into rectangles, $(R_{ij})_{ij}$. We consider the "boundary" of $P_{y,z}(R_{ij})$ as a directed curve in $\mathbb{R}^2_{(y,z)}$ "parameterized" by $P_{y,z}$ restricted to the boundary of the anticlockwise oriented rectangle R_{ij} in the uv-plane. Hence $P_{y,z}(R_{ij})$ will have positive (or at least non-negative) area if it is oriented in an anticlockwise fashion in $\mathbb{R}^2_{(y,z)}$. This gives us the approximation

$$\text{Area}\big(P_{y,z}(R_{ij})\big) \text{ in } \mathbb{R}^2_{(y,z)} \approx \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} (u_i, v_j)\Delta u_i \Delta v_j.$$

Next we transfer the partition of U to $P_{y,z}(\mathbf{S})$ by using $P_{y,z}$ and ϕ and form the Riemann sum of f with respect to this partition. This analysis shows that the Riemann sum

$$\sum_i \sum_j f\big(\phi(u_i, v_j)\big) \times \Big[\text{ Area }\big(P_{y,z}(R_{ij})\big) \text{ in } \mathbb{R}^2_{(y,z)}\Big]$$

is approximately equal to

$$\sum_i \sum_j f\big(\phi(u_i, v_j)\big) \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} (u_i, v_j)\Delta u_i \Delta v_j$$

and on taking a limit we get

$$\iint_{\mathbf{S}} (f, 0, 0) = \iint_U f\big(\phi(u, v)\big) \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} dudv.$$

We proceed in the same way with $(0, g, 0)$ and obtain the diagram shown in Fig. 12.9. This implies that z precedes x (see also Fig. 12.7c) and so the correctly oriented "parametrization" in this case is

$$(u, v) \longrightarrow \big(z(u, v), x(u, v)\big)$$

and we obtain

$$\iint_{\mathbf{S}} (0, g, 0) = \iint_U g\big(\phi(u, v)\big) \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} dudv.$$

Similarly we are led to

$$\iint_{\mathbf{S}} (0, 0, h) = \iint_U h\big(\phi(u, v)\big) \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} dudv.$$

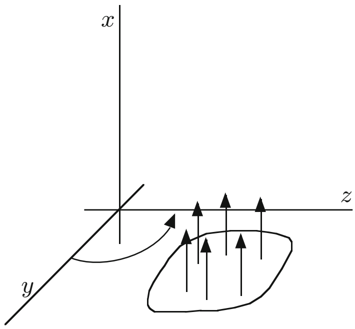

Fig. 12.9

Although each of these integrals looks rather complicated when we add them together we obtain a familiar expression. Since

$$\phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

we have $\phi_u = (x_u, y_u, z_u)$ and $\phi_v = (x_v, y_v, z_v)$. Hence

$$\phi_u \times \phi_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left(\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, -\begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix}, \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right)$$

$$\begin{aligned} \mathbf{F} \cdot \phi_u \times \phi_v &= f \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} - g \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} + h \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \\ &= \begin{vmatrix} f & g & h \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \det \begin{pmatrix} \mathbf{F} \\ \phi_u \\ \phi_v \end{pmatrix} \end{aligned}$$

and

$$\iint_{\mathbf{S}} (f,0,0) + \iint_{\mathbf{S}} (0,g,0) + \iint_{\mathbf{S}} (0,0,h)$$

$$= \iint_U \left(f \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} + g \begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix} + h \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \right) dudv \qquad (12.2)$$

$$= \iint_U \mathbf{F}\big(\phi(u,v)\big) \cdot \phi_u \times \phi_v(u,v) dudv \qquad (12.3)$$

$$= \iint_U \det \begin{pmatrix} \mathbf{F} \\ \phi_u \\ \phi_v \end{pmatrix} dudv. \qquad (12.4)$$

Formula (12.3) is our original definition of the integral of $\mathbf{F}$ over $\mathbf{S}$ and thus (12.2) and (12.4) are new expressions for $\iint_{\mathbf{S}} \mathbf{F}$. We have also shown, provided we use the correct orientations on $P_{y,z}(\mathbf{S})$, $P_{z,x}(\mathbf{S})$ and $P_{x,y}(\mathbf{S})$, that

$$\iint_{\mathbf{S}} \mathbf{F} = \iint_{P_{y,z}(\mathbf{S})} f + \iint_{P_{z,x}(\mathbf{S})} g + \iint_{P_{x,y}(\mathbf{S})} h.$$

In certain cases these projections map $\mathbf{S}$ onto geometrically nice domains in the coordinate planes and this may lead to simpler calculations -see for instance our next example, Example 13.2 and Exercises 12.1, 12.2 and 12.8.

From (12.3) and the results of the previous chapter we also have

$$\begin{aligned}\iint_{\mathbf{S}} \mathbf{F} &= \iint_{U} (\mathbf{F}\cdot\mathbf{n})\|\phi_u \times \phi_v\| du dv \\ &= \iint_{U} (\mathbf{F}\cdot\mathbf{n})\sqrt{EG - F^2} du dv. \qquad (12.5)\end{aligned}$$

Using the notation

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}, \quad \text{etc.}$$

we obtain yet another formula for the integral:

$$\iint_{\mathbf{S}} \mathbf{F} = \iint_{U} (\mathbf{F}\cdot\mathbf{n}) \left[\left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2\right]^{1/2} du dv. \quad (12.6)$$

In view of formula (12.5) the notation $\iint_{S} \langle \mathbf{F}, \mathbf{n} \rangle dA$ is also used in place of $\iint_{(S,\mathbf{n})} \mathbf{F}$ where dA denotes surface area.

We have established five formulae for calculating surface integrals. Furthermore, our excursion into oriented planes has led to geometric insights on the construction of the integral and to a method of evaluating integrals by projecting onto the coordinate planes. There are still a number of topics to be sorted out, e.g. independence of the parametrization, and we discuss these in examples as we proceed.

Example 12.4 In this simple example we use projections to calculate the area of the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The fact that we can easily find the answer independently allows us to check our solution.

From Fig. 12.10, $P = (1/2, 1/2, 0)$ and

$$\|(1/2, 1/2, 0) - (0, 0, 1)\| = \left(\frac{1}{4} + \frac{1}{4} + 1\right)^{1/2} = \sqrt{\frac{3}{2}}.$$

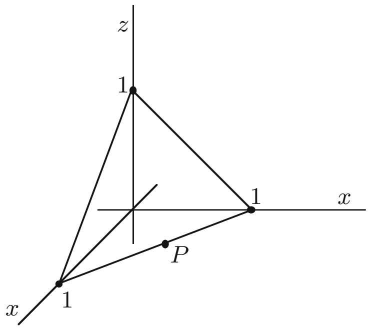

Fig. 12.10

Since $\|(1, 0, 0) - (0, 1, 0)\| = \sqrt{2}$ the (surface) area is $1/2 \cdot \sqrt{2} \cdot \sqrt{3/2} = \sqrt{3}/2$. We now calculate the area using $\iint_{(S,\mathbf{n})} \mathbf{n}$. Since the triangle is part of a *plane* the normal $\mathbf{n}$ is constant on $\mathbf{S}$. By symmetry it is easily seen that the triangle lies in the plane $x + y + z = 1$ and hence the unit normals are $\pm(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Let us take $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ as our unit normal. By symmetry we only have to calculate $\iint_{(S,\mathbf{n})} \big(1/\sqrt{3}, 0, 0\big)$. Clearly $P_{y,z}$ projects onto the anticlockwise oriented triangle $A_1 \subset \mathbb{R}^2_{(y,z)}$ with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$ and

$$\iint_{(S,\mathbf{n})} \big(\frac{1}{\sqrt{3}}, 0, 0\big) = \frac{1}{\sqrt{3}} \text{ Area } (A_1) = \frac{1}{\sqrt{3}} \cdot \frac{1}{2}.$$

Hence

$$\iint_{(S,\mathbf{n})} \mathbf{n} = 3 \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$$

and this agrees with our earlier result.

Exercises

12.1 Find $\iint_{\mathbf{S}} \mathbf{F}$ where $\mathbf{F}(x, y, z) = (1, 2, 3)$ and $\mathbf{S}$ is the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$ oriented so that the origin is on the negative side.

12.2 By using projections and the portion of the sphere that lies in the first octant calculate the area of a sphere of radius r.

12.3 Express as an integral over a region in $\mathbb{R}^2$ the integral $\iint_{\mathbf{S}} \mathbf{F}$ where $\mathbf{F}(x, y, z) = \big(-2/y^3, -6xy^2, 2z^3/x^3\big)$, $\mathbf{S}$ is the graph of $f(u, v) = uv^3$, $(u, v) \in [1, 2] \times [1, 2]$ and the parametrization $\phi(u, v) = (u, v, uv^3)$ is consistent with the orientation. Evaluate the integral.

12.4 Let $0 < a < b$ and let Γ denote the circle of centre $(b, 0)$ and radius a in the xz-plane. Let S denote the surface obtained by rotating Γ about the z-axis. If S is oriented with outward normal and

$$\mathbf{F}(x, y, z) = \left(x - \frac{bx}{\sqrt{x^2 + y^2}}, y - \frac{by}{\sqrt{x^2 + y^2}}, z\right)$$

show that

$$\iint_{\mathbf{S}} \mathbf{F} = 4\pi^2 a^2 b.$$

12.5 Evaluate

(a) $\iint_{\mathbf{S}} y^2 + z^2$

(b) $\iint_{\mathbf{S}} \frac{1}{\left(x^2 + y^2 + (z + a)^2\right)^{1/2}}$

where $\mathbf{S}$ is the portion of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane oriented by the outward normal.

12.6 Describe the surface $S = \{(x, y, z) : z^2 = x^2 + y^2,\ 1 \le z \le 3\}$. If $\mathbf{n}$ is the outward pointing normal to S find

$$\iint_{(S,\mathbf{n})} \frac{(-xz, -yz, x^2 + y^2)}{x^2 + y^2}.$$

12.7 Let S denote the portion of the level set $z = \tan^{-1}(y/x)$ which lies between the planes $z = 0$ and $z = 2\pi$, inside the cone $x^2 + y^2 = z^2$ and outside the cylinder $x^2 + y^2 = \pi^2$. Let

$$\mathbf{G}(x, y, z) = \frac{(-x, -y, -z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Show that

$$F(r, \theta) = (r\cos\theta, r\sin\theta, \theta), \quad \pi < r < \theta,\ \pi < \theta < 2\pi$$

is a parametrization of S. If $\mathbf{n} = \dfrac{F_r \times F_\theta}{\|F_r \times F_\theta\|}$ evaluate $\iint_{(S,\mathbf{n})} \mathbf{G}$.

12.8 Evaluate $\iint_{\mathbf{S}} \mathbf{F}$ where $\mathbf{F}(x, y, z) = (y^2 + z^2, \tan^{-1}(x/z), z\exp(x^2 + y^2))$ and $\mathbf{S}$ is the part of the sphere of radius a centered at the origin in the first octant oriented outwards by (a) using a parametrization and (b) projecting onto the coordinate planes.