Chapter 7
The Frenet–Serret Equations

Summary *We discuss curvature and torsion of directed curves and derive the Frenet–Serret equations. Vector-valued differentiation and orthonormal bases are the main tools used.*

In this chapter we define geometric concepts associated with a directed curve and derive a set of equations—the *Frenet–Serret equations*—which capture the fundamental relationships between them.

We begin with directed curves in $\mathbb{R}^2$ since this particular case exhibits special features not present in higher dimensions. These are due to considering a *one-dimensional* object (the directed curve) in *two-dimensional* space, $\mathbb{R}^2$. The same phenomena appear in Chap. 12 when we examine a *two-dimensional* object (an oriented surface) in *three-dimensional* space, $\mathbb{R}^3$, and the same underlying principles are present when we introduce *torsion* later in this chapter. Moreover, our motivation and interpretation of normal curvature (Chap. 16) and geodesic curvature (Chap. 18) are based on our study of curves in $\mathbb{R}^2$. This special straightforward case deserves particular attention because of the insight it provides into later developments.

Let $P : [a, b] \to \mathbb{R}^2$ denote a *unit speed* parametrization of the directed curve Γ and let $P(t) = \big(x(t), y(t)\big)$ for all t in $[a, b]$. At $P(t) \in \Gamma$ the unit tangent, $T(t)$, is given by

$$T(t) = P'(t) = \big(x'(t), y'(t)\big).$$

The special features, mentioned above, imply that there are just *two* unit vectors in $\mathbb{R}^2$ perpendicular to $T(t)$ and using the *anticlockwise* (or *counterclockwise*) *orientation* of $\mathbb{R}^2$ we can distinguish between them. If we rotate $T(t)$ through $+\pi/2$ in an anticlockwise direction we obtain a unit vector on the *left-hand side* of the direction of motion along Γ (Fig. 7.1). We call this unit vector the *unit normal* to Γ at $P(t)$ and denote it by $N(t)$. In coordinates

$$N(t) = \big(-y'(t), x'(t)\big).$$

We have $\langle T(t), T(t)\rangle = 1$ and differentiating we get, by the product rule,

S. Dineen, *Multivariate Calculus and Geometry*,
Springer Undergraduate Mathematics Series, DOI: 10.1007/978-1-4471-6419-7_7,

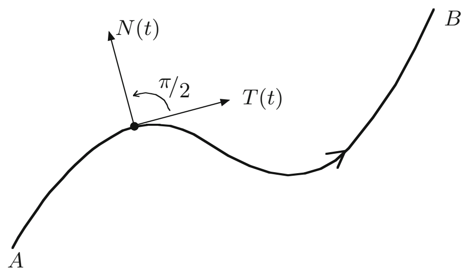

Fig. 7.1

$$\frac{d}{dt}\langle T(t), T(t)\rangle = 0 = \langle T'(t), T(t)\rangle + \langle T(t), T'(t)\rangle.$$

Since

$$\langle T'(t), T(t)\rangle = \langle T(t), T'(t)\rangle$$

this implies

$$\langle T'(t), T(t)\rangle = 0.$$

Hence $T'(t)$ is perpendicular to $T(t)$ and, using once more the fact that $\mathbb{R}^2$ is two-dimensional, we see that $T'(t)$ is parallel to $N(t)$. The *curvature* of Γ at $P(t)$ is defined as the unique scalar, $\kappa(t)$, satisfying

$$T'(t) = \kappa(t)N(t). \tag{7.1}$$

In terms of coordinates

$$\begin{aligned}\kappa(t) &= \langle \kappa(t)N(t), N(t)\rangle = \langle T'(t), N(t)\rangle \\ &= \big(x''(t), y''(t)\big)\cdot\big(-y'(t), x'(t)\big) \\ &= y''(t)x'(t) - x''(t)y'(t)\end{aligned} \tag{7.2}$$

for all $t \in [a, b]$. We call $|\kappa(t)|$ the *absolute curvature* of Γ at $P(t)$ and note that

$$|\kappa(t)| = \|T'(t)\| = \|P''(t)\|. \tag{7.3}$$

Example 7.1 Let $P\colon [a, b] \to \mathbb{R}^2$ denote an *arbitrary* parametrization of the directed curve Γ. We recall from Chap. 5 that $P \circ s^{-1}\colon [0, l] \to \Gamma$ is a unit speed parametrization of Γ where l is the length of Γ and s is the length function. If $P(t) = \big(x(t), y(t)\big)$, $t \in [a, b]$, then

$$P \circ s^{-1}(t) = \big(x \circ s^{-1}(t), y \circ s^{-1}(t)\big), \quad t \in [0, l]$$

and, moreover,

$$(s^{-1})'(t) = \frac{1}{\|P'\big(s^{-1}(t)\big)\|} = \frac{1}{\Big(x'\big(s^{-1}(t)\big)^2 + y'\big(s^{-1}(t)\big)^2\Big)^{1/2}}.$$

We have

$$(x \circ s^{-1})'(t) = x'\big(s^{-1}(t)\big) \cdot (s^{-1})'(t)$$

and

$$(x \circ s^{-1})''(t) = x''\big(s^{-1}(t)\big)\big((s^{-1})'(t)\big)^2 + x'\big(s^{-1}(t)\big) \cdot (s^{-1})''(t)$$

and analogous formulae for $(y \circ s^{-1})'(t)$ and $(y \circ s^{-1})''(t)$. Substituting these into (7.2) and simplifying we obtain the curvature at the point $P(t)$, $t \in [a, b]$,

$$\kappa(t) = \frac{y''(t)x'(t) - x''(t)y'(t)}{\Big(\big(x'(t)\big)^2 + \big(y'(t)\big)^2\Big)^{3/2}}. \tag{7.4}$$

If Γ is the graph of a smooth function $f : [a, b] \to \mathbb{R}$ directed from left to right then $P(t) = \big(t, f(t)\big)$, $t \in [a, b]$, is a parametrization of Γ. This parametrization is only unit speed in the trivial case of a horizontal line (why?). Since $x(t) = t$ we have $x'(t) = 1$ and $x''(t) = 0$ and as $y(t) = f(t)$ we obtain $y'(t) = f'(t)$ and $y''(t) = f''(t)$. Hence the curvature at $\big(t, f(t)\big)$ is

$$\kappa(t) = \frac{f''(t)}{\big(1 + f'(t)^2\big)^{3/2}}.$$

We now discuss the geometric significance of curvature. Let P denote a unit speed parametrization of the directed curve Γ. For simplicity we suppose $0 \in [a, b]$, the domain of definition of P. If t is close to zero then

$$P(t) = P(0) + P'(0)t + P''(0)\frac{t^2}{2} + g(t)t^2 \tag{7.5}$$

where $g(t) \to 0$ as $t \to 0$. Since $P'(0) = T(0)$ and $P''(0) = T'(0) = \kappa(0)N(0)$ we can rewrite this as

$$P(t) = P(0) + T(0)t + \kappa(0)N(0)\frac{t^2}{2} + g(t)t^2.$$

The function

$$Q(t) = P(0) + T(0)t + \kappa(0)N(0)\frac{t^2}{2} \tag{7.6}$$

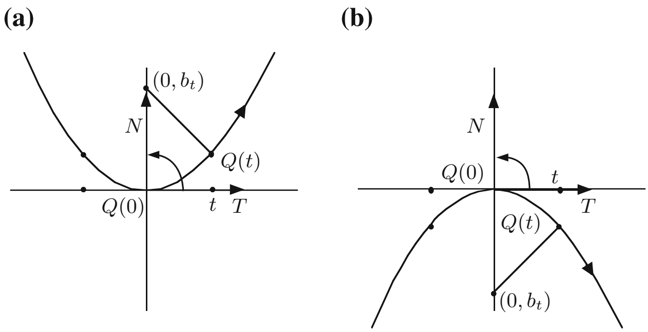

Fig. 7.2

is the best *quadratic approximation* to Γ near $P(0)$ and the curve, parametrized by Q, has the *same* tangent, the *same* normal and the *same* curvature as Γ at $P(0)$. By translating and rotating the plane, if necessary, we can suppose $P(0) = (0, 0)$, $T(0) = (1, 0)$ and $N(0) = (0, 1)$. This implies $Q(t) = (t, \kappa(0)t^2/2)$ and, if $\kappa(0) \neq 0$, then one of the two situations portrayed in Fig. 7.2 holds.

Let C_t denote the circle with centre (a_t, b_t) which passes through the points $Q(-t)$, $Q(0)$, $Q(t)$. By symmetry (a_t, b_t) lies on the y-axis, hence $a_t = 0$ and $|b_t|$ is the radius of C_t. As t tends to zero the circles C_t converge to a circle C with centre $(0, b)$ and radius $|b|$. This is the *circle of curvature* at $Q(0)$ to the directed curve parametrized by Q. Since $\frac{P(t)-Q(t)}{t^2} \rightarrow 0$ as $t \rightarrow 0$ it can easily be shown that C is also the circle of curvature to Γ at $P(0)$, i.e. the circle that fits closest to Γ near $P(0)$. The centre $(0, b)$ is called the *centre of curvature* of Γ at $P(0)$ and $|b|$ is the *radius of curvature*.

We have

$$b_t^2 = \|(0, b_t) - (t, \kappa(0)t^2/2)\|^2 = t^2 + (b_t - \kappa(0)t^2/2)^2.$$

Hence

$$b_t^2 = t^2 + b_t^2 - b_t\kappa(0)t^2 + \kappa(0)^2 t^4/4$$

and

$$b_t\kappa(0) = 1 + \frac{\kappa(0)^2 t^2}{4}.$$

Letting t tend to zero we get $b\kappa(0) = 1$. We interpret $|\kappa(0)|$ as

$$\frac{1}{|b|} = \frac{1}{\text{radius of circle of curvature}}.$$

Since the sign of b tells us on which side of Γ the circle of curvature lies and $b\kappa(0) = 1$ we have

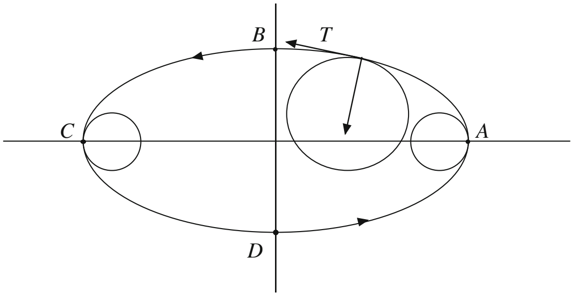

Fig. 7.3

$$\kappa(0) > 0 \iff \text{the circle of curvature and the normal are on the } \textit{same} \text{ side of } \Gamma \text{ (Fig.7.2a)}$$

$$\kappa(0) < 0 \iff \text{the circle of curvature and the normal lie on } \textit{opposite} \text{ sides of } \Gamma \text{(Fig.7.2b).}$$

If $\kappa(0) = 0$ then Γ is rather flat near $P(0)$ and the circle of curvature has infinite radius and thus is a straight line—in our case the x-axis.

We have thus established a geometric interpretation for both absolute curvature and the sign of curvature in $\mathbb{R}^2$ and this often yields immediate and useful information. For example, consider the ellipse in Fig. 7.3, oriented in an anticlockwise direction. The normal will always point into the ellipse and is called, for this reason, the *inner normal*. Since the circle of curvature at any point has finite radius and lies on the same side of the tangent line as the curve the curvature is always strictly positive.

At the points A and C the circles of closest fit to the ellipse have minimum radii among all points on the ellipse. Hence we have maximum curvature at A and C and, similarly, minimum curvature at B and D.

Now consider a directed curve Γ in $\mathbb{R}^n$, $n > 2$, with unit speed parametrization $P\colon [a, b] \to \Gamma$. As before the unit tangent to Γ at $P(t)$ is $P'(t) = T(t)$. We cannot, however, define the unit normal to Γ at $P(t)$ by rotating $T(t)$ since there are not just *two* but an *infinite* number of sides to Γ and thus an infinite number of ways of choosing a unit vector perpendicular to $T(t)$. Even if there were only two directions we would require some concept of *anticlockwise* direction in $\mathbb{R}^n$ to distinguish between them.

To define a normal we need to choose a direction perpendicular to T which is associated in some way to the curve Γ. We still have $\langle T(t), T(t)\rangle = 1$ for all t in $[a, b]$ and differentiating, as we did previously, we get $\langle T'(t), T(t)\rangle = 0$. We have found a special vector, $T'(t)$, perpendicular to $T(t)$ and if this is non-zero (or equivalently if $P''(t) \neq 0$) we define the *unit normal*, $N(t)$, by

$$N(t) = \frac{T'(t)}{\|T'(t)\|}.$$

We define the *curvature* $\kappa(t)$ by $\kappa(t) = \|T'(t)\|$. Thus the normal is *only* defined at points of non-zero curvature and at such points we obtain the equation

$$T'(t) = \kappa(t)N(t). \tag{7.1'}$$

Note that (7.1′) and (7.1) are the *same* equation. However, (7.1) applies to a curve in $\mathbb{R}^2$ while (7.1′), which is the first of the *Frenet–Serret equations* when $n = 3$, applies to a curve in $\mathbb{R}^n$. The definitions of curvature and normal are *different* in these two equations. The technique of using an equation in a simple setting to extend a definition to a more general setting is standard and useful in mathematics.

Now that we have defined κ and N in $\mathbb{R}^n$ we must investigate their properties as they could well be different to those in $\mathbb{R}^2$. Using the same terminology is an expression of our aspirations but does not qualify as a proof. We note first that curvature in $\mathbb{R}^n$ is *always* defined and always non-negative but certain curves, such as straight lines, do not have a normal.

Equation (7.5) and the approximation (7.6) are still valid for curves in $\mathbb{R}^n$ and the same analysis shows that $\kappa(t)$ can be interpreted as the reciprocal of the radius of the circle of curvature to Γ at $P(t)$. Hence curvature in $\mathbb{R}^n$, $n > 2$, has the *same* geometrical interpretation as absolute curvature in $\mathbb{R}^2$. We have seen that the sign of curvature in $\mathbb{R}^2$ was related to the different sides of a curve and, in view of our previous remarks, it is not surprising that it does not feature when $n > 2$.

From now on we restrict ourselves to curves in $\mathbb{R}^3$. The approximation

$$Q(t) = P(0) + T(0)t + \kappa(0)N(0)\frac{t^2}{2}$$

near $0 \in [a, b]$ is still valid and shows that the plane (or two-dimensional subspace) in $\mathbb{R}^3$ closest to the curve near $P(t)$ is the plane through $P(t)$ spanned by $\{T(t), N(t)\}$. We call this the *osculating plane* of the curve at $P(t)$. We may consider the osculating plane as the two-dimensional analogue of the *tangent line*. As we move along the curve the osculating plane will generally change and the more it changes the more twisted the curve. We measure this by defining a new concept—*torsion*—which we denote by τ. We define torsion and a third unit vector, the *binormal* in $\mathbb{R}^3$, in a fashion similar to the way we introduced curvature and the normal for curves in $\mathbb{R}^2$. At each point on the directed curve Γ in $\mathbb{R}^3$ we have obtained two perpendicular unit vectors T and N and these span a two-dimensional subspace of $\mathbb{R}^3$. Hence there are precisely *two* unit vectors perpendicular to T and N. To choose one of these *unambiguously* we require a sense of *direction* or *orientation* in $\mathbb{R}^3$. This will also be important in integration theory. The basis of this sense of direction is known as *"the right-hand rule"* and we describe it in the special case in which we are interested. Use the right thumb as the vector T and the first finger in place of N. Then the second finger will, when put perpendicular to T and N, give the direction of the *binormal*, B (Fig. 7.4).

Think of T and N as determining the flat plane of this page. This page has two sides and thus two unit vectors perpendicular to it. Since the vector N is obtained in

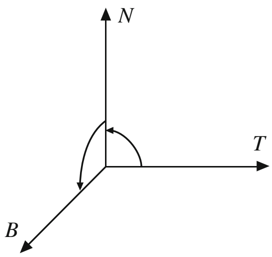

Fig. 7.4

Fig. 7.4 by rotating T in an anticlockwise direction, B will be the unit vector on *this* side of the page. Note that we placed T *before* N in this construction.

Mathematically, we can find the binormal by using the cross product in $\mathbb{R}^3$. For a directed curve Γ in $\mathbb{R}^3$ with parametrization P, unit tangent $T(t)$ and unit normal $N(t)$ the binormal at the point $P(t)$ is given by

$$B(t) = T(t) \times N(t).$$

To derive further results we list standard properties of the *cross product*—all of which follow easily from well-known results about *determinants*.

Let $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$ be vectors in $\mathbb{R}^3$ then

(a) $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$
(b) $\mathbf{v} \times \mathbf{w} \neq 0 \iff \mathbf{v}$ and $\mathbf{w}$ are linearly independent
(c) $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$
(d) $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot |\sin\theta|$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$
(e)

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

and

$$\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \mathbf{w} \cdot \mathbf{u} \times \mathbf{v} = \mathbf{v} \cdot \mathbf{w} \times \mathbf{u}$$

(f) $\|\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}\| =$ volume of the parallelepiped with adjacent sides $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$
(g) $\frac{1}{2}\|(\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})\| =$ area of triangle with vertices $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$.

By (c), $B(t)$ is perpendicular to both $T(t)$ and $N(t)$ and, by (d),

$$\begin{aligned} \|B(t)\| &= \|T(t)\| \cdot \|N(t)\| \cdot |\sin(\pi/2)| \quad (\text{since} T(t) \perp N(t)) \\ &= 1. \end{aligned}$$

Hence $\{T(t), N(t), B(t)\}$ consists of three mutually perpendicular unit vectors in $\mathbb{R}^3$. In particular, they are linearly independent and so form a basis for $\mathbb{R}^3$. We call

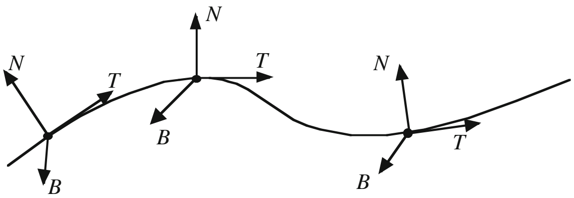

Fig. 7.5

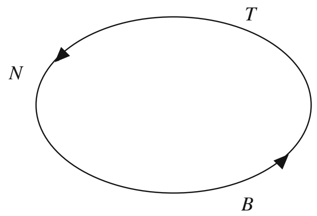

Fig. 7.6

them an *orthonormal basis* ("ortho" comes from orthogonal or perpendicular and normal comes from the fact that they are unit vectors). Another orthonormal basis for $\mathbb{R}^3$ is the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. We consider $\{T(t), N(t), B(t)\}$ as a special basis which is *adapted* to studying the curve Γ near $P(t)$. As t varies the basis $\{T(t), N(t), B(t)\}$ changes and is called a *moving frame* along the curve (see Fig. 7.5).

Since $N(t) \times B(t)$ is easily seen to be a unit vector perpendicular to both N and B we must have

$$N(t) \times B(t) = \pm T(t).$$

Hence

$$\begin{array}{c} T(t) \cdot N(t) \times B(t) = \pm T(t) \cdot T(t) = \pm 1 \\ \parallel \text{ by (e)} \\ B(t) \cdot T(t) \times N(t) = B(t) \cdot B(t) = 1 \end{array}$$

and $N(t) \times B(t) = T(t)$. Similarly $B(t) \times T(t) = N(t)$ and, using (a), the remaining cross products involving T, N and B can be found.

The simplest way to remember these is to use the diagram shown in Fig. 7.6. The cross product of any two taken in an anticlockwise direction is the one that follows it, e.g. $N \times B = T$. If we work in a clockwise direction we get, from (a), the negative of the following one, e.g. $N \times T = -B$.

We are now in a position to make effective use of the orthonormal basis $\{T, N, B\}$. Since it is a basis any vector $\mathbf{v}$ can be written in the form

$$\mathbf{v} = \alpha T + \beta N + \gamma B$$

for some real numbers α, β and γ. If we take the inner product of both sides with respect to T then

$$\langle \mathbf{v}, T\rangle = \alpha\underbrace{\langle T, T\rangle}_{=1} + \beta\underbrace{\langle N, T\rangle}_{=0} + \gamma\underbrace{\langle B, T\rangle}_{=0} = \alpha$$

Similarly $\beta = \langle \mathbf{v}, N\rangle$ and $\gamma = \langle \mathbf{v}, B\rangle$ and thus

$$\mathbf{v} = \langle \mathbf{v}, T\rangle T + \langle \mathbf{v}, N\rangle N + \langle \mathbf{v}, B\rangle B. \tag{7.7}$$

We also note, although we do not require it here, that Pythagoras' Theorem implies

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, T\rangle^2 + \langle \mathbf{v}, N\rangle^2 + \langle \mathbf{v}, B\rangle^2.$$

We use vector-valued differentiation to find $B'(t)$. Since $\|B(t)\| = 1$ we have $\langle B(t), B(t)\rangle = 1$ and hence

$$\begin{aligned} 0 = \frac{d}{dt}\langle B(t), B(t)\rangle &= \langle B'(t), B(t)\rangle + \langle B(t), B'(t)\rangle \\ &= 2\langle B'(t), B(t)\rangle. \end{aligned} \tag{7.8}$$

Since $\langle B(t), T(t)\rangle = 0$ we get, in the same way,

$$\begin{aligned} 0 &= \frac{d}{dt}\langle B(t), T(t)\rangle \\ &= \langle B'(t), T(t)\rangle + \langle B(t), T'(t)\rangle \\ &= \langle B'(t), T(t)\rangle + \langle B(t), \kappa(t)N(t)\rangle \quad \text{(by (7.1}'\text{))} \\ &= \langle B'(t), T(t)\rangle + \kappa(t)\langle B(t), N(t)\rangle \\ &= \langle B'(t), T(t)\rangle \quad (\text{since} B \perp N). \end{aligned} \tag{7.9}$$

Replacing $\mathbf{v}$ by $B'(t)$ in (7.7) and using (7.8) and (7.9) we have

$$B'(t) = \langle B'(t), N(t)\rangle N(t)$$

i.e. $B'(t)$ is parallel to $N(t)$. We use this equation to define the *torsion* of Γ at $P(t)$, $\tau(t)$, by letting[1]

$$\tau(t) = -\langle B'(t), N(t)\rangle$$

and obtain another of the Frenet–Serret equations

$$B'(t) = -\tau(t)N(t). \tag{7.10}$$

[1] By introducing one minus sign here we avoid many more minus signs later.

We discuss the geometric significance of torsion in Chap. 8 but note at this point that $B(t)$ is normal to the osculating plane and hence torsion is a measure of the rate of change of the osculating plane.

We have found T' and B' in Eqs. (7.1′) and (7.10) for a directed curve in $\mathbb{R}^3$ and the remaining Frenet–Serret equation is an expression for N' in terms of the basis $\{T, N, B\}$. To find N' we differentiate the equation $N = B \times T$ using the product rule. We have

$$N' = B' \times T + B \times T' = -\tau N \times T + B \times \kappa N = \tau T \times N - \kappa N \times B.$$

Since $T \times N = B$ and $N \times B = T$ we obtain for all t the equation

$$N'(t) = -\kappa(t)T(t) + \tau(t)B(t). \tag{7.11}$$

Equations (7.1′), (7.10) and (7.11) which express T', N' and B' in terms of T, N and B are known as the *Frenet–Serret equations* and contain, for all practical purposes, complete information on the curve (see also Example 8.5). The set $\{T, N, B, \kappa, \tau\}$ is known as the *Frenet–Serret apparatus* of the curve Γ. It is important to remember that whenever we discuss normals to curves in $\mathbb{R}^3$ or the Frenet–Serret equations for a curve we assuming the curvature is *strictly positive*.

The Frenet–Serret equations are easily remembered when expressed in matrix form

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Certain classical curves are not covered by Definition 5.1 and we encounter one of these—the helix—in our next example. In its natural state the helix does not have an initial or final point and is usually parametrized over $\mathbb{R}$. The reader should have little difficulty analysing such curves by the methods we have already developed. Essentially one examines different finite parts of the curve in turn—for example consider how we treated a part of the helix in Example 5.2. For the sake of completeness, however, we mention how our definition of parametrized curve can be extended in the following natural way to include such curves:

a continuous mapping $P : I \longrightarrow \mathbb{R}^n$ *, I an interval in $\mathbb{R}$, is a parametrized curve if for every closed interval $[a, b] \subset I$ the restriction of P to $[a, b]$ satisfies* (*a*), (*c*) *and* (*d*) *of Definition 5.1.*

Example 7.2 Let Γ denote the helix parametrized by

$$P(t) = (r\cos\omega t, r\sin\omega t, h\omega t), \quad -\infty < t < +\infty$$

where $\omega = (r^2 + h^2)^{-1/2}$ and r, h and ω are all positive. Since

$$P'(t) = (-r\omega \sin \omega t, r\omega \cos \omega t, h\omega)$$

and

$$\begin{aligned}\|P'(t)\| &= (r^2\omega^2 \sin^2 \omega t + r^2\omega^2 \cos^2 \omega t + h^2\omega^2)^{1/2} \\ &= (r^2\omega^2 + h^2\omega^2)^{1/2} = \omega(r^2 + h^2)^{1/2} = 1\end{aligned}$$

the parametrization is unit speed and $T(t) = P'(t)$. We have

$$P''(t) = T'(t) = (-r\omega^2 \cos \omega t, -r\omega^2 \sin \omega t, 0)$$

and

$$\kappa(t) = \|T'(t)\| = (r^2\omega^4 \cos^2 \omega t + r^2\omega^4 \sin^2 \omega t)^{1/2} = \omega^2 r.$$

Note that Γ is *not* a circle but has constant curvature. Hence $\kappa(t) > 0$ and

$$\begin{aligned}N(t) &= \frac{T'(t)}{\|T'(t)\|} = \frac{1}{\omega^2 r}(-r\omega^2 \cos \omega t, -r\omega^2 \sin \omega t, 0) \\ &= (-\cos \omega t, -\sin \omega t, 0).\end{aligned}$$

We have

$$\begin{aligned}B(t) = T(t) \times N(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\omega \sin \omega t & r\omega \cos \omega t & h\omega \\ -\cos \omega t & -\sin \omega t & 0 \end{vmatrix} \\ &= (h\omega \sin \omega t, -h\omega \cos \omega t, r\omega)\end{aligned}$$

and

$$B'(t) = (h\omega^2 \cos \omega t, h\omega^2 \sin \omega t, 0).$$

Since $\tau(t) = -\langle B'(t), N(t)\rangle$ this implies

$$\begin{aligned}\tau(t) &= -\Big\langle (h\omega^2 \cos \omega t, h\omega^2 \sin \omega t, 0),\ (-\cos \omega t, -\sin \omega t, 0)\Big\rangle \\ &= h\omega^2 \cos^2 \omega t + h\omega^2 \sin^2 \omega t = h\omega^2.\end{aligned}$$

We have calculated the Frenet–Serret apparatus for the helix. In doing so we used two of the Frenet–Serret equations. The third equation

$$N' = -\kappa T + \tau B$$

may be used to check our calculations. The Frenet–Serret apparatus was found in the following sequence: first check that P is unit speed then

$$\left\{ \begin{array}{ccccccccc} T(t) & , & \kappa(t) & , & N(t) & , & B(t) & , & \tau(t) \\ \| & & \| & & \| & & \| & & \| \\ P'(t) & & \|T'(t)\| & & T'(t)/\kappa(t) & & T(t) \times N(t) & & -\langle B'(t), N(t)\rangle \end{array} \right\}$$

Other sequences are also possible but the above appear to be generally more direct (for unit speed curves).

Exercises

7.1 Parametrize the curve $x^2 + (y/3)^2 = 9$ with an anticlockwise orientation and hence find its curvature. Find the points where the curvature is a maximum

(a) by inspecting a sketch;
(b) by differentiating the curvature function;
(c) by inspection of the curvature function.

7.2 Let $f\colon U$ (open) $\subset \mathbb{R}^2 \to \mathbb{R}$ and suppose $f^{-1}(0)$ has full rank at each point. If $f^{-1}(0)$ can be parametrized as a directed curve in $\mathbb{R}^2$ show that its absolute curvature equals

$$\frac{|f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2|}{(f_x^2 + f_y^2)^{3/2}}.$$

Using this result find the curvature at all points on the ellipse $(x/a)^2 + (y/b)^2 = 1$ directed so that the normal points outwards. Verify your answer when $a = 3$ and $b = 9$ using Exercise 7.1.

7.3 Let Γ denote the plane curve parametrized by

$$P(t) = (t, \log \cos t), \quad -\pi/4 \le t \le \pi/4.$$

Show that Γ has curvature $-\cos t$ at $P(t)$.

7.4 Show that a directed curve in $\mathbb{R}^3$ is a straight line if and only if all its tangent lines are parallel.

7.5 Show that each of the following gives a unit speed parametrization of a curve Γ in $\mathbb{R}^3$. Calculate the Frenet–Serret apparatus of the curve and verify that $N' = -\kappa T + \tau B$.

(a) $P(t) = \Big(\dfrac{(1+t)^{3/2}}{3}, \dfrac{(1-t)^{3/2}}{3}, \dfrac{t}{\sqrt{2}}\Big), \quad 0 \le t \le 1/2$

(b) $P(t) = \dfrac{1}{2}\Big(\cos^{-1}(t) - t\sqrt{1-t^2},\ 1 - t^2,\ 0\Big), \quad 0 \le t \le 1/2$

(c) $P(t) = \Big(\dfrac{(1+t^2)^{1/2}}{\sqrt{5}}, \dfrac{2t}{\sqrt{5}}, \dfrac{\log(t + \sqrt{1+t^2})}{\sqrt{5}}\Big), \quad t \in \mathbb{R}.$

7.6 Let $P(t) = (a\cos t, a\sin t, at\tan\alpha)$ denote a parametrization of a helix where $0 < \cos\alpha < a < 1$. Find a unit speed parametrization. Show that the centre of curvature also moves on a helix and find the cylinder on which this helix lies.

7.7 If a and b are positive real numbers show that the curve parametrized by

$$P(t) = \left(a\cos t, a\sin t, b\cosh\frac{at}{b}\right), \quad t \in \mathbb{R}$$

lies on the cylinder $x^2 + y^2 = a^2$. Show that the osculating plane at any point on the curve makes a constant angle with the tangent plane to the cylinder at that point.

7.8 If $P\colon [a,b] \to \Gamma$ is a unit speed parametrized curve show that

$$\langle P' \times P'', P'''\rangle = \kappa^2\tau$$

and if $\tau \neq 0$ show that

$$\tau = \frac{\langle P' \times P'', P'''\rangle}{\langle P'', P''\rangle}.$$

7.9 A plane in $\mathbb{R}^3$ (or a cross-section of $\mathbb{R}^3$) consists of all points (x, y, z) satisfying a linear equation $ax + by + cz = d$ where at least one of a, b, c is non-zero. Find $A \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$ such that the plane above coincides with

$$\{A \in \mathbb{R}^3; A \cdot X = \alpha\}.$$

Find α when the plane goes through the origin.

7.10 Let P denote a unit speed parametrization of a directed curve in $\mathbb{R}^3$ with non-zero curvature at $P(0)$. Show that the equation of the osculating plane at $P(0)$ is

$$\{X \in \mathbb{R}^3 : (X - P(0)) \times P'(0) \cdot P''(0) = 0\}.$$

7.11 Let $P\colon [0,l] \to \mathbb{R}^3$ denote a unit speed parametrization of a directed curve Γ, with positive curvature and torsion. For $t \in [0,l]$ let

$$Q(t) = \int_0^t B(x)dx.$$

Show that Q defines a unit speed parametrization of a directed curve $\tilde{\Gamma}$. If $\{T, N, B, \kappa, \tau\}$ is the Frenet–Serret apparatus for Γ show that $\{B, -N, T, \tau, \kappa\}$ is the Frenet–Serret apparatus for $\tilde{\Gamma}$.

7.12 Let Γ denote a directed curve in $\mathbb{R}^3$ with positive curvature at all points and suppose $P : [a,b] \longrightarrow \mathbb{R}^3$ is a unit speed parametrization of Γ. Using the Frenet-Serret equations and the identity $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \times \mathbf{a} \cdot \mathbf{b}$ find a mapping $X : [a,b] \longrightarrow \mathbb{R}^3$ such that $T'(t) = X(t) \times T(t)$, $N'(t) = X(t) \times N(t)$ and $B'(t) = X(t) \times B(t)$ for all $t \in [a,b]$.