Chapter 14
Triple Integrals

Summary *We define triple integrals of scalar-valued functions over open subsets of* $\mathbb{R}^3$, *discuss coordinate systems in* $\mathbb{R}^3$, *justify a change of variable formula and use Fubini's theorem to evaluate integrals.*

Let f be a real-valued function defined on an open subset U of $\mathbb{R}^3$. By using partitions of the coordinate axes to draw planes parallel to the coordinate planes (Fig. 14.1) we obtain a *grid* which partitions $\mathbb{R}^3$ into cubes. Let $(\overline{x}_i, \overline{y}_j, \overline{z}_k)$ denote a typical point in the cube $[x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]$. The Riemann sum of f with respect to this grid

$$\sum_i \sum_j \sum_k f(\overline{x}_i, \overline{y}_j, \overline{z}_k)(x_{i+1} - x_i)\,(y_{j+1} - y_j)\,(z_{k+1} - z_k)$$

is formed by summing over all cubes that lie in U. If the Riemann sums converge, as we take finer and finer partitions and grids, to a limit then f is said to be *Riemann integrable* and the limit

$$\iiint\limits_U f(x, y, z)\,dxdydz$$

is called the (Riemann) integral of f over U.

If U is a bounded open subset of $\mathbb{R}^3$ with smooth boundary and f is the restriction to U of a continuous function $\overline{f}$ on $\overline{U}$ then f is integrable over U. This result, proved using *uniform continuity* of $\overline{f}$ on the *compact* subset $\overline{U}$ of $\mathbb{R}^3$, implies the existence of an abundance of integrable functions.

If $f(x, y, z) \equiv 1$ the Riemann sum is the volume of all cubes inside U and in the limit equals the volume of U, $\text{Vol}(U)$. Thus

$$\text{Vol}(U) = \iiint\limits_U dxdydz.$$

S. Dineen, *Multivariate Calculus and Geometry*,
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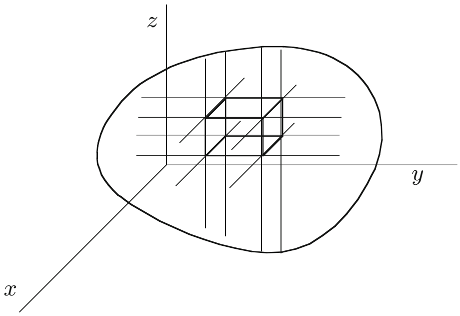

Fig. 14.1

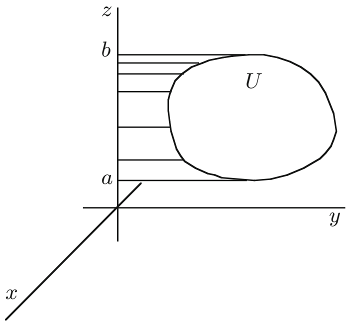

Fig. 14.2

In evaluating triple integrals we use an extension of *Fubini's theorem*. This is obtained from the Riemann sum by first summing over i, taking a limit, then summing over j and taking a limit and finally summing over k and taking a limit. To justify this process it is usual to assume that the domain of integration has a "box-like" appearance, i.e. it is bounded above and below by surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, front and rear by surfaces $x = g_1(y, z)$ and $x = g_2(y, z)$ and on the left and right by surfaces $y = h_1(x, z)$ and $y = h_2(x, z)$. The situations we discuss are of this type but by no means reflect the full range of examples to which Fubini's theorem applies. Many open sets can be partitioned into a finite union of sets and Fubini's theorem applies to each of these. We refer to our remarks on Green's theorem in Chap. 9 for further details.

To apply Fubini's theorem we must examine various cross-sections of the domain of integration U. First suppose that the set of *non-empty cross-sections* of U parallel to the xy-plane, i.e. those obtained by fixing the z-coordinate, determine an interval (a, b) on the z-axis (Fig. 14.2).

Let $A(z)$ denote the cross-section defined by fixing z in (a, b), i.e.

$$A(z) = \{(x, y) \in \mathbb{R}^2_{(x,y)} : (x, y, z) \in U\}.$$

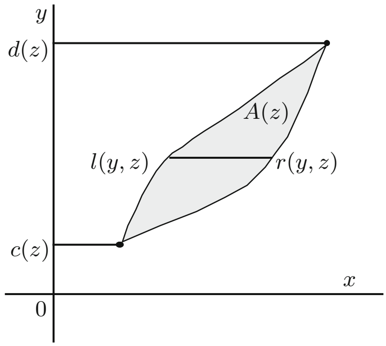

Fig. 14.3

This means

$$\iiint\limits_{U} f(x, y, z)\, dxdydz = \int\limits_{a}^{b} \left\{ \iint\limits_{A(z)} f(x, y, z)\, dxdy \right\} dz$$

and we now have to evaluate the inner integral of *two* variables. In some cases it is possible to do this directly, see for instance Example 14.1, but usually we apply the two-variable Fubini's theorem to each $A(z)$. For this we assume that the region $A(z) \subset \mathbb{R}^2_{(x,y)}$ is bounded on the left and right by the graphs of functions of y over some interval (Fig. 14.3). Both the functions and the interval will depend on z and we denote the interval by $\big(c(z), d(z)\big)$ and the functions on the left and right by $y \to l(y, z)$ and $y \to r(y, z)$ respectively.

This implies

$$\iint\limits_{A(z)} f(x, y, z)\, dxdy = \int\limits_{c(z)}^{d(z)} \left\{ \int\limits_{l(y,z)}^{r(y,z)} f(x, y, z) dx \right\} dy$$

and

$$\iiint\limits_{U} f(x, y, z)\, dxdydz = \int\limits_{a}^{b} \left\{ \int\limits_{c(z)}^{d(z)} \left\{ \int\limits_{l(y,z)}^{r(y,z)} f(x, y, z)\, dx \right\} dy \right\} dz.$$

Thus to evaluate triple integrals it is necessary to identify, by sketching, cross-sections of the open set U. Once this has been achieved and the result compared with the abstract figures above it is a matter of writing down the iterated integrals and evaluating them using one-variable integration theory. We have, as in the two-dimensional case a choice in the order of integration—in fact a total of $3! = 6$ choices, some may be easy, others difficult and some impossible. There are no definite rules.

A particularly simple situation occurs when $U = (a_1, b_1) \times (a_2, b_2) \times (a_3, b_3)$ and $f(x, y, z) = g(x)h(y)k(z)$ where g, h and k are functions of a single variable. In this case

$$\iiint_U f(x, y, z)\, dxdydz = \Big(\int_{a_1}^{b_1} f(x)\, dx\Big)\Big(\int_{a_2}^{b_2} g(y)\, dy\Big)\Big(\int_{a_3}^{b_3} k(z)\, dz\Big).$$

For example, if $U = (0, a) \times (0, b) \times (0, c)$ then

$$\iiint_U xy^2z^3\, dxdydz = \int_0^a xdx \cdot \int_0^b y^2dy \cdot \int_0^c z^3dz = \frac{a^2b^3c^4}{24}.$$

Example 14.1 Let B denote the solid ball of radius r with centre at the origin, i.e. $B = \{(x, y, z) : x^2 + y^2 + z^2 < r^2\}$. We calculate the volume of B by integrating the function $f \equiv 1$ over B. Figure 14.4a shows clearly that the values of z which give non-zero cross-sections lie in the interval $(-r, r)$ and the cross-section for fixed z is the disc $x^2 + y^2 \le r^2 - z^2$ (Fig. 14.4b).

In this special case a direct computation is possible since

$$\iint_{A(z)} 1\, dxdy = \text{Area}\,(A(z)) = \pi(r^2 - z^2)$$

and

$$\text{Vol}(B) = \int_{-r}^{r} \Big\{\int\int_{A(z)} 1\, dxdy\Big\}\, dz \tag{14.1}$$

$$= \int_{-r}^{r} \pi(r^2 - z^2)dz = \pi\left(r^2z - \frac{z^3}{3}\right)\Big|_{-r}^{r} = \frac{4}{3}\pi r^3. \tag{14.2}$$

We now consider the more typical approach to evaluating the inner integral over $A(z)$ by applying Fubini's theorem in two variables. In Fig. 14.4b we have sketched the cross-section $A(z)$ in the xy-plane. The equation $x^2 + y^2 = r^2 - z^2$ (z fixed) has two solutions

$$x = \pm\sqrt{r^2 - z^2 - y^2}.$$

These give the total variation of x and the boundary functions, l and r. We thus have

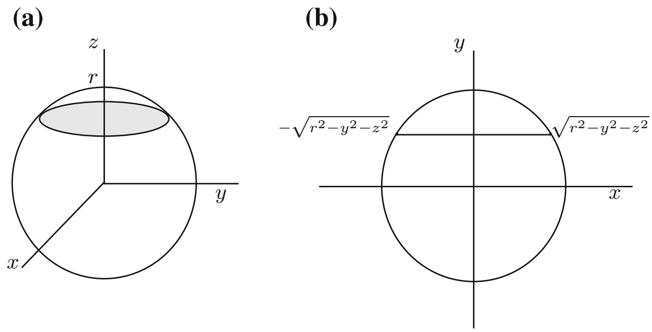

Fig. 14.4

$$\begin{aligned}
\text{Vol}(B) &= \int_{-r}^{r}\left\{\int_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}}\left\{\int_{-\sqrt{r^2-y^2-z^2}}^{+\sqrt{r^2-y^2-z^2}} dx\right\}dy\right\}dz \\
&= \int_{-r}^{r}\left\{\int_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}} 2\sqrt{r^2-y^2-z^2}\,dy\right\}dz \\
&= 2\int_{-r}^{r}\left\{\int_{-\pi/2}^{+\pi/2}(r^2-z^2)\cos^2\theta d\theta\right\}dz \\
&= \pi\int_{-r}^{r}(r^2-z^2)dz = \pi\left(r^2 z - \frac{z^3}{3}\right)\Big|_{-r}^{r} = \frac{4\pi r^3}{3}
\end{aligned}$$

where we let $y = (r^2 - z^2)^{1/2}\sin\theta$, $dy = (r^2 - z^2)^{1/2}\cos\theta$.

The geometry of the previous example was reasonably straightforward. Many examples appear initially to involve a rather complicated geometric shape. However, we are usually dealing with a limited number of objects, mainly conic sections, mixed together and once familiarity with these has been established and sufficiently many cross-sections sketched—with the help of the defining inequalities—the correct approach often presents itself.

Example 14.2 We wish to find the volume of the region V lying below the plane $z = 3 - 2y$ and above the paraboloid $z = x^2 + y^2$, i.e. the set $\{(x, y, z) : x^2 + y^2 < z < 3 - 2y\}$. We begin by considering a full sketch (Fig. 14.5).

The coordinates of P and Q are found by solving between the equations $z = x^2 + y^2$ and $z = 3 - 2y$. These imply $x^2 + y^2 = 3 - 2y$, i.e. $x^2 + y^2 + 2y + 1 = 4$. Hence $x^2 + (y + 1)^2 = 2^2$. The extreme values of y are obtained by letting $x = 0$. This gives $(y + 1)^2 = 2^2$, i.e. $y + 1 = \pm 2$. Hence $y = -3$ or $+1$. The coordinates of P and Q are $(0, -3, 9)$ and $(0, 1, 1)$ respectively. Hence $-3 \le y \le +1$. From

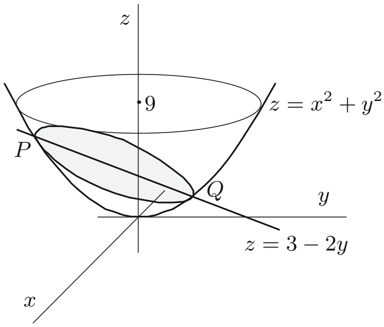

Fig. 14.5

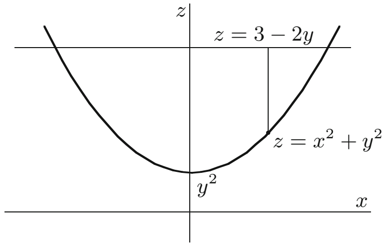

Fig. 14.6

Fig. 14.5 we see $0 \le z \le 9$ and from the equation $x^2 + (y+1)^2 = 2$ we deduce that $-2 \le x \le 2$. Having found the extremal values for non-empty sections we sketch the corresponding cross-sections.

First fix y, $-3 \le y \le 1$. In the zx-plane, $z = y^2 + x^2$ is a parabola and $z = 3 - 2y$ is a straight line. From Fig. 14.6 it follows that for fixed y,

$$-(3 - 2y - y^2)^{1/2} \le x \le (3 - 2y - y^2)^{1/2}$$

and for fixed x and y

$$x^2 + y^2 \le z \le 3 - 2y.$$

We have

$$\begin{aligned} \mathrm{Vol}(V) &= \int_{-3}^{1} \left\{ \int_{-(3-2y-y^2)^{1/2}}^{(3-2y-y^2)^{1/2}} \left\{ \int_{x^2+y^2}^{3-2y} dz \right\} dx \right\} dy \\ &= \int_{-3}^{1} \left\{ \int_{-(3-2y-y^2)^{1/2}}^{(3-2y-y^2)^{1/2}} \Big[z \Big]_{x^2+y^2}^{3-2y} dx \right\} dy \end{aligned}$$

$$= \int_{-3}^{1} \left\{ \int_{-(3-2y-y^2)^{1/2}}^{(3-2y-y^2)^{1/2}} (3 - 2y - y^2 - x^2)\, dx \right\} dy$$

$$= \int_{-3}^{1} \left[(3 - 2y - y^2)x - \frac{x^3}{3} \right]_{-(3-2y-y^2)^{1/2}}^{(3-2y-y^2)^{1/2}} dy$$

$$= \int_{-3}^{1} \frac{4}{3}(3 - 2y - y^2)^{3/2} dy = \frac{4}{3} \int_{-3}^{1} \left(4 - (y+1)^2\right)^{3/2} dy$$

Let $y + 1 = 2\sin\theta$, then $dy = 2\cos\theta d\theta$, $4 - (y+1)^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$, so

$$\begin{aligned} \mathrm{Vol}(V) &= \frac{4}{3} \int_{-\pi/2}^{\pi/2} (4\cos^2\theta)^{3/2} \cdot 2\cos\theta d\theta \\ &= \frac{64}{3} \int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta \\ &= \frac{64}{3} \int_{-\pi/2}^{\pi/2} \frac{1}{4} \left(1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\right) d\theta \\ &= \frac{16}{3} \cdot \frac{3\pi}{2} = 8\pi. \end{aligned}$$

We choose to take y as the final variable in our order of integration since it is clear from Fig. 14.5 that all cross-sections parallel to the xz-plane are of the same type whereas cross-sections parallel to the xy-plane, i.e. fixing z, are different for $z < 1$ and $z \geq 1$. A different order of integration can be used as a second opinion.

From the two previous examples we see that the evaluation of triple integrals proceeded in three stages. First we chose an order of integration and next examined the geometry of the domain of integration in order to determine the limits of integration in the inner integrals. Finally we evaluated a sequence of one variable integrals. The alternatives at each stage in the process point towards a useful technique commonly called *change of variable*. In place of a detailed motivation we briefly mention three pertinent ideas.

(a) The domain of integration U was partitioned into cubes, Fig. 14.7a, with element of volume ΔV easily calculated using $\Delta x \times \Delta y \times \Delta z$. We could use instead a grid based on spheres centred at the origin and planes through the origin to obtain a different element of volume, Fig. 14.7b, or a grid based on cylinders parallel to the z-axis and planes perpendicular and parallel to the z-axis (Fig. 14.7c).

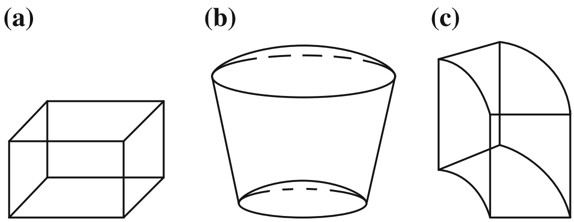

Fig. 14.7

These alternatives lead to more complicated formulae for ΔV but hopefully the new limits of integration and the resulting one-variable integrals are less complicated.

(b) We used Cartesian coordinates (x, y, z) to denote a typical point in the domain of integration U but we may consider other methods of identifying points in U. For instance, if U is the solid sphere of radius 1 then each point in U lies in a sphere of radius r, $0 \leq r \leq 1$, and using the parametrization of the sphere of radius r given in Example 10.5 we can identify points of U by means of r (the distance to the origin), θ the angle of latitude and ψ (the angle of longitude). In terms of the coordinates (r, θ, ψ) the domain U becomes the parallelepiped $(0, 1) \times (-\pi/2, \pi/2) \times (0, 2\pi)$ and, as previously noted, integration in this case is much more pleasant. The new set of coordinates gives a correspondence F between a domain in $\mathbb{R}^3_{(r,\theta,\psi)}$ and the original domain U in $\mathbb{R}^3_{(x,y,z)}$ (Fig. 14.8). By Example 10.5, $F(r, \theta, \psi) = (r\cos\theta\cos\psi, r\cos\theta\sin\psi, r\sin\theta)$. The idea now is to transfer the cubical grid on U, using F, to a grid, which is usually not cubical, on $F(U)$ and hence to evaluate the integral. In carrying out this operation it will be necessary to calculate

$$\text{Vol}\,(F(\Delta V)) = \text{Vol}\,(F(\Delta r \times \Delta\theta \times \Delta\psi))$$

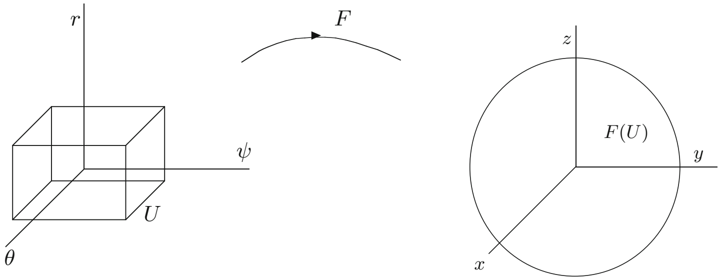

Fig. 14.8

and this is the problem that also arises in (a). The mapping F has many of the features that we have previously associated with a parametrization and, by now, the following definition should appear natural.

Definition 14.3 A parametrization of an open set V in $\mathbb{R}^n$ is a bijective differentiable mapping F from an open subset U of $\mathbb{R}^n$ onto V such that $F'(X)$ is an invertible linear operator for all X in U.

This definition contains the essential properties that we used to parametrize curves and surfaces and may also be regarded as a method of providing V with a *new coordinate system*. The requirement that $F'(X)$ be invertible (or equivalently that $\det\big(F'(X)\big) \neq 0$) is the three-dimensional analogue of the condition $P'(t) \neq 0$ for curves and $\phi_x \times \phi_y \neq 0$ for surfaces.

(c) A third, perhaps more obscure, approach is motivated by the substitutions that may arise in the one-variable integrals in the final stage. In Example 14.2 we needed one such change. Working backwards it may be possible to choose initially a coordinate system which does not require a change of variable in the iterated integrals.

We return now to (b) above to work out the formula for the change of variable. Let U denote an open subset of $\mathbb{R}^3_{(r,s,t)}$ and let $g\colon U \rightarrow g(U) = V$ denote a parametrization of the open subset V of $\mathbb{R}^3_{(u,v,w)}$. Note that g^{-1} is a parametrization of U. To avoid confusion we think of U and V as lying in different copies of $\mathbb{R}^3$ each with their own set of coordinates, (r, s, t) and (u, v, w) respectively. This explains the terminology "change of variables".

Let f denote an integrable function on $g(U)$ (Fig. 14.9). Take a cubical grid on U, transfer it by g to a grid on $g(U)$ and then form a Riemann sum of f. A typical term in this Riemann sum is

$$f\left(g(\overline{r}_i, \overline{s}_j, \overline{t}_k)\right) \text{Vol}\left(g(\Delta r_i \times \Delta s_j \times \Delta t_k)\right).$$

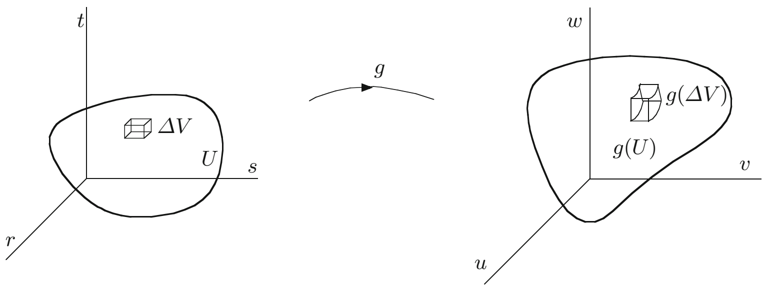

Fig. 14.9

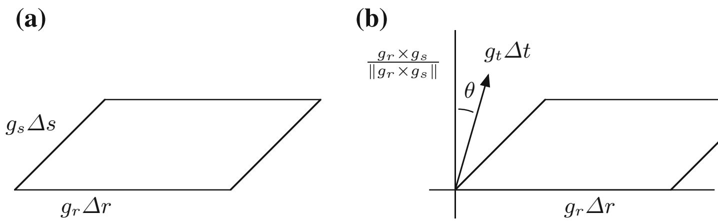

Fig. 14.10

Now $g(\Delta r_i \times \Delta s_j \times \Delta t_k)$ is approximately a parallelepiped with sides $g_r \Delta r$, $g_s \Delta s$ and $g_t \Delta t$. We have already discussed the volumes of parallelepipeds while introducing Stokes' theorem in Chap. 12. The area of the base of the parallelepiped is $\|g_r \times g_s \Delta r_i \cdot \Delta s_j\|$ (Fig. 14.10a) where the partial derivatives of g are evaluated at $(\overline{r}_i, \overline{s}_j, \overline{t}_k)$. The height of the parallelepiped is the length of the projection of $g_t \Delta t_k$ onto a direction perpendicular to the base. Since $g_r \times g_s$ is perpendicular to the base the required height is $\left(g_r \times g_s \cdot g_t \;\big/\; \|g_r \times g_s\|\right) \Delta t_k = \|g_t\| \cos\theta \, \Delta t_k$ (Fig. 14.10b).

Hence

$$\text{Vol}\left(g(\Delta r_i \times \Delta s_j \times \Delta t_k)\right) \approx \|g_r \times g_s \cdot g_t\| \Delta r_i \Delta s_j \Delta t_k.$$

By Exercise 6.5(a)

$$\|g_r \times g_s \cdot g_t\| = \left|\det \begin{pmatrix} g_r \\ g_s \\ g_t \end{pmatrix}\right| = \left|\det(g')\right|$$

and

$$\text{Vol}\left(g(\Delta r_i \times \Delta s_j \times \Delta t_k)\right) \approx \left|\det(g')\right| \Delta r_i \Delta s_j \Delta t_k.$$

Hence the Riemann sum of f over $g(U)$ is approximately

$$\sum_i \sum_j \sum_k f\left(g(\overline{r}_i, \overline{s}_j, \overline{t}_k)\right) \left|\det\left(g'(\overline{r}_i, \overline{s}_j, \overline{t}_k)\right)\right| \Delta r_i \Delta s_j \Delta t_k$$

where we sum over the cubes in the partition of U. In the limit we get the *change of variables formula*

$$\iiint\limits_{g(U)} f(u, v, w)\,dudvdw = \iiint\limits_{U} f\left(g(r, s, t)\right) \left|\det\left(g'(r, s, t)\right)\right| drdsdt.$$

The notations $J(g)$ and $\frac{\partial(u,v,w)}{\partial(r,s,t)}$ are also used in place of $\det(g')$ and this determinant is called the *Jacobian* of g.

If V is an open subset of $\mathbb{R}^2$ then we may identify V with the open subset $\widetilde{V} = V \times (0,1)$ in $\mathbb{R}^3$. A function $f\colon V \to \mathbb{R}$ is integrable over V if and only if $\widetilde{f}$, defined by $\widetilde{f}(x,y,z) = f(x,y)$, is integrable over $\widetilde{V}$ and, moreover,

$$\iiint\limits_{\widetilde{V}} \widetilde{f}(x,y,z)dxdydz = \iint\limits_{V} f(x,y)dxdy.$$

If $g\colon U \subset \mathbb{R}^2_{(r,s)} \to V \subset \mathbb{R}^2_{(u,v)}$ is a mapping between the open sets U and V then it is easily seen that g is a parametrization of V if and only if $\widetilde{g} : \widetilde{U} \subset \mathbb{R}^3_{(r,s,t)} \longrightarrow \widetilde{V} \subset \mathbb{R}^3_{(u,v.w)}$, defined by $\widetilde{g}(r,s,t) = (g(r,s),t)$, is a parametrization of $\widetilde{V}$. Since

$$\widetilde{g}' = \begin{pmatrix} & g' & & 0 \\ & & & 0 \\ 0 & & 0 & 1 \end{pmatrix}$$

we have $\det(g') = \det(\widetilde{g}')$ and

$$\begin{aligned} \iint\limits_{V} f = \iiint\limits_{\widetilde{V}} \widetilde{f} &= \iiint\limits_{\widetilde{U}} \widetilde{f} \circ \widetilde{g} |\det(\widetilde{g}')| \\ &= \iint\limits_{U} f \circ g |\det(g')|. \end{aligned}$$

This justifies the change of variables formula for double integrals (Chap. 9) and yields the following familiar formula

$$\iint\limits_{V=g(U)} f(u,v)dudv = \iint\limits_{U} f\,(g(r,s)) \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} drds$$

where $g(r,s) = (u(r,s), v(r,s))$. In particular, for polar coordinates in the plane, $g\colon (r,\theta) \to (x,y) = (r\cos\theta, r\sin\theta)$, we have

$$|\det(g')| = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

and

$$\iint\limits_{x^2+y^2<1} f(x,y)dxdy = \iint\limits_{\substack{0<r<1 \\ 0<\theta<2\pi}} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

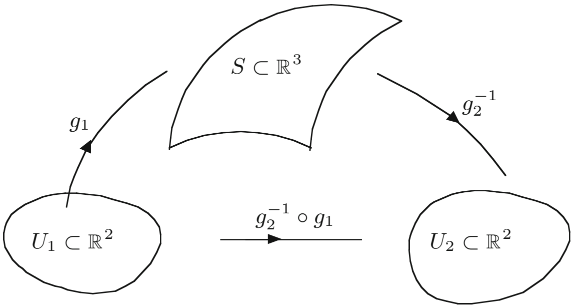

Fig. 14.11

If **S** is a simple oriented surface in $\mathbb{R}^3$ and (g_1, U_1) and (g_2, U_2) are parametrizations of S consistent with the orientation then the bijectivity and smoothness of g_1 and g_2 imply that the mapping $g_2^{-1} \circ g_1$ is a parametrization of U_2 (Fig. 14.11).

The change of variables formula for double integrals may now be used to show that integrals over **S** are *independent* of the parametrization and justifies the notation used in earlier chapters (we remark that g_1 and g_2 map onto the *same* side of **S** if and only if $\det\Big((g_2^{-1} \circ g_1)'\Big)$ is always *strictly positive*).

We have noted above how a solid sphere is a union of surfaces, i.e.

$$\{(x, y, z) : x^2 + y^2 + z^2 < r^2\} = \bigcup_{0 \le s < r} \{(x, y, z) : x^2 + y^2 + z^2 = s^2\}$$

and using parametrization of surfaces we were able to fill in to obtain a parametrization of the solid sphere. In fact, we do not quite get a parametrization of the full solid, since to obtain a bijective mapping, we are forced to miss a small portion of the sphere. This is the same problem that we encountered and discussed fully in parametrizing the classical surfaces (Chap. 10). We do not enter into a full discussion here but remark that after a similar discussion we would arrive at an analogous conclusion for solids. The portion of the solid omitted has volume zero and so for integration purposes we may treat the mappings in the following example as parametrizations of the full solid.

Example 14.4 The procedure outlined above for the sphere can be applied to a number of solids and using Table 11.1 we obtain the following parametrizations.

(a) *Solid sphere of radius a* (spherical polar coordinates)

$$(0, a) \times (0, \pi) \times (0, 2\pi) \in \mathbb{R}^3 \longrightarrow \Big\{(x, y, z) : x^2 + y^2 + z^2 < a^2\Big\}$$
$$(r, \theta, \psi) \longrightarrow (r \sin\theta \cos\psi, r \sin\theta \sin\psi, r \cos\theta)$$

(b) *Solid ellipsoid* (elliptical polar coordinates)

$$(0,1)\times(0,\pi)\times(0,2\pi)\in\mathbb{R}^3 \longrightarrow \left\{(x,y,z) : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}<1\right\}$$
$$(r,\theta,\psi) \longrightarrow (ra\sin\theta\cos\psi, rb\sin\theta\sin\psi, rc\cos\theta)$$

(c) *Solid of revolution* generated by revolving the area beneath the plane curve $P(t)=(x(t),y(t))$, $t\in[a,b]$ and $y(t)>0$, about the x–axis (Example 10.4),

$$(0,1)\times(a,b)\times(0,2\pi)\in\mathbb{R}^3 \longrightarrow \text{Solid of Revolution}$$
$$(r,t,\theta) \longrightarrow (x(t), ry(t)\cos\theta, ry(t)\sin\theta)$$

(d) The *solid cylinder* of radius r and height h parallel to the z-axis is defined by $\{(x,y,z) : x^2+y^2<r^2,\ 0<z<h\}$ and parametrized by

$$(s,\theta,z) \longrightarrow (s\cos\theta, s\sin\theta, z)$$

with domain $(0,r)\times(0,2\pi)\times(0,h)$ (cylindrical coordinates)

(e) The inverted *solid cone* $\{(x,y,z) : x^2+y^2<z^2,\ 0<z<1\}$ is parametrized by

$$(r,\theta,z) \longrightarrow (r\cos\theta, r\sin\theta, z)$$

where $0<r<z, 0<\theta<2\pi, 0<z<1$.

(f) The inverted *solid paraboloid* defined by $\{(x,y,z) : x^2+y^2<z,\ 0<z<1\}$ is parametrized by

$$(r,\theta,z) \longrightarrow (r\cos\theta, r\sin\theta, z)$$

where $0<r<\sqrt{z}, 0<\theta<2\pi, 0<z<1$.

Example 14.5 In this example we calculate the volume of a solid *torus*. We discussed the boundary surface of a torus in Example 11.1 and obtained the parametrization

$$f(\theta,\psi)=((b+r\cos\theta)\cos\psi, (b+r\cos\theta)\sin\psi, r\sin\theta)$$

where $0<\theta<2\pi, 0<\psi<2\pi$ (Fig. 11.4). We generate the *solid torus* by rotating the disc and obtain the parametrization

$$F\colon (0,2\pi)\times(0,2\pi)\times(0,r) \longrightarrow \text{Solid Torus}$$
$$(\theta,\psi,s) \longrightarrow ((b+s\cos\theta)\cos\psi, (b+s\cos\theta)\sin\psi, s\sin\theta)\,.$$

We will use the change of variable formula to calculate the volume but first note how the following geometric observation immediately gives the answer. The solid torus is obtained by rotating a disc of radius r with centre on the y-axis at a distance b

from the origin about the z-axis. Thus the disc is rotated through a distance $2\pi b$ and generates a solid whose volume is

$$\pi r^2 \cdot 2\pi b = 2\pi^2 r^2 b.$$

By the change of variable formula

$$\text{Volume (Torus)} = \iiint\limits_{[0,2\pi]\times[0,2\pi]\times[0,r]} \left|\det\left(F'(\theta,\psi,s)\right)\right| d\theta d\psi ds.$$

From Table 11.1 we obtain, on replacing r by s, the first two columns of F'; $F_\theta = f_\theta$ and $F_\psi = f_\psi$. Hence

$$F'(\theta,\psi,s) = \begin{pmatrix} -s\sin\theta\cos\psi & -(b+s\cos\theta)\sin\psi & \cos\theta\cos\psi \\ -s\sin\theta\sin\psi & (b+s\cos\theta)\cos\psi & \cos\theta\sin\psi \\ s\cos\theta & 0 & \sin\theta \end{pmatrix}.$$

From the matrix representation it is easily seen that F_θ, F_ψ and F_s, i.e. the columns of F', are mutually perpendicular vectors and hence generate a parallelepiped shaped like a rectangular box. In this case the volume is the product of the lengths of the sides. A further application of Table 11.1 gives us $\|F_\theta\| = \|f_\theta\| = \sqrt{E} = s$ and $\|F_\psi\| = \|f_\psi\| = \sqrt{G} = b + s\cos\theta$. Since $\|F_s\| = (\cos^2\theta\cos^2\psi + \cos^2\theta\sin^2\psi + \sin^2\theta)^{1/2} = 1$ we have

$$\left|\det\left(F'(\theta,\psi,s)\right)\right| = \|F_\theta\|\cdot\|F_\psi\|\cdot\|F_s\| = s(b+s\cos\theta).$$

and

$$\begin{aligned}\text{Volume (Torus)} &= \int\limits_{[0,2\pi]}\int\limits_{[0,2\pi]}\int_0^r s(b+s\cos\theta)d\theta d\psi ds \\ &= \int_0^{2\pi} d\psi \cdot \int_0^r \left\{\int_0^{2\pi}(sb+s^2\cos\theta)d\theta\right\}ds \\ &= 2\pi\int_0^r (sb\theta + s^2\sin\theta)\Big|_0^{2\pi} ds \\ &= 2\pi\int_0^r sb2\pi ds = 4\pi^2 b\frac{s^2}{2}\Big|_0^r = 2\pi^2 br^2.\end{aligned}$$

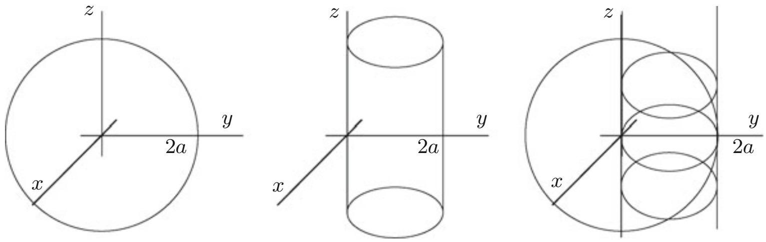

Fig. 14.12

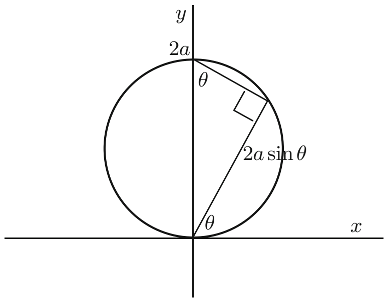

Fig. 14.13

Example 14.6 To find the volume of the solid contained within the sphere $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + (y - a)^2 = a^2$. In Example 12.3 we considered a geometric situation similar to the present one. By not moving the origin we adopt here a slightly different approach. From Fig. 14.12 we see that the solid lies above and below the plane disc $x^2 + (y - a)^2 \leq a^2$ and this immediately suggests polar coordinates.

The volume V is equal to

$$\iiint\limits_{\substack{x^2+y^2+z^2<4a^2 \\ x^2+(y-a)^2<a^2}} 1\, dxdydz.$$

We parametrize the solid using *polar coordinates* in the xy-plane and the usual Cartesian z coordinate, i.e. we use the *cylindrical coordinates*

$$F: (r, \theta, z) \longrightarrow (r\cos\theta, r\sin\theta, z).$$

Since

$$F'(r, \theta, z) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

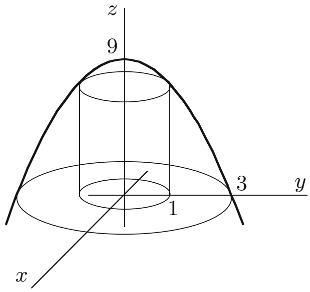

Fig. 14.14

we have $\det\big(F'(r,\theta,z)\big) = r$. To find the limits of integration consider Fig. 14.13 in the xy-plane.

We see that $0 < \theta < \pi$, $0 < r < 2a\sin\theta$ and, from Fig. 14.12, $z^2 < 4a^2 - x^2 - y^2 = 4a^2 - r^2$, i.e. $-\sqrt{4a^2-r^2} < z < \sqrt{4a^2-r^2}$. Hence

$$\text{Volume} = \int_0^{\pi}\left\{\int_0^{2a\sin\theta}\left\{\int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} r\,dz\right\}dr\right\}d\theta$$

$$= 2\int_0^{\pi}\left\{\int_0^{2a\sin\theta} r\sqrt{4a^2-r^2}\,dr\right\}d\theta.$$

Let $s = 4a^2 - r^2$. Then $ds = -2r\,dr$ and

$$\text{Volume} = 2\int_0^{\pi}\left\{\int_{4a^2}^{4a^2\cos^2\theta}\left(-\frac{1}{2}s^{1/2}\right)ds\right\}d\theta$$

$$= -\int_0^{\pi}\left(\frac{2s^{3/2}}{3}\Big|_{4a^2}^{4a^2\cos^2\theta}\right)d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi}(1-|\cos^3\theta|)d\theta = \frac{16a^3}{9}(3\pi-4).$$

Example 14.7 In this example we calculate the volume of the region U bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane and which lies outside the cylinder $x^2 + y^2 = 1$ (Fig. 14.14).

The presence of x^2+y^2 in the defining inequalities suggests cylindrical coordinates (r,θ,z) (the presence of $x^2+y^2+z^2$ would suggest geographical or spherical polar coordinates). From Fig. 14.14 we see that U projects onto $\{(x, y) : 1 \leq x^2+y^2 \leq 9\}$ in the xy-plane. Hence $1 \leq r \leq 3$ and z varies over $0 \leq z \leq 9 - x^2 - y^2 = 9 - r^2$. From the previous example we know that the Jacobian is equal to r. Hence the required volume is

$$\int_0^{2\pi} \left\{ \int_1^3 \left\{ \int_0^{9-r^2} r\,dz \right\} dr \right\} d\theta = \int_0^{2\pi} \left\{ \int_1^3 (9r - r^3)dr \right\} d\theta$$
$$= 2\pi \cdot \left(\frac{9r^2}{2} - \frac{r^4}{4} \right) \Big|_1^3 = 32\pi.$$

Exercises

14.1 Show that the volume of the solid inside the cylinder $x^2 + y^2 - 2ay = 0$ and between the plane $z = 0$ and the paraboloid $4az = x^2 + y^2$ equals $3\pi a^3/8$.

14.2 Find the volume of the solid inside the cylinder $x^2 + y^2 = 2ay$ which lies between the plane $z = 0$ and the cone $x^2 + y^2 = z^2$.

14.3 Show that the volume of the solid defined by the inequalities $x^2 + y^2 \leq 1$ and $\tan^{-1}(y/x) \leq z \leq 2\pi$ equals π^2.

14.4 Let U denote the region above the plane $z = 0$ between the cone $z^2 = x^2 + y^2$ and the paraboloid $z = 2 - x^2 - y^2$. Show that this region projects onto the unit disc in the xy-plane. Using cylindrical coordinates or otherwise show that the volume of U equals $5\pi/6$.

14.5 Evaluate the following integrals:

(a) $$\int_{-3}^{1} \left\{ \int_{y^2}^{3-2y} \left\{ \int_{-(z-y^2)^{1/2}}^{(z-y^2)^{1/2}} dx \right\} dz \right\} dy$$

(b) $$\int_{-2}^{2} \left\{ \int_{-1-(4-x^2)^{1/2}}^{-1+(4-x^2)^{1/2}} \left\{ \int_{x^2+y^2}^{3-2y} dz \right\} dy \right\} dx$$

(c) $$\int_{0}^{9} \left\{ \int_{-\sqrt{z}}^{\sqrt{z}} \left\{ \int_{-(z-x^2)^{1/2}}^{\frac{3-z}{2}} dy \right\} dx \right\} dz.$$

14.6 Use the change of variables formula to calculate the volumes of the solids parametrized in Example 14.4.

14.7 Find the volume of the wedge of the cylinder $\{(x, y, z) : x^2 + y^2 \leq 1\}$ which lies above the xy-plane and between the planes $z = -y$ and $z = 0$.

14.8 Find the volume of the open set which lies above the square $0 < x < 1$, $0 < y < 1$ in the xy-plane and below the surface $z = (1 + x + y)^{1/2}$. Hence write down the volume of the set enclosed by the surfaces $z = (1+|x|+|y|)^{1/2}$, $z = -(1+|x|+|y|)^{1/2}$ and by the planes $x = 1$, $x = -1$, $y = 1$ and $y = -1$.

14.9 Find the volume (of the ice-cream cone) that lies above the xy-plane, inside the cone $3(x^2 + y^2) = z^2$ and the sphere $x^2 + y^2 + z^2 = 4a^2$.

14.10 Evaluate

$$\iiint\limits_V z\,dx\,dy\,dz$$

where $V = \{(x, y, z) : 0 < z < x\sqrt{y},\ 0 < y < 1,\ 1 < x < 2\}$.

14.11 Use the change of variables

$$F : (u, v, w) \longrightarrow \big(u(1 - v), uv(1 - w), uvw\big)$$

to calculate

$$\iiint\limits_V x\,dx\,dy\,dz \quad \text{and} \quad \iiint\limits_V \frac{dx\,dy\,dz}{y + z}$$

where V is the tetrahedron cut from the first octant by the plane $x + y + z = 1$.