Chapter 15
Necessity and Possibility

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Abstract We give a basic introduction to modal logic. This includes possible world semantics, axiom systems, and quantification. Ideas and formal machinery are discussed, but all proofs (and meta-proofs) are omitted. Recommendations are given for those who want more.

15.1 Introduction

Modal operators qualify truth in some way: necessary truth, knowable truth, provable truth, eventual truth, and so on. All these have many formal properties in common while, of course, differing on others. One can abstract these properties and study them for their own sake just as elementary algebra abstracts algebraic equations from natural language problems about weights, measures, distances, and ages. The idea in all cases is that abstraction should provide us with a simple setting in which the formal manipulation of symbols according to precise rules will lead us to results that can be applied back to the complex 'real' world in which the problems arose.

If modal operators are many, what then formally constitutes a modal operator? We do not want to get into the infinite regress of philosophical debate here. A good working definition is, a modal operator is one we can investigate using the formal tools that have been developed for this purpose. Of course this is a time-dependent characterization—tools are human artifacts after all. Here we just consider the core of the subject, *normal modal logics*. These are the best understood using the simplest tools. They do not exhaust the subject.

Modal operators come in dual pairs. Dual to necessity is possibility: X is possibly true if it is not necessary that not-X is true, and X is necessarily true if it is not

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possible that not-X is true. Similarly knowability and consistency (with knowledge) are duals, and so on. Following custom we will use $\Box$ for any necessity-like modal operator and $\Diamond$ for its dual. It will not hurt if you read $\Box X$ as *necessarily* X and $\Diamond X$ as *possibly* X as in the title of this chapter, though you should not think these are the only readings.

(Propositional) formulas are built up from propositional letters using propositional connectives, just as in classical propositional logic, together with a rule of formation saying: if X is a formula so are $\Box X$ and $\Diamond X$. We will be informal about what the propositional connectives are, but it generally is some subset of $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\supset$ (material implication), $\equiv$ (equivalence).

What sort of tools are available for formal modal investigation? Historically, axiom systems came first in modern times, with natural deduction systems, tableau systems, and such things following. Algebraic generalizations of truth tables came along in the 1940s. But ever since *possible world semantics* (relational semantics, Kripke semantics) was developed in the 1960s it has been the common starting point, and it is where we begin.

15.2 Possible World Semantics

What are possible worlds? Don't ask. This is generally a misleading question. One does not need to know what truth is in order to use truth tables—one just needs to know how it behaves with respect to the logical connectives. Likewise one does not need to know what constitutes domains of classical first-order models. It's whatever you like. You get to specify, according to intended application. The logical truths of first-order logic are those that hold no matter what the domain. Well, possible worlds are like that. You get to specify what possible worlds are, according to intended application, and the logical truths of modal logic are those that hold no matter what your specification might have been. For instance, if I am interested in what can be said about a coin flip there are plausibly two possible worlds, one in which the outcome is heads, one in which it is tails. Nothing else matters for this purpose. If I am interested in what is necessary given (our current understanding of) physical laws, possible worlds might be all ways the real universe could be, consistent with those laws. Or it could be all ways some particular experiment might come out. The choice is yours. The question is, what are the laws that hold across all such choices.

Besides possible worlds there is one more essential piece of machinery: an *accessibility relation*. For a particular intended application it may easily be that not all possible worlds are equally possible under all circumstances. For instance suppose the modal operator we have in mind is *from now on*. Then in evaluating the truth of a formula today we must take tomorrow into account, but we can ignore yesterday—tomorrow is accessible from today but yesterday is not. If the modal operator is *has always been* the situation is reversed—yesterday is relevant to today but tomorrow is not.

Definition 2.1 A *frame* is a pair $\langle \mathcal{G}, \mathcal{R} \rangle$ in which $\mathcal{G}$ is a non-empty set and $\mathcal{R}$ is a binary relation on $\mathcal{G}$.

When working with a frame $\langle \mathcal{G}, \mathcal{R} \rangle$ the members of $\mathcal{G}$ are commonly called *possible worlds* or *states*, and for $x, y \in \mathcal{G}$, if $x\mathcal{R}y$ one says that *y is accessible from x* or even *x sees y*. For the time being we will put no constraints on $\mathcal{R}$, though we will consider some later on.

When working with truth tables each line represents an assignment of truth values to propositional letters. We can choose what line to work with—that is arbitrary—but having made such a choice there are fixed rules for evaluating the truth or falsity of more complex formulas. Modal models are like this too, except that truth values for propositional letters can be different at different possible worlds.

Definition 2.2 A *(possible world) model* is a triple, $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ where $\langle \mathcal{G}, \mathcal{R} \rangle$ is a frame and $\mathcal{V}$ assigns a truth value to each propositional letter at each possible world (a *valuation*). That is, if P is a propositional letter and $w \in \mathcal{G}$ then $\mathcal{V}(P, w) \in \{\mathit{true}, \mathit{false}\}$. We say the model $\langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ is *based on* the frame $\langle \mathcal{G}, \mathcal{R} \rangle$.

Given a model, truth values for complex formulas are calculated, world by world, according to certain set rules. At each possible world, propositional connectives behave in their usual truth-table way. But also, $\Box X$ is taken to be true at possible world w if X is true at all possible worlds accessible from w. Similarly $\Diamond X$ is taken to be true at w if X is true at some possible world accessible from w. Thus necessary truth is truth at all possible worlds that are relevant, while possible truth is truth under at least one relevant alternative. Here are the evaluation rules stated precisely. Assume $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ is a model—we write $\mathcal{M}, w \Vdash X$ to indicate that formula X is true at possible world w of model $\mathcal{M}$, and $\mathcal{M}, w \nVdash X$ to indicate that it is not. We give one representative propositional connective case—the others are similar.

$$\mathcal{M}, w \Vdash P \iff \mathcal{V}(P, w) = \mathit{true}, \text{ for } P \text{ a propositional letter}$$
$$\mathcal{M}, w \Vdash X \supset Y \iff \mathcal{M}, w \nVdash X \text{ or } \mathcal{M}, w \Vdash Y$$
$$\mathcal{M}, w \Vdash \Box X \iff \mathcal{M}, z \Vdash X \text{ for every } z \in \mathcal{G} \text{ with } w\mathcal{R}z$$
$$\mathcal{M}, w \Vdash \Diamond X \iff \mathcal{M}, z \Vdash X \text{ for some } z \in \mathcal{G} \text{ with } w\mathcal{R}z$$

Call a formula K-*valid* if it evaluates to *true* at every possible world of every model. The 'K' refers to Kripke, since this is the logic that is given by all possible world models (Kripke models), without any special conditions or restrictions. Typical examples of validities of K are: $\Box(A \wedge B) \equiv (\Box A \wedge \Box B)$, $\Diamond(A \vee B) \equiv (\Diamond A \vee \Diamond B)$, and $(\Box A \wedge \Diamond B) \supset \Diamond(A \wedge B)$. Typical examples of non-validities are: $\Box(A \vee B) \supset (\Box A \vee \Box B)$ and $(\Diamond A \wedge \Diamond B) \supset \Diamond(A \wedge B)$. Think about what these are saying when the modal operators are interpreted in various ways (necessity, knowability, and so on) and you will see that these validities and non-validities are as they ought to be.

15.3 Adding Conditions

Let P be a propositional letter and consider the formula $\Box P \supset P$. One would naturally assume that whatever is necessary is certainly true, so this formula should be valid. But recall that $\Box$ represents many different modalities. Suppose we read $\Box$ as 'is true starting tomorrow.' For this we would not want $\Box P \supset P$, and in fact it is not K-valid. Consider the model $\mathcal{M}_t = \langle \mathcal{G}, \mathcal{R}, \mathcal{V} \rangle$ in which $\mathcal{G}$ consists of two possible worlds; let us call them today and $\mathsf{tomorrow}$, and in which we take $\mathsf{today}\,\mathcal{R}\,\mathsf{tomorrow}$, so that $\mathcal{R}$ represents the passage of time (in a narrow way, of course). Let $\mathcal{V}(P, \mathsf{today}) = \textit{false}$ and $\mathcal{V}(P, \mathsf{tomorrow}) = \textit{true}$. We have $\mathcal{M}_t, \mathsf{today} \Vdash \Box P$ because the only possible world accessible from today is $\mathsf{tomorrow}$, and we have $\mathcal{M}_t, \mathsf{tomorrow} \Vdash P$ because $\mathcal{V}(P, \mathsf{tomorrow}) = \textit{true}$. On the other hand, $\mathcal{M}_t, \mathsf{today} \nVdash P$ because $\mathcal{V}(P, \mathsf{today}) = \textit{false}$. It follows that $\mathcal{M}_t, \mathsf{today} \nVdash \Box P \supset P$, and so indeed $\Box P \supset P$ is not K-valid. If we want $\Box P \supset P$ to hold throughout a model, additional restrictions must be imposed.

Suppose $\Box P$ is read, 'P is known'. One cannot know false things, so we would expect P to be so if $\Box P$ is. But if $\Box P$ is read, 'P is believed,' we would not have the same expectation. One way of thinking about knowledge and belief, involving possible worlds, was explored in detail by Hintikka. We are ignorant in varying ways about the actual state of the world—we may not know if it is snowing at the South Pole, or if it is not, for instance. We may say that the actual world is just one among several possible worlds; in some it is snowing at the South Pole and in others it is not, and so we do not know whether it is or not. But in all of these possible worlds either it is snowing or it is not snowing, and so we know that disjunctive fact. What we know is what is true in all the possible worlds accessible to us. Roughly speaking, the range of relevant possible worlds is a representation of our ignorance. Then the difference between knowledge and belief is that for knowledge the actual world must be among those that are accessible, while for belief it need not be. Beliefs need not be tied to actual facts, merely to possible facts.

A binary relation $\mathcal{R}$ is called *reflexive* if $x\mathcal{R}x$ holds for every x for which the relation is meaningful. Call a frame reflexive if its accessibility relation is reflexive, and likewise for models based on reflexive frames. It would be a good exercise for the reader to show that $\Box P \supset P$ is true at every possible world of any reflexive model. Conversely, if a frame is *not* reflexive, some model based on it will falsify $\Box P \supset P$ at some possible world. The argument goes as follows. Suppose $\langle \mathcal{G}, \mathcal{R} \rangle$ is not reflexive; say for a particular $w \in \mathcal{G}$ we do not have $w\mathcal{R}w$. Let $\mathcal{V}$ be the valuation given by $\mathcal{V}(P, w) = \textit{false}$ and $\mathcal{V}(P, x) = \textit{true}$ for all $x \in \mathcal{G}$ where $x \neq w$. In this model, $\Box P$ is true at w because P is only false at w, which is not accessible from w, while p is true at all other possible worlds, which thus includes all possible worlds accessible from w. But by construction, P is false at w and hence so is $\Box P \supset P$.

Call a model a T-*model* if it, or more properly its frame, is reflexive, and let us say a formula is T-*valid* if it evaluates to *true* at every possible world of every T-model.

Then $\Box X \supset X$ is T-valid for all formulas X and not just for propositional letters. T-validity is appropriate if $\Box$ represents necessity, knowability, provability, being true at every time, and many other modalities. It is not appropriate for believability, obligatory, being true at some time, and so on.

The most commonly investigated propositional modal logics can be captured by putting various conditions on the accessibility relation of frames, just as we did above. These logics have names that are historical—do not look for a pattern. But here are those that are most commonly considered.

$\mathcal{R}$ is *serial* if, for every x there is some y such that $x\mathcal{R}y$. A formula is D-valid if it is true at all possible worlds of every serial model. A typical D validity is $\Box X \supset \Diamond X$. It is easy to see that every T-frame is also a D frame, and so this formula is also a T-validity.

$\mathcal{R}$ is *transitive* if $x\mathcal{R}y$ and $y\mathcal{R}z$ implies $x\mathcal{R}z$. A formula is K4-valid if it is true at all possible worlds of every transitive model. Typical K4 validities are $\Box X \supset \Box\Box X$ and $\Diamond\Diamond X \supset \Diamond X$.

A formula is S4-valid if it is true at all possible worlds of every model that is both reflexive and transitive. Typical S4 validities are $\Box X \equiv \Box\Box X$ and $\Diamond\Diamond X \equiv \Diamond X$.

$\mathcal{R}$ is *symmetric* if $x\mathcal{R}y$ implies $y\mathcal{R}x$. A formula is KB-valid if it is true at all possible worlds of every symmetric model. Typical KB validities are $X \supset \Box\Diamond X$ and $\Diamond\Box X \supset X$.

A formula is S5-valid if it is true at all possible worlds of every model that is reflexive, symmetric, and transitive. Notice that with S4 validity, two consecutive $\Box$ occurrences are equivalent to one. S5 has the property that every string of mixed $\Box$ and $\Diamond$ operators collapses to its last member. For example, $\Diamond\Box\Diamond\Box X \equiv \Box X$ is an S5 validity. This is a very strong property and may or may not be desirable—it depends on the application you have in mind. Investigators in game theory commonly assume the knowledge possessed by agents meets the S5 conditions, for example.

These are hardly all the conditions that have been imposed on models. Further, it is by no means the case that all modal logics that are of interest can be characterized by imposing conditions on frames. Nonetheless, this works for the modal logics that are most commonly used, and it provides a good entry point to a broader subject.

15.4 Axiomatics

Axiomatic formulations of modal logics were investigated long before possible world semantics came along, but today it is common to reverse the historical order. Among the first important results concerning modal semantics was Kripke's proof that several familiar axiomatically formulated modal logics corresponded to logics that had simple semantic characterizations. Here is an outline, for the record. We formulate our axiom systems using axiom *schemes*, without a rule of substitution.

Basic Axiom Schemes

- all classical tautologies (or enough of them)
- $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$
- $\Diamond X \equiv \neg\Box\neg X$

Rules of Inference

- $\frac{X, X \supset Y}{Y}$ (*modus ponens*)
- $\frac{X}{\Box X}$ (*necessitation*)

As usual, an axiomatic proof is a finite sequence of formulas each of which is an instance of an axiom scheme or follows from earlier lines by one of the rules of inference. A proof proves its last line. Call the axiom system above K. It can be shown that the formulas provable in K are exactly the formulas that are K-valid, as defined semantically. We omit the proof here, but it is not difficult.

The modal logics discussed in Sect. 15.3 can be axiomatized by adding schemes to the system for K. We present this as a table. Many more logics can be handled in a similar way—these are merely meant to be representative.

Validity	Axiom Schemes
D	$\Box X \supset \Diamond X$
T	$\Box X \supset X$
K4	$\Box X \supset \Box\Box X$
S4	$\Box X \supset X, \Box X \supset \Box\Box X$
KD	$X \supset \Box\Diamond X$
S5	$\Box X \supset X, \Box X \supset \Box\Box X, X \supset \Box\Diamond X$

In addition to axiom systems many modal logics (including most common ones) have natural deduction systems, tableau (tree) proof systems, and Gentzen sequent calculi. However, there are many modal logics that have axiom systems but not (known) proof systems of these other kinds. Details can be tricky here.

15.5 Quantification

In classical logic, quantification is relatively straightforward. The language is enhanced by adding individual variables and relation symbols (and perhaps also constant and function symbols). Quantifiers, ∀ and ∃ are also added, along with relatively uncomplicated rules of formation. Classical models are introduced consisting of a non-empty domain and an interpretation of relation symbols by relations on that domain. Then machinery is introduced that has the effect of making a universally quantified formula true if every value from the domain makes the formula being quantified true, and an existentially quantified formula true if some value from the domain does so. There is basically only one way of doing all this.

Modally things are not as simple. If one understands quantification from an *actualist* point of view, quantifiers range over what actually exists (whatever that means), while from a *possibilist* point of view quantifiers range over what might exist. (Corresponding temporal versions are *presentist* and *eternalist*.) These differing conceptions can be represented using possible world models in a rather direct way. Of course non-empty domains are involved, and relation symbols are interpreted by relations on domains, but the interpretation might vary from possible world to possible world. For an actualist quantificational semantics we associate with each possible world of a model a non-empty domain, where different worlds can have different domains. These might be disjoint, overlap, or have some other more complex relationship. When evaluating a quantified formula at a possible world, the quantifier is understood to range over the things in the domain of that world only. For a possibilist quantificational semantics, we can think of these separate domains as being combined into one single set, with quantifiers ranging over it no matter at what possible world the truth of a quantified statement is being evaluated. Possibilist semantics validates $(\forall x)\Box A(x) \equiv \Box(\forall x)A(x)$ while actualist semantics does not; in fact it validates neither $(\forall x)\Box A(x) \supset \Box(\forall x)A(x)$ nor $\Box(\forall x)\Box A(x) \supset (\forall x)\Box A(x)$.

The different semantic versions are commonly referred to as *constant domain* (possibilist) and *varying domain* (actualist). It is not the case that one is right and the other wrong, but rather each represents a distinct notion of what quantification in a modal setting is about. In a sense, modal machinery does not dictate, but rather it tells you the consequences of a choice which is made for reasons of philosophical position, taste, or just convenience.

One can even have a formal language with both actualist and possibilist quantifiers, and investigate interactions between the two of them and modal operators. Alternately one could introduce an *existence predicate*, say $E(x)$. Then one could work with an underlying possibilist semantics, think of the things that E is true of at a possible world as the actual existents there, and understand actualist quantification as possibilist quantification relativized to E. The machinery is versatile.

A treatment of equality can be added to the formal machinery. This is closely related to what one imagines the ‘things’ in quantifier domains to be. Suppose we understand domains to consist of objects in some concrete sense, chairs, tables, beer mugs. Objects do not, so to speak, split apart, so we would want the validity of $(x = y) \supset \Box(x = y)$. Likewise neither do they combine so we would also want the validity of $\neg(x = y) \supset \Box\neg(x = y)$. It is rather straightforward to achieve this. On the other hand, we might think of our ‘things’ more intensionally. Are “the tallest tower in Paris” and “the tallest structure built by Eiffel” equal or not? They are in the actual world, but one could certainly create alternate possible worlds in which they are different, or even in which one but not the other is non-existent. These are *non-rigid* designators and their behavior is more complex then that of the objects mentioned earlier. It is possible to introduce quantification over such things too, but it requires more care and nuance.

The quantificational semantics described so far descends directly from the work of Saul Kripke. There is an alternative version due to David Lewis. According to Lewis things cannot exist in the domains of more than one possible world, but they can have *counterparts* in other worlds. Indeed, something existing in one world can have one, many, or no counterparts in an alternate world. Formally, models consist of possible worlds, an alternativeness relation, and a counterpart relation relating quantifier domains. Roughly speaking, something has a property necessarily at a world if at all alternative worlds, all its counterparts have the property. This is an extremely flexible semantics, with relationships to the Kripke-style version. Once again, the machinery provides an array of tools, but it is up to the user to decide what tools to make use of.

15.6 Concluding Comments

The formal machinery of possible worlds, and the accompanying proof procedures, are remarkably plastic. Different conditions on necessity and possibility can be accommodated. Different concepts of existence and quantification can be modeled. Different approaches to identity can be investigated. One should not think of the machinery as settling philosophical problems, but rather as clarifying them. A philosophical hypothesis that can be formalized is coherent. A formalization makes explicit the consequences of adopting that hypothesis. By itself no formalization can say a philosophical position is correct, merely that it is understandable. But to be understandable is almost as good as being true.

Recommended Readings

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