Chapter 14
Constraint Programming

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14.1 Introduction

Constraint satisfaction problems are ubiquitous. A simple example that we will use throughout the first half of this chapter is the following scheduling problem: Choose employees A or B for each of three tasks, X, Y, Z, subject to the work rules that the same employee cannot carry out both tasks X and Y, the same employee cannot carry out both tasks Y and Z, and only employee B is allowed to carry out task Z. (Many readers will recognize this as a simple coloring problem.)

This is an example of a class of problems known as *constraint satisfaction problems* (CSPs). CSPs consist of a set of *variables* (e.g. tasks), a *domain* of *values* (e.g. employees) for each variable, and *constraints* (e.g. work rules) among sets of variables. The constraints specify which combinations of value assignments are allowed (e.g. employee A for task X and employee B for task Y); these allowed combinations *satisfy* the constraints. A *solution* is an assignment of values to each variable such that all the constraints are satisfied (Dechter 2003; Tsang 1993).

We stress that the basic CSP paradigm can be extended in many directions: for example, variables can be added dynamically, domains of values can be continuous, constraints can have priorities, and solutions can be *optimal*, not merely satisfactory.

Examples of constraints are:

- The meeting must start at 6:30.
- The separation between the soldermasks and nets should be at least 0.15 mm.
- This model only comes in blue and green.
- This cable will not handle that much traffic.

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- These sequences should align optimally.
- John prefers not to work on weekends.
- The demand will probably be for more than 5,000 units in August.

Examples of constraint satisfaction or optimization problems are:

- Schedule these employees to cover all the shifts.
- Optimize the productivity of this manufacturing process.
- Configure this product to meet my needs.
- Find any violations of these design criteria.
- Optimize the use of this satellite camera.
- Align these amino acid sequences.

Many application domains (e.g. design) naturally lend themselves to modeling as CSPs. Many forms of reasoning (e.g. temporal reasoning) can be viewed as constraint reasoning. Many disciplines (e.g. operations research) have been brought to bear on these problems. Many computational architectures (e.g. neural networks) have been utilized for these problems. Constraint programming can solve problems in telecommunications, internet commerce, electronics, bioinformatics, transportation, network management, supply chain management, and many other fields.

Some examples of commercial application of constraint technology are:

- Staff planning: BanqueBuxelles Lambert.
- Vehicle production optimization: Chrysler Corporation.
- Planning medical appointments: FREMAP.
- Task scheduling: Optichrome Computer Systems.
- Resource allocation: SNCF (French Railways).
- From push-to-pull manufacturing: Whirlpool.
- Utility service optimization: Long Island Lighting Company.
- Intelligent cabling of big buildings: France Telecom.
- Financial decision support system: Caisse des Dépôts.
- Load capacity constraint regulation: Eurocontrol.
- Planning of satellites missions: Alcatel Espace.
- Optimization of configuration of telecom equipment: Alcatel CIT.
- Production scheduling of herbicides: Monsanto.
- "Just in Time" transport and logistics in food industry: Sun Valley.
- Supply chain management in petroleum industry: ERG Petroli.

CSPs can be represented as *constraint networks*, where the variables correspond to nodes and the constraints to arcs. The constraint network for our sample problem appears in Fig. 14.1. Constraints involving more than two variables can be modeled with hypergraphs, but most basic CSP concepts can be introduced with binary constraints involving two variables, and that is the route we will begin with in this chapter. We will say that a value for one variable is *consistent with* a value for another if the pair of values satisfies the binary constraint between them. (This constraint could be the trivial constraint that allows all pairs of values; such constraints are not represented by arcs in the constraint network.) Note that specifying a domain of values for a variable can be viewed as providing a unary constraint on that single variable.

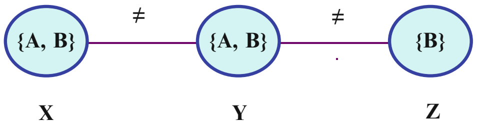


Fig. 14.1 A constraint network representation of a sample constraint satisfaction problem

This chapter focuses on the methods developed in artificial intelligence and the approaches embodied in constraint programming languages. Of course, this brief chapter can only suggest some of the developments in these fields; it is not intended as a survey, only as an introduction. Rather than beginning with formal definitions, algorithms and theorems, we will focus on introducing concepts through examples.

The constraint programming ideal is this: the programming is declarative; we simply state the problem as a CSP and powerful algorithms, provided by a constraint library or language, solve the problem. In practice, this ideal has, of course, been only partially realized, and expert constraint programmers are needed to refine modeling and solving methods for difficult problems.

14.2 Inference

Inference methods make implicit constraint information explicit. Inference can reduce the effort involved in searching for solutions or even synthesize solutions without search. The most common form of inference is known as *arc consistency*. In our sample problem, we can infer that B is not a possible value for Y because there is *no* value for Z that, together with B, satisfies the constraint between Y and Z. This can be viewed as making explicit the fact that the unary constraint on the variable Y does not allow B.

This inference process can *propagate*: after deleting B from the domain of Y, there is no value remaining for Y that together with A for X will satisfy the constraint between X and Y, therefore we can delete A from the domain of X (see Fig. 14.2). If we repeatedly eliminate inconsistent values in this fashion until any value for any variable is consistent with some value for all other variables, we have achieved arc consistency. Many algorithms have been developed to achieve arc consistency efficiently (Bessière and Régin 2001; Mackworth 1977).

Eliminating inconsistent values by achieving arc consistency can greatly reduce the space we must search through for a solution. Arc consistency methods can also be interleaved with search to dynamically reduce the search space, as we shall see in the next section.

Beyond arc consistency lies a broad taxonomy of consistency methods. Many of these can be viewed as some form of *(i, j)-consistency* (Freuder 1985). A CSP is (i, j)-consistent if, given any consistent set of i values for i variables, we can find j values for any other j variables, such that the $i + j$ values together satisfy

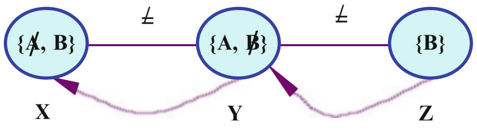


Fig. 14.2 Arc consistency propagation

all the constraints on the $i+j$ variables. Arc consistency is $(1,1)$-consistency. $(k-1,1)$-consistency, or *k-consistency*, for successive values of k constitute an important constraint hierarchy (Freuder 1978).

More advanced forms of consistency processing often prove impractical either because of the processing time involved or because of the space requirements. For example, 3-consistency, otherwise known as *path consistency*, is elegant because it can be shown to ensure that given values for any two variables one can find values that satisfy all the constraints forming any given path between these variables in the constraint network. However, achieving path consistency means making implicit binary constraint information explicit, and storing this information can become too costly for large problems.

For this reason variations on *inverse consistency*, or $(1, j-1)$-consistency, which can be achieved simply by domain reductions, have attracted some interest (Debruyne and Bessière 2001). Various forms of *learning* achieve *partial k*-consistency during search (Dechter 1990; Katsirelos and Bacchus 2005). For example, if we modified our sample problem to allow only A for Z, and we tried assigning B to X and A to Y during a search for a solution to this problem, we would run into a *dead end*: no value would be possible for Z. From that we could learn that the constraint between X and Y should be extended to rule out the pair (B, A), achieving partial path consistency.

Interchangeability (Freuder 1991) provides another form of inference, which can also eliminate values from consideration. Suppose that we modify our sample problem to add employees C and D who can carry out task X. Values C and D would be interchangeable for variable X because in any solution using one we can substitute the other. Thus we can eliminate one in our search for solutions—and if we want to, just substitute it back into any solutions we find. Just as with consistency processing there is a local form of interchangeability that can be efficiently computed. In a sense, inconsistency is an extreme form of interchangeability; all inconsistent values are interchangeable in the null set of solutions that utilize them.

14.3 Modeling

Modeling is a critical aspect of constraint satisfaction. Given a user's understanding of a problem, we must determine how to model the problem as a constraint satisfaction problem. Some models may be better suited for efficient solution than others (Régin 2001).

Experienced constraint programmers may add constraints that are *redundant* in the sense that they do not change the set of solutions to the problem, in the hope that adding these constraints may still be cost effective in terms of reducing problem solving effort. Added constraints that do eliminate some, but not all, of the solutions, may also be useful, e.g. to break symmetries in the problem (Walsh 2012).

Specialized constraints can facilitate the process of modeling problems as CSPs, and associated specialized inference methods can again be cost-effective. For example, imagine that we have a problem with four tasks, two employees who can handle each, but three of these tasks must be undertaken simultaneously. This temporal constraint can be modeled by three separate binary inequality constraints between each pair of these tasks; arc consistency processing of these constraints will not eliminate any values from their domains. On the other hand an *alldifferent* constraint, that can apply to more than two variables at a time, not only simplifies the modeling of the problem, but an associated inference method can eliminate all the values from a variable domain, proving the problem unsolvable. Specialized constraints may be identified for specific problem domains, e.g. scheduling problems.

It has even proven useful to maintain multiple complete models for a problem *channeling* the results of constraint processing between the two (Cheng et al. 1999). As has been noted, a variety of approaches have been brought to bear on constraint satisfaction, and it may prove useful to model part of a problem as, for example, an integer programming problem. Insight is emerging into basic modeling issues, e.g. binary versus non-binary models (Bacchus et al. 2002).

In practice, modeling can be an iterative process. Users may discover that their original specification of the problem was incomplete or incorrect or simply impossible. The problems themselves may change over time.

14.4 Search

In order to find solutions we generally need to conduct some form of search. One family of search algorithms attempts to build a solution by *extending* a set of consistent values for a subset of the problem variables, repeatedly adding a consistent value for one more variable, until a complete solution is reached. Another family of algorithms attempts to find a solution by *repairing* an inconsistent set of values for all the variables, repeatedly changing an inconsistent value for one variable, until a complete solution is reached. (Extension and repair techniques can also be combined.)

Often extension methods are systematic and *complete*, they will eventually try all possibilities, and thus find a solution or determine unsolvability, while often repair methods are stochastic and incomplete. The hope is that completeness can be traded off for efficiency.

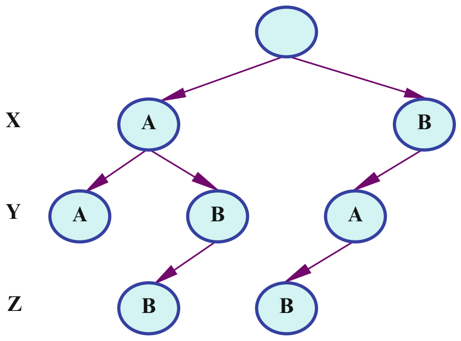


Fig. 14.3 Backtrack search tree for example problem

14.4.1 Extension

The classic extension algorithm is *backtrack search*. Figure 14.3 shows a backtrack search tree representing a trace of a backtrack algorithm solving our sample problem.

A depth-first traversal of this tree corresponds to the order in which the algorithm tried to fit values into a solution. First the algorithm chose to try A for X, then A for Y. At this point it recognized that the choice of A for Y was inconsistent with the choice of A for X: it failed to satisfy the constraint between X and Y. Thus there was no need to try a choice for Z; instead the choice for Y was changed to B. But then B for Z was found to be inconsistent, and no other choice was available, so the algorithm "backed up" to look for another choice for Y. None was available so it backed up to try B for X. This could be extended to A for Y and finally to B for Z, completing the search.

Backtrack search can prune away many potential combinations of values simply by recognizing when an assignment of values to a subset of the variables is already inconsistent and cannot be extended. However, backtrack search is still prone to *thrashing behavior*. A wrong decision early on can require an enormous amount of backing and filling before it is corrected. Imagine, for example, that there were 100 other variables in our example problem, and, after initially choosing A for X and B for Y, the search algorithm tried assigning consistent values to each of those 100 variables before looking at Z. When it proved impossible to find a consistent value for Z (assuming the search was able to get that far successfully) the algorithm would begin trying different combinations of values for all those 100 variables, all in vain.

A variety of modifications to backtrack search address this problem (Kondrak and van Beek 1997). They all come with their own overhead, but the search effort savings can make the overhead worthwhile.

Heuristics can guide the search order. For example, the *minimal domain size* heuristic suggests that as we attempt to extend a partial solution we consider the variables in order of increasing domain size; the motivation there is that we are more likely to fail with fewer values to choose from, and it is better to encounter failure higher in the search tree than lower down when it can induce more thrashing behavior. Using this heuristic in our example we would have first chosen B for Z, then proceeded to a solution without having to back up to a prior level in the search tree. While "fail first" makes sense for the order in which to consider the variables, "succeed first" makes sense for the order in which to try the values for the variables.

Various forms of inference can be used prospectively to prune the search space. For example, search choices can be interleaved with arc consistency maintenance. In our example, if we tried to restore arc consistency after choosing A for X, we would eliminate B from the domain of Z, leaving it empty. At this point we would know that A for X was doomed to failure and could immediately move on to B. Even when failure is not immediate, "look ahead" methods that infer implications of search choices can prune the remaining search space. Furthermore, *dynamic* search order heuristics can be informed by this pruning, e.g. the minimal domain size heuristic can be based on the size of the domains after look-ahead pruning. Maintaining arc consistency is an extremely effective and widely used technique (Sabin and Freuder 1997).

Memory can direct various forms of *intelligent backtracking* (Dechter and Frost 2002). For example, suppose that in our example for some reason our search heuristics directed us to start the search by choosing B for Y followed by A for X. Of course, B the only choice for Z would then fail. Basic backtrack search would back up *chronologically* to then try B for X. However, if the algorithm *remembers* that failure to find a value for Z was based solely on conflict with the choice for Y, it can *jump back* to try the alternative value A at the Y level in the search tree without unnecessarily trying B for X. The benefits of maintaining arc consistency overlap with those of intelligent backtracking, and the former may make the latter unnecessary.

Search can also be reorganized to try alternatives in a top-down as opposed to bottom-up manner. This responds to the observation that heuristic choices made early in the extension process, when the remaining search space is unconstrained by the implications of many previous choices, may be most prone to failure. For example, *limited discrepancy search* iteratively restarts the search process increasing the number of *discrepancies*, or deviations from heuristic advice, that are allowed, until a solution is found (Harvey and Ginsberg 1995). (The search effort at the final discrepancy level dominates the upper-bound complexity computation, so the redundant search effort is not as significant as it might seem.)

Extensional methods can be used in an incomplete manner. As a simple example, *random restart*, starting the search over as soon as a dead end is reached, with a stochastic element to the search order, can be surprisingly successful (Gomes et al. 1997).

14.4.2 Repair

Repair methods start with a complete assignment of values to variables, and work by changing the value assigned to a variable in order to improve the solution. Each such change is called a *move*, and the new assignment is termed a *neighbor* of the previous assignment. Genetic algorithms, which create a new assignment by combining two previous assignments, rather than by moving to a neighbor of a single assignment, can be viewed as a form of repair.

Repair methods utilize a variety of metaphors, physical (hill climbing, simulated annealing) and biological (neural networks, genetic algorithms). For example, we might start a search on our example problem by choosing value A for each variable. Then, seeking to *hill climb* in the search space to an assignment with fewer inconsistencies, we might choose to change the value of Y to B; and we would be done. Hill climbing is a repair-based algorithm in which each move is required to yield a neighbor with a better cost than before. It cannot, in general, guarantee to produce an optimal solution at the point where the algorithm stops because no neighbor has a better cost than the current assignment.

Repair methods can also use heuristics to guide the search. For example, the *min-conflicts* heuristic suggests finding an inconsistent value and then changing it to the alternative value that minimizes the amount of inconsistency remaining (Minton et al. 1992).

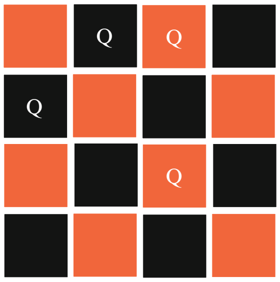
The classic repair process risks getting stuck at a local maximum, where complete consistency has not been achieved, but any single change will only increase inconsistency, or *cycling* through the same set of inconsistent assignments. There are many schemes to cope. A stochastic element can be helpful. When an algorithm has to choose between equally desirable alternatives it may do so randomly. When no good alternative exists it may start over, or jump to a new starting point. Simulated annealing allows moves to neighbors with a worse cost with a given probability. Memory can also be utilized to guide the search and avoid cycling (tabu search).

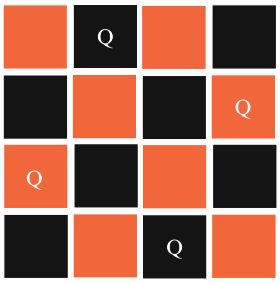
14.5 Example

We illustrate simple modeling, search and inference now with another example. The Queens Problem involves placing queens on a chessboard such that they do not attack one another. A simple version only uses a four-by-four corner of the chessboard to place four queens.

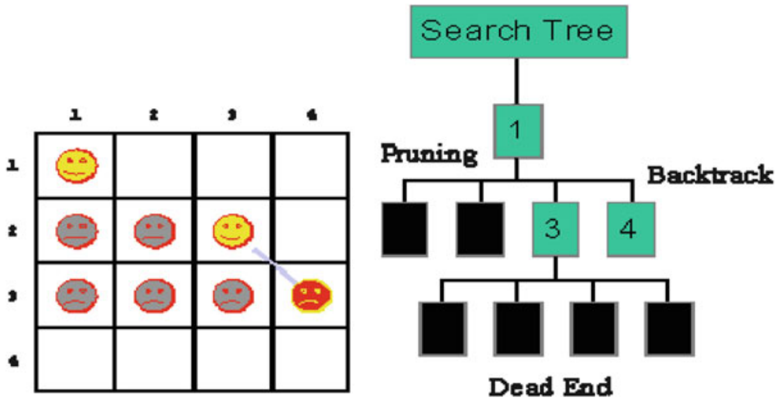
Queens in chess attack horizontally, vertically and diagonally. So, for example, the two queens on the dark squares above attack each other diagonally, the two queens on the light squares attack vertically. One solution is:

If we model this problem as a CSP where the variables are the 4 queens and the values for each queen are the 16 squares, we have 65,536 possible combinations to explore, looking for one where the constraints (the queens do not attack each other) are satisfied. If we observe that we can only have 1 queen per row, and model the problem with a variable corresponding to the queen in each row, each variable having 4 possible values corresponding to the squares in the row, we have only 256 possibilities to search through.

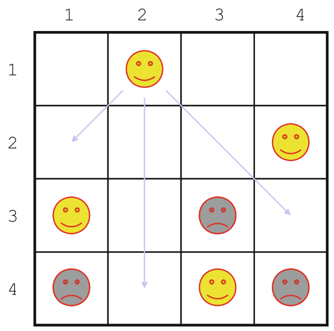




The beginning of the backtrack search tree for this example is shown below. After placing the first row queen in the first column, the first successful spot for the second row queen is in the third column. However, that leaves no successful placement for the third row queen, and we need to backtrack:



In fact, there will be quite a lot of backtracking to do here before we find a solution. However, arc consistency inference can reduce the amount of search we do considerably. Consider what happens if we seek arc consistency after placing a queen in the second column of row 1. This placement directly rules out any square that can be attacked by this queen, of course, and, in fact, arc consistency propagation proceeds to rule out additional possibilities until we are left with a solution. The queens in column 3 of row 3 and column 4 of row 4 are ruled out because they are attacked by the only possibility left for row 2. After the queen in column 3 of row 3 is eliminated, the queen in column 1 of row 4 is attacked by the only remaining possibility for row 3, so it too can be eliminated:



14.6 Tractability

CSPs are in general NP-hard. Analytical and experimental progress has been made in characterizing tractable and intractable problems. The results have been used to inform algorithmic and heuristic methods.

14.6.1 Theory

Tractable classes of CSPs have been identified based on the structure of the constraint network, e.g. tree structure, and on the types of constraints allowed, e.g. max-closed (Jeavons and Cooper 1995). Tractability has been associated with sets of constraints defining a specific class of problems, e.g. temporal reasoning problems defined by *simple temporal networks* (Dechter et al. 1991).

If a constraint network is tree-structured, there will be a *width-one* ordering for the variables in which each variable is directly constrained by at most one variable earlier in the ordering. In our sample problem, which has a trivial tree structure, the ordering X, Y, Z is width-one: Y is constrained by X and Z by Y; the ordering X, Z, Y is not width-one: Y is constrained by both X and Z. If we achieve arc consistency and use a width-one ordering as the order in which we consider variables when trying to extend a partial solution, backtrack search will in fact be *backtrack-free*: for each variable we will be able to find a consistent value without backing up to reconsider a previously instantiated variable (Freuder 1982).

Max-closure requires that if (ab) and (cd) both satisfy the constraint, then also $(\max(ac), \max(bd))$ will satisfy the constraint. If all the constraints in a problem are max-closed, the problem will be tractable. The *less than* constraint is max-closed, e.g. $4 < 6$, $2 < 9$ and $4 < 9$. In fact, simple temporal networks are max-closed.

14.6.2 Experiment

Intuitively it seems natural that many random CSPs would be relatively easy: loosely constrained problems would be easy to solve, highly constrained problems would be easy to prove unsolvable. What is more surprising is the experimental evidence that as we vary the constrainedness of the problems there is a sharp *phase transition* between solvable and unsolvable regions, which corresponds to a sharp spike of *really hard* problems (Cheeseman et al. 1991). ("Phase transition" is a metaphor alluding to physical transitions, such as the one between water and ice.)

14.7 Optimization

Optimization arises in a variety of contexts. If all the constraints in a problem cannot be satisfied, we can seek the *best* partial solution (Freuder and Wallace 1992). If there are many solutions to a problem, we may have some criteria for distinguishing the best one. We can distinguish *hard constraints*, which have to be satisfied, from *soft constraints*, which do not necessarily have to be. Soft constraints can allow us to express probabilities or preferences that make one solution *better* than another.

Again we face issues of modeling, inference and search. What does it mean to be the *best* solution; how do we find the *best* solution? We can assign scores to local elements of our model and combine those to obtain a global score for a proposed solution, then compare these to obtain an optimal solution for the problem.

A simple case is the Max-CSP problem, where we seek a solution that satisfies as many constraints as possible. Here each satisfied constraint scores 1, the score for a proposed solution is the sum of the satisfied constraints, and an optimal solution is one with a maximal score. Many alternatives, such as fuzzy, probabilistic and possibilistic CSPs, have been captured under the general framework of semiring-based or valued CSPs (Bistarelli et al. 1996).

Backtrack search methods can be generalized to branch and bound methods for seeking optimal solutions, where a partial solution is abandoned when it is clear that it cannot be extended to a better solution than one already found. Inference methods can be generalized. Repair methods can often find close to optimal solutions quickly.

14.8 Algorithms

14.8.1 Handling Constraints

Constraint technology is an approach that solves problems by reasoning on the constraints, and when no further inferences can be drawn, making a search step. Thus inference and search are interleaved.

For problems involving hard constraints—ones that must be satisfied in any solution to the problem—reasoning on the constraints is a very powerful technique.

The secret of much of the success of constraint technology comes from its facility to capitalize on the structure of the problem constraints. This enables *global* reasoning to be supported, which can guide a more *intelligent* search procedure than would otherwise be possible.

In this section we introduce some different forms of reasoning and its use in solving problems efficiently.

14.8.2 Domains, and Constraint Propagation

In general, domain constraint propagation algorithms take a set of variables and the original domains as input, and either report inconsistency, or output smaller domains for the variables. Since propagation algorithms can extract more information, each time the input domains become smaller, and since the propagation behavior also makes the domains of the variables smaller, the propagation algorithms can cooperate through these domains. The output from one algorithm is input to another, whose output can in turn be input to the first algorithm. Thus many different propagation algorithms can cooperate, together yielding domain reductions which are much stronger than simply pooling the information from the separate algorithms.

In the following code we constrain two variables, X and Y, to take values in the range 1–10. We constrain X to be greater than Y and (inconsistently!) we also constrain Y to be greater than X:

```
?- X::1..10, Y::1..10
fd:(X>Y), fd:(Y>X)
```

If the propagation algorithm for each constraint makes the bounds consistent, then the first constraint will yield new domains

```
X::2..10, Y::1..9
```

and the second constraint will yield new domains

```
X::1..9, Y::2..10.
```

Pooling the deduced information, we get the intersection of the new domains:

```
X::2..9, Y::2..9.
```

By contrast, if the propagation algorithms communicate through the variable domains, then they will yield new domains, which are then input to the other algorithm until, at the fifth step, the inconsistency between the two constraints is detected.

The interaction of the different algorithms is predictable, even though the algorithms are completely independent, so long as they have certain natural properties. Specifically,

- The output domains must be a subset of the input domains;
- If with input domains ID the algorithm produces output domains OD, then with any input domains which are a subset of ID, the output domains must be a subset of OD.

These properties guarantee that the information produced by the propagation algorithms, assuming they are executed until no new information can be deduced, is guaranteed to be the same, independent of the order in which the different algorithms (or *propagation steps*) are executed.

14.8.3 Constraints and Search

14.8.3.1 Separating Constraint Handling from Search

The constraint programming paradigm supports clarity, correctness and maintainability by separating the statement of the problem as far as possible from the details of the algorithm used to solve it. The problem is stated in terms of its decision variables, the constraints on those variables, and an expression to be optimized.

As a toy example the employee task problem introduced at the beginning of this chapter can be expressed as follows:

```
?- X::[a,b], Y::[a,b], Z::[b], % Set up variables
X= \ =Y, Y= \ =Z. % set up constraints.
```

This states that the variable X can take either of the two values, a or b; similarly Y can take a or b; but Z can only take the value b. Additionally, the value taken by X must be different from that taken by Y; and the values taken by Y and Z must also be different.

To map this problem statement into an algorithm, the developer must

- Choose how to handle each constraint
- Specify the search procedure.

For example, assuming fd is a solver which the developer chooses to handle the constraints, and labeling is a search routine, the whole program is written as follows:

```
?- X::[a,b], Y::[a,b], Z::[b],
fd:(X= \ =Y), fd:(Y= \ =Z),
labeling([X,Y,Z]).
```

This code sends the constraint X= \ =Y to the finite domain solver called fd; and the constraint Y= \ =Z is also sent to fd. The list of variables X, Y and Z is then passed to the labeling routine.

Domain propagation algorithms take as input the domains of the variables, and yield smaller domains. For example, given that the domain of Z is [b], the fd solver for the constraint Y= \ =Z immediately removes b from the domain of Y, yielding a new domain [a]. Every time the domain of one of those variables is reduced by another propagation algorithm, the algorithm *wakes* and reduces the domains further (possibly waking the other algorithm). In the above example, when the domain of Y is reduced to [a], the constraint X= \ =Y wakes, and removes b from the domain of X, reducing the domain of X to [b].

The domain of a variable may be reduced either by propagation, or instead by a search decision. In this case propagation starts as before, and continues until no further information can be derived. Thus search and constraint reasoning are automatically interleaved.

14.8.3.2 Search Heuristics Exploiting Constraint Propagation

Most real applications cannot be solved to optimality because they are simply too large and complex. In such cases it is crucial that the algorithm is directed towards areas of the search space where low-cost feasible solutions are most likely to be found.

Constraint propagation can yield very useful information, which can be used to guide the search for a solution. Not only can propagation exclude impossible choices a priori, but it can also yield information about which choices would be optimal in the absence of certain (awkward) constraints.

Because constraint propagation occurs after each search step, the resulting information is used to support *dynamic* heuristics, where the next choice is contingent upon all the information about the problem gathered during the current search.

In short, incomplete extension search techniques can produce high-quality solutions to large complex industrial applications in areas such as transportation and logistics. The advantage is that the solutions respect all the hard constraints and are therefore applicable in practice.

14.8.4 Global Constraints

In this section we introduce a variety of application-specific "global" constraints. These constraints achieve more, and more efficient, propagation behavior than would be possible using combinations of the standard equality and disequality constraints introduced above. We first outline two global constraints, one called "alldifferent" for handling sets of disequality constraints, and one called "schedule" for handling resource usage by a set of tasks. A complete catalog of global constraints is available at www.emn.fr/z-info/sdemasse/gccat.

14.8.4.1 Alldifferent

Consider the disequality constraint "$= \backslash =$" used in the employee task example at the beginning of this chapter. Perhaps surprisingly, global reasoning on a set of such constraints brings more than local reasoning. Suppose we have a list and want to make local changes until all the elements of the list are distinct. If each element has the same domain (i.e. the same set of possible values), then it suffices to choose any element in conflict (i.e. an element whose value occurs elsewhere in the list), and change it to a new value which doesn't occur elsewhere. If, however, the domains of different elements of the list are different, then there is no guarantee that local improvement will converge to a list whose elements are all different. However, there is a polynomial time graph-based algorithm (Régin 1994) which guarantees to detect if there is no solution to this problem, and otherwise it reduces the domains of all the elements until they contain only values that participate in a solution. The constraint that all the elements of a list must be distinct is usually called alldifferent in constraint programming.

For example, consider the case

```
?- X::[a,b], Y::[a,b], Z::[a,b,c],
alldifferent([X,Y,Z]).
```

In this case Régin's algorithm (Régin 1994) for the alldifferent constraint will reduce the domain of Z to just [c]. As we saw above, the behavior of reducing the variables domains in this manner is called *constraint propagation*.

Global constraint propagation algorithms work co-operatively within a constraint programming framework. In the above example, the propagation algorithm for the alldifferent constraint takes as input the list of variables, and their original domains. As a result it outputs new, smaller, domains for those variables.

14.8.4.2 Schedule

Consider a task scheduling problem comprising a set of tasks with release times (i.e. earliest start times) and due dates (i.e. latest end times), each of which requires a certain amount of resource, running on a set of machines that provide a fixed amount of resource.

The schedule constraint works, in principle, by examining each time period within the time horizon.

- First the algorithm calculates how much resource r_i each task i must necessarily take up within this period.
- If the sum $\sum_i r_i$ exceeds the available resource, then the constraint reports an inconsistency.
- If the sum $\sum_{i \neq j} r_i$ takes up so much resource that task j cannot be scheduled within the period, and the remaining resource within the period is r_T, then task j is constrained only to use an amount r_T of resource within this period.

For non-preemptive scheduling, this constraint may force such a task j to start a certain amount of time before the period begins, or end after it. The information propagated narrows the bounds on the start times of certain tasks.

Whilst this kind of reasoning is expensive to perform, there are many quicker, but theoretically less complete forms of reasoning, such as "edge-finding", which can be implemented in a time quadratic in the number of tasks.

14.8.4.3 Further Global Constraints

The different global constraints outlined above have proven themselves in practice. Using global constraints such as schedule, constraint programming solves benchmark problems in times competitive with the best complete techniques. There are two main advantages of global constraints in constraint programming, in addition to their efficiency in solving standard benchmark problems:

1. They can be augmented by any number of application-specific "side" constraints. The constraint programming framework allows all kinds of constraints to be thrown in, without requiring any change to the algorithms handling the different constraints.
2. They return high-quality information about the problem which can be used to focus the search. Not only do they work with complete search algorithms, but also they guide incomplete algorithms to return good solutions quickly.

Constraint programming systems can include a range of propagation algorithms supporting global reasoning for constraints appearing in different kinds of applications such as rostering, transportation, network optimization, and even bioinformatics.

14.8.4.4 Analysis

One of the most important requirements of a programming system is support for reusability. Many complex models developed by operations researchers have made very little practical impact, because they are so hard to reuse. The concept of a global constraint is inherently quite simple. It is a constraint that captures a class of subproblems, with any number of variables. Global constraints have built-in algorithms, which are specialized for treating the problem class. Any new algorithm can be easily captured as a global constraint and reused. Global constraints have had a major impact, and are used widely and often as a tool in solving complex real-world problems. They are, arguably, the most important contribution that constraint programming has brought to Operations Research (OR).

14.8.5 Different Constraint Behaviors

Constraint reasoning may derive other information than domain reductions. Indeed any valid inference step can be made by a propagation algorithm. For example, from the constraints

$$X > Y, Y > Z$$

propagation can derive that

$$X > Z.$$

To achieve co-operation between propagation algorithms, and termination of propagation sequences, constraint programming systems typically require propagation algorithms to behave in certain standard ways. Normally they are required to produce information of a certain kind, for example domain reductions.

An alternative to propagating domain reductions is to propagate new linear constraints. Just as domain propagation ideally yields the smallest domains which include all values that could satisfy the constraint, so linear propagation ideally yields the convex hull of the set of solutions. In this ideal case, linear propagation is stronger than domain propagation, because the convex hull of the set of solutions is contained in the smallest set of variable domains (termed the *box*) that contains them.

14.8.6 Extension and Repair Search

Extension search is conservative in that, at every node of the search tree, all the problem constraints are satisfied. Repair search is optimistic, in the sense that a variable assignment at a search node may be, albeit promising, actually inconsistent with one or more problem constraints.

14.8.6.1 Constraint Reasoning and Extension Search

Constraint reasoning, in the context of extension search, corresponds to logical deduction. The domain reductions, or new linear constraints yielded by propagation, are indeed a consequence of the constraint in the input state.

14.8.6.2 Constraint Reasoning and Repair Search

Constraint reasoning can also be applied in the context of repair search. The standard method for handling constraints in repair search is to check them in each search state. If a constraint is violated then the "cost" of the state is increased by an amount computed by the constraint checker (either simply counting the number of violated constraints, or assessing the degree of violation of each constraint and penalizing it accordingly). More radically, any state which includes a violated constraint can be rejected, and the assignment leading to this state immediately undone. These methods of handling violated constraints are, of course, fully covered in other chapters of this book.

There are, however, other methods of moving forward from a search state in which constraints are violated. These methods were pioneered in the mathematical programming community, for handling integer-linear problems. They are applicable in case the violated constraint is—for whatever reason—too difficult to directly enforce during search. When such a violation occurs, these methods return to a previous search state (in which no constraints were violated) and add new easier constraints which preclude the violation. Like propagation, the method of dealing with a (violated difficult) constraint yields new information, in the form of easy constraints that can be dealt with actively by the search procedure. We distinguish two ways of choosing the easy constraints: constraint generation and separation. These are best illustrated by example.

1. *Constraint generation.* For an example of constraint generation, consider a traveling salesman problem (TSP) which is being solved by integer/linear programming. At each search node an integer/linear problem is solved, which only approximates the actual TSP constraint. It generates "routes" in which every location has a predecessor and a successor, but unfortunately the route is sometimes composed of two or more unconnected cycles. Consider a search node which represents a route with a detached cycle. This violates the TSP constraint in a way that can be fixed by adding a linear constraint enforcing a unit flow out from the set of *cities* in the detached cycle. This is the *generated* constraint, at the given search node. The search is complete at the first node where the TSP constraint is no longer violated. Constraint generation can be used in cases where the awkward constraints can be expressed by a conjunction of easy constraints, although the number of such easy constraints in the conjunction may be too large for them all to be imposed.
2. *Separation.* Separation behavior is required to fix any violated constraint which cannot be expressed as a conjunction of easy constraints (however large). If the awkward constraint can be approximated, arbitrarily closely, by a (conjunction of) disjunction(s) of easy constraints, then separation can be used. Constraint reasoning yields one of the easy constraints—one that is violated by the current search node—and imposes it so that the algorithm which produces the next search node is guaranteed to satisfy this easy constraint. Completeness is maintained by imposing the other easy constraints in the disjunction on other branches of the search tree.

14.8.6.3 Languages and Systems

One drawback of the logical basis of Constraint Programming is that search methods are not naturally expressed in logic, and repair-based search includes many states which violate constraints and are therefore logically inconsistent. Depth-first search with backtracking is inbuilt in constraint logic programming (CLP, see next section), and therefore concise and natural to express and control (e.g. Refalo 2004). Other forms of tree search, such as best-first, do not fit so well, and repair-based search is not even expressible in most implementations of CLP. For controlling tree search a number of search control languages have been developed, including SALSA (Laburthe and Caseau 1998), ToOLS (de Givry and Jeannin 2006) and search combinators (Schrijvers et al. 2011). For repair search in a constraint programming framework the Comet language (Van Hentenryck and Michel 2009) has been highly influential. More recently Comet has been extended to support both repair-based search and extension search.

A key feature of Comet is the concept of an "invariant", which is a constraint that retains information used during search. When implementing GSAT (Selman et al. 1992), by way of example, an invariant is used to record, for each problem variable in each search state, the change in the number of satisfied propositions that would occur if the variable's value were to be changed. The invariant is specified as a constraint, but maintained by an efficient incremental algorithm.

14.9 Constraint Languages

14.9.1 Constraint Logic Programming

The earliest constraint programming languages, such as *Ref-Arf* and *Alice*, were specialized to a particular class of algorithms. The first general-purpose constraint programming languages were constraint handling systems embedded in logic programming (Jaffar and Lassez 1987; Van Hentenryck et al. 1992), called *constraint logic programming* (CLP). Examples are CLP(fd), HAL, SICStus and ECLiPSe. Certainly logic programming is an ideal host programming paradigm for constraints, and CLP systems are widely used in industry and academia.

Logic programming is based on relations. In fact, every procedure in a logic program can be read as a relation. However, the definition of a constraint is exactly the same thing—a *relation*. Consequently, the extension of logic programming to CLP is entirely natural. Logic programming also has backtrack search built-in, and this is easily modified to accommodate constraint propagation. CLP has been enhanced with some high-level control primitives, allowing active constraint behaviors to be expressed with simplicity and flexibility. The direct representation of the application in terms of constraints, together with the high-level control, results in short simple programs. Since it is easy to change the model and, separately, the behavior of a program, the paradigm supports experimentation with problem solving methods. In the context of a rapid application methodology, it even supports experimentation with the problem (model) itself.

14.9.2 Modeling Languages

On the other hand, operations researchers have introduced a wide range of highly sophisticated specialized algorithms for different classes of problems. These algorithms are only applicable to special classes of problem models expressible in modeling languages such as AMPL (Fourer et al. 2002) and AIMMS.

For many OR researchers CLP and Localizer are too powerful—they seek a modeling language rather than a computer programming language in which to encode their problems. Traditional mathematical modeling languages used by OR researchers have offered little control over the search and the constraint propagation. OPL (Van Hentenryck 1999) and its more expressive successors Essence (Frisch et al. 2007) and Zinc (Marriott et al. 2008) give more control to the algorithm developer. They represent a step towards a full constraint programming language.

By contrast, a number of application development environments (e.g. Visual CHIP) have appeared recently that allow the developer to define and apply constraints graphically, rather than by writing a program. This represents a step in the other direction!

14.9.3 Constraint Satisfaction and Optimization Systems

As constraint technology has matured the community has recognized that it is not a standalone technology, but a weapon in an armory of mathematical tools for tackling complex problems. Indeed, an emerging role for constraint programming is as a framework for combining techniques such as constraint propagation, integer/linear and quadratic programming, interval reasoning, global optimization and metaheuristics.

Several systems are now available that support such a combination of tools and techniques, including G12 (Stuckey et al. 2005), Comet (Van Hentenryck and Michel 2009), IBM ILOG Optimization Studio, Microsoft Solver Foundation and NumberJack, from the Cork Constraint Computation Center. These systems support alternative interfaces to enable their users to exploit the underlying tools in the combination most suited to their particular applications.

A hybridization that has recently proven very successful is the combination of CP with propositional satisfiability solvers (Nieuwenhuis et al. 2005; Stuckey 2010).

14.10 Applications

14.10.1 Current Areas of Application

Constraint programming is based on logic. Consequently any formal specification of an industrial problem can be directly expressed in a constraint program. The drawbacks of earlier declarative programming paradigms have been

- That the programmer had to encode the problem in a way that was efficient to execute on a computer;
- That the end user of the application could not understand the formal specification.

The first breakthrough of constraint programming has been to separate the logical representation of the problem from the efficient encoding in the underlying constraint solvers. This separation of logic from implementation has opened up a range of applications in the area of control, verification and validation.

The second breakthrough of constraint programming has been in the area of software engineering. The constraint paradigm has proven to accommodate a wide variety of problem-solving techniques, and has enabled them to be combined into hybrid techniques and algorithms, suited to whatever problem is being tackled.

As important as the algorithms to the success of constraint technology, has been the facility to link models and solutions to a graphical user interface that makes sense to the end user. Having developers display the solutions in a form intelligible to the end users, forces the developers to put themselves into the shoes of the users.

Moreover, not only are the final solutions displayed to the user: it is also possible to display intermediate solutions found during search, or even partial solutions. The ability to animate the search in a way that is intelligible to the end user means the users can put themselves into the shoes of the developers. In this way the crucial relationship and understanding between developers and end users is supported and users feel themselves involved in the development of the software that will support them in the future.

As a consequence, constraint technology has been applied very successfully in a range of combinatorial problem-solving applications, extending those traditionally tackled using operations research.

The two main application areas of constraint programming are, therefore,

1. Control, verification, and validation;
2. Combinatorial problem solving.

14.10.2 Applications in Control, Verification and Validation

Engineering relies increasingly on software, not only at the design stage, but also during operation. Consider the humble photocopier. Photocopiers are not as humble as they used to be—each system comprises a multitude of components, such as feeders, sorters, staplers and so on. The next generation of photocopiers will have orders of magnitude more components than now. The challenge of maintaining compatibility between the different components, and different versions of the components, has become unmanageable.

Xerox has turned to constraint technology to specify the behavior of the different components in terms of constraints. If a set of components are to be combined in a system, constraint technology is applied to determine whether the components will function correctly and coherently. The facility to specify behavior in terms of constraints has enabled engineers at Xerox not only to simulate complex systems in software but also to revise their specifications before constructing anything and achieve compatibility first time.

Control software has traditionally been expressed in terms of finite-state machines. Proofs of safety and reachability are necessary to ensure that the system only moves between safe states (e.g. the lift never moves while the door is open) and that required states are reached (the lift eventually answers every request). Siemens has applied constraint technology to validate control software, using techniques such as Boolean unification to detect any errors. Similar techniques are also used by Siemens to verify integrated circuits.

Constraint technology is also used to prove properties of software. For example, abstract interpretation benefits from constraint technology in achieving the performance necessary to extract precise information about concrete program behavior.

Finally, constraints are being used not only to verify software but to monitor and restrict its behavior at runtime. *Guardian Agents* ensure that complex software, in medical applications for example, never behaves in a way that contravenes the certain safety and correctness requirements.

For applications in control, validation and verification, the role of constraints is to model properties of complex systems in terms of logic, and then to prove theorems about the systems. The main constraint reasoning used in this area is propositional theorem proving. For many applications, even temporal properties are represented in a form such that they can be proved using propositional satisfiability.

Nevertheless, the direct application of abstract interpretation to concurrent constraint programs offers another way to prove properties of complex dynamic systems.

14.10.3 Combinatorial Problem Solving

Commercially, constraint technology has made a huge impact in problem-solving areas such as transportation, logistics, network optimization, scheduling and timetabling, production control, configuration and design, and it is also showing tremendous potential in new application areas such as bioinformatics and virtual-reality systems.

Starting with applications to transportation, constraint technology is now used by airline, bus and railway companies all over the world. Applications include timetabling, fleet scheduling, crew scheduling and rostering, stand, slot and platform allocation.

Constraints have been applied in the logistics area for parcel delivery, food, chilled goods, and even nuclear waste. As in other application areas, the major IT system suppliers (such as SAP and I2) are increasingly adopting constraint technology.

Constraints have been applied for Internet service planning and scheduling, for minimizing traffic in banking networks, and for optimization and control of distribution and maintenance in water and gas pipe networks. Constraints are used for network planning (bandwidth, routing, peering points), optimizing network flow and pumping energy (for gas and water), and assessing user requirements.

Constraint technology appears to have established itself as the technology of choice in the areas of short-term scheduling, timetabling and rostering. The flexibility and scalability of constraints was proven in the European market (for example at Dassault and Monsanto), but is now used worldwide.

It has been used for timetabling activities in schools and universities, for rostering staff at hospitals, call centers, banks and even radio stations. An interesting and successful application is the scheduling of satellite operations.

The chemical industry has an enormous range of complex production processes whose scheduling and control is a major challenge, currently being tackled with constraints. Oil refineries and steel plants also use constraints in controlling their production processes. Indeed, many applications of constraints to production scheduling also include production monitoring and control.

The majority of commercial applications of constraint technology have, to date, used finite-domain propagation. Finite domains are a very natural way to represent the set of machines that can carry out a task, the set of vehicles that can perform a delivery, or the set of rooms/stands/platforms where an activity can be carried out. Making a choice for one task, precludes the use of the same resource for any other task which overlaps it, and propagation captures this easily and efficiently.

Naturally, most applications involve many groups of tasks and resources with possibly complex constraints on their availability (for example, personnel regulations may require that staff have 2 weekends off in 3, that they must have a day off after each sequence of night-shifts, and that they must not work more than 40 h a week). For complex constraints like this a number of special constraints have been introduced which not only enable these constraints to be expressed quite naturally, but also associate highly efficient specialized forms of finite-domain propagation with each constraint.

14.10.4 Other Applications

14.10.4.1 Constraints and Graphics

An early use of constraints was for building graphical user interfaces. Now these interfaces are highly efficient and scalable, allowing a diagram to be specified in terms of constraints so that it still carries the same impact and meaning whatever the size or shape of the display hardware. The importance of this functionality in the context of the Internet and mobile computing is very clear, and constraint-based graphics are likely to have a major impact in the near future. Constraints are also used in design, involving both spatial constraints and, in the case of real-time systems design, temporal constraints.

14.10.4.2 Constraint Databases

Constraint databases have not yet made a commercial impact, but it is a good bet that future information systems will store constraints as well as data values. The first envisaged application of constraint databases is to geographical information systems. Environmental monitoring will follow, and subsequently design databases supporting both the design and maintenance of complex artifacts such as airplanes.

14.11 Potpourri

There are many more topics that could merit an additional section of their own. Here we briefly sample a few of these. O'Sullivan (2012) discusses some opportunities and challenges for constraint programming.

14.11.1 Dynamic Constraint Problems and Soft Constraints

We may need to handle problems that change over time (for example due to machine breakdown, newly placed priority orders, or late running). *Dynamic CSPs* add or delete constraints to produce a sequence of problems (Dechter and Dechter 1988). After the problem changes, we might want to find a solution close to the solution to the previous problem, or we may simply be interested in finding a new solution as quickly as possible. We may seek robust solutions that are more likely to remain solutions even if the problem does change.

14.11.2 Explanation

Users may feel more comfortable when an explanation can accompany a solution. Explanation is particularly important when a problem is unsolvable. The user wants to know why, and can use advice on modifying the problem to permit a solution, or preferences can be codified a priori to guide subsequent search (Amilhastre et al. 2002). A related set of problems confronts the need constraint programmers have to better understand the solution process. Explanation and visualization of this process can assist in debugging constraint programs, computing solutions more quickly, and finding solutions closer to optimal (Deransart et al. 2000; Junker 2004).

14.11.3 Synthesizing Models and Algorithms

Ideally people with constraints to satisfy or optimize would simply state their problems, in a form congenial to the problem domain, and from this statement a representation suited to efficient processing and an appropriate algorithm to do the processing would be synthesized automatically. In practice, considerable human expertise is often needed to perform this synthesis. The challenge is to automate the modelling and solving process (O'Sullivan 2010).

14.11.4 Distributed Processing

Distributed constraint processing arises in many contexts. There are parallel algorithms for constraint satisfaction and concurrent constraint programming languages. There are applications where the problem itself is distributed in some manner. There are computing architectures that are naturally distributed, e.g. neural networks. There is synergy between constraint processing and software agents. Agents have issues that are naturally viewed in constraint-based terms, e.g. negotiation. Agents can be used to solve constraint satisfaction problems (Yokoo et al. 1998).

14.11.5 Uncertainty

Real-world problems may contain elements of uncertainty. Data may be problematic. The future may not be known. For example, decisions about fuel purchases may need to be made based on uncertain demand dependent on future weather patterns. We want to model and compute with constraints in the presence of such uncertainty (Walsh 2002). Many problems exhibit a form of symmetry (Cohen et al. 2006). For example, in the Queens problem discussed earlier, the two solutions are symmetric in the sense that one is the "mirror image" of the other. Reducing or "breaking" symmetries can save search. Avoiding symmetries is one consideration in choosing how to model a problem. Given a model with symmetry, one can seek to add constraints that break symmetry (Walsh 2012). For example, one could add a constraint to the Queens problem requiring the queen in the first row to be placed in one of the first two columns. There are also methods that avoid symmetry dynamically during search by recognizing "symmetric" portions of the search space.

14.12 Tricks of the Trade

The constraints community uses a variety of different tools to solve complex problems. There are a number of constraint programming systems available, which support constraint propagation, search and a variety of other techniques. For pedagogical purposes we will simply show the solution of a simple problem, solved using one constraint programming system, MiniZinc. This system is free for research use, and can be downloaded from www.g12.csse.unimelb.edu.au/minizinc.

We consider a one-machine scheduling problem. The requirement is to schedule a set of tasks on a machine. Each task has a fixed duration, and each has an earliest start time (the *release date*) and a latest end time (the *due date*). How should we schedule the tasks so as to finish soonest?

In constraint programming a problem is handled in three stages:

1. Initialize the problem variables;
2. Constrain the variables;
3. Search for values for the variables that satisfy the constraints.

14.12.1 Initializing Variables

For a scheduling problem, we introduce a time horizon: all tasks will start and end within this horizon. A variable is declared to represent the start time of each task and this variable can only take a value between 0 (the start of our schedule) and the time horizon. A completion time variable is also declared. This is the time when the last task finishes, and also takes a value between 0 and the time horizon. In this model we have fixed the number of tasks as 5 and the time horizon as 50. These variables are encoded as follows:

```
int: Horizon = 50 ;
int: NTasks = 5 ;

array[1..NTasks] of var 0..Horizon: start ;
var 0..Horizon: all_completed ;
```

In this model we have chosen to represent time as a sequence of time points each represented by an integer between 0 and 50.

14.12.2 Constrain the Variables

The start time of each task is constrained to be after the release date of the task, and the end time before the due date. We write “forall (t in 1..NTasks)” in order to apply the constraints to each of the five tasks.

```
array[1..NTasks] of int: Release_date = [0,10,11,20,25] ;
array[1..NTasks] of int: Duration = [12,10,8,10,5] ;
array[1..NTasks] of int: Due_date = [17,40,30,35,45] ;
```

```
constraint   forall (t in 1..NTasks)
                 ( start[t] >= Release_date[t] ) ;

constraint    forall (t in 1..NTasks)
                 ( start[t]+Duration[t] <= Due_date[t] ) ;
```

The completion time is constrained to be greater than or equal to the end of each task:

```
constraint    forall (t in 1..NTasks)
                 ( start[t] + Duration[t] <= all_completed ) ;
```

(Later, during search, MiniZinc will find the minimum possible completion time, which will ensure that it is actually equal to the end of one of the tasks.)

The interesting constraints are the ones that prevent two tasks running on the machine at the same time. To express this condition, we define a "predicate", which states that either the second task starts after the first plus its duration or else (written "\/") the first task starts after the second plus its duration.

```
predicate not_at_same_time(1..NTasks: t1, 1..NTasks: t2) =
               start[t1] + Duration[t1] <= start[t2]
          \/   start[t2] + Duration[t2] <= start[t1] ;
```

The constraint enforces that this predicate holds for every pair of tasks:

```
constraint   forall (t1,t2 in 1..NTasks where t1 < t2)
                  ( not_at_same_time(t1,t2) ) ;
```

Such a constraint, whose definition is an arbitrary predicate, cannot be expressed in a mathematical modelling language.

14.12.3 Search and Propagation

MiniZinc has a default search where the user need only specify what expression should be minimized, namely:

```
solve
     minimize all_completed ;
```

However, it is normal in constraint programming for the user to specify the order in which variables are assigned a value during search, and an order in which values are tried. A static variable order can be specified for example as follows:

```
solve  ::  int_search(start, input_order, indomain_min, complete)
     minimize all_completed ;
```

The part of the line after the "::" is termed an "annotation". Annotations do not change the logic of a model, but only add control. This annotation specifies that the start variables should be assigned a value in the order that they appear in the array. The first value to try is the earliest possible time point, and then later and later time points when exploring other alternatives during search. The search should be "complete" meaning every possible alternative should be explored until it is proven that the best solution has been found.

Naturally MiniZinc does not have to try every time point for every task start time because some values are ruled out by constraints (for example if the first task has been assigned a start time of 0, then other tasks will only have possible start times starting from the end of the first task—that is, time point 15).

Moreover, once a solution has been found more time points will be removed from the set of alternatives for each task. For example if a solution has been found with a completion time of 45 then, for the start time of the last task, all time points after 40 are ruled out.

The search control annotations offer many other possibilities. Instead of a static order, the next variable to be assigned might be chosen on the basis of the current search state. A common dynamic variable choice heuristic is to select next the variable with the fewest alternative values remaining, which is called "first_fail". Also instead of selecting a value the next variable may simply have its set of alternatives reduced by simply ruling out the higher valued ones as a group (this is termed "domain splitting"). For scheduling tasks, domain splitting is an effective search method. Thus we can control the search as follows:

```
solve  ::  int_search(start, first_fail, indomain_split, complete)
        minimize all_completed ;
```

The whole MiniZinc program for solving the one-machine scheduling problem is as follows. To run it load MiniZinc and run it on a file called "your_name.mzn" containing this model.

```
int: Horizon = 50 ;
int: NTasks = 5 ;

array[1..NTasks] of var 0..Horizon: start ;
var 0..Horizon: all_completed ;

array[1..NTasks] of int: Release_date = [0,10,11,20,25] ;
array[1..NTasks] of int: Duration = [12,10,8,10,5] ;
array[1..NTasks] of int: Due_date = [17,40,30,35,45] ;

constraint   forall (t in 1..NTasks)
```

```
                ( start[t] >= Release_date[t] ) ;

constraint    forall (t in 1..NTasks)
                ( start[t]+Duration[t] <= Due_date[t] ) ;

constraint    forall (t in 1..NTasks)
                ( start[t] + Duration[t] <= all_completed ) ;

predicate not_at_same_time(1..NTasks: t1, 1..NTasks: t2) =
              start[t1] + Duration[t1] <= start[t2]
          \/  start[t2] + Duration[t2] <= start[t1] ;

constraint   forall (t1,t2 in 1..NTasks where t1 < t2)
                 ( not_at_same_time(t1,t2) ) ;

solve  ::  int_search(start, first_fail, indomain_split, complete)
           minimize all_completed ;
```

14.12.4 Introducing Redundant Constraints

The first way to enhance this algorithm is by adding a global constraint, specialized for scheduling problems (see *Global Constraints* above). The new constraint does not remove any solutions: it is logically redundant. However, its powerful propagation behavior enables parts of the search space, where no solutions lie, to be pruned. Consequently, the number of search steps is reduced—dramatically for larger problems! The algorithm was devised by operations researchers, but it has been encapsulated by constraint programmers as a single constraint.

14.12.5 Adding Search Heuristics

The next enhancement is to choose at each search step, first the task with the earliest due date. Whilst this does tend to yield feasible solutions, it does not necessarily produce good solutions, until the end time constraints become tight.

14.12.6 Using an Incomplete Search Technique

For very large problems, complete search may not be possible. In this case the algorithm may be controlled so as to limit the effort wasted in exploring unpromising parts of the search space. This can be done simply by limiting the number of times a non-preferred ordering of tasks is imposed during search and backtracking.

The above techniques combine very easily, and the combination is very powerful indeed. As a result, constraint programming is currently the technology of choice for operational scheduling problems where task orderings are significant.

Sources of Additional Information

Sources of information about constraint programming include:

- The International Conferences on Principles and Practice of Constraint Programming, whose proceedings are available in the Springer LNCS series;
- The International Conference on Integration of AI and OR Techniques in Constraint Programming for Combinatorial Optimization Problems (CPAIOR), whose proceedings are also published in Springer LNCS;
- The *Constraints* journal published by Springer;
- *Handbook of Constraint Programming*, ed. Rossi et al., Elsevier, 2006;
- *Constraint Processing*, Rina Dechter. Morgan Kaufmann, 2003;
- *Programming with Constraints: an Introduction*, Kim Marriott and Peter Stuckey. MIT Press, 1998;
- Online Guide to Constraint Programming, maintained by Roman Barták: http://kti.ms.mff.cuni.cz/~bartak/constraints/
- Constraints groups:
  - http://tech.groups.yahoo.com/group/constraints/
  - http://groups.google.ie/group/comp.constraints
- The Association for Constraint Programming: http://4c.ucc.ie/a4cp/
- Constraint Programming Online: http://4c.ucc.ie/cponline/

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