LECTURE 2

Characters

This lecture contains the heart of our treatment of the representation theory of finite groups: the definition in §2.1 of the character of a representation, and the main theorem (proved in two steps in §2.2 and §2.4) that the characters of the irreducible representations form an orthonormal basis for the space of class functions on G. There will be more examples and more constructions in the following lectures, but this is what you need to know.

§2.1. Characters

As we indicated in the preceding section, there is a remarkably effective tool for understanding the representations of a finite group G, called *character theory*. This is in some ways motivated by the example worked out in the last section where we saw that a representation of $\mathfrak{S}_3$ was determined by knowing the eigenvalues of the action of the elements τ and $\sigma \in \mathfrak{S}_3$. For a general group G, it is not clear what subgroups and/or elements should play the role of $\mathfrak{A}_3$, τ, and σ; but the example certainly suggests that knowing all the eigenvalues of each element of G should suffice to describe the representation.

Of course, specifying all the eigenvalues of the action of each element of G is somewhat unwieldy; but fortunately it is redundant as well. For example, if we know the eigenvalues $\{\lambda_i\}$ of an element $g \in G$, then of course we know the eigenvalues $\{\lambda_i^k\}$ of g^k for each k as well. We can thus use this redundancy to simplify the data we have to specify. The key observation here is it is enough to give, for example, just the *sum* of the eigenvalues of each element of G, since knowing the sums $\sum \lambda_i^k$ of the kth powers of the eigenvalues of a given element $g \in G$ is equivalent to knowing the eigenvalues $\{\lambda_i\}$ of g themselves. This then suggests the following:

Definition. If V is a representation of G, its *character* χ_V is the complex-valued function on the group defined by

$$\chi_V(g) = \mathrm{Tr}(g|_V),$$

the trace of g on V.

In particular, we have

$$\chi_V(hgh^{-1}) = \chi_V(g),$$

so that χ_V is constant on the conjugacy classes of G; such a function is called a *class function*. Note that $\chi_V(1) = \dim V$.

Proposition 2.1. *Let V and W be representations of G. Then*

$$\chi_{V \oplus W} = \chi_V + \chi_W, \qquad \chi_{V \otimes W} = \chi_V \cdot \chi_W,$$
$$\chi_{V^*} = \overline{\chi}_V \quad \textit{and} \quad \chi_{\wedge^2 V}(g) = \tfrac{1}{2}[\chi_V(g)^2 - \chi_V(g^2)].$$

Proof. We compute the values of these characters on a fixed element $g \in G$. For the action of g, V has eigenvalues $\{\lambda_i\}$ and W has eigenvalues $\{\mu_i\}$. Then $\{\lambda_i\} \cup \{\mu_j\}$ and $\{\lambda_i \cdot \mu_j\}$ are eigenvalues for $V \oplus W$ and $V \otimes W$, from which the first two formulas follow. Similarly $\{\lambda_i^{-1} = \overline{\lambda}_i\}$ are the eigenvalues for g on V^*, since all eigenvalues are nth roots of unity, with n the order of g. Finally, $\{\lambda_i\lambda_j | i < j\}$ are the eigenvalues for g on $\wedge^2 V$, and

$$\sum_{i<j} \lambda_i \lambda_j = \frac{(\sum \lambda_i)^2 - \sum \lambda_i^2}{2};$$

and since g^2 has eigenvalues $\{\lambda_i^2\}$, the last formula follows. □

Exercise 2.2. For $\mathrm{Sym}^2 V$, verify that

$$\chi_{\mathrm{Sym}^2 V}(g) = \tfrac{1}{2}[\chi_V(g)^2 + \chi_V(g^2)].$$

Note that this is compatible with the decomposition

$$V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V.$$

Exercise 2.3*. Compute the characters of $\mathrm{Sym}^k V$ and $\wedge^k V$.

Exercise 2.4*. Show that if we know the character χ_V of a representation V, then we know the eigenvalues of each element g of G, in the sense that we know the coefficients of the characteristic polynomial of $g: V \to V$. Carry this out explicitly for elements $g \in G$ of orders 2, 3, and 4, and for a representation of G on a vector space of dimension 2, 3, or 4.

Exercise 2.5. (*The original fixed-point formula*). If V is the permutation representation associated to the action of a group G on a finite set X, show that $\chi_V(g)$ is the number of elements of X fixed by g.

As we have said, the character of a representation of a group G is really a function on the set of conjugacy classes in G. This suggests expressing the basic information about the irreducible representations of a group G in the form of a *character table*. This is a table with the conjugacy classes $[g]$ of G listed across the top, usually given by a representative g, with (for reasons that will become apparent later) the number of elements in each conjugacy class over it; the irreducible representations V of G listed on the left; and, in the appropriate box, the value of the character χ_V on the conjugacy class $[g]$.

Example 2.6. We compute the character table of $\mathfrak{S}_3$. This is easy: to begin with, the trivial representation takes the values (1, 1, 1) on the three conjugacy classes [1], [(12)], and [(123)], whereas the alternating representation has values $(1, -1, 1)$. To see the character of the standard representation, note that the permutation representation decomposes: $\mathbb{C}^3 = U \oplus V$; since the character of the permutation representation has, by Exercise 2.5, the values (3, 1, 0), we have $\chi_V = \chi_{\mathbb{C}^3} - \chi_U = (3, 1, 0) - (1, 1, 1) = (2, 0, -1)$. In sum, then, the character table of $\mathfrak{S}_3$ is

	1	3	2
$\mathfrak{S}_3$	1	(12)	(123)
trivial U	1	1	1
alternating U'	1	-1	1
standard V	2	0	-1

This gives us another solution of the basic problem posed in Lecture 1: if W is any representation of $\mathfrak{S}_3$ and we decompose W into irreducible representations $W \cong U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$, then $\chi_W = a\chi_U + b\chi_{U'} + c\chi_V$. In particular, since the functions χ_U, $\chi_{U'}$ and χ_V are independent, we see that *W is determined up to isomorphism by its character χ_W.*

Consider, for example, $V \otimes V$. Its character is $(\chi_V)^2$, which has values 4, 0, and 1 on the three conjugacy classes. Since $V \oplus U \oplus U'$ has the same character, this implies that $V \otimes V$ decomposes into $V \oplus U \oplus U'$, as we have seen directly. Similarly, $V \otimes U'$ has values 2, 0, and -1, so $V \otimes U' \cong V$.

Exercise 2.7*. Find the decomposition of the representation $V^{\otimes n}$ using character theory.

Characters will be similarly useful for larger groups, although it is rare to find simple closed formulas for decomposing tensor products.

§2.2. The First Projection Formula and Its Consequences

In the last lecture, we asked (among other things) for a way of locating explicitly the direct sum factors in the decomposition of a representation into irreducible ones. In this section we will start by giving an explicit formula for the projection of a representation onto the direct sum of the trivial factors in this decomposition; as it will turn out, this formula alone has tremendous consequences.

To start, for any representation V of a group G, we set

$$V^G = \{v \in V: gv = v \quad \forall g \in G\}.$$

We ask for a way of finding V^G explicitly. The idea behind our solution to this is already implicit in the previous lecture. We observed there that for any representation V of G and any $g \in G$, the endomorphism $g: V \to V$ is, in general, not a G-module homomorphism. On the other hand, if we take the *average* of all these endomorphisms, that is, we set

$$\varphi = \frac{1}{|G|} \sum_{g \in G} g \in \operatorname{End}(V),$$

then the endomorphism φ will be G-linear since $\sum g = \sum hgh^{-1}$. In fact, we have

Proposition 2.8. *The map φ is a projection of V onto V^G.*

Proof. First, suppose $v = \varphi(w) = (1/|G|)\sum gw$. Then, for any $h \in G$,

$$hv = \frac{1}{|G|}\sum hgw = \frac{1}{|G|}\sum gw,$$

so the image of φ is contained in V^G. Conversely, if $v \in V^G$, then $\varphi(v) = (1/|G|)\sum v = v$, so $V^G \subset \operatorname{Im}(\varphi)$; and $\varphi \circ \varphi = \varphi$. □

We thus have a way of finding explicitly the direct sum of the trivial subrepresentations of a given representation, although the formula can be hard to use if it does not simplify. If we just want to know the number m of copies of the trivial representation appearing in the decomposition of V, we can do this numerically, since this number will be just the trace of the projection φ. We have

$$m = \dim V^G = \operatorname{Trace}(\varphi)$$
$$= \frac{1}{|G|}\sum_{g \in G} \operatorname{Trace}(g) = \frac{1}{|G|}\sum_{g \in G} \chi_V(g). \tag{2.9}$$

In particular, we observe that for an irreducible representation V other than the trivial one, the sum over all $g \in G$ of the values of the character χ_V is zero.

We can do much more with this idea, however. The key is to use Exercise 1.2: if V and W are representations of G, then with $\operatorname{Hom}(V, W)$, the representation defined in Lecture 1, we have

$$\operatorname{Hom}(V, W)^G = \{G\text{-module homomorphisms from } V \text{ to } W\}.$$

If V is irreducible then by Schur's lemma $\dim \operatorname{Hom}(V, W)^G$ is the multiplicity of V in W; similarly, if W is irreducible, $\dim \operatorname{Hom}(V, W)^G$ is the multiplicity of W in V, and in the case where both V and W are irreducible, we have

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W. \end{cases}$$

But now the character $\chi_{\operatorname{Hom}(V,W)}$ of the representation $\operatorname{Hom}(V, W) = V^* \otimes W$ is given by

$$\chi_{\operatorname{Hom}(V,W)}(g) = \overline{\chi_V(g)} \cdot \chi_W(g).$$

We can now apply formula (2.9) in this case to obtain the striking

$$\frac{1}{|G|}\sum_{g \in G} \overline{\chi_V(g)}\chi_W(g) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W. \end{cases} \tag{2.10}$$

To express this, let

$$\mathbb{C}_{\text{class}}(G) = \{\text{class functions on } G\}$$

and define an Hermitian inner product on $\mathbb{C}_{\text{class}}(G)$ by

$$(\alpha, \beta) = \frac{1}{|G|}\sum_{g \in G} \overline{\alpha(g)}\beta(g). \tag{2.11}$$

Formula (2.10) then amounts to

Theorem 2.12. *In terms of this inner product, the characters of the irreducible representations of G are orthonormal.*

For example, the orthonormality of the three irreducible representations of $\mathfrak{S}_3$ can be read from its character table in Example 2.6. The numbers over each conjugacy class tell how many times to count entries in that column.

Corollary 2.13. *The number of irreducible representations of G is less than or equal to the number of conjugacy classes.*

We will soon show that there are no nonzero class functions orthogonal to the characters, so that equality holds in Corollary 2.13.

Corollary 2.14. *Any representation is determined by its character.*

Indeed if $V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$, with the V_i distinct irreducible representations, then $\chi_V = \sum a_i \chi_{V_i}$, and the χ_{V_i} are linearly independent.

Corollary 2.15. *A representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$.*

In fact, if $V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$ as above, then $(\chi_V, \chi_V) = \sum a_i^2$.
The multiplicities a_i can be calculated via

Corollary 2.16. *The multiplicity a_i of V_i in V is the inner product of χ_V with χ_{V_i}, i.e., $a_i = (\chi_V, \chi_{V_i})$.*

We obtain some further corollaries by applying all this to the regular representation R of G. First, by Exercise 2.5 we know the character of R; it is simply

$$\chi_R(g) = \begin{cases} 0 & \text{if } g \neq e \\ |G| & \text{if } g = e. \end{cases}$$

Thus, we see first of all that R is not irreducible if $G \neq \{e\}$. In fact, if we set $R = \bigoplus V_i^{\oplus a_i}$, with V_i distinct irreducibles, then

$$a_i = (\chi_{V_i}, \chi_R) = \frac{1}{|G|} \chi_{V_i}(e) \cdot |G| = \dim V_i. \tag{2.17}$$

Corollary 2.18. *Any irreducible representation V of G appears in the regular representation* dim *V times.*

In particular, this proves again that there are only finitely many irreducible representations. As a numerical consequence of this we have the formula

$$|G| = \dim(R) = \sum_i \dim(V_i)^2. \tag{2.19}$$

Also, applying this to the value of the character of the regular representation on an element $g \in G$ other than the identity, we have

$$0 = \sum (\dim V_i) \cdot \chi_{V_i}(g) \quad \text{if } g \neq e. \tag{2.20}$$

These two formulas amount to the Fourier inversion formula for finite groups, cf. Example 3.32. For example, if all but one of the characters is known, they give a formula for the unknown character.

Exercise 2.21. The orthogonality of the rows of the character table is equivalent to an orthogonality for the columns (assuming the fact that there are as many rows as columns). Written out, this says:

(i) For $g \in G$,

$$\sum_{\chi} \overline{\chi(g)}\chi(g) = \frac{|G|}{c(g)},$$

where the sum is over all irreducible characters, and $c(g)$ is the number of elements in the conjugacy class of g.

(ii) If g and h are elements of G that are not conjugate, then

$$\sum_{\chi} \overline{\chi(g)}\chi(h) = 0.$$

Note that for $g = e$ these reduce to (2.19) and (2.20).

§2.3. Examples: $\mathfrak{S}_4$ and $\mathfrak{A}_4$

To see how the analysis of the characters of a group actually goes in practice, we now work out the character table of $\mathfrak{S}_4$. To start, we list the conjugacy classes in $\mathfrak{S}_4$ and the number of elements of $\mathfrak{S}_4$ in each. As with any symmetric group $\mathfrak{S}_d$, the conjugacy classes correspond naturally to the *partitions* of d, that is, expressions of d as a sum of positive integers $a_1, a_2, \ldots, a_k$, where the correspondence associates to such a partition the conjugacy class of a permutation consisting of disjoint cycles of length $a_1, a_2, \ldots, a_k$. Thus, in $\mathfrak{S}_4$ we have the classes of the identity element 1 $(4 = 1 + 1 + 1 + 1)$, a transposition such as (12), corresponding to the partition $4 = 2 + 1 + 1$; a three-cycle (123) corresponding to $4 = 3 + 1$; a four-cycle (1234) $(4 = 4)$; and the product of two disjoint transpositions (12)(34) $(4 = 2 + 2)$.

Exercise 2.22. Show that the number of elements in each of these conjugacy classes is, respectively, 1, 6, 8, 6, and 3.

As for the irreducible representations of $\mathfrak{S}_4$, we start with the same ones that we had in the case of $\mathfrak{S}_3$: the trivial U, the alternating U', and the standard representation V, i.e., the quotient of the permutation representation associated to the standard action of $\mathfrak{S}_4$ on a set of four elements by the trivial subrepresentation. The character of the trivial representation on the five conjugacy classes is of course (1, 1, 1, 1, 1), and that of the alternating representation is $(1, -1, 1, -1, 1)$. To find the character of the standard representation, we observe that by Exercise 2.5 the character of the permutation representation on $\mathbb{C}^4$ is $\chi_{\mathbb{C}^4} = (4, 2, 1, 0, 0)$ and, correspondingly,

$$\chi_V = \chi_{\mathbb{C}^4} - \chi_U = (3, 1, 0, -1, -1).$$

Note that $|\chi_V| = 1$, so V is irreducible. The character table so far looks like

	1	6	8	6	3
$\mathfrak{S}_4$	1	(12)	(123)	(1234)	(12)(34)
trivial U	1	1	1	1	1
alternating U'	1	−1	1	−1	1
standard V	3	1	0	−1	−1

Clearly, we are not done yet: since the sum of the squares of the dimensions of these three representations is $1 + 1 + 9 = 11$, by (2.19) there must be additional irreducible representations of $\mathfrak{S}_4$, the squares of whose dimensions add up to $24 - 11 = 13$. Since there are by Corollary 2.13 at most two of them, there must be exactly two, of dimensions 2 and 3. The latter of these is easy to locate: if we just tensor the standard representation V with the alternating one U', we arrive at a representation V' with character $\chi_{V'} = \chi_V \cdot \chi_{U'} = (3, -1, 0, 1, -1)$. We can see that this is irreducible either from its character (since $|\chi_{V'}| = 1$) or from the fact that it is the tensor product of an irreducible representation with a one-dimensional one; since its character is not equal to that of any of the first three, this must be one of the two missing ones. As for the remaining representation of degree two, we will for now simply call it W; we can determine its character from the orthogonality relations (2.10). We obtain then the complete character table for $\mathfrak{S}_4$:

	1	6	8	6	3
$\mathfrak{S}_4$	1	(12)	(123)	(1234)	(12)(34)
trivial U	1	1	1	1	1
alternating U'	1	−1	1	−1	1
standard V	3	1	0	−1	−1
$V' = V \otimes U'$	3	−1	0	1	−1
Another W	2	0	−1	0	2

Exercise 2.23. Verify the last row of this table from (2.10) or (2.20).

We now get a dividend: we can take the character of the mystery representation W, which we have obtained from general character theory alone, and use it to describe the representation W explicitly! The key is the 2 in the last column for χ_W: this says that the action of (12)(34) on the two-dimensional vector space W is an involution of trace 2, and so must be the identity. Thus, W is really a representation of the quotient group[1]

[1] If N is a normal subgroup of a group G, a representation $\rho: G \to GL(V)$ is trivial on N if and only if it factors through the quotient

$$G \to G/N \to GL(V).$$

Representations of G/N can be identified with representations of G that are trivial on N.

$$\mathfrak{S}_4/\{1, (12)(34), (13)(24), (14)(23)\} \cong \mathfrak{S}_3.$$

[One may see this isomorphism by letting $\mathfrak{S}_4$ act on the elements of the conjugacy class of (12)(34); equivalently, if we realize $\mathfrak{S}_4$ as the group of rigid motions of a cube (see below), by looking at the action of $\mathfrak{S}_4$ on pairs of opposite faces.] W must then be just the standard representation of $\mathfrak{S}_3$ pulled back to $\mathfrak{S}_4$ via this quotient.

Example 2.24. As we said above, the group of rigid motions of a cube is the symmetric group on four letters; $\mathfrak{S}_4$ acts on the cube via its action on the four long diagonals. It follows, of course, that $\mathfrak{S}_4$ acts as well on the set of faces, of edges, of vertices, etc.; and to each of these is associated a permutation representation of $\mathfrak{S}_4$. We may thus ask how these representations decompose; we will do here the case of the faces and leave the others as exercises.

We start, of course, by describing the character χ of the permutation representation associated to the faces of the cube. Rotation by 180° about a line joining the midpoints of two opposite edges is a transposition in $\mathfrak{S}_4$ and fixes no faces, so $\chi(12) = 0$. Rotation by 120° about a long diagonal shows $\chi(123) = 0$. Rotation by 90° about a line joining the midpoints of two opposite faces shows $\chi(1234) = 2$, and rotation by 180° gives $\chi((12)(34)) = 2$. Now $(\chi, \chi) = 3$, so χ is the sum of three distinct irreducible representations. From the table, $(\chi, \chi_U) = (\chi, \chi_{V'}) = (\chi, \chi_W) = 1$, and the inner products with the others are zero, so this representation is $U \oplus V' \oplus W$. In fact, the sums of opposite faces span a three-dimensional subrepresentation which contains U (spanned by the sum of all faces), so this representation is $U \oplus W$. The differences of opposite faces therefore span V'.

Exercise 2.25*. Decompose the permutation representation of $\mathfrak{S}_4$ on (i) the vertices and (ii) the edges of the cube.

Exercise 2.26. The alternating group $\mathfrak{A}_4$ has four conjugacy classes. Three representations U, U', and U'' come from the representations of

$$\mathfrak{A}_4/\{1, (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}/3,$$

so there is one more irreducible representation V of dimension 3. Compute the character table, with $\omega = e^{2\pi i/3}$:

	1	4	4	3
$\mathfrak{A}_4$	1	(123)	(132)	(12)(34)
U	1	1	1	1
U'	1	ω	ω^2	1
U''	1	ω^2	ω	1
V	3	0	0	-1

Exercise 2.27. Consider the representations of $\mathfrak{S}_4$ and their restrictions to $\mathfrak{A}_4$. Which are still irreducible when restricted, and which decompose? Which pairs of nonisomorphic representations of $\mathfrak{S}_4$ become isomorphic when restricted? Which representations of $\mathfrak{A}_4$ arise as restrictions from $\mathfrak{S}_4$?

§2.4. More Projection Formulas; More Consequences

In this section, we complete the analysis of the characters of the irreducible representations of a general finite group begun in §2.2 and give a more general formula for the projection of a general representation V onto the direct sum of the factors in V isomorphic to a given irreducible representation W. The main idea for both is a generalization of the "averaging" of the endomorphisms $g: V \to V$ used in §2.2, the point being that instead of simply averaging all the g we can ask the question: what linear combinations of the endomorphisms $g: V \to V$ are G-linear endomorphisms? The answer is given by

Proposition 2.28. *Let $\alpha: G \to \mathbb{C}$ be any function on the group G, and for any representation V of G set*

$$\varphi_{\alpha,V} = \sum \alpha(g) \cdot g: V \to V.$$

Then $\varphi_{\alpha,V}$ is a homomorphism of G-modules for all V if and only if α is a class function.

Proof. We simply write out the condition that $\varphi_{\alpha,V}$ be G-linear, and the result falls out: we have

$$\begin{aligned}\varphi_{\alpha,V}(hv) &= \sum \alpha(g) \cdot g(hv) \\ &= \sum \alpha(hgh^{-1}) \cdot hgh^{-1}(hv)\end{aligned}$$

(substituting hgh^{-1} for g)

$$\begin{aligned} &= h\left(\sum \alpha(hgh^{-1}) \cdot g(v)\right) \\ &= h\left(\sum \alpha(g) \cdot g(v)\right)\end{aligned}$$

(if α is a class function)

$$= h(\varphi_{\alpha,V}(v)).$$

Exercise 2.29*. Complete this proof by showing that conversely if α is not a class function, then there exists a representation V of G for which $\varphi_{\alpha,V}$ fails to be G-linear. □

As an immediate consequence of this proposition, we have

Proposition 2.30. *The number of irreducible representations of G is equal to the number of conjugacy classes of G. Equivalently, their characters $\{\chi_V\}$ form an orthonormal basis for $\mathbb{C}_{\text{class}}(G)$.*

Proof. Suppose $\alpha: G \to \mathbb{C}$ is a class function and $(\alpha, \chi_V) = 0$ for all irreducible representations V; we must show that $\alpha = 0$. Consider the endomorphism

$$\varphi_{\alpha,V} = \sum \alpha(g) \cdot g: V \to V$$

as defined above. By Schur's lemma, $\varphi_{\alpha,V} = \lambda \cdot \text{Id}$; and if $n = \dim V$, then

$$\begin{aligned}\lambda &= \frac{1}{n} \cdot \text{trace}(\varphi_{\alpha,V}) \\ &= \frac{1}{n} \cdot \sum \alpha(g)\chi_V(g) \\ &= \frac{|G|}{n}\overline{(\alpha, \chi_{V^*})} \\ &= 0.\end{aligned}$$

Thus, $\varphi_{\alpha,V} = 0$, or $\sum \alpha(g) \cdot g = 0$ on any representation V of G; in particular, this will be true for the regular representation $V = R$. But in R the elements $\{g \in G\}$, thought of as elements of $\text{End}(R)$, are linearly independent. For example, the elements $\{g(e)\}$ are all independent. Thus $\alpha(g) = 0$ for all g, as required. □

This proposition completes the description of the characters of a finite group in general. We will see in more examples below how we can use this information to build up the character table of a given group. For now, we mention another way of expressing this proposition, via the *representation ring* of the group G.

The representation ring $R(G)$ of a group G is easy to define. First, as a group we just take $R(G)$ to be the free abelian group generated by all (isomorphism classes of) representations of G, and mod out by the subgroup generated by elements of the form $V + W - (V \oplus W)$. Equivalently, given the statement of complete reducibility, we can just take all integral linear combinations $\sum a_i \cdot V_i$ of the irreducible representations V_i of G; elements of $R(G)$ are correspondingly called *virtual representations*. The ring structure is then given simply by tensor product, defined on the generators of $R(G)$ and extended by linearity.

We can express most of what we have learned so far about representations of a finite group G in these terms. To begin, the character defines a map

$$\chi: R(G) \to \mathbb{C}_{\text{class}}(G)$$

from $R(G)$ to the ring of complex-valued functions on G; by the basic formulas of Proposition 2.1, this map is in fact a ring homomorphism. The statement that a representation in determined by its character then says that χ is injective;

the images of χ are called *virtual characters* and correspond thereby to virtual representations. Finally, our last proposition amounts to the statement that *χ induces an isomorphism*

$$\chi_{\mathbb{C}}: R(G) \otimes \mathbb{C} \to \mathbb{C}_{\text{class}}(G).$$

The virtual characters of G form a lattice $\Lambda \cong \mathbb{Z}^c$ in $\mathbb{C}_{\text{class}}(G)$, in which the actual characters sit as a cone $\Lambda_0 \cong \mathbb{N}^c \subset \mathbb{Z}^c$. We can thus think of the problem of describing the characters of G as having two parts: first, we have to find Λ, and then the cone $\Lambda_0 \subset \Lambda$ (once we know Λ_0, the characters of the irreducible representations will be determined). In the following lecture we will state theorems of Artin and Brauer characterizing $\Lambda \otimes \mathbb{Q}$ and Λ.

The argument for Proposition 2.30 also suggests how to obtain a more general projection formula. Explicitly, if W is a fixed irreducible representation, then for any representation V, look at the weighted sum

$$\psi = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot g \in \operatorname{End}(V).$$

By Proposition 2.28, ψ is a G-module homomorphism. Hence, if V is irreducible, we have $\psi = \lambda \cdot \mathrm{Id}$, and

$$\begin{aligned} \lambda &= \frac{1}{\dim V} \operatorname{Trace} \psi \\ &= \frac{1}{\dim V} \cdot \frac{1}{|G|} \sum \overline{\chi_W(g)} \cdot \chi_V(g) \\ &= \begin{cases} \dfrac{1}{\dim V} & \text{if } V = W \\ 0 & \text{if } V \neq W. \end{cases} \end{aligned}$$

For arbitrary V,

$$\psi_V = \dim W \cdot \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot g : V \to V \tag{2.31}$$

is the projection of V onto the factor consisting of the sum of all copies of W appearing in V. In other words, if $V = \bigoplus V_i^{\oplus a_i}$, then

$$\pi_i = \dim V_i \cdot \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{V_i}(g)} \cdot g \tag{2.32}$$

is the projection of V onto $V_i^{\oplus a_i}$.

Exercise 2.33*. (a) In terms of representations V and W in $R(G)$, the inner product on $\mathbb{C}_{\text{class}}(G)$ takes the simple form

$$(V, W) = \dim \operatorname{Hom}_G(V, W).$$

(b) If $\chi \in \mathbb{C}_{\text{class}}(G)$ is a virtual character, and $(\chi, \chi) = 1$, then either χ or $-\chi$ is the character of an irreducible representation, the plus sign occurring when $\chi(1) > 0$. If $(\chi, \chi) = 2$, and $\chi(1) > 0$, then χ is either the sum or the difference of two irreducible characters.

(c) If U, V, and W are irreducible representations, show that U appears in $V \otimes W$ if and only if W occurs in $V^* \otimes U$. Deduce that this cannot occur unless $\dim U \geq \dim W/\dim V$.

We conclude this lecture with some exercises that use characters to work out some standard facts about representations.

Exercise 2.34*. Let V and W be irreducible representations of G, and L_0: $V \to W$ any linear mapping. Define L: $V \to W$ by

$$L(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot L_0(g \cdot v).$$

Show that $L = 0$ if V and W are not isomorphic, and that L is multiplication by $\text{trace}(L_0)/\dim(V)$ if $V = W$.

Exercise 2.35*. Show that, if the irreducible representations of G are represented by unitary matrices [cf. Exercise 1.14], the matrix entries of these representations form an orthogonal basis for the space of *all* functions on G [with inner product given by (2.11)].

Exercise 2.36*. If G_1 and G_2 are groups, and V_1 and V_2 are representations of G_1 and G_2, then the tensor product $V_1 \otimes V_2$ is a representation of $G_1 \times G_2$, by $(g_1 \times g_2) \cdot (v_1 \otimes v_2) = g_1 \cdot v_1 \otimes g_2 \cdot v_2$. To distinguish this "external" tensor product from the internal tensor product—when $G_1 = G_2$—this *external tensor product* is sometimes denoted $V_1 \boxtimes V_2$. If χ_i is the character of V_i, then the value of the character χ of $V_1 \boxtimes V_2$ is given by the product:

$$\chi(g_1 \times g_2) = \chi_1(g_1)\chi_2(g_2).$$

If V_1 and V_2 are irreducible, show that $V_1 \boxtimes V_2$ is also irreducible and show that every irreducible representation of $G_1 \times G_2$ arises this way. In terms of representation rings,

$$R(G_1 \times G_2) = R(G_1) \otimes R(G_2).$$

In these lectures we will often be given a subgroup G of a general linear group $\text{GL}(V)$, and we will look for other representations inside tensor powers of V. The following problem, which is a theorem of Burnside and Molien, shows that for a finite group G, all irreducible representations can be found this way.

Problem 2.37*. Show that if V is a faithful representation of G, i.e., $\rho: G \to \mathrm{GL}(V)$ is injective, then any irreducible representation of G is contained in some tensor power $V^{\otimes n}$ of V.

Problem 2.38*. Show that the dimension of an irreducible representation of G divides the order of G.

Another challenge:

Problem 2.39*. Show that the character of any irreducible representation of dimension greater than 1 assumes the value 0 on some conjugacy class of the group.