LECTURE 23

Complex Lie Groups; Characters

This lecture serves two functions. First and foremost, we make the transition back from Lie algebras to Lie groups: in §23.1 we classify the groups having a given semisimple Lie algebra, and say which representations of the Lie algebra, as described in the preceding lectures, lift to which groups. Secondly, we introduce in §23.2 the notion of *character* in the context of Lie theory; this gives us another way of describing the representations of the classical groups, and also provides a necessary framework for the results of the following two lectures. Then in §23.3 we sketch the beautiful interrelationships among Dynkin diagrams, compact homogeneous spaces and the irreducible representations of a Lie group. The first two sections are elementary modulo a little topology needed to calculate the fundamental groups of the classical groups in §23.1. The third section, by contrast, may appear impossible: it involves, at various points, projective algebraic geometry, holomorphic line bundles, and their cohomology. In fact, a good deal of §23.3 can be understood without these notions; the reader is encouraged to read as much of the section as seems intelligible. A final section §23.4 gives a very brief introduction to the related Bruhat decomposition, which is included because of its ubiquity in the literature.

§23.1. Representations of Complex Simple Lie Groups

In Lecture 21 we classified all simple Lie algebras over $\mathbb{C}$. This in turn yields a classification of simple complex Lie groups: as we saw in Lecture 7, for any Lie algebra $\mathfrak{g}$ there is a unique simply connected group G, and all other (connected) complex Lie groups with Lie algebra $\mathfrak{g}$ are quotients of G by

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discrete subgroups of the center $Z(G)$. In this section, we will first describe the groups associated to the classical Lie algebras, and then proceed to describe which of the representations of the classical algebras we have described in Part III lift to which of the groups. We start with

Proposition 23.1. *For all $n \geq 1$, the Lie groups $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$ are connected and simply connected. For $n \geq 1$, $\mathrm{SO}_n\mathbb{C}$ is connected, with $\pi_1(\mathrm{SO}_2\mathbb{C}) = \mathbb{Z}$, and $\pi_1(\mathrm{SO}_n\mathbb{C}) = \mathbb{Z}/2$ for $n \geq 3$.*

Proof. The main tool needed from topology is the long exact homotopy sequence of a fibration. If the Lie group G acts transitively on a manifold M, and H is the isotropy group of a point P_0 of M, then $G/H = M$, and the map $G \to M$ by $g \mapsto g \cdot P_0$ is a fibration with fiber H. The resulting long exact sequence is, assuming the spaces are connected,

$$\cdots \to \pi_2(M) \to \pi_1(H) \to \pi_1(G) \to \pi_1(M) \to \{1\}. \tag{23.2}$$

(The base points, which are omitted in this notation, can be taken to be the identity elements of H and G, and the point P_0 in M.) In practice we will know M and H are connected, from which it follows that G is also connected. From this exact sequence, if M and H are also simply connected, the same follows for G.

To apply the long exact homotopy sequence in our present circumstance we argue by induction, noting first that $\mathrm{SL}_1\mathbb{C} = \mathrm{SO}_1\mathbb{C} = \{1\}$. Now consider the action of $G = \mathrm{SL}_n\mathbb{C}$ on the manifold $M = \mathbb{C}^n \backslash \{0\}$. The subgroup H fixing the vector $P_0 = (1, 0, \ldots, 0)$ consists of matrices whose first column is $(1, 0, \ldots, 0)$ and whose lower right $(n-1)$ by $(n-1)$ matrix is in $\mathrm{SL}_{n-1}\mathbb{C}$; it follows that as topological spaces $H \cong \mathrm{SL}_{n-1}\mathbb{C} \times \mathbb{C}^{n-1}$. Since M is simply connected for $n \geq 2$ (having the sphere S^{2n-1} as a deformation retract), and H has $\mathrm{SL}_{n-1}\mathbb{C}$ as a deformation retract, the claim for $\mathrm{SL}_n\mathbb{C}$ follows from (23.2) by induction on n.

The group $\mathrm{SO}_2\mathbb{C}$ is isomorphic to the multiplicative group $\mathbb{C}^*$, which has the circle as a deformation retract, so $\pi_1(\mathrm{SO}_2\mathbb{C}) = \mathbb{Z}$. The group $G = \mathrm{SO}_n\mathbb{C}$ acts transitively on $M = \{v \in \mathbb{C}^n \colon Q(v, v) = 1\}$, where Q is the symmetric bilinear form preserved by G. (The transitivity of the action is more or less equivalent to knowing that all nondegenerate symmetric bilinear forms are equivalent.) For explicit calculations take the standard Q for which the standard basis $\{e_i\}$ of $\mathbb{C}^n$ is an orthonormal basis. This time the subgroup H fixing e_1 is $\mathrm{SO}_{n-1}\mathbb{C}$. From the following exercise, it follows that M has the sphere S^{n-1} as a deformation retract. By (23.2) the map

$$\pi_1(\mathrm{SO}_{n-1}\mathbb{C}) \to \pi_1(\mathrm{SO}_n\mathbb{C})$$

is an isomorphism for $n \geq 4$. So it suffices to look at $\mathrm{SO}_3\mathbb{C}$. This could be done by looking at the maps in the sme exact sequence, but we saw in Lecture 10 that $\mathrm{SO}_3\mathbb{C}$ has a two-sheeted covering by $\mathrm{SL}_2\mathbb{C}$, which is simply connected by the preceding paragraph, so $\pi_1(\mathrm{SO}_3\mathbb{C}) = \mathbb{Z}/2$, as required.

The group $G = \mathrm{Sp}_{2n}\mathbb{C}$ acts transitively on

$$M = \{(v, w) \in \mathbb{C}^{2n} \times \mathbb{C}^{2n} : Q(v, w) = 1\},$$

where Q is the skew form preserved by G, and the isotropy group is $\mathrm{Sp}_{2n-2}\mathbb{C}$. Since $\mathrm{Sp}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}$, the first case is known. By the following exercise, since M is defined in $\mathbb{C}^{4n}$ by a nondegenerate quadratic form, M has S^{4n-1} as a deformation retract, so we conclude again by induction. □

Exercise 23.3*. Show that $\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum z_i^2 = 1\}$ is homeomorphic to the tangent bundle to the $(n-1)$-sphere, i.e., to

$$T_{S^{n-1}} = \{(u, v) \in S^{n-1} \times \mathbb{R}^n : u \cdot v = 0\}.$$

Using the exact sequence $\{1\} \to \mathrm{SL}_n\mathbb{C} \to \mathrm{GL}_n\mathbb{C} \to \mathbb{C}^* \to \{1\}$ we deduce from the proposition and (23.2) that

$$\pi_1(\mathrm{GL}_n\mathbb{C}) = \mathbb{Z}. \tag{23.4}$$

Exercise 23.5. Show that for all the above groups G, the second homotopy groups $\pi_2(G)$ are trivial.

We digress a moment here to mention a famous fact. Each of the above groups G has an associated compact subgroup: $\mathrm{SU}(n) \subset \mathrm{SL}_n\mathbb{C}$, $\mathrm{Sp}(n) \subset \mathrm{Sp}_{2n}\mathbb{C}$, and $\mathrm{SO}(n) \subset \mathrm{SO}_n\mathbb{C}$. In fact, each of these subgroups is connected, and these inclusions induce isomorphisms of their fundamental groups.

Exercise 23.6. Prove these assertions by finding compatible actions of the subgroups on appropriate manifolds. Alternatively, observe that in each case the compact subgroup in question is just the subgroup of G preserving a Hermitian form on $\mathbb{C}^n$ or $\mathbb{C}^{2n}$, and use Gram–Schmidt to give a retraction of G onto the subgroup.

Now, by Proposition 23.1 the simply-connected complex Lie groups corresponding to the Lie algebras $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_m\mathbb{C}$ are

$$\tilde{G} = \mathrm{SL}_n\mathbb{C}, \quad \mathrm{Sp}_{2n}\mathbb{C}, \quad \text{and } \mathrm{Spin}_m\mathbb{C}.$$

We also know the center $Z(\tilde{G})$ of each of these groups. From Lecture 7 we also know the other connected groups with these Lie algebras:

- The complex Lie groups with Lie algebra $\mathfrak{sl}_n\mathbb{C}$ are $\mathrm{SL}_n\mathbb{C}$ and quotients of $\mathrm{SL}_n\mathbb{C}$ by subgroups of the form $\{e^{2\pi l i/m} \cdot I\}_l$ for m dividing n (in particular, if n is prime the only such groups are $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{PSL}_n\mathbb{C}$).
- The complex Lie groups with Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ are $\mathrm{Sp}_{2n}\mathbb{C}$ and $\mathrm{PSp}_{2n}\mathbb{C}$.
- The complex Lie groups with Lie algebra $\mathfrak{so}_{2n+1}\mathbb{C}$ are $\mathrm{Spin}_{2n+1}\mathbb{C}$ and $\mathrm{SO}_{2n+1}\mathbb{C}$.
  and
- The complex Lie groups with Lie algebra $\mathfrak{so}_{2n}\mathbb{C}$ are $\mathrm{Spin}_{2n}\mathbb{C}$, $\mathrm{SO}_{2n}\mathbb{C}$ and $\mathrm{PSO}_{2n}\mathbb{C}$; in addition, if n is even, there are two other groups covered doubly by $\mathrm{Spin}_{2n}\mathbb{C}$ and covering doubly $\mathrm{PSO}_{2n}\mathbb{C}$ [cf. Exercise 20.36].

These are called the *classical groups*. In the cases where we have observed coincidences of Lie algebras, we have the following isomorphisms of groups:

$$\mathrm{Spin}_3\mathbb{C} \cong \mathrm{SL}_2\mathbb{C} \quad \text{and} \quad \mathrm{SO}_3\mathbb{C} \cong \mathrm{PSL}_2\mathbb{C};$$

$$\mathrm{Spin}_4\mathbb{C} \cong \mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C} \quad \text{and} \quad \mathrm{PSO}_4\mathbb{C} \cong \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C};$$

$$\mathrm{Spin}_5\mathbb{C} \cong \mathrm{Sp}_4\mathbb{C} \quad \text{and} \quad \mathrm{SO}_5\mathbb{C} \cong \mathrm{PSp}_4\mathbb{C};$$

and

$$\mathrm{Spin}_6\mathbb{C} \cong \mathrm{SL}_4\mathbb{C} \quad \text{and} \quad \mathrm{PSO}_6\mathbb{C} \cong \mathrm{PSL}_4\mathbb{C}.$$

Note that in the first case $n = 4$ where there is an intermediate subgroup between $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{PSL}_n\mathbb{C}$, the subgroup in question is interesting: it turns out to be $\mathrm{SO}_6\mathbb{C}$. In general, however, these intermediate groups seldom arise.

Consider now representations of these classical groups. According to the basic result of Lecture 7, representations of a complex Lie algebra $\mathfrak{g}$ will correspond exactly to representations of the associated simply connected Lie group $\tilde{G}$: specifically, for any representation

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$$

of $\mathfrak{g}$, setting

$$\tilde{\rho}(\exp(X)) = \exp(\rho(X))$$

determines a well-defined homomorphism

$$\tilde{\rho}\colon \tilde{G} \to \mathrm{GL}(V).$$

For any other group with algebra $\mathfrak{g}$, given as the quotient $\tilde{G}/C$ of $\tilde{G}$ by a subgroup $C \subset Z(\tilde{G})$, the representations of G are simply the representations of $\tilde{G}$ trivial on C. It is therefore enough to see which of the representations of the classical Lie algebras described in Part III are trivial on which subgroups $C \subset Z(\tilde{G})$.

This turns out to be very straightforward. To begin with, we observe that the center of each group G with Lie algebra $\mathfrak{g}$ lies in the image of the chosen Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ under the exponential map. It will therefore be enough to know when $\exp(\rho(X)) = I$ for $X \in \mathfrak{h}$; and since the representations ρ of $\mathfrak{g}$ are particularly simple on $\mathfrak{h}$ this presents no difficulty.

What we do have to do first is to describe the restriction of the exponential map to $\mathfrak{h}$, so that we can say which elements of $\mathfrak{h}$ exponentiate to elements of $Z(\tilde{G})$. For the groups that are given as matrix groups, this will all be perfectly obvious, but for the spin groups we will need to do a little calculation. We will also want to describe the *Cartan subgroup* H of each of the classical groups G, which is the connected subgroup whose Lie algebra is the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For $G = \mathrm{SL}_n\mathbb{C}$, H is just the diagonal matrices in G, i.e.,

$$H = \{\mathrm{diag}(z_1, \ldots, z_n)\colon z_1 \cdot \ldots \cdot z_n = 1\}.$$

Similarly in $\mathrm{Sp}_{2n}\mathbb{C}$ or $\mathrm{SO}_{2n}\mathbb{C}$, $H = \{\mathrm{diag}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1})\}$, whereas in $\mathrm{SO}_{2n+1}\mathbb{C}$, $H = \{\mathrm{diag}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1)\}$. In each of these cases the exponential mapping from $\mathfrak{h}$ to H is just the usual exponentiation of diagonal matrices.

To calculate the exponential mapping for $\mathrm{Spin}_m\mathbb{C}$, we need to describe the elements in $\mathrm{Spin}_m\mathbb{C}$ that lie over the diagonal matrices in $\mathrm{SO}_m\mathbb{C}$. This is not a difficult task. Calculating as in §20.2, we find that for any nonzero complex number z and any $1 \leq j \leq n$, and with $m = 2n + 1$ or $m = 2n$, the elements

$$w_j(z) = \frac{1}{2}(ze_j \cdot e_{n+j} + z^{-1}e_{n+j} \cdot e_j) = z^{-1} + \left(\frac{z - z^{-1}}{2}\right)e_j \cdot e_{n+j} \qquad (23.7)$$

in the Clifford algebra are in fact elements of $\mathrm{Spin}_m\mathbb{C}$. Moreover, if $\rho\colon \mathrm{Spin}_m\mathbb{C} \to \mathrm{SO}_m\mathbb{C}$ is the covering, the image $\rho(w_j(z))$ is the diagonal matrix whose jth entry is z^2, $(n + j)$th entry is z^{-2}, and other diagonal entries are 1. These elements $w_j(z)$ also commute with each other, so for any nonzero complex numbers $z_1, \ldots, z_n$ we can define

$$w(z_1, \ldots, z_n) = w_1(z_1) \cdot w_2(z_2) \cdot \ldots \cdot w_n(z_n). \qquad (23.8)$$

Then $\rho(w(z_1, \ldots, z_n)) = \mathrm{diag}(z_1^2, \ldots, z_n^2, z_1^{-2}, \ldots, z_n^{-2})$ if $m = 2n$, while if $m = 2n + 1$, we get the same diagonal matrix but with a 1 at the end.

Let $H_i = E_{i,i} - E_{n+i,n+i}$, the usual basis for $\mathfrak{h} \subset \mathfrak{so}_m\mathbb{C}$.

Lemma 23.9. *For any complex numbers* $a_1, \ldots, a_n$,

$$\exp(a_1 H_1 + \cdots + a_n H_n) = w(e^{a_1/2}, \ldots, e^{a_n/2})$$

in $\mathrm{Spin}_m\mathbb{C}$.

Proof. Since the map $\exp\colon \mathfrak{h} \to \mathrm{Spin}_m\mathbb{C}$ is determined by the facts that it is continuous, it takes 0 to 1, and its composite with ρ is the exponential for $\mathrm{SO}_m\mathbb{C}$, this follows from the preceding formulas. □

Exercise 23.10*. Show that $\exp(\sum a_j H_j) = 1$ if and only if each a_j is in $2\pi i\mathbb{Z}$ and $\sum a_j \in 4\pi i\mathbb{Z}$.

We see also that $\exp(\mathfrak{h})$ contains the center of $\mathrm{Spin}_m\mathbb{C}$. Indeed, $-1 = w(-1, 1, \ldots, 1)$, and if m is even, the other central elements are $\pm\omega$, with $\omega = w(i, \ldots, i)$, as we calculated in Exercise 20.36. (This, of course, also contains the fact that there is a path between 1 and -1, proving again that $\mathrm{Spin}_m\mathbb{C}$ is connected.)

Exercise 23.11*. Verify for all the classical groups G that: (i) $H = \exp(\mathfrak{h})$ is a closed subgroup of G that contains the center of G; (ii) the map of fundamental groups $\pi_1(H, e) \to \pi_1(G, e)$ is surjective; (iii) for any connected covering $\pi\colon G' \to G$, $\pi^{-1}(H)$ is connected and is the Cartan subgroup of G'.

Now let $G = \tilde{G}/C$ be a semisimple Lie group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h}$. Choose an ordering of the roots, and let Γ_λ be the irreducible representation of $\mathfrak{g}$ with highest weight λ. The basic fact that we need is

Lemma 23.12. *The representation Γ_λ is a representation of $G = \tilde{G}/C$ if and only if*

$$\lambda(X) \in 2\pi i\mathbb{Z} \quad \textit{whenever } \exp(X) \in C.$$

Proof. The representation Γ_λ is a representation of G when $g \cdot v = v$ for all $g \in C$, where v is a highest weight vector in Γ_λ. Since $\exp(\mathfrak{h})$ contains C, this says $\exp(X) \cdot v = v$ for all $X \in \mathfrak{h}$ such that $\exp(X) \in C$. Now by the naturality of the exponential map, and since $X \cdot v = \lambda(X)v$ for $X \in \mathfrak{h}$, we have $\exp(X) \cdot v = e^{\lambda(X)}v$. Hence the condition is that $e^{\lambda(X)}v = v$, or that $e^{\lambda(X)} = 1$ if $\exp(X) \in C$, which is the displayed criterion. □

Let us work this out explicitly for each of the classical groups. It may help to introduce a notation for the irreducible representations which, among other virtues, allows some common terminology in the various cases. Note that for each of $\mathfrak{sl}_{n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, and $\mathfrak{so}_{2n+1}$ the root space $\mathfrak{h}^*$ is spanned by weights we have called $L_1, \ldots, L_n$, so a weight can be written uniquely in form $\lambda_1 L_1 + \cdots + \lambda_n L_n$. We may sometimes write λ in place of the weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$. In the rest of this lecture at least, we write Γ_λ for the irreducible representation with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$. Note that by our choice of Weyl chambers the highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ that arise satisfy

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \quad \text{for } \mathfrak{sl}_{n+1}, \mathfrak{sp}_{2n}, \text{ and } \mathfrak{so}_{2n+1},$$

where the λ_i are all integers in the first two cases, and for $\mathfrak{so}_{2n+1}$ they are either all integers or all half-integers; and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \geq 0 \quad \text{for } \mathfrak{so}_{2n},$$

with the λ_i all integers or all half-integers.

Proposition 23.13. *For each subgroup C of the center of $\tilde{G}$, the representation Γ_λ is a representation of $\tilde{G}/C$ precisely under the following conditions:*

(i) $\tilde{G} = \mathrm{SL}_{n+1}\mathbb{C}$, *$C$ has order m dividing $n + 1$: $\sum \lambda_j \equiv 0 \bmod (m)$.*
(ii) $\tilde{G} = \mathrm{Sp}_{2n}\mathbb{C}$, $C = \{\pm 1\}$: *$\sum \lambda_j$ is even.*
(iii) $\tilde{G} = \mathrm{Spin}_{2n}\mathbb{C}$ *or* $\mathrm{Spin}_{2n+1}\mathbb{C}$, $C = \{\pm 1\}$: *all λ_i are integers.*
(iv) $\tilde{G} = \mathrm{Spin}_{2n}\mathbb{C}$, $C = \{\pm 1, \pm\omega\}$: *all λ_i are integers, $\sum \lambda_j$ is even.*
(v) $\tilde{G} = \mathrm{Spin}_{2n}\mathbb{C}$, *$n$ even, $C = \{1, \omega\}$: $\sum \lambda_j$ is an even integer; and for $C = \{1, -\omega\}$: $\sum \lambda_j - n/2$ is an odd integer.*

In particular, representations of $\mathrm{PSL}_{n+1}\mathbb{C}$ are given by partitions λ with $\sum \lambda_j \equiv 0 \bmod (n + 1)$, and those for $\mathrm{PSp}_{2n}\mathbb{C}$ have $\sum \lambda_j$ even. Case (iii) verifies what we saw in Lecture 19 about representations of $\mathrm{SO}_m\mathbb{C}$. Representations of $\mathrm{PSO}_m\mathbb{C}$ correspond to integral partitions λ with $\sum \lambda_j$ even.

Proof. With the preceding lemma and the explicit description of everything in sight, the calculations are routine. In case (i), for example, a generator for

C is of the form $\exp(X)$, with

$$X = (2\pi i/m)\left(\sum_{j=1}^{n} E_{j,j} - nE_{n+1,n+1}\right),$$

and so $\lambda(X) = (2\pi i/m)(\sum \lambda_j)$ will be a multiple of $2\pi i$ exactly when $\sum \lambda_j$ is divisible by m. For $\mathrm{Sp}_{2n}\mathbb{C}$, $\exp(X) = -1$ when $X = \pi i(\sum H_j)$, so $\lambda(X) = \pi i \sum \lambda_j$, and (ii) follows. The calculations are similar for $\mathrm{Spin}_m\mathbb{C}$, noting that $\exp(2\pi i(H_1) = -1$ and $\exp(\pi i(\sum H_j)) = \omega$. □

By way of an example, recall that any irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is of the form $\mathrm{Sym}^k V$, where V is the standard two-dimensional representation. Any such representation, of course, lifts to the group $\mathrm{SL}_2\mathbb{C}$; but it lifts to $\mathrm{PSL}_2\mathbb{C} \cong \mathrm{SO}_3\mathbb{C}$ if and only if k is even (in particular, the "standard" representation of $\mathrm{SO}_3\mathbb{C}$ on $\mathbb{C}^3$ is the symmetric square $\mathrm{Sym}^2 V$). For another example, we have seen that any irreducible representation of $\mathfrak{sp}_4\mathbb{C}$ may be found in a tensor product $\mathrm{Sym}^k V \otimes \mathrm{Sym}^l W$, where V is the standard four-dimensional representation of $\mathfrak{sp}_4\mathbb{C}$ and $W \subset \wedge^2 V$ the complement of the trivial one-dimensional representation. All such representations lift to $\mathrm{Sp}_4\mathbb{C}$, but they lift to $\mathrm{PSp}_4\mathbb{C} \cong \mathrm{SO}_5\mathbb{C}$ if and only if k is even—equivalently, if they are contained in a representation of the form $\mathrm{Sym}^l W \otimes \mathrm{Sym}^k(\wedge^2 W)$, where W is the "standard" representation of $\mathrm{SO}_5\mathbb{C}$.

Exercise 23.14. Show that each of these semisimple complex Lie groups G has a finite-dimensional faithful representation.

The result of the proposition can be put in a more formal setting, which brings out a feature that our alert reader has surely noticed: the center of the simply-connected form of $\mathfrak{g}$ is isomorphic to the quotient group Λ_W/Λ_R of the weight lattice modulo the root lattice. We note first that this abelian group Λ_W/Λ_R is *finite*. We have seen this for the classical Lie algebras. In general, we have

Lemma 23.15. *The group Λ_W/Λ_R is finite, of order equal to the determinant of the Cartan matrix.*

Proof. The simple roots α form a basis for the root lattice Λ_R. The corresponding elements H_α form a basis for

$$\Gamma_R = \mathbb{Z}\{H_\gamma : \gamma \in R\},$$

a lattice in $\mathfrak{h}$; this is proved in Appendix D.4. Since Λ_W is defined to be the lattice of elements of $\mathfrak{h}^*$ that take integral values on Γ_R, the determinant

$$\det(\alpha(H_\beta)) = \det(n_{\alpha\beta})$$

is the index $[\Lambda_W : \Lambda_R]$. □

In particular, for the exceptional groups, Λ_W/Λ_R is trivial for (G_2), (F_4), and (E_8), and cyclic of order two for (E_7) and order three for (E_6).

In fact, the center of the simply-connected group is naturally isomorphic to the *dual* of Λ_W/Λ_R. To express this, consider the natural dual of this last group. The lattice Γ_R defined in the preceding proof is a sublattice of the lattice

$$\Gamma_W = \{X \in \mathfrak{h}: \alpha(X) \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

Note that Λ_W was defined to be the lattice of elements of $\mathfrak{h}^*$ that take integral values on Γ_R. It follows formally from the definitions and the fact that Λ_W/Λ_R is finite that we have a perfect pairing

$$\Gamma_W/\Gamma_R \times \Lambda_W/\Lambda_R \to \mathbb{Q}/\mathbb{Z}, \qquad (X, \alpha) \mapsto \alpha(X).$$

The claim is that there is a natural isomorphism from Γ_W/Γ_R to the center of $\tilde{G}$, which is given by the exponential. More precisely, let $e_G: \mathfrak{h} \to H \subset G$ be the homomorphism defined by

$$e_G(X) = \exp(2\pi i X).$$

We claim that when $G = \tilde{G}$ is the simply-connected group, $\text{Ker}(e_{\tilde{G}}) = \Gamma_R$ and $e_{\tilde{G}}(\Gamma_W)$ is the center of $\tilde{G}$, from which it follows that $e_{\tilde{G}}$ induces an isomorphism

$$\Gamma_W/\Gamma_R \cong Z(\tilde{G}).$$

More generally, for any $G = \tilde{G}/C$, define a lattice $\Gamma(G)$ between Γ_R and Γ_W by

$$\Gamma(G) = \text{Ker}(e_G).$$

Then e_G determines an isomorphism

$$\Gamma_W/\Gamma(G) \cong Z(G).$$

We may thus state our result as

Theorem 23.16. *There is a one-to-one correspondence between connected Lie groups G with the Lie algebra $\mathfrak{g}$ and lattices $\Lambda \subset \mathfrak{h}^*$ such that*

$$\Lambda_R \subset \Lambda \subset \Lambda_W.$$

The correspondence is given by associating to a group G the lattice dual to the kernel of the exponential map exp: $\mathfrak{g} \to G$; *in particular, the largest lattice Λ_W corresponds to the simply-connected group, the smallest Λ_R to the adjoint group with no center. In terms of this correspondence, the irreducible representation V_λ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ will lift to a representation of the group G corresponding to $\Lambda \subset \mathfrak{h}^*$ if and only if $\lambda \in \Lambda$.*

Note also that

$$H = \mathfrak{h}/\Gamma(G) \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*,$$

with $n = \dim_{\mathbb{C}}\mathfrak{h}$ copies of $\mathbb{C}^*$.

Exercise 23.17*. Show that these claims follow formally from what we have seen: that the image of the exponential map contains the center, and that for any weight α there is a representation V of $\mathfrak{g}$ whose weight space V_α is not zero. Show also that e_G determines an isomorphism $\Gamma(G)/\Gamma_R \cong \pi_1(G)$. In diagram form,

$$\begin{array}{ccc} \Gamma_W & & G_0 \\ \cup & \Big\} \ \text{Center}(G) & \uparrow \\ \Gamma(G) & & G \\ \cup & \Big\} \ \pi_1(G) & \uparrow \\ \Gamma_R & & \tilde{G} \end{array}$$

Exercise 23.18. Find the kernels of each of the spin and half-spin representations $\text{Spin}_m\mathbb{C} \to \text{GL}(S)$ and $\text{Spin}_m\mathbb{C} \to \text{GL}(S^\pm)$.

Exercise 23.19*. Classify the irreducible representations of the full orthogonal group $\text{O}_m\mathbb{C}$.

Note that by our analysis of the Lie algebra $\mathfrak{g}_2$ there is a unique group G_2 with this Lie algebra, which is simultaneously the simply-connected and adjoint forms; the representations of this group are exactly those of the algebra $\mathfrak{g}_2$. The same is true for the Lie algebras of type (F_4) and (E_8), while (E_7) and (E_6) each have two associated groups, an adjoint one with fundamental group $\mathbb{Z}/2$ and $\mathbb{Z}/3$, and a simply-connected form with center $\mathbb{Z}/2$ and $\mathbb{Z}/3$ respectively.

It may be worth pointing out that each complex simple Lie group G can be realized as a closed subgroup defined by polynomial equations in some general linear group, i.e., that G is an *affine algebraic group.* Every irreducible representation $G \to \text{GL}(V)$ is also defined by polynomials in appropriate coordinates. This explains why the whole subject can be developed from the point of view of algebraic groups, as in [Bor1] and [Hu2].

The Weyl group $\mathfrak{W}$, which we defined as a subgroup of $\text{Aut}(\mathfrak{h}^*)$, can be interpreted in terms of any connected Lie group G with Lie algebra $\mathfrak{g}$. Let H be the Cartan subgroup corresponding to $\mathfrak{h}$, and let $N(H)$ be the normalizer:

$$N(H) = \{g \in G \colon gHg^{-1} = H\}.$$

We have homomorphisms:

$$N(H) \to \text{Aut}(H) \to \text{Aut}(\mathfrak{h}) \to \text{Aut}(\mathfrak{h}^*),$$

the first defined by conjugation, the second by differentiation at the identity, and the third using the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Killing form. Fact 14.11 can be sharpened to the claim that this map determines an *isomorphism*

$$N(H)/N \xrightarrow{\cong} \mathfrak{W}. \tag{23.20}$$

When G is the adjoint form of the Lie algebra, this isomorphism is proved in Appendix D. The general case follows, using:

Exercise 23.21. Show that if $\pi\colon G' \to G$ is a connected covering, with Cartan subgroups $H' = \pi^{-1}(H)$, then the induced map $N(H')/H' \to N(H)/H$ is an isomorphism.

Exercise 23.22. For each of the classical groups, and each simple root α, find an element in $N(H)$ that maps to the reflection W_α in $\mathfrak{W}$.

§23.2. Representation Rings and Characters

Just as with finite groups, we can form the representation ring R of a semisimple Lie algebra or Lie group: take the free abelian group on the isomorphism classes $[V]$ of finite-dimensional representations V, and divide by the relations $[V] = [V'] + [V'']$ whenever $V \cong V' \oplus V''$. By the complete reducibility of representations, it follows as before that R is a free abelian group on the classes $[V]$ of irreducible representations. Again, the tensor product of representations makes R into a ring: $[V]\cdot[W] = [V \otimes W]$. Many of our questions about decomposing representations and tensor products of representations can be nicely encoded by describing R more fully. We do this first for the Lie algebras.

For a semisimple Lie algebra $\mathfrak{g}$, let $\Lambda = \Lambda_W$ be the weight lattice, and let $\mathbb{Z}[\Lambda]$ be the integral group ring on the abelian group Λ. We write $e(\lambda)$ for the basis element of $\mathbb{Z}[\Lambda]$ corresponding to the weight λ; for now at least these are just formal symbols, having nothing to do with exponentials (but see (23.40)). Elements of $\mathbb{Z}[\Lambda]$ are expressions of the form $\sum n_\lambda e(\lambda)$, i.e., they assign an integer n_λ to each weight λ, with all but a finite number being zero. So $\mathbb{Z}[\Lambda]$ is a natural carrier for the information about multiplicities of representations. Define a *character homomorphism*

$$\text{Char}\colon R(\mathfrak{g}) \to \mathbb{Z}[\Lambda] \tag{23.23}$$

by the formula $\text{Char}[V] = \sum \dim(V_\lambda)e(\lambda)$, where V_λ is the weight space of V for the weight λ and $\dim(V_\lambda)$ its multiplicity. This is clearly an additive homomorphism.

The first assertion about this character map is that it is *injective*. This comes down to the fact that a representation is determined by the multiplicities of its weight spaces, which is something we saw in Lecture 14.

The product in the group ring $\mathbb{Z}[\Lambda]$ is determined by $e(\alpha)\cdot e(\beta) = e(\alpha + \beta)$. We claim next that Char is a *ring homomorphism*. This comes from the familiar fact that

$$(V \otimes W)_\lambda = \bigoplus_{\mu+\nu=\lambda} V_\mu \otimes W_\nu.$$

The Weyl group $\mathfrak{W}$ acts on $\mathbb{Z}[\Lambda]$, and a third simple claim is that the image of Char is contained in the ring of invariants $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$. This comes down to the fact that, for an irreducible (and hence for any) representation V, the weight spaces obtained by reflecting in walls of the Weyl chambers all have the same dimension.

Let $\omega_1, \ldots, \omega_n$ be a set of fundamental weights; as we have seen, these are the first weights along edges of a Weyl chamber, and they are free generators for the lattice Λ. Let $\Gamma_1, \ldots, \Gamma_n$ be the classes in $R(\mathfrak{g})$ of the irreducible representations with highest weights $\omega_1, \ldots, \omega_n$.

Theorem 23.24. (a) *The representation ring $R(\mathfrak{g})$ is a polynomial ring on the variables $\Gamma_1, \ldots, \Gamma_n$.*

(b) *The homomorphism $R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}}$ is an isomorphism.*

In particular, this says that $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ is a polynomial ring on the variables $\mathrm{Char}(\Gamma_1), \ldots, \mathrm{Char}(\Gamma_n)$. In fact, the theorem is equivalent to this assertion, since if we take variables $U_1, \ldots, U_n$ and map the polynomial ring on the U_i to $R(\mathfrak{g})$ by sending U_i to Γ_i, we have

$$\mathbb{Z}[U_1, \ldots, U_n] \to R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}}.$$

If the composite is an isomorphism, the second being injective, both must be isomorphisms, which is what the theorem says.

In spite of its fancy appearance, we will see that the theorem follows quite easily from what we know about the action of the Weyl group $\mathfrak{W}$ on the weights.

For any $P \in \mathbb{Z}[\Lambda]$ let us say that α is a *highest weight* for P if the coefficient of $e(\alpha)$ in P is nonzero, and, with a chosen ordering of weights as before, α is the largest such weight. We first observe that if P is invariant under $\mathfrak{W}$, then the highest weight for P is in $\mathscr{W} \cap \Lambda$, where $\mathscr{W}$ is our chosen (closed) Weyl chamber. In general, weights in $\mathscr{W} \cap \Lambda$ are often referred to as *dominant weights.*

Now suppose $\{P_\lambda\}$ is any collection of elements in $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$, one for each dominant weight λ, such that P_λ has highest weight λ and the coefficient of $e(\lambda)$ is 1. We claim that the P_λ form an additive basis for $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ over $\mathbb{Z}$. This is easy to see and is the same argument used in the theory of symmetric polynomials in any algebra text: given P with highest weight λ, if the coefficient of $e(\lambda)$ is m, then $P - mP_\lambda$ is invariant whose highest weight is lower, and one continues inductively until one reaches weight zero, i.e., the constants.

Let $P_i = \mathrm{Char}(\Gamma_i)$, which has highest weight ω_i, and suppose the coefficient of $e(\omega_i)$ is 1. Since any weight $\lambda \in \mathscr{W} \cap \Lambda$ can be uniquely expressed in the form $\lambda = \sum m_i\omega_i$, for some non-negative integers m_i, and the highest weight of $\prod (P_i)^{m_i}$ is $\sum m_i\omega_i$, it follows that the monomials $\prod (P_i)^{m_i}$ in $P_1, \ldots, P_n$ form an additive basis for $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$. This says precisely that $\mathbb{Z}[P_1, \ldots, P_n] = \mathbb{Z}[\Lambda]^{\mathfrak{W}}$, and completes the proof. □

Let us work this out concretely for each of our cases $\mathfrak{sl}_{n+1}\mathbb{C}$, $\mathfrak{sp}_n\mathbb{C}$, $\mathfrak{so}_{2n+1}\mathbb{C}$, and $\mathfrak{so}_{2n}\mathbb{C}$. Each lattice Λ contains weights we have called $L_1, \ldots, L_n$; in the first case we also have L_{n+1} with $L_1 + \cdots + L_{n+1} = 0$. We set

$$x_i = e(L_i), \qquad x_i^{-1} = e(-L_i) \in \mathbb{Z}[\Lambda]. \tag{23.25}$$

Note that in case $L_1, \dots, L_n$ is a basis for Λ, then

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}] = \mathbb{Z}[x_1, \dots, x_n, (x_1 \cdot \ldots \cdot x_n)^{-1}]$$

as a subring of the field $\mathbb{Q}(x_1, \dots, x_n)$.

(A_n) For $\mathfrak{sl}_{n+1}\mathbb{C}$, fundamental weights are

$$L_1, \quad L_1 + L_2, \quad L_1 + L_2 + L_3, \dots, L_1 + \cdots + L_n,$$

corresponding to the irreducible representations $V, \wedge^2 V, \dots, \wedge^n V$, with $V = \mathbb{C}^{n+1}$ the standard representation. The character of $\wedge^k V$ is $\sum e(\alpha)$, the sum over all α that are sums of k different L_i for $1 \le i \le n+1$. So $\mathrm{Char}(\wedge^k V) = A_k$, where A_k is the kth elementary symmetric function of $x_1, \dots, x_{n+1}$. The Weyl group is the symmetric group $\mathfrak{S}_{n+1}$, acting by permutation on the indices, so the theorem in this case says that

$$R(\mathfrak{sl}_{n+1}) = \mathbb{Z}[\Lambda]^{\mathfrak{S}_{n+1}} = \mathbb{Z}[A_1, \dots, A_n]. \tag{23.26}$$

Note that $\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, \dots, x_n, x_{n+1}]/(x_1 \cdot \ldots \cdot x_{n+1} - 1)$, so $\mathbb{Z}[\Lambda]$ has an additive basis consisting of all monomials x^α, with α an n-tuple of non-negative integers, but with not all α_i positive.

(C_n) For $\mathfrak{sp}_{2n}\mathbb{C}$, the lattice Λ and fundamental weights have the same description as in the preceding case. The corresponding irreducible representations are the kernels $V^{(k)}$ of the contraction maps $\wedge^k V \to \wedge^{k-2} V$, with now $V = \mathbb{C}^{2n}$ the standard representation, $k = 1, \dots, n$. The character of $\wedge^k V$ is $\sum e(\alpha)$, the sum over all α that are sums of k different $\pm L_i$ for $1 \le i \le n$. The character $\mathrm{Char}(\wedge^k V)$ is thus the elementary symmetric polynomial C_k in the variables $x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}$. The theorem then says that

$$\begin{aligned} R(\mathfrak{sp}_{2n}\mathbb{C}) &= \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[C_1, C_2 - 1, C_3 - C_1, \dots, C_n - C_{n-2}] \\ &= \mathbb{Z}[C_1, C_2, C_3, \dots, C_n]. \end{aligned} \tag{23.27}$$

(B_n) For $\mathfrak{so}_{2n+1}\mathbb{C}$, Λ is spanned by the L_i together with $\frac{1}{2}(L_1 + \cdots + L_n)$. The fundamental representations are $V, \wedge^2 V, \dots, \wedge^{n-1} V$, and the spin representation S. The character of $\wedge^k V$ is the kth elementary symmetric function of the $2n+1$ elements $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$, and 1; denote this by B_k. The character of S, which we denote by B, is the sum $\sum x_1^{\pm 1/2} \cdot \ldots \cdot x_n^{\pm 1/2}$, where

$$x_i^{+1/2} = e(L_i/2), \qquad x_i^{-1/2} = e(-L_i/2). \tag{23.28}$$

So B is the nth elementary symmetric polynomial in the variables $x_i^{+1/2} + x_i^{-1/2}$. Therefore,

$$R(\mathfrak{so}_{2n+1}\mathbb{C}) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[B_1, \dots, B_{n-1}, B]. \tag{23.29}$$

(D_n) For $\mathfrak{so}_{2n}\mathbb{C}$, Λ and $\mathbb{Z}[\Lambda]$ are the same as in the preceding case. The fundamental representations are $V, \wedge^2 V, \dots, \wedge^{n-2} V$, and the half-spin representations S^+ and S^-. The character of $\wedge^k V$, denoted D_k, is the kth elementary symmetric function of the $2n$ elements $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$. The character $D^{\pm}$ of $S^{\pm}$ is the sum $\sum x_1^{\pm 1/2} \cdot \ldots \cdot x_n^{\pm 1/2}$, where the number of plus signs is even or odd according to the sign. We have

$$R(\mathfrak{so}_{2n}\mathbb{C}) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[D_1, \ldots, D_{n-2}, D^+, D^-]. \tag{23.30}$$

Exercise 23.31*. (a) Prove the following relation in $R(\mathfrak{so}_{2n+1}\mathbb{C})$:

$$B^2 = B_n + \cdots + B_1 + 1,$$

corresponding to the isomorphism

$$S \otimes S \cong \wedge^n V \oplus \cdots \oplus \wedge^1 V \oplus \wedge^0 V.$$

This describes $R(\mathfrak{so}_{2n+1}\mathbb{C})$ as a quadratic extension of the ring $\mathbb{Z}[B_1, \ldots, B_n]$.

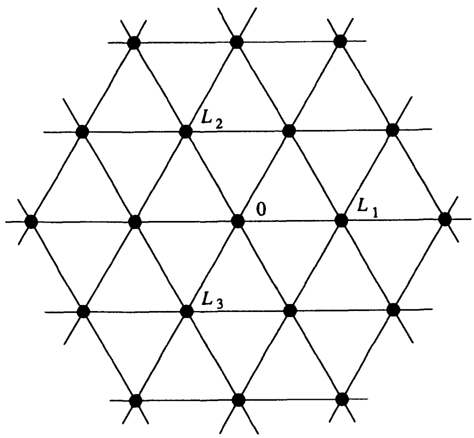
(b) Let D_n^+ (respectively, D_n^-) be the character of the representation whose highest weight is twice that of D^+ (resp., D^-), so that, for example, the sum of the representations D_n^+ and D_n^- is $\wedge^n V$. Prove the relations in $R(\mathfrak{so}_{2n}\mathbb{C})$:

$$D^+ \cdot D^+ = D_n^+ + D_{n-2} + D_{n-4} + \cdots,$$

$$D^- \cdot D^- = D_n^- + D_{n-2} + D_{n-4} + \cdots,$$

$$D^+ \cdot D^- = D_{n-1} + D_{n-3} + D_{n-5} + \cdots.$$

We can likewise describe the representation ring for $\mathfrak{g}_2$. Here, we may take as generators for the weight lattice the weights L_1 and L_2 as pictured in the diagram



and correspondingly write $\mathbb{Z}[\Lambda]$ as $\mathbb{Z}[x_1, x_1^{-1}, x_2, x_2^{-1}]$, where $x_i = e(L_i)$. It will be a little more symmetric to introduce $L_3 = -L_1 - L_2$ as pictured and $x_3 = x_1^{-1} \cdot x_2^{-1} = e(L_3)$, and write

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, x_2, x_3]/(x_1 x_2 x_3 - 1).$$

In these terms the Weyl group is the group $\mathfrak{W}$ generated by the symmetric group $\mathfrak{S}_3$ permuting the variables x_i and the involution sending each x_i to x_i^{-1}. The standard representation has weights $\pm L_i$ and 0, and so has character

$$A = A(x_1, x_2, x_3) = 1 + x_1 + x_1^{-1} + x_2 + x_2^{-1} + x_3 + x_3^{-1}.$$

Similarly, the adjoint representation has weights $\pm L_i$, $\pm(L_i - L_j)$, and 0 (taken twice); its character is

$$B = A(x_1, x_2, x_3) + A(x_1/x_2, x_2/x_3, x_3/x_1).$$

The theorem thus implies in this case the equality

$$R(\mathfrak{g}_2) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[A, B]. \tag{23.32}$$

Exercise 23.33. Verify directly the statement that any element of $\mathbb{Z}[x_1, x_2, x_3]/(x_1 x_2 x_3 - 1)$ invariant under the group $\mathfrak{W}$ as described is in fact a polynomial in A and B.

Similarly we can define the representation ring $R(G)$ of a semisimple group G. When G is the simply-connected form of its Lie algebra $\mathfrak{g}$, $R(G) = R(\mathfrak{g})$, so $R(\mathrm{SL}_n\mathbb{C})$, $R(\mathrm{Sp}_{2n}\mathbb{C})$, $R(\mathrm{Spin}_{2n+1}\mathbb{C})$, and $R(\mathrm{Spin}_{2n}\mathbb{C})$ are given by (23.26), (23.27), (23.29), and (23.30). In general, $R(G)$ is a subring of $R(\mathfrak{g})$; we can read off which subring by looking at Proposition 23.13. We have, in fact,

$$R(\mathrm{SO}_{2n+1}\mathbb{C}) = \mathbb{Z}[B_1, \ldots, B_n]; \tag{23.34}$$

$$R(\mathrm{SO}_{2n}\mathbb{C}) = \mathbb{Z}[D_1, \ldots, D_{n-1}, D_n^+, D_n^-], \tag{23.35}$$

with D_n^+ and D_n^- as in Exercise 23.31. But this time there is one relation:

$$(D_n^+ + D_{n-2} + D_{n-4} + \cdots + 1)(D_n^- + D_{n-2} + D_{n-4} + \cdots + 1)$$
$$= (D_{n-1} + D_{n-3} + \cdots +)^2.$$

Exercise 23.36*.

(a) Prove (23.34).
(b) Show that the relation in (23.35) comes from Exercise 23.31(b). Show that $R(\mathrm{SO}_{2n}\mathbb{C})$ is the polynomial ring in the $n + 1$ generators shown, modulo the ideal generated by the one polynomial indicated.
(c) Describe the representation rings for the other groups with these simple Lie algebras.
(d) Prove the isomorphism

$$R(\mathrm{GL}_n\mathbb{C}) = \mathbb{Z}[E_1, \ldots, E_n, E_n^{-1}],$$

where the E_k are the elementary symmetric functions of $x_1, \ldots, x_n$.

Exercise 23.37*. (a) Show that the image of $R(\mathrm{O}_m\mathbb{C})$ in $R(\mathrm{SO}_m\mathbb{C})$ is the polynomial ring $\mathbb{Z}[B_1, \ldots, B_n]$ if $m = 2n + 1$, and $\mathbb{Z}[D_1, \ldots, D_n]$ if $m = 2n$.

(b) Show that

$$R(\mathrm{O}_{2n+1}\mathbb{C}) = R(\mathrm{SO}_{2n+1}\mathbb{C}) \otimes R(\mathbb{Z}/2)$$
$$= \mathbb{Z}[B_1, \ldots, B_n, B_{2n+1}]/((B_{2n+1})^2 - 1)$$

and

$$R(\mathrm{O}_{2n}\mathbb{C}) = \mathbb{Z}[D_1, \ldots, D_n, D_{2n}]/I,$$

where I is the ideal generated by $(D_{2n})^2 - 1$ and $D_n D_{2n} - D_n$.

Exercise 23.38*. The mapping that takes a representation V to its dual V^* induces an involution of the representation ring: $[V]^* = [V^*]$. The ring $\mathbb{Z}[\Lambda]$ has an involution determined by $(e(\lambda))^* = e(-\lambda)$. Show that the character homomorphism commutes with these involutions. Show that for $\mathfrak{sl}_{n+1}$, $(A_k)^* = A_{n+1-k}$; for $\mathfrak{so}_{2n+1}\mathbb{C}$, and $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_{2n}\mathbb{C}$ for n even, the involution is the identity; while for $\mathfrak{so}_{2n}\mathbb{C}$ with n odd, $(D_k)^* = D_k$, $(D^+)^* = D^-$, $(D^-)^* = D^+$. Deduce that all representations of all symplectic and orthogonal groups are self-dual. Note that when $*$ is the identity, all representations are self-dual. In the other cases, compute the duals of irreducible representations with given highest weight.

The following exercise deals with a special property of the representation rings of semisimple Lie groups and algebras.

Exercise 23.39*. The representation rings $R = R(\mathfrak{g})$ and $R(G)$ have another important structure: they are *λ-rings*. There are operators

$$\lambda^i\colon R(G) \to R(G), \quad i = 0, 1, 2, \ldots,$$

determined by $\lambda^i([V]) = [\Lambda^i V]$ for any representation V.

(a) Show that this determines well-defined maps, satisfying $\lambda^0 = 1$, $\lambda^1 = \mathrm{Id}$, and

$$\lambda^i(x + y) = \sum_{i+j=k} \lambda^i(x) \cdot \lambda^j(y)$$

for any x and y in R. In fact, R is what is called a *special* λ-ring: there are formulas for $\lambda^i(x \cdot y)$ and $\lambda^i(\lambda^j(x))$, valid as if x and y could be written as sums of one-dimensional representations (see, e.g., [A-T]).

(b) Show that λ^i extends to $\mathbb{Z}[\Lambda]$, and use this to verify that $R(G)$ is a special λ-ring.

Define *Adams operators* $\psi^k\colon R \to R$ by $\psi^k(x) = P_k(\lambda^1 x, \ldots, \lambda^n x)$, where P_k is the expression for the kth power sum (cf. Exercise A.32) in terms of the elementary symmetric functions, $n \geq k$. Equivalently,

$$\psi^k(x) - \psi^{k-1}(x)\lambda^1(x) + \cdots + (-1)^k k \lambda^k(x) = 0.$$

(c) Show that, regarding R as the ring of functions on the group G, $(\psi^k x)(g) = x(g^k)$. Equivalently, $\psi^k(e(\lambda)) = e(k\lambda)$.

(d) Show that each ψ^k is a ring homomorphism, and $\psi^k \circ \psi^l = \psi^{k+l}$.
(e) Show that for a representation V,

$$\operatorname{Char}(\operatorname{Sym}^2 V) = \tfrac{1}{2}\operatorname{Char}(V)^2 + \tfrac{1}{2}\psi^2(\operatorname{Char}(V)),$$
$$\operatorname{Char}(\Lambda^2 V) = \tfrac{1}{2}\operatorname{Char}(V)^2 - \tfrac{1}{2}\psi^2(\operatorname{Char}(V)).$$

Show that $\operatorname{Char}(\operatorname{Sym}^d V)$ and $\operatorname{Char}(\Lambda^d V)$ can be written as polynomials in $\psi^k(\operatorname{Char}(V))$, $1 \le k \le d$.

Formal Characters and Actual Characters

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any representation V of $\mathfrak{g}$, the image of $[V] \in R(\mathfrak{g})$ in $\mathbb{Z}[\Lambda]$ is called the *formal character* of V. As it turns out, this formal character can be identified with the honest character of the corresponding representation of the group G, restricted to the Cartan subgroup H:

(23.40) *If* $\operatorname{Char}(V) = \sum m_\alpha e(\alpha)$ *is the formal character, and* $\exp(X)$ *is an element of* H, *then the trace of* $\exp(X)$ *on* V *is* $\sum m_\alpha e^{\alpha(X)}$.

This is simply because $\exp(X)$ acts on the weight space V_μ by multiplication by $e^{\mu(X)}$, as we have seen. In particular, a representation is determined by the character of its restriction to a Cartan subgroup.

Another common notation for this is to set $e(X) = \exp(2\pi i X)$, and $e(z) = \exp(2\pi i z)$. Then the trace of $e(X)$ is $\sum m_\alpha e(\alpha(X))$.

Exercise 23.41. As a function on H, the character of a representation is invariant under the Weyl group $\mathfrak{W} = N(H)/H$. Describe $R(G)$ as a ring of $\mathfrak{W}$-invariant functions on H.

This is also compatible with our descriptions of elements of $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$ as Laurent polynomials in variables x_i or $x_i^{1/2}$. For $\mathrm{SL}_{n+1}\mathbb{C}$, for example, if the character $\operatorname{Char}(W)$ of a representation W is $P(x_1, \ldots, x_{n+1})$, the trace of the matrix $\operatorname{diag}(z_1, \ldots, z_{n+1})$ on V is $P(z_1, \ldots, z_{n+1})$. Similarly for the other groups, using the diagonal matrices described in the first section of this lecture. For the spin groups, the element $w(z_1, \ldots, z_n)$ defined in (23.8) has trace given by substituting z_i for $x_i^{1/2}$, and z_i^{-1} for $x_i^{-1/2}$ in the corresponding Laurent polynomial.

Exercise 23.42*. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two semisimple Lie algebras, show that

$$R(\mathfrak{g}_1 \times \mathfrak{g}_2) = R(\mathfrak{g}_1) \otimes R(\mathfrak{g}_2).$$

Exercise 23.43*. (a) For the natural inclusion $\mathfrak{sl}_n\mathbb{C} \subset \mathfrak{sl}_{n+1}\mathbb{C}$, restriction of representations gives a homomorphism $R(\mathfrak{sl}_{n+1}\mathbb{C}) \to R(\mathfrak{sl}_n\mathbb{C})$, which can be described by saying what happens to the polynomial generators. Since $\bigwedge^k(\mathbb{C}^n \oplus \mathbb{C}) = \bigwedge^k(\mathbb{C}^n) \oplus \bigwedge^{k-1}(\mathbb{C}^n)$, this is

$$A_k \mapsto A_k + A_{k-1}.$$

Give the analogous descriptions for the following inclusions:

$$\mathfrak{sp}_{2n-2}\mathbb{C} \subset \mathfrak{sp}_{2n}\mathbb{C}, \qquad \mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}, \qquad \mathfrak{so}_{2n-1}\mathbb{C} \subset \mathfrak{so}_{2n}\mathbb{C};$$

$$\mathfrak{sl}_n\mathbb{C} \subset \mathfrak{sp}_{2n}\mathbb{C}, \qquad \mathfrak{sl}_n\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}, \qquad \mathfrak{sl}_n\mathbb{C} \subset \mathfrak{so}_{2n}\mathbb{C};$$

$$\mathfrak{sp}_{2n}\mathbb{C} \subset \mathfrak{sl}_{2n}\mathbb{C}, \qquad \mathfrak{so}_{2n+1}\mathbb{C} \subset \mathfrak{sl}_{2n+1}\mathbb{C}, \qquad \mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{sl}_{2n}\mathbb{C}.$$

(b) The inclusion $\mathfrak{sl}_n\mathbb{C} \times \mathfrak{sl}_m\mathbb{C} \subset \mathfrak{sl}_{n+m}\mathbb{C}$ determines a restriction homomorphism $R(\mathfrak{sl}_{n+m}\mathbb{C}) \to R(\mathfrak{sl}_n\mathbb{C} \times \mathfrak{sl}_m\mathbb{C}) = R(\mathfrak{sl}_n\mathbb{C}) \otimes R(\mathfrak{sl}_m\mathbb{C})$, which takes polynomial generators A_k to $A_k \otimes 1 + A_{k-1} \otimes A_1 + \cdots + 1 \otimes A_k$. Compute analogously for

$$\mathfrak{sp}_{2n}\mathbb{C} \times \mathfrak{sp}_{2m}\mathbb{C} \subset \mathfrak{sp}_{2n+2m}\mathbb{C}, \qquad \mathfrak{so}_n\mathbb{C} \times \mathfrak{so}_m\mathbb{C} \subset \mathfrak{so}_{n+m}\mathbb{C}.$$

Which of these inclusions correspond to removing nodes from the Dynkin diagrams?

Exercise 23.44. Compute the isomorphisms of representation rings corresponding to the isomorphisms $\mathfrak{sl}_2\mathbb{C} \cong \mathfrak{so}_3\mathbb{C}$, $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$, and $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$.

§23.3. Homogeneous Spaces

In this section we will introduce and describe the compact homogeneous spaces associated to the classical groups. As we will see, these are classified neatly in terms of Dynkin diagrams, and are, in turn, closely related to the representation theory of the groups acting on them. Unfortunately, we are unable to give here more than the barest outline of this beautiful subject; but we will at least try to say what the principal objects are, and what connections among them exist. In particular, we give at the end of the section a diagram (23.58) depicting these objects and correspondences to which the reader can refer while reading this section.

We begin by introducing the notion of Borel subalgebras and Borel subgroups. Recall first that a choice of Cartan subalgebra $\mathfrak{h}$ in a semisimple Lie algebra $\mathfrak{g}$ determines, as we have seen, a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. To each choice of ordering of the root system $R = R^+ \cup R^-$, we can associate a subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha,$$

called a *Borel subalgebra*. Note that $\mathfrak{b}$ is solvable, since $\mathscr{D}\mathfrak{b} \subset \bigoplus \mathfrak{g}_\alpha$, $\mathscr{D}^2\mathfrak{b} \subset \bigoplus \mathfrak{g}_{\alpha+\beta}$, etc. In fact, $\mathfrak{b}$ is a maximal solvable subalgebra (Exercise 14.35).

If G is a Lie group with semisimple Lie algebra $\mathfrak{g}$, the connected subgroup B of G with Lie algebra $\mathfrak{b}$ is called a *Borel subgroup.*

Claim 23.45. *B is a closed subgroup of G, and the quotient G/B is compact.*

Proof. Consider the adjoint representation of G on $\mathfrak{g}$. The action of the Borel subalgebra $\mathfrak{b}$ obviously preserves the subspace $\mathfrak{b} \subset \mathfrak{g}$, and, in fact, $\mathfrak{b}$ is just the inverse image of the subalgebra of $\mathfrak{gl}(\mathfrak{g})$ preserving this subspace: if $X = \sum X_\alpha$ is any element of $\mathfrak{g}$ with $X_\alpha \in \mathfrak{g}_\alpha$ and $X_\alpha \neq 0$ for some $\alpha \in R^-$, we could find an element H of $\mathfrak{h} \subset \mathfrak{b}$ with $\mathrm{ad}(X)(H) \notin \mathfrak{b}$—any H not in the annihilator of $\alpha \in \mathfrak{h}^*$ would do. B is thus (the connected component of the identity in) the inverse image in G of the subgroup of $\mathrm{GL}(\mathfrak{g})$ carrying $\mathfrak{b}$ into itself. It follows that B is closed; and the quotient G/B is contained in a Grassmannian and hence compact. (Alternatively, we could consider the action of G on the projective space $\mathbb{P}(\wedge^m \mathfrak{g})$, where m is the number of positive roots, and observe that B is the stabilizer of the point corresponding to the exterior product of the positive root spaces.)

In fact, in the case of the classical groups, it is easy to describe the Borel subgroups and the corresponding quotients.

For $G = \mathrm{SL}_{n+1}\mathbb{C}$, B is the group of all upper-triangular matrices in G, i.e., those automorphisms preserving the standard flag. It follows that G/B is the usual (complete) flag manifold, i.e., the variety of all flags

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{n+1}\}$$

of subspaces with $\dim(V_r) = r$.

For $G = \mathrm{SO}_{2n+1}\mathbb{C}$ the orthogonal group of automorphisms of $\mathbb{C}^{2n+1}$ preserving a quadratic form Q, B is the subgroup of automorphisms which preserve a fixed flag $V_1 \subset \cdots \subset V_n$ of isotropic subspaces with $\dim(V_r) = r$. All such flags being conjugate, G/B is the variety of all such flags, i.e.,

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^{2n+1}: Q(V_n, V_n) \equiv 0\}.$$

Note that B automatically preserves the flag of orthogonal subspaces, so that we could also characterize G/B as the space of complete flags equal to their orthogonal complements, i.e.,

$$G/B = \{V_1 \subset \cdots \subset V_{2n} \subset \mathbb{C}^{2n+1}: Q(V_i, V_{2n+1-i}) = 0\}.$$

The same holds for $\mathrm{Sp}_{2n}\mathbb{C}$: the Borel subgroups $B \subset \mathrm{Sp}_{2n}\mathbb{C}$ are just the subgroups preserving a half-flag of isotropic subspaces, or equivalently a full flag of pairwise complementary subspaces; and the quotient G/B is correspondingly the variety of all such flags.

For $G = \mathrm{SO}_{2n}\mathbb{C}$, B fixes an isotropic flag $V_1 \subset \cdots \subset V_{n-1}$, and

$$G/B = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^{2n}: Q(V_{n-1}, V_{n-1}) = 0\}.$$

Exercise 23.46. With our choice of basis $\{e_i\}$, let V_r be the subspace spanned by the first r basic vectors. If B is defined to be the subgroup that preserves V_r for $1 \leq r \leq n$, verify that the Lie algebra of B is spanned by the Cartan subalgebra and the positive root spaces described in Lectures 17 and 19.

We now want to consider more general quotients of a semisimple complex group G. To begin with, we say that a (closed, complex analytic, and connected[1]) subgroup P of G is *parabolic* if the quotient G/P can be realized as the orbit of the action of G on $\mathbb{P}(V)$ for some representation V of G. In particular, G/P is a projective algebraic variety. It follows from the proof of Claim 23.45 that any Borel subgroup B of G is parabolic. The following two claims characterize parabolic subgroups as those containing a Borel subgroup, i.e., the Borel subgroups are exactly the *minimal* parabolic subgroups.

Claim 23.47. *If B is a Borel subgroup and P a parabolic subgroup of G, then there is an $x \in G$ with*

$$B \subset xPx^{-1}.$$

Claim 23.48. *If a subgroup P of G contains a Borel subgroup B, then P is parabolic.*

The first claim is deduced from a version of *Borel's fixed point theorem*: if B is a connected solvable group, V a representation of B and $X \subset \mathbb{P}V$ a projective variety carried into itself under the action of B on $\mathbb{P}V$, then B must have a fixed point on X. This is straightforward: we observe (by Lie's theorem (9.11)) that the action of the solvable group B on V must preserve a flag of subspaces

$$0 \subset V_1 \subset \cdots \subset V_n = V$$

with $\dim(V_i) = i$. We can thus find a subspace $V_i \subset V$ fixed by B such that X intersects $\mathbb{P}V_i$ in a finite collection of points, which must then be fixed points for the action of B on X. As for Claim 23.48 we will soon see directly how G/P is a projective variety whenever P is a subgroup containing B.

We can now completely classify the parabolic subgroups of a simple group, up to conjugacy. By the above, we may assume that P contains a Borel subgroup B. Correspondingly, its Lie algebra $\mathfrak{p}$ is a subspace of $\mathfrak{g}$ containing $\mathfrak{b}$ and invariant under the action of B on $\mathfrak{g}$; i.e., it is a direct sum

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in T} \mathfrak{g}_\alpha$$

for some subset T of R that contains all positive roots. Now, in order for $\mathfrak{p}$ to be a subalgebra of $\mathfrak{g}$, the subset T must be closed under addition (that is, if two roots are in T, then either their sum is in T or is not a root). Since, in addition, T contains all the positive roots, we may observe that if α, β, and γ are positive roots with $\alpha = \beta + \gamma$, then we must have

$$-\alpha \in T \Rightarrow -\beta \in T \text{ and } -\gamma \in T.$$

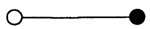
[1] It is a general fact that P must be connected if G/P is a projective variety.

Clearly, any such subset T must be generated by R^+ together with the negatives of a subset Σ of the set of simple roots. Thus, if for each subset Σ of the set of simple roots we let $T(\Sigma)$ consist of all roots which can be written as sums of negatives of the roots in Σ, together with all positive roots, and form the subalgebra

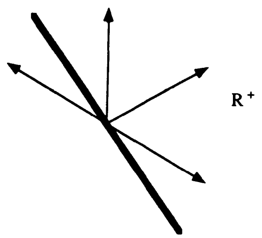
$$\mathfrak{p}(\Sigma) = \mathfrak{h} \oplus \bigoplus_{\alpha \in T(\Sigma)} \mathfrak{g}_\alpha, \tag{23.49}$$

then $\mathfrak{p}(\Sigma)$ is a parabolic subalgebra, the corresponding Lie group $P(\Sigma)$ is a parabolic subgroup containing B, and we obtain in this way all the parabolic subgroups of G. We can express this as the observation that, up to conjugacy, *parabolic subgroups of the simple group G are in one-to-one correspondence with subsets of the nodes of the Dynkin diagram, i.e., with subsets of the set of simple roots.*

Examples. In the case of $\mathfrak{sl}_3\mathbb{C}$, there is a symmetry in the Dynkin diagram, so that there is only one parabolic subgroup other than the Borel, corresponding to the diagram



This, in turn, gives the subset of the root system



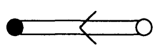
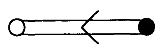
corresponding to the subgroup

$$P = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$$

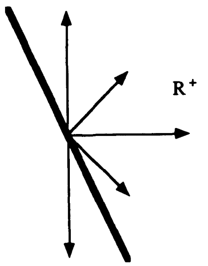
and the homogeneous space

$$G/P = \mathbb{P}^2.$$

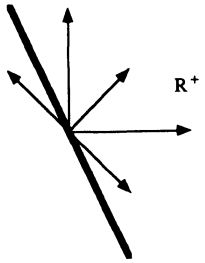
In the case of $\mathfrak{sp}_4\mathbb{C}$, there are two subdiagrams of the Dynkin diagram:

 and 

these correspond to the subsets of the root system



and

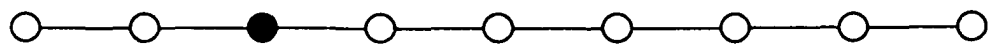


(Here we are using a black dot to indicate an omitted simple root, a white dot to indicate an included one.) The corresponding subgroups of $Sp_4\mathbb{C}$ are those preserving the vector e_1, and preserving the subspace spanned by e_1 and e_2, respectively. The quotients G/B are thus the variety of one-dimensional isotropic subspaces (i.e., the variety $\mathbb{P}^3$ of all the one-dimensional spaces) and the variety of two-dimensional isotropic subspaces.

Exercise 23.50. Interpret the diagrams above as giving rise to parabolic subgroups of the group $SO_5\mathbb{C}$ of automorphisms of $\mathbb{C}^5$ preserving a symmetric bilinear form. Show that the corresponding homogeneous spaces are the variety of isotropic planes and lines in $\mathbb{C}^5$, respectively. In particular, deduce the classical algebraic geometry facts that:

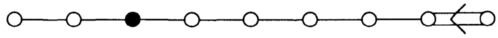
(i) The variety of isotropic 2-planes for a nondegenerate skew-symmetric bilinear form on $\mathbb{C}^4$ is isomorphic to a quadric hypersurface in $\mathbb{P}^4$.
(ii) The variety of isotropic 2-planes for a nondegenerate symmetric bilinear form on $\mathbb{C}^5$ (equivalently, lines on a smooth quadric hypersurface in $\mathbb{P}^4$) is isomorphic to $\mathbb{P}^3$.

In general, it is not hard to see that any parabolic subgroup P in a classical group G may be described as the subgroup that preserves a partial flag in the standard representation. In particular, a maximal parabolic subgroup, corresponding to omitting one node of the Dynkin diagram, may be described as the subgroup of G preserving a single subspace. Thus, for $G = SL_m\mathbb{C}$, the kth node of the Dynkin diagram

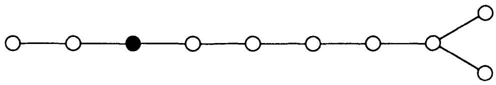


corresponds to the Grassmannian $G(k, m)$ of k-dimensional subspaces of $\mathbb{C}^m$. (Note that the symmetry of the diagram reflects the isomorphism of the Grassmannians $G(k, m)$ and $G(m - k, m)$.)

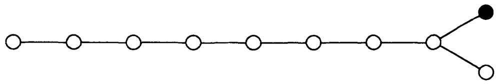
For $Sp_{2n}\mathbb{C}$, the kth node of the Dynkin diagram



corresponds to the *Lagrangian Grassmannian* of isotropic k-planes, for $k = 1, 2, \ldots, n$. Similarly, for $G = SO_{2n+1}\mathbb{C}$, the kth node of the Dynkin diagram corresponds to the *orthogonal Grassmannian* of isotropic k-planes in $\mathbb{C}^{2n+1}$. Finally, for $SO_{2n}\mathbb{C}$, for $k = 1, 2, \ldots, n-2$ the kth node of the Dynkin diagram



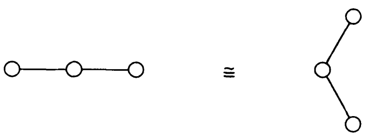
yields the orthogonal Grassmannian of isotropic k-planes in $\mathbb{C}^{2n}$, but there is one anomaly: either of the last two nodes



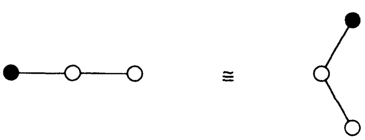
gives one of the two connected components of the Grassmannian of isotropic n-planes.

Exercise 23.51*. Compute $\mathfrak{p}(\Sigma)$ directly for each of the classical groups, and verify the above statements. Why is the orthogonal Grassmannian of isotropic $(n-1)$-planes in $\mathbb{C}^{2n}$ not included on the list?

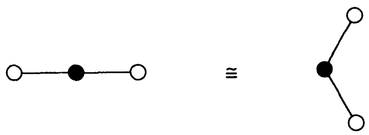
As we saw already in Exercise 23.50, the low-dimensional coincidences between Dynkin diagrams can be used to recover some facts we have seen before. For example, the coincidence $(D_2) = (A_1) \times (A_1)$ identifies the two family of lines on a quadratic surface in $\mathbb{P}^3$ with two copies of $\mathbb{P}^1$. The coincidence $(A_3) = (D_3)$



gives rise to two identifications of marked diagrams: we have



corresponding to the isomorphism between the Grassmann varieties $\mathbb{P}^3 = G(1, 4)$, $\check{\mathbb{P}}^3 = G(3, 4)$ and the two components of the family of 2-planes on a quadric hypersurface Q in $\mathbb{P}^5$; and



corresponding to the isomorphism of the Grassmannian $G(2, 4)$ with the quadric hypersurface Q itself. Finally, an observation that is not quite so elementary, but which we saw in §20.3: the identification of the diagrams

says that *either connected component of the variety of* 3*-planes on a smooth quadric hypersurface* Q *in* $\mathbb{P}^7$ *is isomorphic to the quadric* Q *itself.*

There is another way to realize the compact homogeneous spaces associated to a simple group G. Let $V = \Gamma_\lambda$ be an irreducible representation of G with highest weight λ, and consider the action of G on the projective space $\mathbb{P}V$. Let $p \in \mathbb{P}V$ be the point corresponding to the eigenspace with eigenvalue λ. We have then

Claim 23.52. *The orbit $G \cdot p$ is the unique closed orbit of the action of G on $\mathbb{P}V$.*

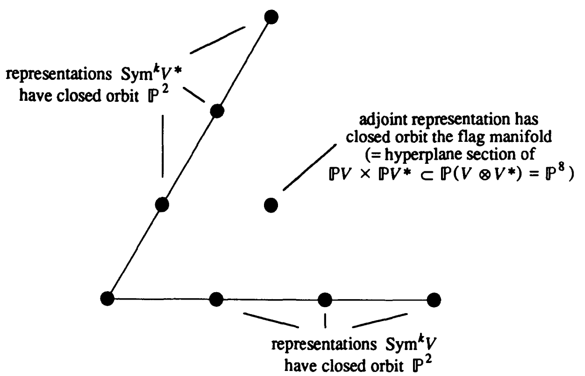
Proof. The point p is fixed under the Borel subgroup B, so that the stabilizer of p is a parabolic subgroup P_λ; the orbit G/P_λ is thus compact and hence closed. Conversely, by the Borel fixed point theorem, any closed orbit of G contains a fixed point for the action of B; but p is the unique point in $\mathbb{P}V$ fixed by B. □

In fact, it is not hard to say which parabolic subgroup P_λ is, in terms of the classification above: *it is the parabolic subgroup corresponding to the subset of simple roots that are perpendicular to the weight* λ. Now, sets Σ of simple roots correspond to faces of the Weyl chamber, namely, the face that is the intersection of all hyperplanes perpendicular to all roots in Σ.

We thus have a correspondence between faces of the Weyl chamber and parabolic subgroups P, such that if $V = \Gamma_\lambda$ is the irreducible representation with highest weight λ, then the unique closed orbit of the action of G on $\mathbb{P}V$

is of the form G/P, where P is the parabolic subgroup corresponding to the open face of $\mathscr{W}$ containing λ. In particular, weights in the interior of the Weyl chamber correspond to $P_\lambda = B$, and so determine the full flag manifold G/B, whereas weights on the edges give rise to the quotients of G by maximal parabolics. Note that we do obtain in this way all compact homogeneous spaces for G.

For example, we have the representations of $\mathrm{SL}_3\mathbb{C}$: as we have seen, the representations $\mathrm{Sym}^k V$ and $\mathrm{Sym}^k V^*$, with highest weights on the boundaries of the Weyl chamber, have closed orbits $\{v^k\}_{v \in V}$ and $\{l^k\}_{l \in V^*}$, isomorphic to $\mathbb{P}V$ and $\mathbb{P}V^*$. By contrast, the adjoint representation—the complement of the trivial representation in $\mathrm{Hom}(V, V) = V \otimes V^*$—has as closed orbit the variety of traceless rank 1 homomorphisms, which is isomorphic to the flag manifold via the map sending a homomorphism φ to the pair $(\mathrm{Im}\ \varphi, \mathrm{Ker}\ \varphi)$. The picture is

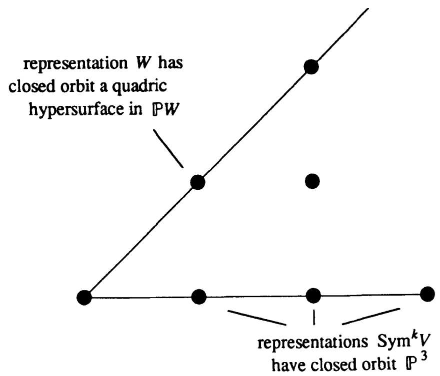


In general, if V is the standard representation of $\mathrm{SL}_n\mathbb{C}$, in the representations of $\mathrm{SL}_n\mathbb{C}$ of the form $W = \mathrm{Sym}^k V$ we saw that the vectors of the form $\{v^k\}_{v \in V}$ formed a closed orbit in $\mathbb{P}W$, called the Veronese embedding of $\mathbb{P}^{n-1}$. Likewise, in representations of the form $W = \wedge^k V$ the decomposable vectors $\{v_1 \wedge v_2 \wedge \cdots \wedge v_k\}$ formed a closed orbit in $\mathbb{P}W$; this is the Plücker embedding of the Grassmannian.

Similarly, we may identify the closed orbits in representations of $\mathrm{Sp}_4\mathbb{C}$. Recall here that the basic representations of $\mathrm{Sp}_4\mathbb{C}$ are the standard representation $V \cong \mathbb{C}^4$ and the complement W of the trivial representation in the exterior square $\wedge^2 V$; all other representations are contained in a tensor product of symmetric powers of these. Now, $\mathrm{Sp}_4\mathbb{C}$ acts transitively on $\mathbb{P}V$; the closed orbit is all of $\mathbb{P}^3$. In general, in $\mathbb{P}(\mathrm{Sym}^k V)$ the closed orbit is just the set of vectors $\{v^k\}_{v \in V} \cong \mathbb{P}^3$. By contrast, the closed orbit in $\mathbb{P}W$ is just the intersection of the hyperplane $\mathbb{P}W \subset \mathbb{P}(\wedge^2 V)$ with the locus of decomposable vectors $\{v \wedge w\}_{v, w \in V}$; this is the variety

$$X = \{v \wedge w: Q(v, w) = 0\}$$

of isotropic 2-planes $\Lambda \subset V$ for the skew form Q.



For the group $\mathrm{Spin}_{2n+1}\mathbb{C}$, the closed orbit of the spin representation S is the orthogonal Grassmannian of n-dimensional isotropic subspaces of $\mathbb{C}^{2n+1}$. The corresponding subvariety

$$G/P \hookrightarrow \mathbb{P}(S)$$

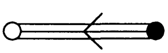
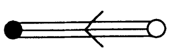
is a variety of dimension $(n+1)n/2$ in $\mathbb{P}^N$, $N = 2^n - 1$, called the *spinor variety*, or the variety of *pure spinors*. Similarly for $\mathrm{Spin}_{2n}\mathbb{C}$, the two spin representations S^+ and S^- give embeddings of the two components of the orthogonal Grassmannian of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$, one in $\mathbb{P}(S^+)$, one in $\mathbb{P}(S^-)$. These spinor varieties have dimension $n(n-1)/2$ in projective spaces of dimension $2^{n-1} - 1$.

Exercise 23.53. Show that the spinor variety for $\mathrm{Spin}_{2n-1}\mathbb{C}$ is isomorphic to each of the spinor varieties for $\mathrm{Spin}_{2n}\mathbb{C}$. In fact they are projectively equivalent as subvarieties of projective space $\mathbb{P}^N$, $N = 2^{n-1} - 1$.

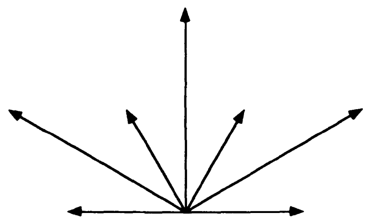
It follows that, for $m \leq 8$, the spinor varieties for $\mathrm{Spin}_m\mathbb{C}$ are isomorphic to homogeneous spaces we have described by other means. The first new one is the 10-dimensional variety in $\mathbb{P}^{15}$, which comes from $\mathrm{Spin}_9\mathbb{C}$ or $\mathrm{Spin}_{10}\mathbb{C}$.

It is worth going back to interpret some of the "geometric plethysm" of earlier lectures (e.g., Exercises 11.36 and 13.24) in this light.

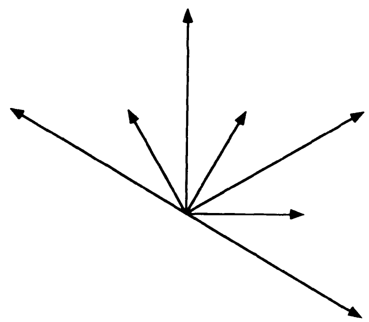
Finally, we can describe (at least one of) the compact homogeneous spaces for the group G_2 in this way. To begin with, G_2 has two maximal parabolic subgroups, corresponding to the diagrams

 and 

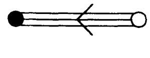
These are the groups whose Lie algebras are the parabolic subalgebras spanned by the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ together with the root spaces corresponding to the roots in the diagrams



and



In particular, each of these parabolic subgroups will have dimension 9, so that both the corresponding homogeneous spaces will be five-dimensional varieties. We can use this to identify one of these spaces: if V is the standard seven-dimensional representation of G_2, the closed orbit in $\mathbb{P}V \cong \mathbb{P}^6$ will be a hypersurface, which (since it is homogeneous) can only be a quadric hypersurface. Thus, the homogeneous space for G_2 corresponding to the diagram



is a quadric hypersurface in $\mathbb{P}^6$. In particular, we see again that the action of G_2 on V preserves a nondegenerate bilinear form, i.e., we have an inclusion

$$G_2 \hookrightarrow SO_7\mathbb{C}.$$

The other homogeneous space Y of $\mathfrak{g}_2$ is less readily described. One way to describe it is to use the fact that the adjoint representation W of $\mathfrak{g}_2$ is contained in the exterior square $\wedge^2 V$ of the standard. Since the Grassmannian $\mathbb{G}(1, 7) \subset \mathbb{P}(\wedge^2 V)$ of lines in $\mathbb{P}V$ is closed and invariant in $\mathbb{P}(\wedge^2 V)$, it follows that Y is contained in the intersection of $\mathbb{G}$ with the subspace $\mathbb{P}W \subset \mathbb{P}(\bigwedge^2 V)$. In other words, in terms of the skew-symmetric trilinear form ω on V preserved by the action of G_2, we can say that Y is contained in the locus

$$\Sigma = \{\Lambda \subset V: \omega(\Lambda, \Lambda, \cdot) \equiv 0\} \subset G(2, V).$$

Problem 23.54. Is $Y = \Sigma$?

Exercise 23.55. Show that the representation of E_6 whose highest weight is the first fundamental weight ω_1 determines a 16-dimensional homogeneous space in $\mathbb{P}^{26}$.

These homogeneous spaces have an amazing way of showing up as extremal examples of subvarieties of projective spaces, starting with a discovery of Severi that the Veronese surface in $\mathbb{P}^5$ is the only surface in $\mathbb{P}^5$ (nonsingular and not contained in a hyperplane) whose chords do not fill up $\mathbb{P}^5$. For recent work along these lines, see [L-VdV], with its appendix by Zak on interesting projective varieties that arise from representation theory.

Although we have described homogeneous spaces only for semisimple Lie groups, this is no real loss of generality: any irreducible representation V of a Lie group G comes from a representation of its semisimple quotient, up to multiplying by a character (see Proposition 9.17), and this character does not change the orbits in $\mathbb{P}(V)$.

It is possible to take this whole correspondence one step further and use it to give a construction of the irreducible representations of G; this is the modern approach to constructing the irreducible representations, due primarily to Borel, Weil, Bott, and, in a more general setting, Schmid. We do not have the means to do this in detail in the present circumstances, but we will sketch the construction.

The idea is very straightforward. We have just seen that for every irreducible representation V of G there is a unique closed orbit $X = G/P$ of the action of G on $\mathbb{P}V$. We obtain in this way from V a projective variety X together with a line bundle L on X invariant under the action of G (the restriction of the universal bundle from $\mathbb{P}V$). In fact, we may recover V from this data simply as the vector space of holomorphic sections of the line bundle L on X. What ties this all together is the fact that this gives us a one-to-one correspondence between irreducible representations of G and ample (positive) line bundles on compact homogeneous spaces G/P. More generally, using the projection maps $G/B \to G/P$, we may pull back all these line bundles to line bundles on G/B. This then extends to give an isomorphism between the weight lattice of $\mathfrak{g}$ and the group of line bundles on G/B, with the wonderful property that for dominant weights λ, the space of holomorphic sections of the associated line bundle L_λ is the irreducible representation of G with highest weight λ.

The point of all this, apart from its intrinsic beauty, is that we can go backward: starting with just the group G, we can construct the homogeneous space G/B, and then realize all the irreducible representations of G as cohomology groups of line bundles on G/B. To carry this out, start with a weight $\lambda \in \mathfrak{h}^*$ for $\mathfrak{g}$. We have seen that λ exponentiates to a homomorphism $H \to \mathbb{C}^*$, i.e., it gives a one-dimensional representation $\mathbb{C}_\lambda$ of H. We want to induce this representation from H to G. If $H \subset B \subset G$ is a Borel subgroup, the representation extends trivially to B, since B is a semidirect product of H and the nilpotent subgroup N whose Lie algebra is the direct sum of those $\mathfrak{g}_\alpha$ for positive roots α. Then we can form

$$\begin{aligned} L_\lambda &= G \times_B \mathbb{C}_\lambda \\ &= (G \times \mathbb{C}_\lambda)/\{(g, v) \sim (gx, x^{-1}v), x \in B\}, \end{aligned}$$

which, with its natural projection to G/B, is a holomorphic line bundle on the projective variety G/B. The cohomology groups of such a line bundle are finite dimensional, and since G acts on L_λ, these cohomology groups are representations of G.

We have Bott's theorem for the vanishing of the cohomology of this line bundle:

Claim 23.56. $H^i(G/B, L_\lambda) = 0$ *for* $i \neq i(\lambda)$,

where $i(\lambda)$ is an integer depending on which Weyl chamber λ belongs to. If λ is a dominant weight (i.e., belongs to the closure of the positive Weyl chamber for the choice of positive roots used in defining B), then $i(-\lambda) = 0$. In this case the sections $H^0(G/B, L_{-\lambda})$ are a finite-dimensional vector space, on which G acts.

Claim 23.57. *For λ a dominant weight, the space of sections $H^0(G/B, L_{-\lambda})$ is the irreducible representation with highest weight λ.*

In this context the Riemann–Roch theorem can be applied to give a formula for the dimension of the irreducible representation. In fact, the dimension part of Weyl's character formula can be proved this way. More refined analysis, using the Woods Hole fixed point theorem, can be used to get the full character formula (cf. [A-B]). For a very readable introduction to this, see [Bot].

We conclude this discussion by giving a diagram showing the relationships among the various objects associated to an irreducible representation of a semisimple Lie algebra $\mathfrak{g}$. The objects and maps in diagram (23.58) are explained next.

First of all, as we have indicated, the term "Grassmannians" means the ordinary Grassmannians in the case of the groups $SL_n\mathbb{C}$, and the Lagrangian Grassmannians and the orthogonal Grassmannians of isotropic subspaces in the cases of $Sp_{2n}\mathbb{C}$ and $SO_m\mathbb{C}$, respectively. Likewise, "flag manifolds" refers to the spaces parametrizing nested sequences of such subspaces. In the cases

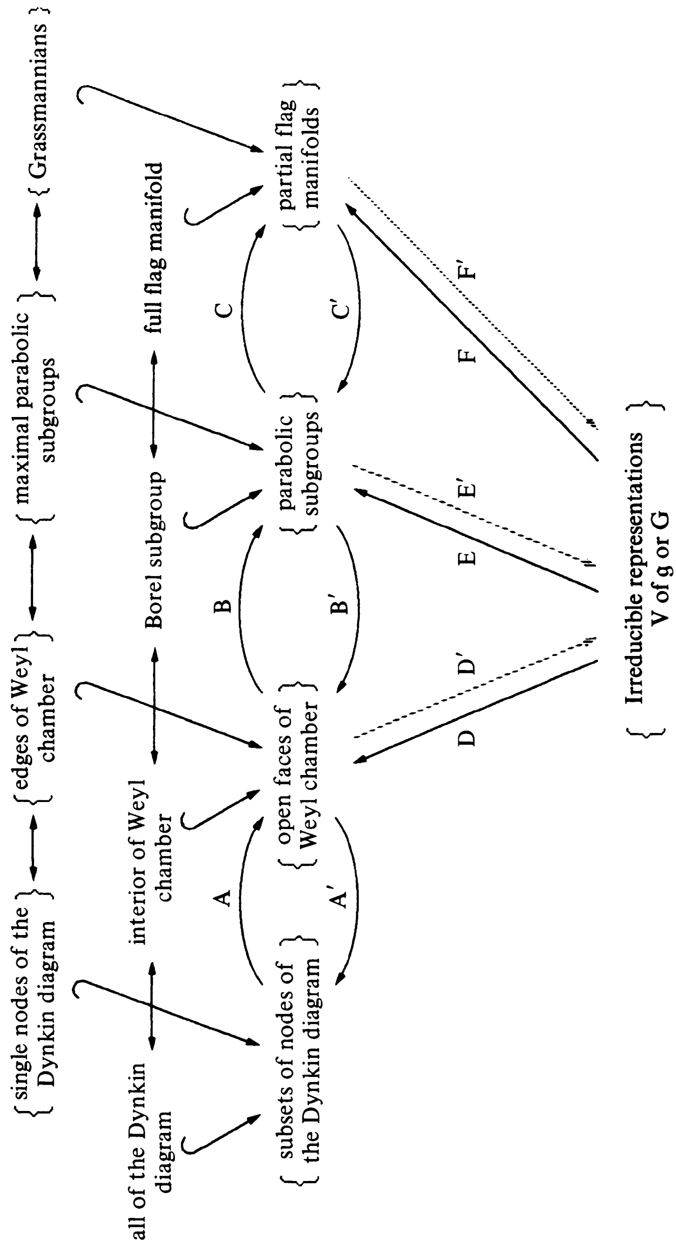


Diagram 23.58

of the exceptional Lie algebras, the term "Grassmannian" should just be ignored; except for the quotient of G_2 by one of its two maximal parabolic subgroups, the homogeneous spaces for the exceptional groups are not varieties with which we are likely to be a priori familiar.

With this said, we may describe the maps A, B, etc., as follows:

A, A': the map A associates to a subset of the nodes of the Dynkin diagram (equivalently, a subset S of the set of simple roots) the face of the Weyl chamber described by

$$\mathscr{W}_S = \left\{\lambda: \begin{matrix} (\lambda, \alpha) > 0, \forall \alpha \in S; \\ (\lambda, \alpha) = 0, \forall \alpha \notin S \end{matrix}\right\},$$

where (,) is the Killing form; the inverse is clear.

B, B': the map B associates to a face $\mathscr{W}_S$ of the Weyl chamber the subalgebra $\mathfrak{g}_S$ spanned by the Cartan subalgebra $\mathfrak{h}$, the positive root spaces $\mathfrak{g}_\alpha$, $\alpha \in R^+$, and the root spaces $\mathfrak{g}_{-\alpha}$ corresponding to those positive roots α perpendicular to $\mathscr{W}_S$. Equivalently, in terms of the corresponding subset S of the simple roots, $\mathfrak{g}_S$ will be generated by the Borel subalgebra, together with the root spaces $\mathfrak{g}_{-\alpha}$ for $\alpha \notin S$. Again, since every parabolic subalgebra is conjugate to one of this form, the inverse map is clear.

C, C': The map C simply associates to a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ the quotient G/P of G by the corresponding parabolic subgroup $P \subset G$. In the other direction, given the homogeneous space $X = G/P$, with the action of G, the group P is just the stabilizer of a point in X. Note that the connected component of the identity in the automorphism group of G/P may be strictly larger: for example, $\mathbb{P}^{2n-1}$ is a compact homogeneous space for $Sp_{2n}\mathbb{C}$, and we have seen that a quadric hypersurface in $\mathbb{P}^6$ is a homogeneous space for G_2.

D, D': The map D associates to the irreducible representation V of $\mathfrak{g}$ with highest weight λ the open face of the Weyl chamber containing λ. In the other direction, given an open face $\mathscr{W}_S$ of $\mathscr{W}$, choose a lattice point $\lambda \in \mathscr{W}_S \cap \Lambda_W$ and take $V = \Gamma_\lambda$.

E: We send the representation V to the subalgebra or subgroup fixing the highest weight vector $v \in V$.

F, F': We associate to the representation V the (unique) closed orbit of the corresponding action of the group G on the projective space $\mathbb{P}V$. Going in the other direction, we have to choose an ample line bundle L on the space G/P, and then take its vector space of holomorphic sections.

§23.4. Bruhat Decompositions

We end this lecture with a brief introduction to the *Bruhat decomposition* of a semisimple complex Lie group G, and the related *Bruhat cells* in the flag manifold G/B. These ideas are not used in this course, but they appear so often elsewhere that it may be useful to describe them in the language we have developed in this lecture. We will give the general statements, but verify them only for the classical groups. General proofs can be found in [Bor1] or [Hu2].

As we have seen, a choice of positive roots determines a Borel subgroup B and Cartan subgroup H, with normalizer $N(H)$, so $N(H)/H$ is identified with the Weyl group $\mathfrak{W}$. For each $W \in \mathfrak{W}$ fix a representative n_W in $N(H)$. The double coset $B \cdot n_W \cdot B$ is clearly independent of choice of n_W, and will be denoted $B \cdot W \cdot B$.

Theorem 23.59 (Bruhat Decomposition). *The group G is a disjoint union of the $|\mathfrak{W}|$ double cosets $B \cdot W \cdot B$, as W varies over the Weyl group.*

Let us first see this explicitly for $G = \mathrm{SL}_m\mathbb{C}$. Here $N(H)$ consists of all monomial matrices in $\mathrm{SL}_m\mathbb{C}$, i.e., matrices with exactly one nonzero entry in each row and each column, and $\mathfrak{W} = \mathfrak{S}_m$; a monomial matrix with nonzero entry in the $\sigma(j)$th row of the jth column maps to the permutation σ. To see that the double cosets cover G, given $g \in G$, use elementary row operations by left multiplication by elements in B to get an element $b \cdot g^{-1}$, with $b \in B$ chosen so that the total number of zeros appearing at the left in the rows in $b \cdot g^{-1}$ is as large as possible. If two rows of $b \cdot g^{-1}$ had the same number of zeros at the left, one could increase the total by an elementary row operation. Since all the rows of $b \cdot g^{-1}$ start with different numbers of zeros, this matrix can be put in upper-triangular form by left multiplication by a monomial matrix; therefore, there is a permutation σ so that $b' = n_\sigma \cdot b \cdot g^{-1}$ is upper triangular, i.e., $g = (b')^{-1} \cdot n_\sigma \cdot b$ is in $B \cdot \sigma \cdot B$. To see that the double cosets are disjoint, suppose $n_{\sigma'} = b' \cdot n_\sigma \cdot b$ for some b and b' in B. From the equation $b = (n_\sigma)^{-1} \cdot (b')^{-1} \cdot n_{\sigma'}$ one sees that b must have nonzero entries in each place where $(n_\sigma)^{-1} \cdot n_{\sigma'}$ does, from which it follows that $\sigma' = \sigma$.

In fact, this can be strengthened as follows. Let U (resp. U^-) be the subgroup of G whose Lie algebra is the sum of all root spaces $\mathfrak{g}_\alpha$ for all positive (resp. negative) roots α. For $G = \mathrm{SL}_m\mathbb{C}$, U (resp. U^-) consists of upper- (resp. lower-) triangular matrices with 1's on the diagonal. For W in the Weyl group, define subgroups

$$U(W) = U \cap n_W \cdot U^- \cdot n_W^{-1}, \qquad U(W)' = U \cap n_W \cdot U \cdot n_W^{-1}$$

of U, which are again independent of the choice of representative n_W for W.

Corollary 23.60. *Every element in $B \cdot W \cdot B$ can be written $u \cdot n_W \cdot b$ for unique elements u in $U(W)$ and b in B.*

To see the existence of such an expression, note first that the Lie algebra of $U(W)$ is the sum of all root spaces $\mathfrak{g}_\alpha$ for which α is positive and $W^{-1}(\alpha)$ is negative; and the Lie algebra of $U(W)'$ is the sum of all root spaces $\mathfrak{g}_\alpha$ for which α and $W^{-1}(\alpha)$ are positive. One sees from this that $U(W) \cdot U(W)' \cdot H$ is the entire Borel group B. Since $H \cdot n_W = n_W \cdot H$ and $U(W)' \cdot n_W = n_W \cdot U$, and H and U are subgroups of B,

$$\begin{aligned} B \cdot n_W \cdot B &= U(W) \cdot U(W)' \cdot H \cdot n_W \cdot B \\ &= U(W) \cdot U(W)' \cdot n_W \cdot B \\ &= U(W) \cdot n_W \cdot B. \end{aligned}$$

To see the uniqueness, suppose that $n_W = u \cdot n_W \cdot b$ for some u in $U(W)$ and b in B. Then $n_W^{-1} \cdot u \cdot n_W$ is in $U^- \cap B = \{1\}$, so $u = 1$, as required.

Note in particular that the dimension of $U(W)$ is the cardinality of $R^+ \cap W(R^-)$, where R^+ and R^- are the positive and negative roots; this is also the minimum number $l(W)$ of reflections in simple roots whose product is W, cf. Exercise D.30. It is a general fact, which we will see for the classical groups, that $U(W)$ is isomorphic to an affine space $\mathbb{C}^{l(W)}$.

It follows from the Bruhat decomposition that G/B is a disjoint union of the cosets $X_W = B \cdot n_W \cdot B/B$, again with W varying over the Weyl group. These X_W are called *Bruhat cells*. From the corollary we see that X_W is isomorphic to the affine space $U(W) \cong \mathbb{C}^{l(W)}$.

For $G = \mathrm{SL}_m\mathbb{C}$ and σ in $\mathfrak{W} = \mathfrak{S}_m$, the group $U(\sigma)$ consists of matrices with 1's on the diagonal, and zero entry in the i, j place whenever either $i > j$ or $\sigma^{-1}(i) < \sigma^{-1}(j)$, which is an affine space of dimension $l(\sigma) = \#\{(i, j): i > j \text{ and } \sigma(i) < \sigma(j)\}$.

Exercise 23.61. Identifying $\mathrm{SL}_m\mathbb{C}/B$ with the space of all flags, show that X_σ consists of those flags $0 \subset V_1 \subset V_2 \subset \cdots$ such that the dimensions of intersections with the standard flag are governed by σ, in the following sense: for each $1 \le k \le m$, the set of k numbers d such that $V_k \cap \mathbb{C}^{d-1} \subsetneqq V_k \cap \mathbb{C}^d$ is precisely the set $\{\sigma(1), \sigma(2), \ldots, \sigma(k)\}$.

We will verify the Bruhat decomposition for $\mathrm{Sp}_{2n}\mathbb{C}$ by regarding it as a subgroup of $\mathrm{SL}_{2n}\mathbb{C}$ and using what we have just seen for $\mathrm{SL}_{2n}\mathbb{C}$, following [Ste2]. Our description of $\mathrm{Sp}_{2n}\mathbb{C}$ in Lecture 16 amounts to saying that it is the fixed point set of the automorphism φ of $\mathrm{SL}_{2n}\mathbb{C}$ given by $\varphi(A) = M^{-1} \cdot {}^tA^{-1} \cdot M$, with $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The Borel subgroup of $\mathrm{Sp}_{2n}\mathbb{C}$ will be the intersection of the Borel subgroup B of $\mathrm{SL}_{2n}\mathbb{C}$ with $\mathrm{Sp}_{2n}\mathbb{C}$, provided we change the order of the basis of $\mathbb{C}^{2n}$ to $e_1, \ldots, e_n, e_{2n}, \ldots, e_{n+1}$, so that B consists of matrices whose upper left block is upper triangular, whose lower left block is zero, and whose lower right block is lower triangular. The automorphism φ maps this B to itself, and also preserves the diagonal subgroup H and its normalizer $N(H)$, and the groups U and U^-. The Weyl group of $\mathrm{Sp}_{2n}\mathbb{C}$ can be identified with the permutations in $\mathfrak{S}_{2n}$ such that $\sigma(n + i) = \sigma(i) \pm n$ for all $1 \le i \le n$, and it is exactly for these σ for which one can choose a monomial representative n_σ in $\mathrm{Sp}_{2n}\mathbb{C}$. Now if g is any element in $\mathrm{Sp}_{2n}\mathbb{C}$, write $g = u \cdot n_\sigma \cdot b$ according to the above corollary. Then

$$g = \varphi(g) = \varphi(u) \cdot \varphi(n_\sigma) \cdot \varphi(b),$$

and by uniqueness of the decomposition we must have $\varphi(u) = u$, $\varphi(n_\sigma) = n_\sigma \cdot h$, $h \in H$, and $\varphi(b) = h^{-1} \cdot b$. It follows that σ belongs to the Weyl group of $\mathrm{Sp}_{2n}\mathbb{C}$. This gives the Bruhat decomposition, and, moreover, a unique decomposition of $g \in \mathrm{Sp}_{2n}\mathbb{C}$ into $u \cdot n_\sigma \cdot b$, with u in $U(\sigma) \cap \mathrm{Sp}_{2n}\mathbb{C}$. Since this latter is an affine space, this shows that the corresponding Bruhat cell in the symplectic flag manifold is an affine space.

Exactly the same idea works for the orthogonal groups $\mathrm{SO}_m\mathbb{C}$, by realizing them as fixed points of automorphisms of $\mathrm{SL}_m\mathbb{C}$ of the form $A \mapsto M^{-1} \cdot {}^tA^{-1} \cdot M$, with M the matrix giving the quadratic form.

Note finally that if W' is the element in the Weyl group that takes each Weyl chamber to its negative, then $B \cdot W' \cdot B$ is a dense open subset of G, a fact which is evident for the classical groups by the above discussion. The corresponding Bruhat cell $X_{W'}$ is the image of U^- in G/B, which is also a dense open set. It follows that a function or section of a line bundle on G/B is determined by its values on U^-. For treatises developing representation theory via functions on U^-, see [N-S] or [Žel].

The following exercise uses these ideas to sketch a proof of Claim 23.57 that the sections of the bundle $L_{-\lambda}$ on G/B form the irreducible representation with highest weight λ:

Exercise 23.62*. (a) Show that sections s of $L_{-\lambda}$ are all of the form $s(gB) = (g, f(g))$, where f is a holomorphic function on G satisfying

$$f(g \cdot x) = \lambda(x) f(g) \quad \text{for all } x \in B.$$

(b) Let $n' \in N(H)$ be a representative of the element W' in the Weyl group which takes each element to its negative. Show that f is determined by its value at n'.

(c) Show that any highest weight for f must be λ, and conclude that $H^0(G/B, L_{-\lambda})$ is the irreducible representation Γ_λ with highest weight λ.

The holomorphic functions f of this exercise are functions on the space G/U. In other words, all irreducible representations of G can be found in spaces of functions on G/U. This is one common approach to the study of representations, especially by the Soviet school, cf. [N-S], [Žel].

Functions on G/U form a commutative ring, which indicates how to make the sum of all the irreducible representations into a commutative ring. In fact, for the classical groups, these rings are the algebras $\mathbb{S}^{\bullet}$, $\mathbb{S}^{\langle\bullet\rangle}$, and $\mathbb{S}^{[\bullet]}$ constructed in Lectures 15, 17, and 19, cf. [L-T]. They are also coordinate rings for natural embeddings of flag manifolds in products of projective spaces.