LECTURE 3

Examples; Induced Representations; Group Algebras; Real Representations

This lecture is something of a grabbag. We start in §3.1 with examples illustrating the use of the techniques of the preceding lecture. Section 3.2 is also by way of an example. We will see quite a bit more about the representations of the symmetric groups in general later; §4 is devoted to this and will certainly subsume this discussion, but this should provide at least a sense of how we can go about analyzing representations of a class of groups, as opposed to individual groups. In §§3.3 and 3.4 we introduce two basic notions in representation theory, induced representations and the group algebra. Finally, in §3.5 we show how to classify representations of a finite group on a real vector space, given the answer to the corresponding question over $\mathbb{C}$, and say a few words about the analogous question for subfields of $\mathbb{C}$ other than $\mathbb{R}$. Everything in this lecture is elementary except Exercises 3.9 and 3.32, which involve the notions of Clifford algebras and the Fourier transform, respectively (both exercises, of course, can be skipped).

§3.1. Examples: $\mathfrak{S}_5$ and $\mathfrak{A}_5$

We have found the representations of the symmetric and alternating groups for $n \leq 4$. Before turning to a more systematic study of symmetric and alternating groups, we will work out the next couple of cases.

Representations of the Symmetric Group $\mathfrak{S}_5$

As before, we start by listing the conjugacy classes of $\mathfrak{S}_5$ and giving the number of elements of each: we have 10 transpositions, 20 three-cycles, 30 four-cycles and 24 five-cycles; in addition, we have 15 elements conjugate to (12)(34) and 10 elements conjugate to (12)(345). As for the irreducible representations, we have, of course, the trivial representation U, the alternating representation U', and the standard representation V; also, as in the case of $\mathfrak{S}_4$ we can tensor the standard representation V with the alternating one to obtain another irreducible representation V' with character $\chi_{V'} = \chi_V \cdot \chi_{U'}$.

Exercise 3.1. Find the characters of the representations V and V'; deduce in particular that V and V' are distinct irreducible representations.

The first four rows of the character table are thus

	1	10	20	30	24	15	20
$\mathfrak{S}_5$	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1

Clearly, we need three more irreducible representations. Where should we look for these? On the basis of our previous experience (and Problem 2.37), a natural place would be in the tensor products/powers of the irreducible representations we have found so far, in particular in $V \otimes V$ (the other two possible products will yield nothing new: we have $V' \otimes V = V \otimes V \otimes U'$ and $V' \otimes V' = V \otimes V$). Of course, $V \otimes V$ breaks up into $\wedge^2 V$ and $\mathrm{Sym}^2 V$, so we look at these separately. To start with, by the formula

$$\chi_{\wedge^2 V}(g) = \tfrac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$$

we calculate the character of $\wedge^2 V$:

$$\chi_{\wedge^2 V} = (6, 0, 0, 0, 1, -2, 0);$$

we see from this that it is indeed a fifth irreducible representation (and that $\wedge^2 V \otimes U' = \wedge^2 V$, so we get nothing new that way).

We can now find the remaining two representations in either of two ways. First, if n_1 and n_2 are their dimensions, we have

$$5! = 120 = 1^2 + 1^2 + 4^2 + 4^2 + 6^2 + n_1^2 + n_2^2,$$

so $n_1^2 + n_2^2 = 50$. There are no more one-dimensional representations, since these are trivial on normal subgroups whose quotient group is cyclic, and $\mathfrak{A}_5$

is the only such subgroup. So the only possibility is $n_1 = n_2 = 5$. Let W denote one of these five-dimensional representations, and set $W' = W \otimes U'$. In the table, if the row giving the character of W is

$$(5 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6),$$

that of W' is $(5 \quad -\alpha_1 \quad \alpha_2 \quad -\alpha_3 \quad \alpha_4 \quad \alpha_5 \quad -\alpha_6)$. Using the orthogonality relations or (2.20), one sees that $W' \not\cong W$; and with a little calculation, up to interchanging W and W', the last two rows are as given:

	1	10	20	30	24	15	20
$\mathfrak{S}_5$	1	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
U	1	1	1	1	1	1	1
U'	1	-1	1	-1	1	1	-1
V	4	2	1	0	-1	0	-1
V'	4	-2	1	0	-1	0	1
$\wedge^2 V$	6	0	0	0	1	-2	0
W	5	1	-1	-1	0	1	1
W'	5	-1	-1	1	0	1	-1

From the decomposition $V \oplus U = \mathbb{C}^5$, we have also $\wedge^4 V = \wedge^5 \mathbb{C}^5 = U'$, and $V^* = V$. The perfect pairing[1]

$$V \times \wedge^3 V \to \wedge^4 V = U',$$

taking $v \times (v_1 \wedge v_2 \wedge v_3)$ to $v \wedge v_1 \wedge v_2 \wedge v_3$ shows that $\wedge^3 V$ is isomorphic to $V^* \otimes U' = V'$.

Another way to find the representations W and W' would be to proceed with our original plan, and look at the representation $\mathrm{Sym}^2 V$. We will leave this in the form of an exercise:

Exercise 3.2. (i) Find the character of the representation $\mathrm{Sym}^2 V$.

(ii) Without using any knowledge of the character table of $\mathfrak{S}_5$, use this to show that $\mathrm{Sym}^2 V$ is the direct sum of three distinct irreducible representations.

(iii) Using our knowledge of the first five rows of the character table, show that $\mathrm{Sym}^2 V$ is the direct sum of the representations U, V, and a third irreducible representation W. Complete the character table for $\mathfrak{S}_5$.

Exercise 3.3. Find the decomposition into irreducibles of the representations $\wedge^2 W$, $\mathrm{Sym}^2 W$, and $V \otimes W$.

[1] If V and W are n-dimensional vector spaces, and U is one dimensional, a *perfect pairing* is a bilinear map $\beta \colon V \times W \to U$ such that no nonzero vector v in V has $\beta(v, W) = 0$. Equivalently, the map $V \to \mathrm{Hom}(W, U) = W^* \otimes U$, $v \mapsto (w \mapsto \beta(v, w))$, is an isomorphism.

Representations of the Alternating Group $\mathfrak{A}_5$

What happens to the conjugacy classes above if we replace $\mathfrak{S}_d$ by $\mathfrak{A}_d$? Obviously, all the odd conjugacy classes disappear; but at the same time, since conjugation by a transposition is now an outer, rather than inner, automorphism, some conjugacy classes may break into two.

Exercise 3.4. Show that the conjugacy class in $\mathfrak{S}_d$ of permutations consisting of products of disjoint cycles of lengths $b_1, b_2, \ldots$ will break up into the union of two conjugacy classes in $\mathfrak{A}_d$ if all the b_k are odd and distinct; if any b_k are even or repeated, it remains a single conjugacy class in $\mathfrak{A}_d$. (We consider a fixed point as a cycle of length 1.)

In the case of $\mathfrak{A}_5$, this means we have the conjugacy class of three-cycles (as before, 20 elements), and of products of two disjoint transpositions (15 elements); the conjugacy class of five-cycles, however, breaks up into the conjugacy classes of (12345) and (21345), each having 12 elements.

As for the representations, the obvious first place to look is at restrictions to $\mathfrak{A}_5$ of the irreducible representations of $\mathfrak{S}_5$ found above. An irreducible representation of $\mathfrak{S}_5$ may become reducible when restricted to $\mathfrak{A}_5$; or two distinct representations may become isomorphic, as will be the case with U and U', V and V', or W and W'. In fact, U, V, and W stay irreducible since their characters satisfy $(\chi, \chi) = 1$. But the character of $\wedge^2 V$ has values $(6, 0, -2, 1, 1)$ on the conjugacy classes listed above, so $(\chi, \chi) = 2$, and $\wedge^2 V$ is the sum of two irreducible representations, which we denote by Y and Z. Since the sums of the squares of all the dimensions is 60, $(\dim Y)^2 + (\dim Z)^2 = 18$, so each must be three dimensional.

Exercise 3.5*. Use the orthogonality relations to complete the character table of $\mathfrak{A}_5$:

	1	20	15	12	12
$\mathfrak{A}_5$	1	(123)	(12)(34)	(12345)	(21345)
U	1	1	1	1	1
V	4	1	0	-1	-1
W	5	-1	1	0	0
Y	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
Z	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

The representations Y and Z may in fact be familiar: $\mathfrak{A}_5$ can be realized as the group of motions of an icosahedron (or, equivalently, of a dodecahedron) and Y is the corresponding representation. Note that the two representations $\mathfrak{A}_5 \to GL_3(\mathbb{R})$ corresponding to Y and Z have the same image, but (as you can see from the fact that their characters differ only on the conjugacy classes of (12345) and (21345)) differ by an *outer* automorphism of $\mathfrak{A}_5$.

Note also that $\wedge^2 V$ does not decompose over $\mathbb{Q}$; we could see this directly from the fact that the vertices of a dodecahedron cannot all have rational coordinates, which follows from the analogous fact for a regular pentagon in the plane.

Exercise 3.6. Find the decomposition of the permutation representation of $\mathfrak{A}_5$ corresponding to the (i) vertices, (ii) faces, and (iii) edges of the icosahedron.

Exercise 3.7. Consider the dihedral group D_{2n}, defined to be the group of isometries of a regular n-gon in the plane. Let $\Gamma \cong \mathbb{Z}/n \subset D_{2n}$ be the subgroup of rotations. Use the methods of Lecture 1 (applied there to the case $\mathfrak{S}_3 \cong D_6$) to analyze the representations of D_{2n}: that is, restrict an arbitrary representation of D_{2n} to Γ, break it up into eigenspaces for the action of Γ, and ask how the remaining generator of D_{2n} acts of these eigenspaces.

Exercise 3.8. Analyze the representations of the dihedral group D_{2n} using the character theory developed in Lecture 2.

Exercise 3.9. (a) Find the character table of the group of order 8 consisting of the quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$ under multiplication. This is the case $m = 3$ of a collection of groups of order 2^m, which we denote H_m. To describe them, let C_m denote the complex Clifford algebra generated by $v_1, \ldots, v_m$ with relations $v_i^2 = -1$ and $v_i \cdot v_j = -v_j \cdot v_i$, so C_m has a basis $v_I = v_{i_1} \cdot \ldots \cdot v_{i_r}$, as $I = \{i_i < \cdots < i_r\}$ varies over subsets of $\{1, \ldots, m\}$. (See §20.1 for notation and basic facts about Clifford algebras). Set

$$H_m = \{\pm v_I \colon |I| \text{ is even}\} \subset (C_m^{\text{even}})^*.$$

This group is a 2-to-1 covering of the abelian 2-group of $m \times m$ diagonal matrices with ± 1 diagonal entries and determinant 1. The center of H_m is $\{\pm 1\}$ if m is odd and is $\{\pm 1, \pm v_{\{1,\ldots,m\}}\}$ if m is even. The other conjugacy classes consist of pairs of elements $\{\pm v_I\}$. The isomorphisms of C_m^{even} with a matrix algebra or a product of two matrix algebras give a 2^n-dimensional "spin" representation S of H_{2n+1}, and two 2^{n-1}-dimensional "spin" or "half-spin" representations S^+ and S^- of H_{2n}.

(b) Compute the characters of these spin representations and verify that they are irreducible.

(c) Deduce that the spin representations, together with the 2^{m-1} one-dimensional representations coming from the abelian group $H_m/\{\pm 1\}$ give a complete set of irreducible representations, and compute the character table for H_m.

For odd m the groups H_m are examples of *extra-special* 2-*groups*, cf. [Grie], [Qu].

Exercise 3.10. Find the character table of the group $\mathrm{SL}_2(\mathbb{Z}/3)$.

Exercise 3.11. Let $H(\mathbb{Z}/3)$ be the *Heisenberg group* of order 27:

$$H(\mathbb{Z}/3) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{Z}/3 \right\} \subset \mathrm{SL}_3(\mathbb{Z}/3).$$

Analyze the representations of $H(\mathbb{Z}/3)$, first by the methods of Lecture 1 (restricting in this case to the center

$$Z = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b \in \mathbb{Z}/3 \right\} \cong \mathbb{Z}/3$$

of $H(\mathbb{Z}/3)$), and then by character theory.

§3.2. Exterior Powers of the Standard Representation of $\mathfrak{S}_d$

How should we go about constructing representations of the symmetric groups in general? The answer to this is not immediate; it is a subject that will occupy most of the next lecture (where we will produce all the irreducible representations of $\mathfrak{S}_d$). For now, as an example of the elementary techniques developed so far we will analyze directly one of the obvious candidates:

Proposition 3.12. *Each exterior power $\wedge^k V$ of the standard representation V of $\mathfrak{S}_d$ is irreducible,* $0 \le k \le d - 1$.

Proof. From the decomposition $\mathbb{C}^d = V \oplus U$, we see that V is irreducible if and only if $(\chi_{\mathbb{C}^d}, \chi_{\mathbb{C}^d}) = 2$. Similarly, since

$$\wedge^k \mathbb{C}^d = (\wedge^k V \otimes \wedge^0 U) \oplus (\wedge^{k-1} V \otimes \wedge^1 U) = \wedge^k V \oplus \wedge^{k-1} V,$$

it suffices to show that $(\chi, \chi) = 2$, where χ is the character of the representation $\wedge^k \mathbb{C}^d$. Let $A = \{1, 2, \ldots, d\}$. For a subset B of A with k elements, and $g \in G = \mathfrak{S}_d$, let

$$\{g\}_B = \begin{cases} 0 & \text{if } g(B) \neq B \\ 1 & \text{if } g(B) = B \text{ and } g|_B \text{ is an even permutation} \\ -1 & \text{if } g(B) = B \text{ and } g|_B \text{ is odd.} \end{cases}$$

Here, if $g(B) = B$, $g|_B$ denotes the permutation of the set B determined by g. Then $\chi(g) = \sum \{g\}_B$, and

$$\begin{aligned}(\chi, \chi) &= \frac{1}{d!}\sum_{g\in G}\left(\sum_B \{g\}_B\right)^2 \\ &= \frac{1}{d!}\sum_{g\in G}\sum_B\sum_C \{g\}_B\{g\}_C \\ &= \frac{1}{d!}\sum_B\sum_C\sum_g (\text{sgn } g|_B)\cdot(\text{sgn } g|_C),\end{aligned}$$

where the sums are over subsets B and C of A with k elements, and in the last equation, the sum is over those g with $g(B) = B$ and $g(C) = C$. Such g is given by four permutations: one of $B \cap C$, one of $B\setminus B \cap C$, one of $C\setminus B \cap C$, and one of $A\setminus B \cup C$. Letting l be the cardinality of $B \cap C$, this last sum can be written

$$\begin{aligned}&\frac{1}{d!}\sum_B\sum_C\sum_{a\in\mathfrak{S}_l}\sum_{b\in\mathfrak{S}_{k-l}}\sum_{c\in\mathfrak{S}_{k-l}}\sum_{h\in\mathfrak{S}_{d-2k+l}}(\text{sgn } a)^2(\text{sgn } b)(\text{sgn } c) \\ &\quad = \frac{1}{d!}\sum_B\sum_C l!(d-2k+l)!\left(\sum_{b\in\mathfrak{S}_{k-l}}\text{sgn } b\right)\left(\sum_{c\in\mathfrak{S}_{k-l}}\text{sgn } c\right).\end{aligned}$$

These last sums are zero unless $k - l = 0$ or 1. The case $k = l$ gives

$$\frac{1}{d!}\sum_B k!(d-k)! = \frac{1}{d!}\binom{d}{k}k!(d-k)! = 1.$$

Similarly, the terms with $k - l = 1$ also add up to 1, so $(\chi, \chi) = 2$, as required. □

Note by way of contrast that the symmetric powers of the standard representation of $\mathfrak{S}_d$ are almost never irreducible. For example, we already know that the representation $\text{Sym}^2 V$ contains one copy of the trivial representation: this is just the statement that every irreducible real representation (such as V) admits an inner product (unique, up to scalars) invariant under the group action; nor is the quotient of $\text{Sym}^2 V$ by this trivial subrepresentation necessarily irreducible, as witness the case of $\mathfrak{S}_5$.

§3.3. Induced Representations

If $H \subset G$ is a subgroup, any representation V of G restricts to a representation of H, denoted $\text{Res}_H^G V$ or simple Res V. In this section, we describe an important construction which produces representations of G from representations of H. Suppose V is a representation of G, and $W \subset V$ is a subspace which is H-invariant. For any g in G, the subspace $g \cdot W = \{g \cdot w\colon w \in W\}$ depends only on the left coset gH of g modulo H, since $gh \cdot W = g \cdot (h \cdot W) = g \cdot W$; for a coset

σ in G/H, we write $\sigma \cdot W$ for this subspace of V. We say that V is *induced* by W if every element in V can be written uniquely as a sum of elements in such translates of W, i.e.,

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

In this case we write $V = \text{Ind}_H^G W = \text{Ind}\, W$.

Example 3.13. The permutation representation associated to the left action of G on G/H is induced from the trivial one-dimensional representation W of H. Here V has basis $\{e_\sigma : \sigma \in G/H\}$, and $W = \mathbb{C} \cdot e_H$, with H the trivial coset.

Example 3.14. The regular representation of G is induced from the regular representation of H. Here V has basis $\{e_g : g \in G\}$, whereas W has basis $\{e_h : h \in H\}$.

We claim that, given a representation W of H, such V exists and is unique up to isomorphism. Although we will later give several fancier ways to see this, it is not hard to do it by hand. Choose a representative $g_\sigma \in G$ for each coset $\sigma \in G/H$, with e representing the trivial coset H. To see the uniqueness, note that each element of V has a unique expression $v = \sum g_\sigma w_\sigma$, for elements w_σ in W. Given g in G, write $g \cdot g_\sigma = g_\tau \cdot h$ for some $\tau \in G/H$ and $h \in H$. Then we must have

$$g \cdot (g_\sigma w_\sigma) = (g \cdot g_\sigma) w_\sigma = (g_\tau \cdot h) w_\sigma = g_\tau (h w_\sigma).$$

This proves the uniqueness and tells us how to construct $V = \text{Ind}(W)$ from W. Take a copy W^σ of W for each left coset $\sigma \in G/H$; for $w \in W$, let $g_\sigma w$ denote the element of W^σ corresponding to w in W. Let $V = \bigoplus_{\sigma \in G/H} W^\sigma$, so every element of V has a unique expression $v = \sum g_\sigma w_\sigma$ for elements w_σ in W. Given $g \in G$, define

$$g \cdot (g_\sigma w_\sigma) = g_\tau (h w_\sigma) \quad \text{if } g \cdot g_\sigma = g_\tau \cdot h.$$

To show that this defines as action of G on V, we must verify that $g' \cdot (g \cdot (g_\sigma w_\sigma)) = (g' \cdot g) \cdot (g_\sigma w_\sigma)$ for another element g' in G. Now if $g' \cdot g_\tau = g_\rho \cdot h'$, then

$$g' \cdot (g \cdot (g_\sigma w_\sigma)) = g' \cdot (g_\tau (h w_\sigma)) = g_\rho (h' (h w_\sigma)).$$

Since $(g' \cdot g) \cdot g_\sigma = g' \cdot (g \cdot g_\sigma) = g' \cdot g_\tau \cdot h = g_\rho \cdot h' \cdot h$, we have

$$(g' \cdot g) \cdot (g_\sigma w_\sigma) = g_\rho ((h' \cdot h) w_\sigma) = g_\rho (h' \cdot (h w_\sigma)),$$

as required.

Example 3.15. If $W = \bigoplus W_i$, then $\text{Ind}\, W = \bigoplus \text{Ind}\, W_i$.

The existence of the induced representation follows from Examples 3.14 and 3.15 since any W is a direct sum of summands of the regular representation.

Exercise 3.16. (a) If U is a representation of G and W a representation of H, show that (with all tensor products over $\mathbb{C}$)

$$U \otimes \operatorname{Ind} W = \operatorname{Ind}(\operatorname{Res}(U) \otimes W).$$

In particular, $\operatorname{Ind}(\operatorname{Res}(U)) = U \otimes P$, where P is the permutation representation of G on G/H. For a formula for $\operatorname{Res}(\operatorname{Ind}(W))$, for W a representation of H, see [Se2, p. 58].

(b) Like restriction, induction is transitive: if $H \subset K \subset G$ are subgroups, show that

$$\operatorname{Ind}_H^G(W) = \operatorname{Ind}_K^G(\operatorname{Ind}_H^K W).$$

Note that Example 3.15 says that the map Ind gives a group homomorphism between the representation rings $R(H)$ and $R(G)$, in the opposite direction from the ring homomorphism Res: $R(G) \to R(H)$ given by restriction; Exercise 3.16(a) says that this map satisfies a "push–pull" formula $\alpha \cdot \operatorname{Ind}(\beta) = \operatorname{Ind}(\operatorname{Res}(\alpha) \cdot \beta)$ with respect to the restriction map.

Proposition 3.17. *Let W be a representation of H, U a representation of G, and suppose $V = \operatorname{Ind} W$. Then any H-module homomorphism $\varphi\colon W \to U$ extends uniquely to a G-module homomorphism $\tilde{\varphi}\colon V \to U$. i.e.,*

$$\operatorname{Hom}_H(W, \operatorname{Res} U) = \operatorname{Hom}_G(\operatorname{Ind} W, U).$$

In particular, this universal property determines Ind W *up to canonical isomorphism.*

Proof. With $V = \bigoplus_{\sigma \in G/H} \sigma \cdot W$ as before, define $\tilde{\varphi}$ on $\sigma \cdot W$ by

$$\sigma \cdot W \xrightarrow{g_\sigma^{-1}} W \xrightarrow{\varphi} U \xrightarrow{g_\sigma} U,$$

which is independent of the representative g_σ for σ since φ is H-linear. □

To compute the character of $V = \operatorname{Ind} W$, note that $g \in G$ maps σW to $g\sigma W$, so the trace is calculated from those cosets σ with $g\sigma = \sigma$, i.e., $s^{-1}gs \in H$ for $s \in \sigma$. Therefore,

$$\chi_{\operatorname{Ind} W}(g) = \sum_{g\sigma=\sigma} \chi_W(s^{-1}gs) \qquad (s \in \sigma \text{ arbitrary}). \tag{3.18}$$

Exercise 3.19. (a) If C is a conjugacy class of G, and $C \cap H$ decomposes into conjugacy classes $D_1, \ldots, D_r$ of H, (3.18) can be rewritten as: the value of the character of Ind W on C is

$$\chi_{\operatorname{Ind} W}(C) = \frac{|G|}{|H|} \sum_{i=1}^{r} \frac{|D_i|}{|C|} \chi_W(D_i).$$

(b) If W is the trivial representation of H, then

$$\chi_{\operatorname{Ind} W}(C) = \frac{[G:H]}{|C|} \cdot |C \cap H|.$$

Corollary 3.20 (Frobenius Reciprocity). *If W is a representation of H, and U a representation of G, then*

$$(\chi_{\mathrm{Ind}\, W}, \chi_U)_G = (\chi_W, \chi_{\mathrm{Res}\, U})_H.$$

PROOF. It suffices by linearity to prove this when W and U are irreducible. The left-hand side is the number of times U appears in Ind W, which is the dimension of $\mathrm{Hom}_G(\mathrm{Ind}\, W, U)$. The right-hand side is the dimension of $\mathrm{Hom}_H(W, \mathrm{Res}\, U)$. These dimensions are equal by the proposition. □

If W and U are irreducible, Frobenius reciprocity says: *the number of times U appears in* Ind W *is the same as the number of times W appears in* Res U.

Frobenius reciprocity can be used to find characters of G if characters of H are known.

Example 3.21. We compute $\mathrm{Ind}_H^G W$, when $H = \mathfrak{S}_2 \subset G = \mathfrak{S}_3$, $W = V_2$ (the standard representation) $= U_2'$ (the alternating representation). We know the irreducible represenatations of $\mathfrak{S}_3$: U_3, U_3', V_3, which restrict to U_2, $U_2' = V_2$, $U_2 \oplus U_2'$, respectively. Thus, by Frobenius, $\mathrm{Ind}\, V_2 = U_3' \oplus V_3$.

Example 3.22. Consider next $H = \mathfrak{S}_3 \subset G = \mathfrak{S}_4$, $W = V_3$. Again we know the irreducible representations, and $\mathrm{Res}\, U_4 = U_3$, $\mathrm{Res}\, U_4' = U_3'$, $\mathrm{Res}\, V_4 = U_3 \oplus V_3$ [the vector

$$(1, 1, 1, -3) \in V_4 = \{(x_1, x_2, x_3, x_4) \colon \textstyle\sum x_i = 0\}$$

is fixed by H], $\mathrm{Res}\, V_4' = U_3' \oplus V_3'$, with $V_3' = V_3$, and $\mathrm{Res}\, W_4 = V_3$ (as one may see directly). Hence, $\mathrm{Ind}\, V_3 = V_4 \oplus V_4' \oplus W_4$. (Note that the isomorphism $\mathrm{Res}\, W_4 = V_3$ actually follows, since one W_4 is all that could be added to $V_4 \oplus V_4'$ to get $\mathrm{Ind}\, V_3$.)

Exercise 3.23. Determine the isomorphism classes of the representations of $\mathfrak{S}_4$ induced by (i) the one-dimensional representation of the group generated by (1234) in which $(1234)\cdot v = iv$, $i = \sqrt{-1}$; (ii) the one-dimensional representation of the group generated by (123) in which $(123)\cdot v = e^{2\pi i/3}v$.

Exercise 3.24. Let $H = \mathfrak{A}_5 \subset G = \mathfrak{S}_5$. Show that $\mathrm{Ind}\, U = U \oplus U'$, $\mathrm{Ind}\, V = V \oplus V'$, and $\mathrm{Ind}\, W = W \oplus W'$, whereas $\mathrm{Ind}\, Y = \mathrm{Ind}\, Z = \wedge^2 V$.

Exercise 3.25*. Which irreducible representations of $\mathfrak{S}_d$ remain irreducible when restricted to $\mathfrak{A}_d$? Which are induced from $\mathfrak{A}_d$? How much does this tell you about the irreducible representations of $\mathfrak{A}_d$?

Exercise 3.26*. There is a unique nonabelian group of order 21, which can be realized as the group of affine transformations $x \mapsto \alpha x + \beta$ of the line over the field with seven elements, with α a cube root of unity in that field. Find the irreducible representations and character table for this group.

Now that we have introduced the notion of induced representation, we can state two important theorems describing the characters of representations of a finite group. In the preceding lecture we mentioned the notion of *virtual character*; this is just an element of the image Λ of the character map

$$\chi\colon R(G) \to \mathbb{C}_{\text{class}}(G)$$

from the representation ring $R(G)$ of virtual representations. The following two theorems both state that in order to generate $\Lambda \otimes \mathbb{Q}$ (resp. Λ) it is enough to consider the simplest kind of induced representations, namely, those induced from cyclic (respective elementary) subgroups of G. For the proofs of these theorems we refer to [Se2, §9, 10]. We will not need them in these lectures.

Artin's Theorem 3.27. *The characters of induced representations from cyclic subgroups of G generate a lattice of finite index in* Λ.

A subgroup H of G is *p-elementary* if $H = A \times B$, with A cyclic of order prime to p and B a p-group.

Brauer's Theorem 3.28. *The characters of induced representations from elementary subgroups of G generate the lattice* Λ.

§3.4. The Group Algebra

There is an important notion that we have already dealt with implicitly but not explicitly; this is the group algebra $\mathbb{C}G$ associated to a finite group G. This is an object that for all intents and purposes can completely replace the group G itself; any statement about the representations of G has an exact equivalent statement about the group algebra. Indeed, to a large extent the choice of language is a matter of taste.

The underlying vector space of the group algebra of G is the vector space with basis $\{e_g\}$ corresponding to elements of the group G, that is, the underlying vector space of the regular representation. We define the algebra structure on this vector space simply by

$$e_g \cdot e_h = e_{gh}.$$

By a representation of the algebra $\mathbb{C}G$ on a vector space V we mean simply an algebra homomorphism

$$\mathbb{C}G \to \text{End}(V),$$

so that a representation V of $\mathbb{C}G$ is the same thing as a left $\mathbb{C}G$-module. Note that a representation $\rho\colon G \to \text{Aut}(V)$ will extend by linearity to a map $\tilde{\rho}\colon \mathbb{C}G \to \text{End}(V)$, so that representations of $\mathbb{C}G$ correspond exactly to representations of G; the left $\mathbb{C}G$-module given by $\mathbb{C}G$ itself corresponds to the regular representation.

If $\{W_i\}$ are the irreducible representations of G, then we have seen that the regular representation R decomposes

$$R = \bigoplus (W_i)^{\oplus \dim(W_i)}.$$

We can now refine this statement in terms of the group algebra: we have

Proposition 3.29. *As algebras,*

$$\mathbb{C}G \cong \bigoplus \operatorname{End}(W_i).$$

Proof. As we have said, for any representation W of G, the map $G \to \operatorname{Aut}(W)$ extends by linearity to a map $\mathbb{C}G \to \operatorname{End}(W)$; applying this to each of the irreducible representations W_i gives us a canonical map

$$\varphi\colon \mathbb{C}G \to \bigoplus \operatorname{End}(W_i).$$

This is injective since the representation on the regular representation is faithful. Since both have dimension $\sum(\dim W_i)^2$, the map is an isomorphism. □

A few remarks are in order about the isomorphism φ of the proposition. First, φ can be interpreted as the Fourier transform, cf. Exercise 3.32. Note also that Proposition 2.28 has a natural interpretation in terms of the group algebra: it says that the center of $\mathbb{C}G$ consists of those $\sum \alpha(g)e_g$ for which α is a class function.

Next, we can thinkof φ as the decomposition of the semisimple algebra $\mathbb{C}G$ into a product of matrix algebras. It implies that the matrix entries of the irreducible representations give a basis for the space of all functions on G, cf. Exercise 2.35.

Note in particular that any irreducible representation is isomorphic to a (minimal) left ideal in $\mathbb{C}G$. These left ideals are generated by idempotents. In fact, we can interpret the projection formulas of the last lecture in the language of the group algebra: the formulas say simply that the elements

$$\dim W \cdot \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot e_g \in \mathbb{C}G$$

are the idempotents in the group algebra corresponding to the direct sum factors in the decomposition of Proposition 3.29. To locate the irreducible representations W_i of a group G [not just a direct sum of $\dim(W_i)$ copies], we want to find other idempotents of $\mathbb{C}G$. We will see this carried out for the symmetric groups in the following lecture.

The group algebra also gives us another description of induced representations: if W is a representation of a subgroup H of G, then the induced representation may be constructed simply by

$$\operatorname{Ind} W = \mathbb{C}G \otimes_{\mathbb{C}H} W,$$

where G acts on the first factor: $g\cdot(e_{g'}\otimes w) = e_{gg'}\otimes w$. The isomorphism of the reciprocity theorem is then a special case of a general formula for a change of rings $\mathbb{C}H \to \mathbb{C}G$:

$$\mathrm{Hom}_{\mathbb{C}H}(W, U) = \mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{C}H} W, U).$$

Exercise 3.30*. The induced representation $\mathrm{Ind}(W)$ can also be realized concretely as a space of W-valued functions on G, which can be useful to produce matrix realizations, or when trying to decompose $\mathrm{Ind}(W)$ into irreducible pieces. Show that $\mathrm{Ind}(W)$ is isomorphic to

$$\mathrm{Hom}_H(\mathbb{C}G, W) \cong \{f\colon G \to W : f(hg) = hf(g), \quad \forall h \in H, g \in G\},$$

where G acts by $(g'\cdot f)(g) = f(gg')$.

Exercise 3.31. If $\mathbb{C}G$ is identified with the space of functions on G, the function φ corresponding to $\sum_{g\in G}\varphi(g)e_g$, show that the product in $\mathbb{C}G$ corresponds to the convolution $*$ of functions:

$$(\varphi * \psi)(g) = \sum_{h\in G}\varphi(h)\psi(h^{-1}g).$$

(With integration replacing summation, this indicates how one may extend the notion of regular representation to compact groups.)

Exercise 3.32*. If $\rho\colon G \to \mathrm{GL}(V_\rho)$ is a representation, and φ is a function on G, define the *Fourier transform* $\hat{\varphi}(\rho)$ in $\mathrm{End}(V_\rho)$ by the formula

$$\hat{\varphi}(\rho) = \sum_{g\in G}\varphi(g)\cdot\rho(g).$$

(a) Show that $\widehat{\varphi * \psi}(\rho) = \hat{\varphi}(\rho)\cdot\hat{\psi}(\rho)$.

(b) Prove the *Fourier inversion formula*

$$\varphi(g) = \frac{1}{|G|}\sum \dim(V_\rho)\cdot\mathrm{Trace}(\rho(g^{-1})\cdot\hat{\varphi}(\rho)),$$

the sum over the irreducible representations ρ of G. This formula is equivalent to formulas (2.19) and (2.20).

(c) Prove the *Plancherel formula* for functions φ and ψ on G:

$$\sum_{g\in G}\varphi(g^{-1})\psi(g) = \frac{1}{|G|}\sum_{\rho}\dim(V_\rho)\cdot\mathrm{Trace}(\hat{\varphi}(\rho)\hat{\psi}(\rho)).$$

Our choice of left action of a group on a space has been perfectly arbitrary, and the entire story is the same if G acts on the *right* instead. Moreover, there is a standard way to change a right action into a left action, and vice versa: Given a right action of G on V, define the left action by

$$g\cdot v = v\cdot(g^{-1}), \qquad g \in G, v \in V.$$

If $A = \mathbb{C}G$ is the group algebra, a right action of G on V makes V a right A-module. To turn right modules into left modules, we can use the anti-involution $a \mapsto \hat{a}$ of A defined by $(\sum a_g e_g)^\wedge = \sum a_g e_{g^{-1}}$. A right A-module is then turned into a left A-module by setting $a \cdot v = v \cdot \hat{a}$.

The following exercise will take you back to the origins of representation theory in the 19th century, when Frobenius found the characters by factoring this determinant.

Exercise 3.33*. Given a finite group G of order n, take a variable x_g for each element g in G, and order the elements of G arbitrarily. Let F be the determinant of the $n \times n$ matrix whose entry in the row labeled by g and column labeled by h is $x_{g \cdot h^{-1}}$. This is a form of degree n in the n variables x_g, which is independent of the ordering. Normalize the factors of F to take the value 1 when $x_e = 1$ and $x_g = 0$ for $g \neq e$. Show that the irreducible factors of F correspond to the irreducible representations of G. Moreover, if F_ρ is the factor corresponding to the representation ρ, show that the degree of F_ρ is the degree $d(\rho)$ of the representation ρ, and that each F_ρ occurs in F $d(\rho)$ times. If χ_ρ is the character of ρ, and $g \neq e$, show that $\chi_\rho(g)$ is the coefficient of $x_g \cdot x_e^{d(\rho)-1}$ in F_ρ.

§3.5. Real Representations and Representations over Subfields of $\mathbb{C}$

If a group G acts on a real vector space V_0, then we say the corresponding complex representation of $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ is *real*. To the extent that we are interested in the action of a group G on real rather than complex vector spaces, the problem we face is to say which of the complex representations of G we have studied are in fact real.

Our first guess might be that a representation is real if and only if its character is real-valued. This turns out not to be the case: the character of a real representation is certainly real-valued, but the converse need not be true. To find an example, suppose $G \subset \mathrm{SU}(2)$ is a finite, nonabelian subgroup. Then G acts on $\mathbb{C}^2 = V$ with a real-valued character since the trace of any matrix in $\mathrm{SU}(2)$ is real. If V were a real representation, however, then G would be a subgroup of $\mathrm{SO}(2) = S^1$, which is abelian. To produce such a group, note that $\mathrm{SU}(2)$ can be identified with the unit quaternions. Set $G = \{\pm 1, \pm i, \pm j, \pm k\}$. Then $G/\{\pm 1\}$ is abelian, so has four one-dimensional representations, which give four one-dimensional representations of G. Thus, G has one irreducible two-dimensional representation, whose character is real, but which is not real.

Exercise 3.34*. Compute the character table for this quaternion group G, and compare it with the character table of the dihedral group of order 8.

A more successful approach is to note that if V is a real representation of G, coming from V_0 as above, then one can find a positive definite symmetric bilinear form on V_0 which is preserved by G. This gives a symmetric bilinear form on V which is preserved by G. Not every representation will have such a form since degeneracies may arise when one tries to construct one following the construction of Proposition 1.5. In fact,

Lemma 3.35. *An irreducible representation V of G is real if and only if there is a nondegenerate symmetric bilinear form B on V preserved by G.*

Proof. If we have such B, and an arbitrary nondegenerate Hermitian form H, also G-invariant, then

$$V \xrightarrow{B} V^* \xrightarrow{H} V$$

gives a conjugate linear isomorphism φ from V to V: given $x \in V$, there is a unique $\varphi(x) \in V$ with $B(x, y) = H(\varphi(x), y)$, and φ commutes with the action of G. Then $\varphi^2 = \varphi \circ \varphi$ is a complex linear G-module homomorphism, so $\varphi^2 = \lambda \cdot \mathrm{Id}$. Moreover,

$$H(\varphi(x), y) = B(x, y) = B(y, x) = H(\varphi(y), x) = \overline{H(x, \varphi(y))},$$

from which it follows that $H(\varphi^2(x), y) = H(x, \varphi^2(y))$, and therefore λ is a positive real number. Changing H by a scalar, we may assume $\lambda = 1$, so $\varphi^2 = \mathrm{Id}$. Thus, V is a sum of real eigenspaces V_+ and V_- for φ corresponding to eigenvalues 1 and -1. Since φ commutes with G, V_+ and V_- are G-invariant subspaces. Finally, $\varphi(ix) = -i\varphi(x)$, so $iV_+ = V_-$, and $V = V_+ \otimes \mathbb{C}$. □

Note from the proof that a real representation is also characterized by the existence of a conjugate linear endomorphism of V whose square is the identity; if $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$, it is given by conjugation: $v_0 \otimes \lambda \mapsto v_0 \otimes \bar{\lambda}$.

A warning is in order here: an irreducible representation of G on a vector space over $\mathbb{R}$ may become reducible when we extend the group field to $\mathbb{C}$. To give the simplest example, the representation of $\mathbb{Z}/n$ on $\mathbb{R}^2$ given by

$$\rho \colon k \mapsto \begin{pmatrix} \cos\dfrac{2\pi k}{n} & -\sin\dfrac{2\pi k}{n} \\ \sin\dfrac{2\pi k}{n} & \cos\dfrac{2\pi k}{n} \end{pmatrix}$$

is irreducible over $\mathbb{R}$ for $n > 2$ (no line in $\mathbb{R}^2$ is fixed by the action of $\mathbb{Z}/n$), but will be reducible over $\mathbb{C}$. Thus, classifying the irreducible representations of G over $\mathbb{C}$ that are real does not mean that we have classified all the irreducible real representations. However, we will see in Exercise 3.39 below how to finish the story once we have found the real representations of G that are irreducible over $\mathbb{C}$.

Suppose V is an irreducible representation of G with χ_V real. Then there is a G-equivariant isomorphism $V \cong V^*$, i.e., there is a G-equivariant (nondegenerate) bilinear form B on V; but, in general, B need not be symmetric. Regarding B in

$$V^* \otimes V^* = \mathrm{Sym}^2 V^* \oplus \wedge^2 V^*,$$

and noting the uniqueness of B up to multiplication by scalars, we see that B is either symmetric or skew-symmetric. If B is skew-symmetric, proceeding as above one can scale so $\varphi^2 = -\mathrm{Id}$. This makes V "quaternionic," with φ becoming multiplication[2] by j:

Definition 3.36. A *quaternionic* representation is a (complex) representation V which has a G-invariant homomorphism $J\colon V \to V$ that is conjugate linear, and satisfies $J^2 = -\mathrm{Id}$. Thus, a skew-symmetric nondegenerate G-invariant B determines a quaternionic structure on V.

Summarizing the preceding discussion we have the

Theoem 3.37. *An irreducible representation V is one and only one of the following*:

(1) Complex: *χ_V is not real-valued; V does not have a G-invariant nondegenerate bilinear form.*

(2) Real: *$V = V_0 \otimes \mathbb{C}$, a real representation; V has a G-invariant symmetric nondegenerate bilinear form.*

(3) Quaternionic: *χ_V is real, but V is not real; V has a G-invariant skew-symmetric nondegeneate bilinear form.*

Exercise 3.38. Show that for V irreducible,

$$\frac{1}{|G|}\sum_{g \in G} \chi_V(g^2) = \begin{cases} 0 & \text{if } V \text{ is complex} \\ 1 & \text{if } V \text{ is real} \\ -1 & \text{if } V \text{ is quaternionic.} \end{cases}$$

This verifies that the three cases in the theorem are mutually exclusive. It also implies that if the order of G is odd, all nontrivial representations must be complex.

Exercise 3.39. Let V_0 be a real vector space on which G acts irreducibly, $V = V_0 \otimes \mathbb{C}$ the corresponding real representation of G. Show that if V is not irreducible, then it has exactly two irreducible factors, and they are conjugate complex representations of G.

[2] See §7.2 for more on quaternions and quaternonic representations.

Exercise 3.40. Classify the real representations of $\mathfrak{A}_4$.

Exercise 3.41*. The group algebra $\mathbb{R}G$ is a product of simple $\mathbb{R}$-algebras corresponding to the irreducible representations over $\mathbb{R}$. These simple algebras are matrix algebras over $\mathbb{C}$, $\mathbb{R}$, or the quaternions $\mathbb{H}$ according as the representation is complex, real, or quaternionic.

Exercise 3.42*. (a) Show that all characters of a group are real if and only if every element is conjugate to its inverse.

(b) Show that an element σ in a split conjugacy class of $\mathfrak{A}_d$ is conjugate to its inverse if and only if the number of cycles in σ whose length is congruent to 3 modulo 4 is even.

(c) Show that the only d's for which every character of $\mathfrak{A}_d$ is real-valued are $d = 1, 2, 5, 6, 10$, and 14.

Exercise 3.43*. Show that: (i) the tensor product of two real or two quaternionic representations is real; (ii) for any V, $V^* \otimes V$ is real; (iii) if V is real, so are all $\wedge^k V$; (iv) if V is quaternionic, $\wedge^k V$ is real for k even, quaternionic for k odd.

Representations over Subfields of $\mathbb{C}$ in General

We consider next the generalization of the preceding problem to more general subfields of $\mathbb{C}$. Unfortunately, our results will not be nearly as strong in general, but we can at least express the problem neatly in terms of the representation ring of G.

To begin with, our terminology in this general setting is a little different. Let $K \subset \mathbb{C}$ be any subfield. We define a K-representation of G to be a vector space V_0 over K on which G acts; in this case we say that the complex representation $V = V_0 \otimes \mathbb{C}$ is *defined over K*.

One way to measure how many of the representations of G are defined over a field K is to introduce the *representation ring $R_K(G)$ of G over K*. This is defined just like the ordinary representation ring; that is, it is just the group of formal linear combinations of K-representations of G modulo relations of the form $V + W - (V \oplus W)$, with multiplication given by tensor product.

Exercise 3.44*. Describe the representation ring of G over $\mathbb{R}$ for some of the groups G whose complex representation we have analyzed above. In particular, is the rank of $R_{\mathbb{R}}(G)$ always the same as the rank of $R(G)$?

Exercise 3.45*. (a) Show that $R_K(G)$ is the subring of the ring of class functions on G generated (as an additive group) by characters of representations defined over K.

(b) Show that the characters of irreducible representations over K form an orthogonal basis for $R_K(G)$.

(c) Show that a complex representation of G can be defined over K if and only if its character belongs to $R_K(G)$.

For more on the relation between $R_K(G)$ and $R(G)$, see [Se2].