LECTURE 22

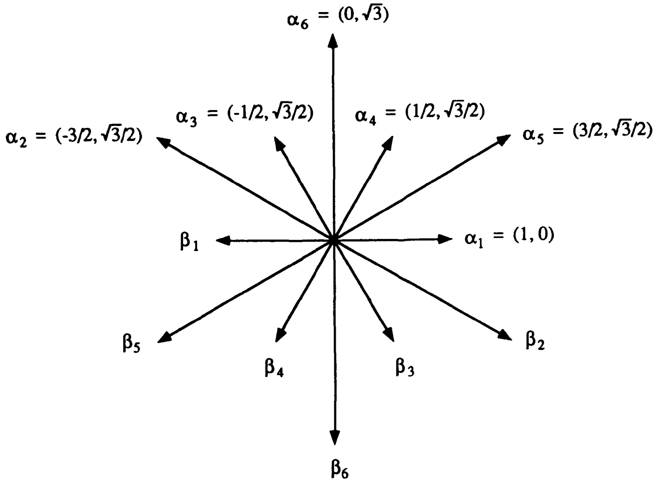
$\mathfrak{g}_2$ and Other Exceptional Lie Algebras

This lecture is mainly about $\mathfrak{g}_2$, with just enough discussion of the algebraic constructions of the other exceptional Lie algebras to give the reader a sense of their complexity. $\mathfrak{g}_2$, being only 14-dimensional, is different: we can reasonably carry out in practice the process described in §21.3 to arrive at an explicit description of the algebra by specifying a basis and all pairwise products; we do this in §22.1 and verify in §22.2 that the result really is a Lie algebra. In §22.3 we analyze the representations of $\mathfrak{g}_2$, and arrive in particular at another description of $\mathfrak{g}_2$: it is the algebra of endomorphisms of a seven-dimensional vector space preserving a general trilinear form. (Note that §22.3 may be read independently of either §22.1, §21.2, or §21.3.) Finally, in the fourth section we will sketch some of the more abstract (i.e., coordinate free) approaches to the construction of the five exceptional Lie algebras. While the first two sections are completely elementary, the constructions given in §22.4 involve some fairly serious algebra.

§22.1. Construction of $\mathfrak{g}_2$ from Its Dynkin Diagram

In this section we will carry out explicitly the process described in the preceding section for the Dynkin diagram (G_2), constructing in this way a Lie algebra $\mathfrak{g}_2$ with diagram (G_2) (and in particular proving its existence).

The first step is to find the root system from the Dynkin diagram. In the case of $\mathfrak{g}_2$ this is immediate; we may draw the root system $R \subset \mathfrak{h}^*$ associated to the diagram G_2 as follows:



Here the positive roots are denoted α_i, with α_1 and α_2 the simple roots. The coordinate system here has no particular significance (in particular, recall that the configuration of roots α_i and β_i is determined only up to a real scalar), but is convenient for calculating inner products. Note that the Weyl group is the dihedral group generated by rotation through an angle of $\pi/3$ and reflection in the horizontal; the Weyl chamber associated to the choice of ordering of the roots given is the cone between the roots α_6 and α_4.

As indicated in the preceding section, we start by letting X_1 be any eigenvector for the action of $\mathfrak{h}$ with eigenvalue α_1, and X_2 any eigenvector for the action of $\mathfrak{h}$ with eigenvalue α_2. We similarly let Y_1 and Y_2 be eigenvectors with eigenvalues β_1 and β_2 and set

$$H_1 = [X_1, Y_1] \quad \text{and} \quad H_2 = [X_2, Y_2].$$

We can choose Y_1 and Y_2 so that the elements $H_i \in \mathfrak{h}$ satisfy $\alpha_1(H_1) = \alpha_2(H_2) = 2$, i.e.,

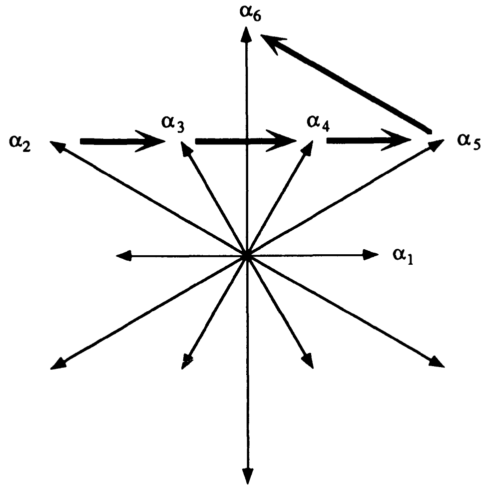
$$[H_1, X_1] = 2 \cdot X_1 \quad \text{and} \quad [H_2, X_2] = 2 \cdot X_2.$$

It follows that

$$[H_1, Y_1] = -2 \cdot Y_1 \quad \text{and} \quad [H_2, Y_2] = -2 \cdot Y_2,$$

i.e., H_i, X_i, and Y_i span a subalgebra $\mathfrak{s}_{\alpha_i} \cong \mathfrak{sl}_2\mathbb{C}$, with H_i, X_i, and Y_i a normalized basis for this copy of $\mathfrak{sl}_2\mathbb{C}$.

Now, it is clear from the diagram above that there is a unique way of writing each positive root α_i as a sum of simple roots $\alpha_{i_1} + \cdots + \alpha_{i_k}$ so that the partial sums $\alpha_{i_1} + \cdots + \alpha_{i_l}$ are roots for each $l \leq k$ (modulo exchanging the first two terms): we go through the root system by the path



i.e., we write

$$\alpha_3 = \alpha_1 + \alpha_2,$$
$$\alpha_4 = \alpha_1 + \alpha_3 = \alpha_1 + \alpha_1 + \alpha_2,$$
$$\alpha_5 = \alpha_1 + \alpha_4 = \alpha_1 + \alpha_1 + \alpha_1 + \alpha_2,$$
$$\alpha_6 = \alpha_2 + \alpha_5 = \alpha_2 + \alpha_1 + \alpha_1 + \alpha_1 + \alpha_2.$$

According to the general recipe, this means we now set

$$X_3 = [X_1, X_2], \qquad X_4 = [X_1, X_3],$$
$$X_5 = [X_1, X_4], \qquad X_6 = [X_2, X_5],$$

and define $Y_3, \ldots, Y_6$ similarly. The elements $H_1, H_2, X_1, \ldots, X_6, Y_1, \ldots, Y_6$ then form a basis for the 14-dimensional $\mathfrak{g}_2$, with H_1 and H_2 a basis for $\mathfrak{h}$, X_i a generator of the eigenspace $\mathfrak{g}_{\alpha_i}$, and Y_i a generator of $\mathfrak{g}_{\beta_i}$ for $i = 1, \ldots, 6$.

The task at hand now is to write down the multiplication table for $\mathfrak{g}_2$ in terms of this basis. Of course, some products are already known: we know, for example, that H_i, X_i, and Y_i form a normalized basis for $\mathfrak{sl}_2\mathbb{C}$ for $i = 1, 2$, and we have the relations defining $X_3, \ldots, X_6$ and $Y_1, \ldots, Y_6$ above. In addition, since we know that the product $[X_i, X_j]$ lies in the root space $\mathfrak{g}_{\alpha_i+\alpha_j}$ for each i and j, we see immediately that $[X_i, X_j] = 0$ whenever $\alpha_i + \alpha_j$ is not a root. We deduce that

$$[X_1, X_5] = [X_1, X_6] = [X_2, X_3] = [X_2, X_4] = [X_2, X_6] = [X_3, X_5]$$
$$= [X_3, X_6] = [X_4, X_5] = [X_4, X_6] = [X_5, X_6] = 0,$$

and likewise

$$[Y_1, Y_5] = [Y_1, Y_6] = [Y_2, Y_3] = [Y_2, Y_4] = [Y_2, Y_6] = [Y_3, Y_5]$$
$$= [Y_3, Y_6] = [Y_4, Y_5] = [Y_4, Y_6] = [Y_5, Y_6] = 0.$$

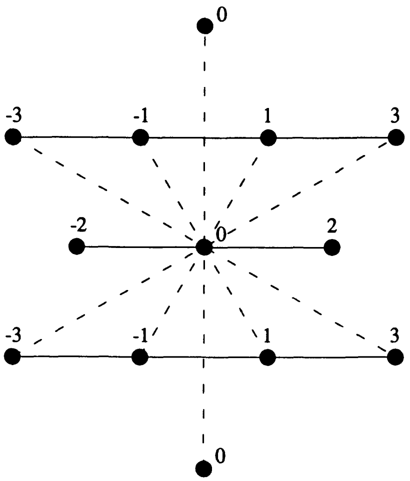
Similarly, we know that $[X_i, Y_j] = 0$ whenever $\alpha_i + \beta_j = \alpha_i - \alpha_j$ is not a root; this tells us as well that

$$[X_1, Y_2] = [X_1, Y_6] = [X_2, Y_1] = [X_2, Y_4] = [X_2, Y_5] = [X_3, Y_5]$$
$$= [X_4, Y_2] = [X_5, Y_2] = [X_5, Y_3] = [X_6, Y_1] = 0.$$

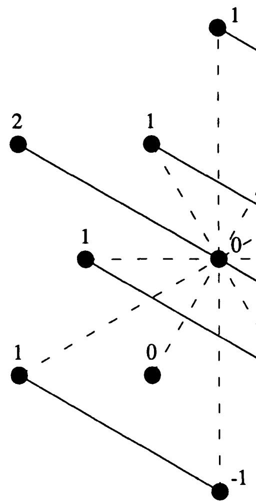
The multiplication table thus far looks like

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	*	*	*	*	*	*	*	*	*	*
H_2		*	*	$2X_2$	$-2Y_2$	*	*	*	*	*	*	*	*
X_1			H_1	X_3	0	X_4	*	X_5	*	0	*	0	0
Y_1				0	Y_3	*	Y_4	*	Y_5	*	0	0	0
X_2					H_2	0	*	0	0	X_6	0	0	*
Y_2						*	0	0	0	0	Y_6	*	0
X_3							*	*	*	0	0	0	*
Y_3								*	*	0	0	*	0
X_4									*	0	*	0	*
Y_4										*	0	*	0
X_5											*	0	*
Y_5												*	0
X_6													*

The next thing to do is to describe the action of H_1 and H_2 on the various vectors X_i and Y_i. This can be done using the inner product on $\mathfrak{h}$, but it is perhaps simpler to go back to the basic idea of restriction to the subalgebras $\mathfrak{s}_{\alpha_1}$ and $\mathfrak{s}_{\alpha_2}$. For example, if we want to determine the action of H_1 on the various X_i, consider how the algebra $\mathfrak{g} = \mathfrak{h} \bigoplus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{\beta_i})$ decomposes as a representation of $\mathfrak{s}_{\alpha_1}$:



We get two trivial representations (the spans of X_6 and Y_6, as already noted); one copy of the adjoint representation $\mathrm{Sym}^2 V$ (the subalgebra $\mathfrak{s}_{\alpha_1}$ itself) spanned by X_1, Y_1, and H_1; and two copies of the irreducible four-dimensional representation $\mathrm{Sym}^3 V$ spanned by X_2, X_3, X_4, and X_5 and Y_5, Y_4, Y_3, and Y_2. In particular, it follows that X_2, X_3, X_4, and X_5 are eigenvectors for the action of H_1 with eigenvalues of -3, -1, 1, and 3, respectively; and likewise Y_5, Y_4, Y_3, and Y_2 are eigenvectors with eigenvalues -3, -1, 1, and 3. In similar fashion, we consider the decomposition of $\mathfrak{g}$ under the action of $\mathfrak{s}_{\alpha_2} = \mathbb{C}\{H_2, X_2, Y_2\}$: diagrammatically, this looks like



Here we have two trivial representations, spanned by X_4 and Y_4, one adjoint ($\mathfrak{s}_{\alpha_2}$ itself), and four copies of the standard two-dimensional representation V, spanned by X_6 and X_5, X_3 and X_1, Y_1 and Y_3, and Y_5 and Y_6. It follows that X_6, X_3, Y_1, and Y_5 are eigenvectors for the action of H_2 with eigenvalue 1, and likewise X_5, X_1, Y_3, and Y_6 are eigenvectors with eigenvalue -1.

Including this information, we can fill in the top two rows of the multiplication table:

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$

Decomposing $\mathfrak{g}_2$ according to the action of $\mathfrak{s}_{\alpha_1}$ and $\mathfrak{s}_{\alpha_2}$ gives us information about the action of X_1, X_2, Y_1, and Y_2 on the other basis vectors as well. For example, we saw a moment ago that X_5 and X_6 together span a subrepresentation of $\mathfrak{g}_2$ under the action of $\mathfrak{s}_{\alpha_2}$, with $\mathrm{ad}(X_2)$ carrying X_5 to X_6. It follows from this that $\mathrm{ad}(Y_2)$ must carry X_6 back to X_5: we have

$$\begin{aligned}\mathrm{ad}(Y_2)(X_6) &= \mathrm{ad}(Y_2)\,\mathrm{ad}(X_2)(X_5)\\ &= \mathrm{ad}(X_2)\,\mathrm{ad}(Y_2)(X_5) - \mathrm{ad}([X_2, Y_2])(X_5)\\ &= 0 - \mathrm{ad}(H_2)(X_5) = X_5.\end{aligned}$$

Similarly, since $\mathrm{ad}(X_2)$ carries X_1 into $-X_3$, which together with X_1 spans a copy of the standard two-dimensional representation of $\mathfrak{s}_{\alpha_2} \cong \mathfrak{sl}_2\mathbb{C}$, it follows that $\mathrm{ad}(Y_2)$ will carry $-X_3$ back to X_1. Likewise from the fact that $\mathrm{ad}(Y_2)$ carries Y_1 to $-Y_3$ we see that $\mathrm{ad}(Y_2)(Y_3) = -Y_1$, and since $\mathrm{ad}(Y_2)\colon Y_5 \mapsto Y_6$, $\mathrm{ad}(X_2)\colon Y_6 \mapsto Y_5$.

We can in the same way use the action of $\mathfrak{s}_{\alpha_1}$ to determine the values of $\mathrm{ad}(X_1)$ and $\mathrm{ad}(Y_2)$ on various basis vectors, though because the representation of $\mathfrak{s}_{\alpha_1}$ on $\mathfrak{g}_2$ has larger-dimensional components this is slightly more complicated. To begin with, consider the representation of $\mathfrak{s}_{\alpha_1}$ on the subspace spanned by X_2, X_3, X_4, and X_5. We know that $\mathrm{ad}(X_1)$ carries X_2 to X_3, and since X_2 is an eigenvector for the action of the commutator $[X_1, Y_1] = H_1$ with eigenvalue -3, it follows that $\mathrm{ad}(Y_1)$ must carry X_3 to $3X_2$: we have

$$\begin{aligned}\mathrm{ad}(Y_1)(X_3) &= \mathrm{ad}(Y_1)\,\mathrm{ad}(X_1)(X_2)\\ &= \mathrm{ad}(X_1)\,\mathrm{ad}(Y_1)(X_2) - \mathrm{ad}([X_1, Y_1])(X_2)\\ &= 0 - \mathrm{ad}(H_1)(X_2) = 3X_2.\end{aligned}$$

Using this, we can next determine the action of Y_1 on X_4:

$$\begin{aligned}\mathrm{ad}(Y_1)(X_4) &= \mathrm{ad}(Y_1)\,\mathrm{ad}(X_1)(X_3)\\ &= \mathrm{ad}(X_1)\,\mathrm{ad}(Y_1)(X_3) - \mathrm{ad}(H_1)(X_3)\\ &= \mathrm{ad}(X_1)(3X_2) + X_3 = 4X_3,\end{aligned}$$

and we calculate likewise that $\mathrm{ad}(Y_1)(X_5) = 3X_4$. Analogously, knowing that $\mathrm{ad}(Y_1)$ carries Y_2 to Y_3 to Y_4 to Y_5 yields the information that $\mathrm{ad}(X_1)$ must carry Y_3, Y_4, and Y_5 to $3Y_2$, $4Y_3$ and, $3Y_4$, respectively. Including all this information in the chart, the next four rows of our multiplication table are

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
X_1			H_1	X_3	0	X_4	$3Y_2$	X_5	$4Y_3$	0	$3Y_4$	0	0
Y_1				0	Y_3	$3X_2$	Y_4	$4X_3$	Y_5	$3X_4$	0	0	0
X_2					H_2	0	$-Y_1$	0	0	X_6	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	X_5	0

We next have to find the commutators of the basis elements X_i and Y_j for $i, j \geq 3$. We cannot do this by looking at the action of the subalgebras generated by X_i and Y_i, since for $i \geq 3$ we do not know the commutator $[X_i, Y_i]$. Rather, the way to do this is outlined in the general proof in the preceding section: we just use the expression of the X_i and Y_j as brackets of the generators X_1, X_2, Y_1, and Y_2 to reduce the problem to brackets with these generators, which we now know. Thus, for example, the first unknown entry in the table at present is the bracket $[X_3, Y_3]$. We calculate this by writing X_3 as $[X_1, X_2]$, so that

$$\begin{aligned}\operatorname{ad}(X_3)(Y_3) &= \operatorname{ad}([X_1, X_2])(Y_3)\\ &= \operatorname{ad}(X_1)\operatorname{ad}(X_2)(Y_3) - \operatorname{ad}(X_2)\operatorname{ad}(X_1)(Y_3)\\ &= \operatorname{ad}(X_1)(-Y_1) - \operatorname{ad}(X_2)(3Y_2)\\ &= -H_1 - 3H_2.\end{aligned}$$

Likewise, to evaluate $[X_3, X_4]$ we have

$$\begin{aligned}\operatorname{ad}(X_3)(X_4) &= \operatorname{ad}([X_1, X_2])(X_4)\\ &= \operatorname{ad}(X_1)\operatorname{ad}(X_2)(X_4) - \operatorname{ad}(X_2)\operatorname{ad}(X_1)(X_4)\\ &= -\operatorname{ad}(X_2)(X_5) = -X_6.\end{aligned}$$

In this way, we can evaluate all brackets with X_3; knowing these, we can reduce any bracket with X_4 to one involving X_1 and X_3 by writing $X_4 = [X_1, X_3]$, and so on. Continuing in this way, we may complete our multiplication table:

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	X_4	$3Y_2$	X_5	$4Y_3$	0	$3Y_4$	0	0
Y_1				0	Y_3	$3X_2$	Y_4	$4X_3$	Y_5	$3X_4$	0	0	0
X_2					H_2	0	$-Y_1$	0	0	X_6	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	X_5	0
X_3							$-H_1$ $-3H_2$	$-X_6$	$4Y_1$	0	0	0	$3Y_4$
Y_3								$4X_1$	$-Y_6$	0	0	$3X_4$	0
X_4									$8H_1$ $+12H_2$	0	$-12Y_1$	0	$12Y_3$
Y_4										$-12X_1$	0	$12X_3$	0
X_5											$-36H_1$ $-36H_2$	0	$36Y_2$
Y_5												$36X_2$	0
X_6													$36H_1$ $+72H_2$

Of course, in retrospect we see that the basis we have chosen is far from the most symmetric one possible: for example, if we divided X_4 and Y_4 by 2 and X_5, X_6, Y_5, and Y_6 by 6, and changed the signs of X_5 and Y_3, the form of the table would be

Table 22.1

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1			H_1	X_3	0	$2X_4$	$-3Y_2$	$-3X_5$	$-2Y_3$	0	Y_4	0	0
Y_1				0	$-Y_3$	$3X_2$	$-2Y_4$	$2X_3$	$3Y_5$	$-X_4$	0	0	0
X_2					H_2	0	Y_1	0	0	$-X_6$	0	0	Y_5
Y_2						$-X_1$	0	0	0	0	Y_6	$-X_5$	0
X_3							H_1+3H_2	$-3X_6$	$2Y_1$	0	0	0	Y_4
Y_3								$-2X_1$	$3Y_6$	0	0	$-X_4$	0
X_4									$2H_1+3H_2$	0	$-Y_1$	0	$-Y_3$
Y_4										X_1	0	X_3	0
X_5											H_1+H_2	0	$-Y_2$
Y_5												X_2	0
X_6													H_1+2H_2

There was another good reason for these changes: now each of the brackets $[X_i, Y_i]$ will be the distinguished element of $\mathfrak{h}$ corresponding to the root α_i. If we denote this element by H_i, then we read off from the table that

$$H_3 = H_1 + 3H_2, \qquad H_4 = 2H_1 + 3H_2,$$
$$H_5 = H_1 + H_2, \qquad H_6 = H_1 + 2H_2, \tag{22.2}$$

and

$$H_i = [X_i, Y_i], \qquad [H_i, X_i] = 2X_i, \qquad [H_i, Y_i] = -2Y_i, \tag{22.3}$$

for $i = 1, 2, 3, 4, 5, 6$.

§22.2. Verifying That $\mathfrak{g}_2$ Is a Lie Algebra

The calculation of the preceding section gives a complete description of what the Lie algebra $\mathfrak{g}_2$ must look like, but there is still some work to be done: unless we know that there is a Lie algebra with diagram (G_2), we do not know that the above multiplication table defines a Lie algebra, let alone a simple one. In fact, the simplicity is not much of a problem (cf. Exercise 14.34), but to know that it is a Lie algebra requires knowing that the Jacobi identity is valid. One could simply check this from the table for all $\binom{14}{3}$ triples of elements from the basis, a rather uninviting task.

There is another way, which gives more structure to the preceding calculations, and which will give a clue for possible constructions of other Lie algebras. The root diagram for (G_2) is made up of two hexagons, one with long arrows, the other with short. This suggests that we should find a copy of the corresponding Lie algebra $\mathfrak{sl}_3\mathbb{C}$ inside $\mathfrak{g}_2$. The subspace spanned by $\mathfrak{h}$ and the root spaces corresponding to the six longer roots is clearly closed under brackets, so is the obvious candidate. The long roots are α_5, α_2, and $\alpha_6 = \alpha_5 + \alpha_2$, and their inverses. So we define $\mathfrak{g}_0$ to be the subspace spanned by the corresponding vectors:

$$\mathfrak{g}_0 = \mathbb{C}\{H_5, H_2, X_5, Y_5, X_2, Y_2, X_6, Y_6\}.$$

The multiplication table for $\mathfrak{g}_0$ is read off from Table 22.1:

	H_2	X_5	Y_5	X_2	Y_2	X_6	Y_6
H_5	0	$2X_5$	$-2Y_5$	$-X_2$	Y_2	X_6	$-Y_6$
H_2		$-X_5$	Y_5	$2X_2$	$-2Y_2$	X_6	$-Y_6$
X_5			H_5	X_6	0	0	$-Y_2$
Y_5				0	$-Y_6$	X_2	0
X_2					H_2	0	Y_5
Y_2						$-X_5$	0
X_6							$H_5 + H_2$

This is exactly the multiplication table for $\mathfrak{sl}_3\mathbb{C}$, with its standard basis (in the same order):

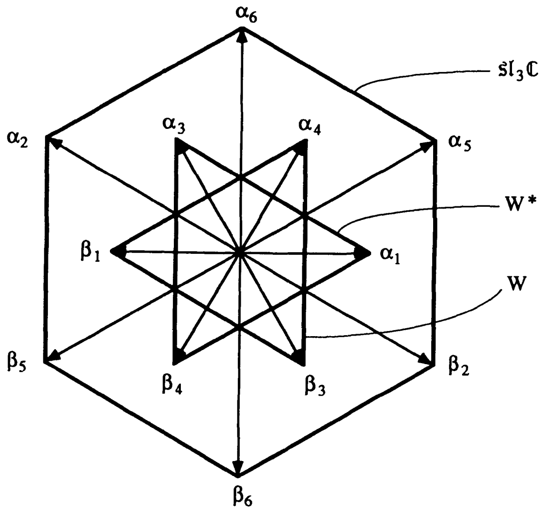
$$\mathfrak{sl}_3\mathbb{C} = \mathbb{C}\{E_{1,1} - E_{2,2}, E_{2,2} - E_{3,3}, E_{1,2}, E_{2,1}, E_{2,3}, E_{3,2}, E_{1,3}, E_{3,1}\}.$$

So we have determined an isomorphism

$$\mathfrak{g}_0 \cong \mathfrak{sl}_3\mathbb{C}.$$

(Note right away that this verifies the Jacobi identity for triples taken from $\mathfrak{g}_0$.)

The rest of the Lie algebra must be a representation of the subalgebra $\mathfrak{g}_0 \cong \mathfrak{sl}_3\mathbb{C}$, and we know what this must be: the smaller hexagon is the union of the two triangles which are the weight diagrams for the standard representation of $\mathfrak{sl}_3$ and its dual, which we denote here by W and W^*; W is the sum of the root spaces for α_4, β_1, and β_3, while W^* is the sum of those for β_4, α_1, and α_3.



Again, a look at the table shows that the vectors X_4, Y_1, and Y_3 form a basis for $W = \mathbb{C}^3$ that corresponds to the standard basis e_1, e_2, and e_3, and similarly Y_4, X_1, and X_3 form a basis for $W^* = (\mathbb{C}^3)^*$ that corresponds to the dual basis e_1^*, e_2^*, and e_3^*: we have

$$W = \mathbb{C}\{X_4, Y_1, Y_3\}; \qquad W^* = \mathbb{C}\{Y_4, X_1, X_3\};$$
$$\mathfrak{g}_2 = \mathfrak{g}_0 \oplus W \oplus W^*.$$

With these isomorphisms, the brackets

$$\mathfrak{g}_0 \times W \to W \quad \text{and} \quad \mathfrak{g}_0 \times W^* \to W^*$$

correspond to the standard operations of $\mathfrak{sl}_3\mathbb{C}$ on $\mathbb{C}^3$ and $(\mathbb{C}^3)^*$.

Next we look at brackets of elements in W. Note that $[W, W]$ is contained in W^*, either by weights or by looking at the table. The table is

	Y_1	Y_3
X_4	$-2X_3$	$2X_1$
Y_1	0	$-2Y_4$

or

	e_2	e_3
e_1	$-2e_3^*$	$2e_2^*$
e_2	0	$-2e_1^*$

Identifying $W = \mathbb{C}^3$, $W^* = (\mathbb{C}^3)^*$ as above, we see that the bracket $W \times W \to W^*$ becomes the map

$$W \times W \to W^* = \Lambda^2 W, \qquad v \times w \mapsto -2 \cdot v \wedge w.$$

Similarly for W^*, we have $[W^*, W^*] \subset W$, and the bracket is identified with the map

$$W^* \times W^* \to W = \Lambda^2 W^*, \qquad \varphi \times \psi \mapsto 2 \cdot \varphi \wedge \psi.$$

Finally we must look at brackets of elements of W with those of W^*, which land in $\mathfrak{g}_0$. Here the table is

	Y_4	X_1	X_3
X_4	$2H_5 + H_2$	$3X_5$	$3X_6$
Y_1	$3Y_5$	$H_2 - H_5$	$3X_2$
Y_3	$3Y_6$	$3Y_2$	$-H_5 - 2H_2$

In terms of the standard bases, $[e_i, e_j^*] = 3E_{i,j} - \delta_{ij}I$. Intrinsically, this mapping

$$[\ ,\ ]\colon W \times W^* \to \mathfrak{sl}_3\mathbb{C} \subset \mathfrak{gl}(W)$$

can be described by the formula

$$[v, \varphi](w) = 3\varphi(w)v - \varphi(v)w \tag{22.4}$$

for $v, w \in W$ and $\varphi \in W^*$.

Exercise 22.5*. Show that $[v, \varphi]$ is the element of $\mathfrak{sl}_3\mathbb{C}$ characterized by the formula

$$B([v, \varphi], Z) = 18\varphi(Z\cdot v) \quad \text{for all } Z \in \mathfrak{sl}_3\mathbb{C},$$

where B is the Killing form on $\mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{C}$. In other words, if we write $v * \varphi$ for the element in $\mathfrak{g}_0 = \mathfrak{sl}_3$ satisfying the identity

$$B(v * \varphi, Z) = \varphi(Z\cdot v) \quad \text{for all } Z \in \mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{C}, \tag{22.6}$$

then the bracket $[v, \varphi]$ can be written in the form

$$[v, \varphi] = 18\cdot v * \varphi. \tag{22.7}$$

It is now a relatively painless task to verify the Jacobi identity, since, rather than having to check it for triples from a basis, it suffices to check it on triples of arbitrary elements of the three spaces $\mathfrak{g}_0$, W, and W^* using the above linear algebra descriptions for the brackets. We will write out this exercise, since the same reasoning will be used later. For example, for three or two elements from $\mathfrak{g}_0$, this amounts to the fact that $\mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{C}$ is a Lie algebra and W and W^* are representations.

For one element Z in $\mathfrak{g}_0$, and two elements v and w in W, the Jacobi identity for these three elements is equivalent to the identity

$$Z\cdot(v \wedge w) = (Z\cdot v) \wedge w + v \wedge (Z\cdot w),$$

which we know for the action of a Lie algebra on an exterior product; and similarly for one element in $\mathfrak{g}_0$ and two in W^*.

The Jacobi identity for $Z \in \mathfrak{g}_0$, $v \in W$, and $\varphi \in W^*$ amounts to

$$[Z, v * \varphi] = (Z\cdot v) * \varphi + v * (Z\cdot\varphi).$$

Applying $B(Y, —)$ to both sides, and using the identity $B(Y, [Z, X]) = B([Y, Z], X)$, this becomes

$$\varphi([Y, Z]\cdot v) = \varphi(Y\cdot(Z\cdot v)) + (Z\cdot\varphi)(Y\cdot v).$$

Since $\varphi([Y, Z]\cdot v) = \varphi(Y\cdot(Z\cdot v)) - \varphi(Z\cdot(Y\cdot v))$, this reduces to

$$(Z\cdot\varphi)(w) = -\varphi(Z\cdot w),$$

for $w = Y\cdot v$, which comes from the fact that W and W^* are dual representations.

For triples u, v, w in W, the Jacobi identity is similarly reduced to the identity

$$(u \wedge v)(Z\cdot w) + (v \wedge w)(Z\cdot u) + (w \wedge u)(Z\cdot v) = 0$$

for all $z \in \mathfrak{g}_0$, which amounts to

$$\begin{aligned} &u \wedge v \wedge (Z\cdot w) + u \wedge (Z\cdot v) \wedge w + (Z\cdot u) \wedge v \wedge w \\ &= Z\cdot(u \wedge v \wedge w) = 0 \quad \text{in } \textstyle\bigwedge^3 W = \mathbb{C}; \end{aligned}$$

and similarly for triples from W^*.

For $v, w \in W$, and $\varphi \in W^*$, noting that

$$[[v, w], \varphi] = -2\cdot[v \wedge w, \varphi] = -4\cdot(v \wedge w) \wedge \varphi = -4\cdot(\varphi(v)w - \varphi(w)v),$$

the Jacobi identity for these elements reads

$$-4\cdot(\varphi(v)w - \varphi(w)v) = -[w, \varphi](v) + [v, \varphi](w). \tag{22.8}$$

The right-hand side is

$$-[w, \varphi](v) + [v, \varphi](w) = -(3\varphi(v)w - \varphi(w)v) + (3\varphi(w)v - \varphi(v)w),$$

which proves this case. (This last line was the only place where we needed to use the definition (22.4) in place of the fancier (22.7).)

The last case is for one element v in W and two elements φ and ψ in W^*. This time identity to be proved comes down to

$$-4\cdot(\psi(v)\varphi - \varphi(v)\psi) = [v, \varphi]\cdot\psi - [v, \psi]\cdot\varphi.$$

Applying both sides to an element w in W, this becomes

$$-4\cdot(\psi(v)\varphi(w) - \varphi(v)\psi(w)) = \varphi([v, \psi]\cdot w) - \psi([v, \varphi]\cdot w).$$

If we apply ψ to the previous case (22.8) we have

$$-4\cdot(\varphi(v)\psi(w) - \varphi(w)\psi(v)) = -\psi([w, \varphi]\cdot v) + \psi([v, \varphi]\cdot w).$$

And these are the same, using the symmetry of the Killing form:

$$18\cdot\varphi([v, \psi]\cdot w) = B([v, \psi], [w, \varphi]) = B([w, \varphi], [v, \psi]) = 18\psi([w, \varphi]\cdot v).$$

This completes the proof that the algebra with multiplication table (22.1) is a Lie algebra. With the hindsight derived from working all this out, of course, we see that there is a quicker way to construct $\mathfrak{g}_2$, without any multiplication table: simply start with $\mathfrak{sl}_3\mathbb{C} \oplus W \oplus W^*$, and define products according to the above rules.

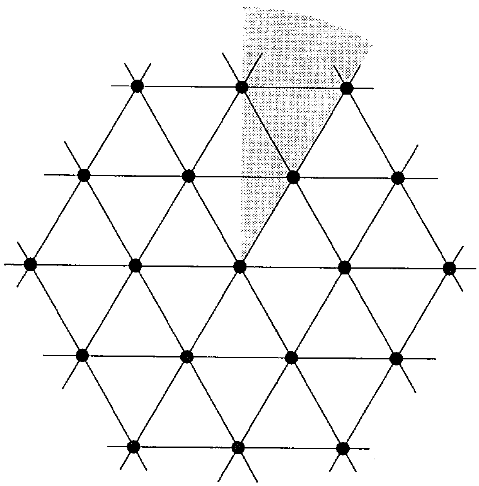
§22.3. Representations of $\mathfrak{g}_2$

We would now like to use the standard procedure, outlined in Lecture 14 (and carried out for the classical Lie algebras in Lectures 15–20) to say something about the representations of $\mathfrak{g}_2$. One nice aspect of this is that, working simply from the root system of $\mathfrak{g}_2$ and analyzing its representations, we will arrive at what is perhaps the simplest description of the algebra: we will see that $\mathfrak{g}_2$ is the algebra of endomorphisms of a seven-dimensional vector space preserving a general trilinear form.

The first step is to find the weight lattice for $\mathfrak{g}_2$. This is the lattice $\Lambda_W \subset \mathfrak{h}^*$ dual to the lattice $\Gamma_W \subset \mathfrak{h}$ generated by the six distinguished elements H_i. By (22.2), Γ_W is generated by H_1 and H_2. Since the values of the eigenvalues α_1 and α_2 on H_1 and H_2 are given by

$$\alpha_1(H_1) = 2, \qquad \alpha_1(H_2) = -1,$$
$$\alpha_2(H_1) = -3, \qquad \alpha_2(H_2) = 2,$$

it follows that the weight lattice is generated by the eigenvalues α_1 and α_2 (and in particular the weight lattice Λ_W is equal to the root lattice Λ_R). The picture is thus

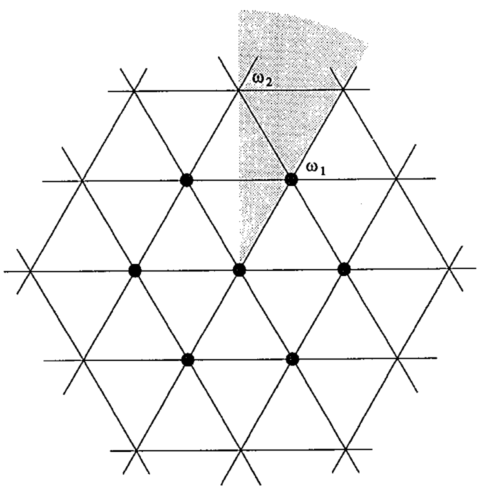


As in the case of the classical Lie algebras, the intersection of the (closed) Weyl chamber $\mathscr{W}$ with the weight lattice is a free semigroup on the two fundamental weights

$$\omega_1 = 2\alpha_1 + \alpha_2 \quad \text{and} \quad \omega_2 = 3\alpha_1 + 2\alpha_2.$$

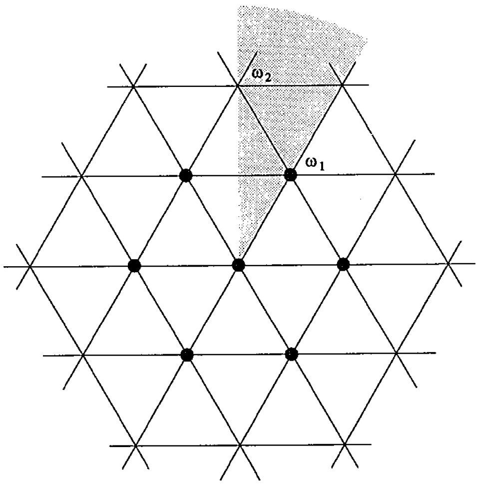
Any irreducible representation of $\mathfrak{g}_2$ will thus have a highest weight vector λ which is a non-negative linear combination of these two. As usual, we write $\Gamma_{a,b}$ for the irreducible representation with highest weight $a\omega_1 + b\omega_2$.

Let us consider first the representation $\Gamma_{1,0}$ with highest weight ω_1. Translating ω_1 around by the action of the Weyl group, we see that the weight diagram of $\Gamma_{1,0}$ looks like



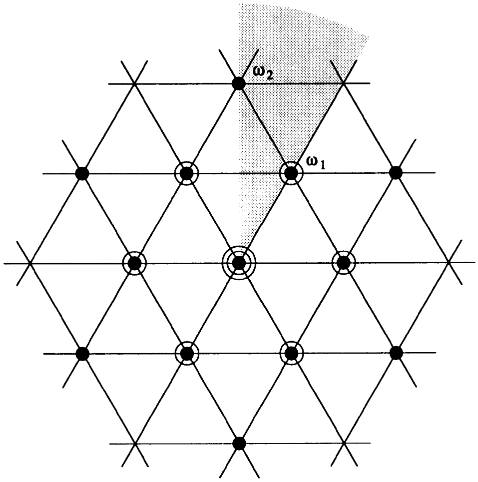
Since there is only one way of getting from the weight ω_1 to the weight 0 by subtraction of simple positive roots, the multiplicity of the weight 0 in $\Gamma_{1,0}$ must be 1. $\Gamma_{1,0}$ is thus a seven-dimensional representation. It is the smallest of the representations of $\mathfrak{g}_2$, and moreover has the property (as we will verify below) that every irreducible representation of $\mathfrak{g}_2$ appears in its tensor algebra; we will therefore call it the *standard* representation of $\mathfrak{g}_2$ and denote it V.

The next smallest representation of $\mathfrak{g}_2$ is the representation $\Gamma_{0,1}$ with highest weight ω_2; this is just the adjoint representation, with weight diagram



Note that the multiplicity of 0 as a weight of $\Gamma_{0,1}$ is 2, and the dimension of $\Gamma_{0,1}$ is 14.

Consider next the exterior square $\wedge^2 V$ of the standard representation $V = \Gamma_{1,0}$ of $\mathfrak{g}_2$. Its weight diagram looks like

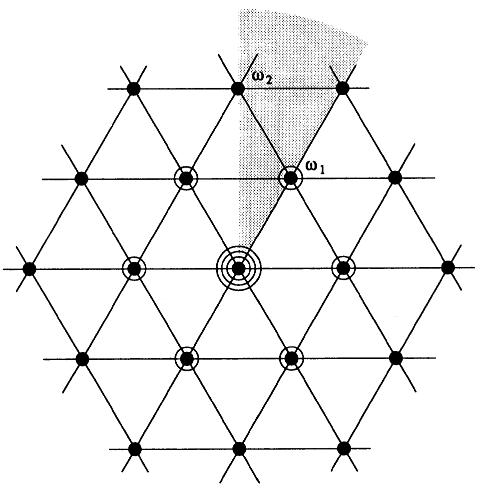


from which we may deduce that

$$\wedge^2 V \cong \Gamma_{0,1} \oplus V.$$

In particular, since the adjoint representation $\Gamma_{0,1}$ of $\mathfrak{g}_2$ is contained in $\wedge^2 V$, and the irreducible representation $\Gamma_{a,b}$ with highest weight $a\omega_1 + b\omega_2$ is contained in the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b \Gamma_{0,1}$, we see that *every irreducible representation of* $\mathfrak{g}_2$ *appears in some tensor power* $V^{\otimes m}$ *of the standard representation*, as stated above.

Next, look at the symmetric square $\mathrm{Sym}^2 V$ of the standard representation. It has weight diagram

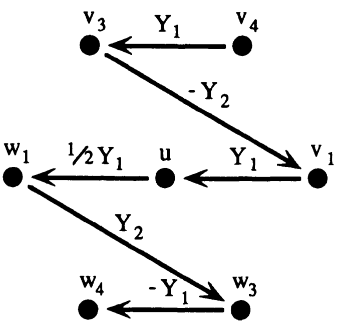


Clearly, this contains a copy of the irreducible representation $\Gamma_{2,0}$ of $\mathfrak{g}_2$ with highest weight $2\omega_1$. Depending on the multiplicities of this representation, it may also contain a copy of V itself, of the trivial representation, or both; or it may be irreducible. To see which is in fact the case, we need to know more about the action of $\mathfrak{g}_2$ on the standard representation V. We will do this in two ways, first by direct calculation, and second using the decomposition of $\mathfrak{g}_2$ into $\mathfrak{sl}_3 \oplus W \oplus W^*$. Although the second approach is shorter, the first illustrates how one can calculate for the exceptional Lie algebras very much as we have been doing in the classical cases.

To describe V explicitly, start with a highest weight vector for V, i.e., any nonzero element v_4 of the eigenspace $V_4 \subset V$ for the action of $\mathfrak{h}$ with eigenvalue α_4. The image of v_4 under the root vector Y_1 will then be a nonzero element of the eigenspace V_3 with eigenvalue α_3 (this follows from the fact that the direct sum $V_3 \oplus V_4$, as a representation of the subalgebra $\mathfrak{s}_{\alpha_1} \subset \mathfrak{g}$, is a copy of the standard representation of $\mathfrak{s}_{\alpha_1} \cong \mathfrak{sl}_2\mathbb{C}$). Similarly, the image of v_3 under Y_2 is a generator v_1 of the eigenspace V_1 with eigenvalue α_1, the image of v_1 under Y_1 is a generator of the eigenspace V_0 with eigenvalue 0, and so on. We may thus choose as a basis for V the vectors

$$v_4, \qquad v_3 = Y_1(v), \qquad v_1 = -Y_2(v_3), \qquad u = Y_1(v_1),$$
$$w_1 = \tfrac{1}{2}Y_1(u), \qquad w_3 = Y_2(w_1), \qquad \text{and} \qquad w_4 = -Y_1(w_3),$$

where v_i (resp. w_i) is an eigenvector with eigenvalue α_i (resp. β_i). (The signs and coefficient $\frac{1}{2}$ in the definition of w_1 are there for reasons of symmetry—see Exercise 22.10.) Diagrammatically, the action of $\mathfrak{g}_2$ may be represented by the arrows



Exercise 22.9. (i) Verify that the vectors v_i, w_i, and u, as defined above, are indeed generators of the corresponding eigenspaces. (ii) Find, in terms of this basis for V, the images of v_4 under the elements Y_3, Y_4, Y_5, and Y_6.

Exercise 22.10. Show that the elements X_i and $Y_i \in \mathfrak{g}_2$ all carry basis vectors v_j and w_j into other basis vectors, up to sign (or to zero, of course), and carry u to twice basis vectors, that is, $X_i u = 2v_i$ and $Y_i u = 2w_i$ for $i = 1, 3, 4$.

Now, the representation $\mathrm{Sym}^2 V$ has, as basis, the pairwise products of the basis vectors for V; and the subrepresentation $\Gamma_{2,0}$ is just the subspace generated by the images of the highest weight vector v_4^2 under (repeated applications of) the generators Y_1, Y_2 of the negative root spaces of $\mathfrak{g}_2$. Thus, for example, the eigenspace in $\mathrm{Sym}^2 V$ with eigenvalue α_4 is the span of the products $u \cdot v_4$ and $v_3 \cdot v_1$; the part of this lying in $\Gamma_{2,0}$ will be the span of the two vectors $Y_2 Y_1 Y_1(v_4^2)$ and $Y_1 Y_2 Y_1(v_4^2)$. We calculate:

$$\begin{aligned} Y_2 Y_1 Y_1(v_4^2) &= Y_2 Y_1(2v_3 \cdot v_4) = Y_2(2v_3^2) \\ &= -4v_1 \cdot v_3 \end{aligned}$$

and

$$\begin{aligned} Y_1 Y_2 Y_1(v_4^2) &= Y_1 Y_2(2v_3 \cdot v_4) = -Y_1(2v_1 \cdot v_4) \\ &= -2v_1 \cdot v_3 - 2u \cdot v_4. \end{aligned}$$

We see, in other words, that $\Gamma_{2,0}$ assumes the weight α_4 with multiplicity 2, so that in particular $\mathrm{Sym}^2 V$ does not contain a copy of V.

Similarly, to see whether or not $\mathrm{Sym}^2 V$ contains a copy of the trivial representation, we have to calculate the multiplicity of the weight 0 in $\Gamma_{2,0}$. Since any path in the weight lattice from the eigenvalue $2\alpha_4$ to 0 obtained by subtracting α_1 and α_2 must pass through α_4, we can do this by evaluating the products of Y_1 and Y_2 on the generators $v_1 \cdot v_3$ and $u \cdot v_4$ of the eigenspace with eigenvalue α_4: we have

$$\begin{aligned} Y_1 Y_1 Y_2(v_1 v_3) &= -Y_1 Y_1(v_1^2) = -Y_1(2u \cdot v_1) \\ &= -4w_1 \cdot v_1 - 2u^2; \\ Y_1 Y_1 Y_2(u \cdot v_4) &= 0; \\ Y_1 Y_2 Y_1(v_1 v_3) &= Y_1 Y_2(u \cdot v_3) = -Y_1(u \cdot v_1) \\ &= -2w_1 \cdot v_1 - u^2; \\ Y_1 Y_2 Y_1(u \cdot v_4) &= Y_1 Y_2(u \cdot v_3 + 2w_1 \cdot v_4) \\ &= Y_1(-u \cdot v_1 + 2w_3 \cdot v_4) \\ &= -2w_1 \cdot v_1 - u^2 - 2w_4 \cdot v_4 + 2w_3 v_3; \\ Y_2 Y_1 Y_1(v_1 v_3) &= Y_2 Y_1(u \cdot v_3) = Y_2(2w_1 \cdot v_3) \\ &= -2w_1 \cdot v_1 + 2w_3 \cdot v_3; \end{aligned}$$

and

$$\begin{aligned} Y_2 Y_1 Y_1(u \cdot v_4) &= Y_2 Y_1(u \cdot v_3 + 2w_1 \cdot v_4) = Y_2(4w_1 \cdot v_3) \\ &= -4w_1 \cdot v_1 + 4w_3 \cdot v_3. \end{aligned}$$

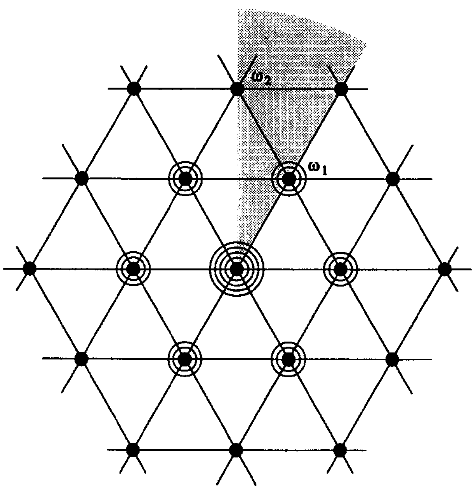
We see from this that the 0-eigenspace of $\Gamma_{2,0}$ is three dimensional; we thus have the decomposition

$$\mathrm{Sym}^2 V \cong \Gamma_{2,0} \oplus \mathbb{C}.$$

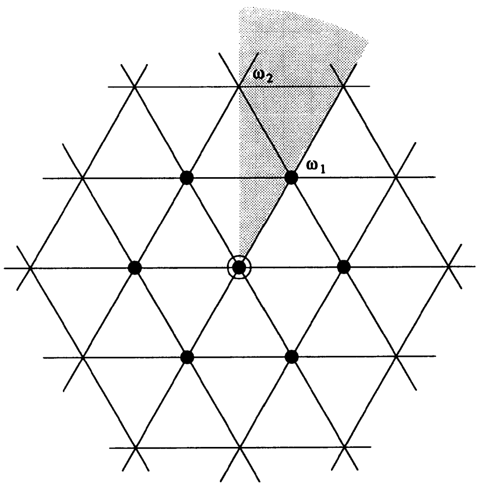
In particular, we deduce that *the action of* $\mathfrak{g}_2$ *on the standard representation* $V = \mathbb{C}^7$ *preserves a quadratic form*; and correspondingly that the subalgebra $\mathfrak{g}_2 \subset \mathfrak{sl}(V) = \mathfrak{sl}_7\mathbb{C}$ is actually contained in the algebra $\mathfrak{so}_7\mathbb{C}$. We will see this again in the following section, where we will give alternative descriptions of the exceptional Lie algebras, and again in §23.3 where we describe compact homogeneous spaces for Lie groups.

Exercise 22.11. Analyze in general the symmetric powers $\mathrm{Sym}^k V$ of the standard representation V of $\mathfrak{g}_2$.

Finally, consider the exterior cube $\bigwedge^3 V$ of the standard representation. The weight diagram is



and after we remove one copy of the representation $\Gamma_{2,0}$ with highest weight $2\omega_1$ (this is the sum of the three highest weights α_4, α_3, and α_1 of V), we are left with



This, by what we have seen, can only be the direct sum of the standard representation V with the trivial representation $\mathbb{C}$. In sum, then, we conclude that

$$\wedge^3 V \cong \Gamma_{2,0} \oplus V \oplus \mathbb{C}.$$

Note in particular that, as a corollary, *the action of* $\mathfrak{g}_2$ *on the standard representation preserves a skew-symmetric trilinear form* ω *on* V. It is not hard to write down this form: it is a linear combination of the five vectors $w_3 \wedge u \wedge v_3, v_4 \wedge u \wedge w_4, w_1 \wedge u \wedge v_1, v_1 \wedge v_3 \wedge w_4$, and $w_1 \wedge w_3 \wedge v_4$; and the fact that it is preserved by X_1 and X_2 is enough to determine the coefficients: we have

$$\omega = w_3 \wedge u \wedge v_3 + v_4 \wedge u \wedge w_4 + w_1 \wedge u \wedge v_1$$
$$+ 2v_1 \wedge v_3 \wedge w_4 + 2w_1 \wedge w_3 \wedge v_4.$$

The fact that the action of $\mathfrak{g}_2$ on V preserves the skew-symmetric cubic form ω takes on additional significance when we make a naive dimension count. The space $\wedge^3 V$ of all such alternating forms has dimension 35, while the algebra $\mathfrak{gl}(V)$ of endomorphisms of V has dimension 49; the difference is exactly the dimension of the algebra $\mathfrak{g}_2$. In fact, we can check directly that the linear map

$$\varphi\colon \mathfrak{gl}(V) \to \wedge^3 V$$

sending $A \in \operatorname{End}(V)$ to $A(\omega)$ is surjective. We deduce that ω is a general cubic alternating form [i.e., an open dense subset of $\wedge^3 V$ corresponds to forms equivalent to ω under $\operatorname{Aut}(V)$], and hence that

Proposition 22.12. *The algebra* $\mathfrak{g}_2$ *is exactly the algebra of endomorphisms of a seven-dimensional vector space* V *preserving a general skew-symmetric cubic form* ω *on* V.

Exercise 22.13*. Verify that the map φ above is surjective by direct calculation of the action of $\mathfrak{gl}(V)$ on $\omega \in \bigwedge^3 V$.

Exercise 22.14. As an alternative to the preceding exercise, analyze skew-symmetric trilinear forms on $\mathbb{C}^n$ to show that for $n \leq 7$ there are only finitely many such forms, up to the action of $\mathrm{GL}_n\mathbb{C}$. Verify that the form ω above is general in $\bigwedge^3\mathbb{C}^7$. (In fact, there are only finitely many cubic alternating forms on $\mathbb{C}^8$ as well, though this is fairly complicated; for $n \geq 9$ a simple dimension count shows that there is a continuously varying family of such forms.)

Note that the cubic form ω preserved by the action of $\mathfrak{g}_2$ gives us explicitly the inclusion

$$V \hookrightarrow \textstyle\bigwedge^2 V$$

deduced earlier from their weight diagrams: this is just the map $V^* \to \bigwedge^2 V$ given by contraction/wedge product with ω, composed with the isomorphism of V with V^*.

Exercise 22.15*. Find the algebra of endomorphisms of a six-dimensional vector space preserving a general skew-symmetric trilinear form.

We will see the form ω again when we describe $\mathfrak{g}_2$ in the following section.

These calculations using the table amount to using all the information that can be extracted from the subalgebras $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$ of $\mathfrak{g}_2$. Using the copy of $\mathfrak{sl}_3\mathbb{C}$ that we found in the second section can make some of this more transparent. Make the identification

$$\mathfrak{g}_2 = \mathfrak{g}_0 \oplus W \oplus W^* = \mathfrak{sl}_3\mathbb{C} \oplus W \oplus W^*.$$

As a representation of $\mathfrak{sl}_3\mathbb{C}$, the seven-dimensional representation V must be the sum of W, W^*, and the trivial representation $\mathbb{C}$. If we make this identification,

$$V = W \oplus W^* \oplus \mathbb{C},$$

it is not hard to work out how the rest of $\mathfrak{g}_2$ acts. This is given in the following table:

		W	W^*	$\mathbb{C}$
		w	ψ	z
$\mathfrak{g}_0$	X	$X \cdot w$	$X \cdot \psi$	0
W	v	$-v \wedge w$	$\psi(v)$	$2z \cdot v$
W^*	φ	$\varphi(w)$	$\varphi \wedge \psi$	$2z \cdot \varphi$

With this identification, we have $u = 1$ in $\mathbb{C}$, and

$$v_4 = e_1, \qquad w_1 = e_2, \qquad w_3 = e_3 \quad \text{in } W = \mathbb{C}^3;$$

$$w_4 = e_1^*, \qquad v_1 = e_2^*, \qquad v_3 = e_3^* \quad \text{in } W^* = (\mathbb{C}^3)^*.$$

Conversely, it is not hard to verify that the above table defines a representation of $\mathfrak{g}_2$, by checking the various cases of the identity $[\xi, \eta]\cdot y = \xi\cdot(\eta\cdot y) - \eta\cdot(\xi\cdot y)$ for ξ, η in $\mathfrak{g}_2$ and y in V. Note that the cubic form ω becomes

$$\omega = \sum_{i=1}^{3} e_i \wedge u \wedge e_i^* + 2(e_1 \wedge e_2 \wedge e_3 + e_1^* \wedge e_2^* \wedge e_3^*).$$

This description of V can be used to verify the calculations made earlier, and also to study its symmetric and exterior powers. For example, $\mathrm{Sym}^2 V$ decomposes over $\mathfrak{sl}_3\mathbb{C}$ into

$$\mathrm{Sym}^2 W \oplus \mathrm{Sym}^2 W^* \oplus \mathrm{Sym}^2\mathbb{C} \oplus W \otimes \mathbb{C} \oplus W^* \otimes \mathbb{C} \oplus W \otimes W^*$$

$$= \mathrm{Sym}^2 W \oplus \mathrm{Sym}^2 W^* \oplus \mathbb{C} \oplus W \oplus W^* \oplus \mathfrak{sl}_3\mathbb{C} \oplus \mathbb{C}.$$

To get the weights around the outside ring, the irreducible representation $\Gamma_{2,0}$ must include $\mathrm{Sym}^2 W$, $\mathrm{Sym}^2 W^*$, and $\mathfrak{sl}_3\mathbb{C}$. Checking that $W \subset \mathfrak{g}_2$ maps $\mathrm{Sym}^2 W^*$ nontrivially to W^* shows that it must also include W and W^*. To finish it suffices to compute the part killed by $\mathfrak{g}_2$, which must lie in the sum of the two components which are trivial for $\mathfrak{sl}_3\mathbb{C}$; checking that this is one dimensional, one recovers the decomposition

$$\mathrm{Sym}^2 V = \Gamma_{2,0} \oplus \mathbb{C}.$$

Exercise 22.16. Use this method to decompose $\bigwedge^3 V$ and $\mathrm{Sym}^3 V$.

§22.4. Algebraic Constructions of the Exceptional Lie Algebras

In this section we will sketch a few of the abstract approaches to the construction of the five exceptional Lie algebras. The constructions are not as easy as you might wish: although the exceptional Lie groups and their Lie algebras have a remarkable way of showing up unexpectedly in many areas of mathematics and physics, they do not have such simple descriptions as the classical series. Indeed, they were not discovered until the classification theorem forced mathematicians to look for them.

To begin with, the method we used to construct $\mathfrak{g}_2$ in the second section of this lecture can be generalized to construct other Lie algebras. This is the construction of Freudenthal, which we do first. It can be used to construct the Lie algebra $\mathfrak{e}_8$ for the diagram (E_8). From $\mathfrak{e}_8$ it is possible to construct $\mathfrak{e}_7$ and $\mathfrak{e}_6$ and $\mathfrak{f}_4$. Then we will present (or at least sketch) several other approaches to their construction. Since it is a rather technical subject, probably not really suited for a first course, we will touch on several approaches rather than give a detailed discussion of one.

The construction of $\mathfrak{g}_2$ as a sum $\mathfrak{g}_0 \oplus W \oplus W^*$ that we found in the second section works more generally, with very little change. Suppose $\mathfrak{g}_0$ is a semi-simple Lie algebra, and W is a representation of $\mathfrak{g}_0$; let W^* be the dual representation, and set

$$\mathfrak{g} = \mathfrak{g}_0 \oplus W \oplus W^*.$$

We also need maps

$$\wedge: \Lambda^2 W \to W^* \quad \text{and} \quad \wedge: \Lambda^2 W^* \to W$$

of representations of $\mathfrak{g}_0$. We assume these are given by trilinear maps of $\mathfrak{g}_0$-representations $T: \Lambda^3 W \to \mathbb{C}$ and $T': \Lambda^3 W^* \to \mathbb{C}$, which means that

$$(u \wedge v)(w) = T(u, v, w) \quad \text{and} \quad \vartheta(\varphi \wedge \psi) = T'(\varphi, \psi, \vartheta).$$

We can then define a bracket on $\mathfrak{g}$ by the same rules as in the second section. To describe it, we let $X, Y, Z, \ldots$ denote arbitrary elements of $\mathfrak{g}_0$, $u, v, w, \ldots$ elements of W, and $\varphi, \psi, \vartheta, \ldots$ elements of W^*. The bracket in $\mathfrak{g}$ is determined by setting:

(i) $[X, Y] = [X, Y]$ (the given bracket in $\mathfrak{g}_0$),
(ii) $[X, v] = X \cdot v$ (the action of $\mathfrak{g}_0$ on W),
(iii) $[X, \varphi] = X \cdot \varphi$ (the canonical action of $\mathfrak{g}_0$ on W^*),
(iv) $[v, w] = a \cdot (v \wedge w)$ (for a scalar a to be determined),
(v) $[\varphi, \psi] = b \cdot (\varphi \wedge \psi)$ (for a scalar b to be determined)
(vi) $[v, \varphi] = c \cdot (v * \varphi)$ (for a scalar c to be determined).

As before, $v * \varphi$ is the element of $\mathfrak{g}_0$ such that

$$B(v * \varphi, Z) = \varphi(Z \cdot v) \quad \text{for all } Z \in \mathfrak{g}_0,$$

where B is the Killing form on $\mathfrak{g}_0$. The rules (i)–(vi) determine a bilinear product [,] on all of $\mathfrak{g}$, and the fact that it is skew follows from the facts that $[X, X] = 0$, $[v, v] = 0$, and $[\varphi, \varphi] = 0$.

The argument that we gave showing that $\mathfrak{g}_2$ satisfies the Jacobi identity works in this general case without essential change, except for the last two cases, where explicit calculation is needed. For $v, w \in W$, and $\varphi \in W^*$, the Jacobi identity is equivalent to the identity

$$ab((v \wedge w) \wedge \varphi) = c((v * \varphi) \cdot w - (w * \varphi) \cdot v). \tag{22.17}$$

For $v \in W$, $\varphi, \psi \in W^*$, the Jacobi identity amounts to

$$ab((\varphi \wedge \psi) \wedge v) = c((v * \psi) \cdot \varphi - (v * \varphi) \cdot \psi). \tag{22.18}$$

We will see in Exercise 22.20 that (22.17) and (22.18) are equivalent. Again, the simplicity of the resulting Lie algebra is easy to see, provided all the weight spaces are one dimensional, using Exercise 14.34, so we have:

Proposition 22.19 (Freudenthal). *Given a representation W of a semisimple Lie algebra $\mathfrak{g}_0$ and trilinear forms T and T' inducing maps $\bigwedge^2 W \to W^*$ and $\bigwedge^2 W^* \to W$, such that (22.17) and (22.18) are satisfied, the above products make*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus W \oplus W^*$$

into a Lie algebra. If the weight spaces of W are all one dimensional, and the weights of W, W^*, and the roots of $\mathfrak{g}_0$ are all distinct, and $abc \neq 0$, then $\mathfrak{g}$ is semisimple, with the same Cartan subalgebra as $\mathfrak{g}_0$.

Exercise 22.20*. (a) Show that the trilinear map T determines a map $\wedge$: $\bigwedge^2 W \to W^*$ of representations if and only if it satisfies the identity

$$T(X \cdot u, v, w) + T(u, X \cdot v, w) + T(u, v, X \cdot w) = 0 \qquad \forall X \in \mathfrak{g}_0,$$

and similarly for T'.

(b) Show that each of (22.17) and (22.18) is equivalent to the identity

$$ab \cdot (v \wedge w)(\varphi \wedge \psi) = c \cdot (B(w * \psi, v * \varphi) - B(w * \varphi, v * \psi)).$$

The Lie algebra $\mathfrak{e}_8$ for (E_8) can be constructed by this method. This time $\mathfrak{g}_0$ is taken to be the Lie algebra $\mathfrak{sl}_9\mathbb{C}$; if $V = \mathbb{C}^9$ is the standard representation of $\mathfrak{sl}_9\mathbb{C}$, let $W = \bigwedge^3 V$, so $W^* = \bigwedge^3 V^*$; the trilinear map is the usual wedge product

$$\textstyle\bigwedge^3 V \otimes \bigwedge^3 V \otimes \bigwedge^3 V \to \bigwedge^9 V = \mathbb{C},$$

and similarly for $\bigwedge^3 V^*$. We leave the verifications to the reader:

Exercise 22.21*. (i) Verify the conditions on the roots of $\mathfrak{sl}_9$ and the weights of $\bigwedge^3 V$ and $\bigwedge^3 V^*$. (ii) Use the fact that $B(X, Y) = 18 \cdot \mathrm{Tr}(XY)$ for $\mathfrak{sl}_9$ to show that (22.17) holds precisely if $c = -18ab$. (iii) Show that the Dynkin diagram of the resulting Lie algebra is (E_8).

Note that the dimension of $\mathfrak{sl}_9\mathbb{C}$ is 80, and that of W and W^* is 84, so the sum has dimension 248, as predicted by the root system of (E_8).

Once the Lie algebra $\mathfrak{e}_8$ is constructed, $\mathfrak{e}_7$ and $\mathfrak{e}_6$ can be found as subalgebras, as follows. Note that removing one or two nodes from the long arm of the Dynkin diagram of (E_8) leads to the Dynkin diagrams (E_7) and (E_6).

In general, if $\mathfrak{g}$ is a simple Lie algebra, with Dynkin diagram D, consider a subdiagram D° of D obtained by removing some subset of nodes, together with all the lines meeting these nodes.[1] Then we can construct a semisimple subalgebra $\mathfrak{g}^\circ$ of $\mathfrak{g}$ with D° as its Dynkin diagram. In fact, $\mathfrak{g}^\circ$ is the subalgebra generated by all the root spaces $\mathfrak{g}_{\pm\alpha}$, where α is a root in D°.

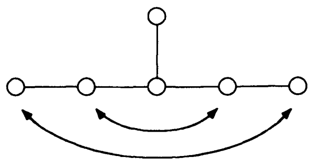
[1] If there are double or triple lines between two nodes, both nodes should be removed or kept together.

Exercise 22.22. (a) Prove this by verifying that the positive roots of $\mathfrak{g}^\circ$ are the positive roots β of $\mathfrak{g}$ that are sums of the roots in D°, and the Cartan subalgebra $\mathfrak{h}^\circ$ is spanned by the corresponding vectors $H_\beta \in \mathfrak{h}$.

(b) Carry this out for $\mathfrak{e}_7$ and $\mathfrak{e}_6$; in particular, show again that $\mathfrak{e}_7$ has 63 positive roots, so dimension $7 + 2(63) = 133$, and $\mathfrak{e}_6$ has 36 positive roots, so dimension $6 + 2(36) = 78$.

Exercise 22.23. For each of the simple Lie algebras, find the subalgebras obtained by removing one node from an end of its Dynkin diagram.

The last exceptional Lie algebra $\mathfrak{f}_4$ can be constructed by taking an invariant subalgebra of $\mathfrak{e}_6$ by an involution. This involution corresponds to the evident symmetry in the Dynkin diagram:



In general, an automorphism of a Dynkin diagram arises from an automorphism of the corresponding semisimple Lie algebra, as follows from the fact that the multiplication table is determined by the Dynkin diagram, cf. Proposition 21.22 and Claim 21.25.

Exercise 22.24*. (a) Show that the invariant subalgebra for the indicated involution of $\mathfrak{e}_6$ is a simple Lie algebra $\mathfrak{f}_4$ with Dynkin diagram (F_4).

(b) Find the invariant subalgebra for the involutions of (A_n) and (D_n), and for an automorphism of order three of (D_4).

Exercise 22.25*. For each automorphism of the Dynkin diagrams (A_n) and (D_n), find an explicit automorphism of $\mathfrak{sl}_{n+1}\mathbb{C}$ and $\mathfrak{so}_{2n}\mathbb{C}$ that induces it.

The exceptional Lie algebras can also be realized as the Lie algebras of derivations of certain nonassociative algebras. This also gives realizations of corresponding Lie groups as groups of automorphism of these algebras (see Exercise 8.28). Some examples of this for associative algebras should be familiar. The group of automorphisms of the algebra $\mathbb{H}$ of (real) quaternions is $O(3)$, so the Lie algebra of derivations is $\mathfrak{so}_3\mathbb{R}$. The Lie algebra of derivations of the complexification $\mathbb{H}_\mathbb{C}$ is $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$.

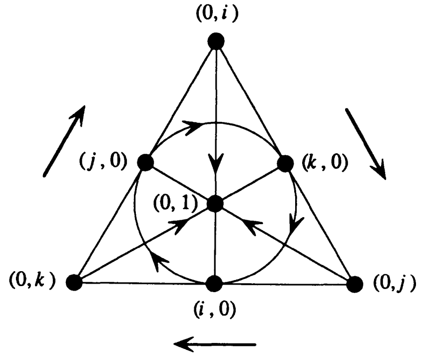
The exceptional group G_2 can be realized as the group of automorphisms of the complexification of the eight-dimensional *Cayley algebra*, or algebra of *octonions*. Recall that the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ can be constructed as the set of pairs (a, b) of complex numbers. In a similar way the Cayley algebra, which we denote by $\mathbb{O}$, can be constructed as the set of pairs (a, b), with a and b quaternions. The addition is componentwise, with multiplication

$$(a, b) \circ (c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

where $^-$ denotes conjugation in $\mathbb{H}$. This algebra $\mathbb{O}$ also has a conjugation, which takes (a, b) to $(\bar{a}, -b)$. It has a basis $1 = (1, 0)$, together with seven elements $e_1, \ldots, e_7$:

$$(i, 0), (j, 0), (k, 0), (0, 1), (0, i), (0, j), (0, k).$$

These satisfy $e_p \circ e_p = -1$ and $e_p \circ e_q = -e_q \circ e_p$ for $p \neq q$, and the conjugate $\overline{e_p}$ of e_p is $-e_p$. The multiplication table can be encoded in the diagram:



Here, if e_p, e_q, and e_r appear on a line in the order shown by the arrow, then

$$e_p \circ e_q = e_r, \qquad e_q \circ e_r = e_p, \qquad e_q \circ e_p = e_r.$$

Note in particular that any two of these basic elements generate a subalgebra of $\mathbb{O}$ isomorphic to $\mathbb{H}$.

Exercise 22.26. Show that the subalgebra of $\mathbb{O}$ generated by any two elements is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Deduce that, although $\mathbb{O}$ is noncommutative and nonassociative, it is "alternative," i.e., it satisfies the identities $(x \circ x) \circ y = x \circ (x \circ y)$ and $y \circ (x \circ x) = (y \circ x) \circ x$.

A trace and norm can be defined on $\mathbb{O}$ by

$$\operatorname{Tr}(x) = \tfrac{1}{2}(x + \bar{x}), \qquad N(x) = x \circ \bar{x};$$

these satisfy the relation $x^2 - 2\operatorname{Tr}(x) + N(x) = 0$. Let $\beta(x, y) = \frac{1}{2}(x \circ \bar{y} + y \circ \bar{x})$ be the bilinear form associated to N; note that the above basis is an orthonormal basis for this inner product.

Let G be the group of algebra automorphisms of the real algebra $\mathbb{O}$. The next exercise sketches a proof that the complexification of G is a Lie group of type (G_2).

Exercise 22.27*. The center of $\mathbb{O}$ is $\mathbb{R}\cdot 1$, which is preserved by G. Let Y be orthogonal space to $\mathbb{R}\cdot 1$ with respect to the quadratic form N. Then G is imbedded in the group $SO(Y)$ of orthogonal transformations of Y.

(a) Define a "cross product" $\times$ on Y by the formula $v \times w = v\cdot w + \beta(v, w)\cdot 1$. Show that G can be identified with the group of orthogonal transformations of Y that preserve the cross product.
(b) Show that $G = \mathrm{Aut}(\mathbb{O})$ acts transitively on the 6-sphere

$$S^6 = \{\sum r_i e_i \colon \sum r_i^2 = 1\},$$

and the subgroup K that fixes $i = e_1$ is mapped onto the 5-sphere in $e_1^\perp$ by the map $g \mapsto g\cdot j$. Conclude from this that G is 14-dimensional and simply connected.
(c) Show that $\{D \in \mathrm{Der}(\mathbb{O})\colon D(i) = 0\}$ is isomorphic to $\mathfrak{su}_3$.
(d) Verify that the Lie algebra of derivations of the complex octonians is the simple Lie algebra of type (G_2).

Exercise 22.28*. The octonions can also be constructed from the Clifford algebra of an eight-dimensional vector space with a nondegenerate quadratic form. With V, S^+, and S^- as in §20.3, with $v_1 \in V$, $s_1 \in S^+$, $t_1 = v_1\cdot s_1 \in S^-$ chosen so the values of the quadratic forms are 1 on each of them as in Exercise 20.50, define a product $V \times V \to V$, $(v, w) \mapsto v \circ w$ by the formula

$$v \circ w = (v\cdot t_1)\cdot(w\cdot s_1).$$

Note that $v\cdot t_1 \in S^+$, $w\cdot s_1 \in S^-$, so their product $(v\cdot t_1)\cdot(w\cdot s_1)$ is back in V.

(a) Show that V with this product is isomorphic to the complex octonians $\mathbb{O}$, with unit v_1, with the map $v \mapsto -\rho(v_1)(v)$ corresponding to conjugation in $\mathbb{O}$.

Conversely, starting with the complex octonians $\mathbb{O}$, one can reconstruct the algebra of §20.3: define $A = \mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$, define an automorphism J of order 3 of A by $J(x, y, z) = (z, x, y)$, and define a product $\cdot$ from each succession of two factors to the third by the formulas $x\cdot y = \bar{x} \circ \bar{y}$, $y\cdot z = \bar{y} \circ \bar{z}$, $z\cdot x = \bar{z} \circ \bar{x}$.

(b) Show that A is isomorphic to the algebra described in §20.3.

(c) Identifying $\mathfrak{so}_8\mathbb{C}$ with the space of skew linear transformations of $\mathbb{O}$, show that for each A in $\mathfrak{so}_8\mathbb{C}$ there are unique B and C in $\mathfrak{so}_8\mathbb{C}$ such that

$$A(x \circ y) = B(x) \circ y + x \circ C(y)$$

for all complex octonions x and y. Equivalently, if one defines a trilinear form $(\ ,\ ,\ )$ on the octonions by $(x, y, z) = \mathrm{Tr}((x \circ y) \circ z) = \mathrm{Tr}(x \circ (y \circ z))$,

$$(Ax, y, z) + (x, By, z) + (x, y, Cz) = 0$$

for all x, y, z. Show that this trilinear form agrees with that defined in Exercise 20.49, and the mapping $A \mapsto B$ determines the triality automorphism j' of $\mathfrak{so}_8\mathbb{C}$ of order three described in Exercise 20.51.

Exercise 22.29. Define three homomorphisms from the real Clifford algebra $C_7 = C(0, 7)$ to $\mathrm{End}_{\mathbb{R}}(\mathbb{O})$ by sending $v \in \mathbb{R}^7 = \sum \mathbb{R}e_i$ to the maps L_v, R_v, and T_v defined by $L_v(x) = v \circ x$, $R_v(x) = x \circ v$, and $T_v(x) = v \circ (x \circ v) = (v \circ x) \circ v$.

(a) Show that these do determine maps of the Clifford algebra, and that the induced maps

$$\mathrm{Spin}_8 \mathbb{R} \hookrightarrow C_8^{\text{even}} = C_7 \to \mathrm{End}_{\mathbb{R}}(\mathbb{O})$$

are the two spin representations and the standard representation, respectively.

(b) Verify that $T_v(x \circ y) = L_v(x) \cdot L_v(y)$ for all v, x, y, and use this to verify the triality formula in (c) of the preceding exercise.

The algebra $\mathfrak{f}_4$ can be realized as the derivation algebra of the complexification of a 27-dimensional *Jordan algebra* $\mathbb{J}$. This can be constructed as the set of matrices of the form

$$\begin{pmatrix} a & \alpha & \beta \\ \bar{\alpha} & b & \gamma \\ \bar{\beta} & \bar{\gamma} & c \end{pmatrix},$$

with a, b, c scalars, and α, β, γ in $\mathbb{O}$. The product $\circ$ in $\mathbb{J}$ is given by

$$x \circ y = \tfrac{1}{2}(xy + yx),$$

where the products on the right-hand side are defined by usual matrix multiplication. This algebra is commutative but not associative, and satisfies the identity $((x \circ x) \circ y) \circ x = (x \circ x) \circ (y \circ x)$. In fact, (F_4) is the group of automorphisms of this 27-dimensional space that preserve the scalar product $(x, y) = \mathrm{Tr}(x \circ y)$ and the scalar triple product $(x, y, z) = \mathrm{Tr}((x \circ y) \circ z)$. The kernel of the trace map is an irreducible 26-dimensional representation of $\mathfrak{f}_4$. For details see [Ch-S], [To], [Pos].

In addition, there is a cubic form "det" on $\mathbb{J}$ such that the linear automorphisms of $\mathbb{J}$ that preserve this form is a group of type (E_6). This again shows $\mathfrak{f}_4$ as a subalgebra of $\mathfrak{e}_6$.

The other exceptional Lie algebras can also be constructed as derivations of appropriate algebras. We refer for this to [Ti2], [Dr], [Fr2], [Jac2], and the references found in these sources. Other constructions were given by Witt, cf. [Wa]. The simple Lie algebras are also constructed explicitly in [S-K, §1]. See also [Ch-S], [Fr1], and [Sc].

What little we will have to say about the representations of the four exceptional Lie algebras besides $\mathfrak{g}_2$ can wait until we have the Weyl character formula.