LECTURE 9

Initial Classification of Lie Algebras

In this lecture we define various subclasses of Lie algebras: nilpotent, solvable, semisimple, etc., and prove basic facts about their representations. The discussion is entirely elementary (largely because the hard theorems are stated without proof for now); there are no prerequisites beyond linear algebra. Apart from giving these basic definitions, the purpose of the lecture is largely to motivate the narrowing of our focus to semisimple algebras that will take place in the sequel. In particular, the first part of §9.3 is logically the most important for what follows.

§9.1. Rough Classification of Lie Algebras

We will give, in this section, a preliminary sort of classification of Lie algebras, reflecting the degree to which a given Lie algebra $\mathfrak{g}$ fails to be abelian. As we have indicated, the goal ultimately is to narrow our focus onto *semisimple* Lie algebras.

Before we begin, two definitions, both completely straightforward: First, we define the *center* $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ to be the subspace of $\mathfrak{g}$ of elements $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. Of course, we say $\mathfrak{g}$ is *abelian* if all brackets are zero.

Exercise 9.1. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. Show that the subgroup of G generated by exponentiating the Lie subalgebra $Z(\mathfrak{g})$ is the connected component of the identity in the center $Z(G)$ of G.

W. Fulton et al., *Representation Theory*

Next, we say that a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is an *ideal* if it satisfies the condition

$$[X, Y] \in \mathfrak{h} \quad \text{for all } X \in \mathfrak{h}, Y \in \mathfrak{g}.$$

Just as connected subgroups of a Lie group correspond to subalgebras of its Lie algebra, the notion of ideal in a Lie algebra corresponds to the notion of normal subgroup, in the following sense:

Exercise 9.2. Let G be a connected Lie group, $H \subset G$ a connected subgroup and $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras. Show that H is a normal subgroup of G if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

Observe also that the bracket operation on $\mathfrak{g}$ induces a bracket on the quotient space $\mathfrak{g}/\mathfrak{h}$ if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

This, in turns, motivates the next bit of terminology: we say that a Lie algebra $\mathfrak{g}$ is *simple* if $\dim \mathfrak{g} > 1$ and it contains no nontrivial ideals. By the last exercise, this is equivalent to saying that the adjoint form G of the Lie algebra $\mathfrak{g}$ has no nontrivial normal Lie subgroups.

Now, to attempt to classify Lie algebras, we introduce two descending chains of subalgebras. The first is the *lower central series* of subalgebras $\mathscr{D}_k\mathfrak{g}$, defined inductively by

$$\mathscr{D}_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

and

$$\mathscr{D}_k\mathfrak{g} = [\mathfrak{g}, \mathscr{D}_{k-1}\mathfrak{g}].$$

Note that the subalgebras $\mathscr{D}_k\mathfrak{g}$ are in fact ideals in $\mathfrak{g}$. The other series is called the *derived series* $\{\mathscr{D}^k\mathfrak{g}\}$; it is defined by

$$\mathscr{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$$

and

$$\mathscr{D}^k\mathfrak{g} = [\mathscr{D}^{k-1}\mathfrak{g}, \mathscr{D}^{k-1}\mathfrak{g}].$$

Exercise 9.3. Use the Jacobi identity to show that $\mathscr{D}^k\mathfrak{g}$ is also an ideal in $\mathfrak{g}$. More generally, if $\mathfrak{h}$ is an ideal in a Lie algebra $\mathfrak{g}$, show that $[\mathfrak{h}, \mathfrak{h}]$ is also an ideal in $\mathfrak{g}$; hence all $\mathscr{D}^k\mathfrak{h}$ are ideals in $\mathfrak{g}$.

Observe that we have $\mathscr{D}^k\mathfrak{g} \subset \mathscr{D}_k\mathfrak{g}$ for all k, with equality when $k = 1$; we often write simply $\mathscr{D}\mathfrak{g}$ for $\mathscr{D}_1\mathfrak{g} = \mathscr{D}^1\mathfrak{g}$ and call this the *commutator subalgebra*. We now make the

Definitions

(i) We say that $\mathfrak{g}$ is *nilpotent* if $\mathscr{D}_k\mathfrak{g} = 0$ for some k.
(ii) We say that $\mathfrak{g}$ is *solvable* if $\mathscr{D}^k\mathfrak{g} = 0$ for some k.

(iii) We say that $\mathfrak{g}$ is *perfect* if $\mathscr{D}\mathfrak{g} = \mathfrak{g}$ (this is not a concept we will use much).
(iv) We say that $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ has no nonzero solvable ideals.

The standard example of a nilpotent Lie algebra is the algebra $\mathfrak{n}_n\mathbb{R}$ of strictly upper-triangular $n \times n$ matrices; in this case the kth subalgebra $\mathscr{D}_k\mathfrak{g}$ in the lower central series will be the subspace $\mathfrak{n}_{k+1,n}\mathbb{R}$ of matrices $A = (a_{i,j})$ such that $a_{i,j} = 0$ whenever $j \leq i + k$, i.e., that are zero below the diagonal and within a distance k of it in each column or row. (In terms of a complete flag $\{V_i\}$ as in §7.2, these are just the endomorphisms that carry V_i into V_{i-k-1}.) It follows also that any subalgebra of the Lie algebra $\mathfrak{n}_n\mathbb{R}$ is likewise nilpotent; we will show later that any nilpotent Lie algebra is isomorphic to such a subalgebra. We will also see that if a Lie algebra $\mathfrak{g}$ is represented on a vector space V, such that each element acts as a nilpotent endomorphism, there is a basis for V such that, identifying $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n\mathbb{R}$, $\mathfrak{g}$ maps to the subalgebra $\mathfrak{n}_n\mathbb{R} \subset \mathfrak{gl}_n\mathbb{R}$.

Similarly, a standard example of a solvable Lie algebra is the space $\mathfrak{b}_n\mathbb{R}$ of upper-triangular $n \times n$ matrices; in this Lie algebra the commutator $\mathscr{D}\mathfrak{b}_n\mathbb{R}$ is the algebra $\mathfrak{n}_n\mathbb{R}$ and the derived series is, thus, $\mathscr{D}^k\mathfrak{b}_n\mathbb{R} = \mathfrak{n}_{2^{k-1},n}\mathbb{R}$. Again, it follows that any subalgebra of the algebra $\mathfrak{b}_n\mathbb{R}$ is likewise solvable; and we will prove later that, conversely, *any* representation of a solvable Lie algebra on a vector space V consists, in terms of a suitable basis, entirely of upper-triangular matrices (i.e., given a solvable Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$, there exists a basis for V such that under the corresponding identification of $\mathfrak{gl}(V)$ with $\mathfrak{gl}_n\mathbb{R}$, the subalgebra $\mathfrak{g}$ is contained in $\mathfrak{b}_n\mathbb{R} \subset \mathfrak{gl}_n\mathbb{R}$).

It is clear from the definitions that the properties of being nilpotent or solvable are inherited by subalgebras or homomorphic images. We will see that the same is true for semisimplicity in the case of homomorphic images, though not for subalgebras.

Note that $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}$ has a sequence of Lie *subalgebras* $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_k = 0$, such that $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian. Indeed, if this is the case, one sees by induction that $\mathscr{D}^i\mathfrak{g} \subset \mathfrak{g}_i$ for all i. (One may also refine such a sequence to one where each quotient $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is one dimensional.) It follows from this description that if $\mathfrak{h}$ is an ideal in a Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ *is solvable if and only if* $\mathfrak{h}$ *and* $\mathfrak{g}/\mathfrak{h}$ *are solvable Lie algebras.* (The analogous assertion for nilpotent Lie algebras is false: the ideal $\mathfrak{n}_n$ is nilpotent in the Lie algebra $\mathfrak{b}_n$ of upper-triangular matrices, and the quotient is the nilpotent algebra $\mathfrak{d}_n$ of diagonal matrices, but $\mathfrak{d}_n$ is not nilpotent.) If $\mathfrak{g}$ is the Lie algebra of a connected Lie group G, then $\mathfrak{g}$ is solvable if and only if there is a sequence of connected subgroups, each normal in G (or in the next in the sequence), such that the quotients are abelian.

In particular, the sum of two solvable ideals in a Lie algebra $\mathfrak{g}$ is again solvable [note that $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$]. It follows that the sum of all solvable ideals in $\mathfrak{g}$ is a maximal solvable ideal, called the *radical* of $\mathfrak{g}$ and denoted $\mathrm{Rad}(\mathfrak{g})$. The quotient $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple. Any Lie algebra $\mathfrak{g}$ thus fits into an exact sequence

$$0 \to \mathrm{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) \to 0 \tag{9.4}$$

where the first algebra is solvable and the last is semisimple. With this somewhat shaky justification (but see Proposition 9.17), we may say that to study the representation theory of an arbitrary Lie algebra, we have to understand individually the representation theories of solvable and semisimple Lie algebras. Of these, the former is relatively easy, at least as regards irreducible representations. The basic fact about them—that any irreducible representation of a solvable Lie algebra is one dimensional—will be proved later in this lecture. The representation theory of semisimple Lie algebras, on the other hand, is extraordinarily rich, and it is this subject that will occupy us for most of the remainder of the book.

Another easy consequence of the definitions is the fact that *a Lie algebra is semisimple if and only if it has no nonzero abelian ideals.* Indeed, the last nonzero term in the derived sequence of ideals $\mathscr{D}^k\mathrm{Rad}(\mathfrak{g})$ would be an abelian ideal in $\mathfrak{g}$ (cf. Exercise 9.3). A semisimple Lie algebra can have no center, so *the adjoint representation of a semisimple Lie algebra is faithful.*

It is a fact that the sequence (9.4) splits, in the sense that there are subalgebras of $\mathfrak{g}$ that map isomorphically onto $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$. The existence of such a *Levi decomposition* is part of the general theory we are postponing. To show that an arbitrary Lie algebra has a faithful representation (*Ado's theorem*), one starts with a faithful representation of the center, and then builds a representation of the radical step by step, inserting a string of ideals between the center and the radical. Then one uses a splitting to get from a faithful representation on the radical to some representation on all of $\mathfrak{g}$; the sum of this representation and the adjoint representation is then a faithful representation. See Appendix E for details.

One reason for the terminology simple/semisimple will become clear later in this lecture, when we show that a semisimple Lie algebra is a direct sum of simple ones.

Exercise 9.5. Every semisimple Lie algebra is perfect. Show that the Lie group of Euclidean motions of $\mathbb{R}^3$ has a Lie algebra $\mathfrak{g}$ which is perfect, i.e., $\mathscr{D}\mathfrak{g} = \mathfrak{g}$, but $\mathfrak{g}$ is not semisimple. More generally, if $\mathfrak{h}$ is semisimple, and V is an irreducible representation of $\mathfrak{h}$, the twisted product

$$\mathfrak{g} = \{(v, X) | v \in V, X \in \mathfrak{h}\} \quad \text{with } [(v, X), (w, Y)] = (Xw - Yv, [X, Y])$$

is a Lie algebra with $\mathscr{D}\mathfrak{g} = \mathfrak{g}$, $\mathrm{Rad}(\mathfrak{g}) = V$ abelian, and $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g}) = \mathfrak{h}$.

Exercise 9.6. (a) Show that the following are equivalent for a Lie algebra $\mathfrak{g}$: (i) $\mathfrak{g}$ is nilpotent. (ii) There is a chain of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$ with $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ contained in the center of $\mathfrak{g}/\mathfrak{g}_{i+1}$. (iii) There is an integer n such that

$$\mathrm{ad}(X_1) \circ \mathrm{ad}(X_2) \circ \cdots \circ \mathrm{ad}(X_n)(Y) = [X_1, [X_2, \ldots, [X_n, Y]\ldots]] = 0$$

for all $X_1, \ldots, X_n, Y$ in $\mathfrak{g}$.

(b) Conclude that a connected Lie group G is nilpotent if and only if it can be realized as a succession of central extensions of abelian Lie groups.

Exercise 9.7*. If G is connected and nilpotent, show that the exponential map $\exp: \mathfrak{g} \to G$ is surjective, making $\mathfrak{g}$ the universal covering space of G.

Exercise 9.8. Show that the following are equivalent for a Lie algebra $\mathfrak{g}$: (i) $\mathfrak{g}$ is solvable. (ii) There is a chain of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$ with $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ abelian. (iii) There is a chain of subalgebras $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$ such that $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$, and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

§9.2. Engel's Theorem and Lie's Theorem

We will now prove the statement made above about representations of solvable Lie algebras always being upper triangular. This may give the reader an idea of how the general theory proceeds, before we go back to the concrete examples that are our main concern. The starting point is

Theorem 9.9 (Engel's Theorem). *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be any Lie subalgebra such that every $X \in \mathfrak{g}$ is a nilpotent endomorphism of V. Then there exists a nonzero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$.*

Note this implies that there exists a basis for V in terms of which the matrix representative of each $X \in \mathfrak{g}$ is strictly upper triangular: since $\mathfrak{g}$ kills v, it will act on the quotient $\bar{V}$ of V by the span of v, and by induction we can find a basis $\bar{v}_2, \ldots, \bar{v}_n$ for $\bar{V}$ in terms of which this action is strictly upper triangular. Lifting $\bar{v}_i$ to any $v_i \in V$ and setting $v_1 = v$ then gives a basis for V as desired.

PROOF OF THEOREM 9.9. One observation before we start is that if $X \in \mathfrak{gl}(V)$ is any nilpotent element, then the adjoint action $\mathrm{ad}(X): \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is nilpotent. This is straightforward: to say that X is nilpotent is to say that there exists a flag of subspaces $0 \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset V_{k+1} = V$ such that $X(V_i) \subset V_{i-1}$; we can then check that for any endomorphism Y of V the endomorphism $\mathrm{ad}(X)^m(Y)$ carries V_i into V_{i+k-m}.

We now proceed by induction on the dimension of $\mathfrak{g}$. The first step is to show that, under the hypotheses of the problem, $\mathfrak{g}$ *contains an ideal* $\mathfrak{h}$ *of codimension one*. In fact, let $\mathfrak{h} \subset \mathfrak{g}$ be any maximal proper subalgebra; we claim that $\mathfrak{h}$ has codimension one and is an ideal. To see this, we look at the adjoint representation of $\mathfrak{g}$; since $\mathfrak{h}$ is a subalgebra the adjoint action $\mathrm{ad}(\mathfrak{h})$ of $\mathfrak{h}$ on $\mathfrak{g}$ preserves the subspace $\mathfrak{h} \subset \mathfrak{g}$ and so acts on $\mathfrak{g}/\mathfrak{h}$. Moreover, by our observation above, for any $X \in \mathfrak{h}$ $\mathrm{ad}(X)$ acts nilpotently on $\mathfrak{gl}(V)$, hence on $\mathfrak{g}$, hence on $\mathfrak{g}/\mathfrak{h}$. Thus, by induction, there exists a nonzero element $\bar{Y} \in \mathfrak{g}/\mathfrak{h}$ killed by $\mathrm{ad}(X)$ for all $X \in \mathfrak{h}$; equivalently, there exists an element $Y \in \mathfrak{g}$ not in $\mathfrak{h}$ such that $\operatorname{ad}(X)(Y) \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. But this is to say that the subspace $\mathfrak{h}'$ of $\mathfrak{g}$ spanned by $\mathfrak{h}$ and Y is a Lie subalgebra of $\mathfrak{g}$, in which $\mathfrak{h}$ sits as an ideal of codimension one; by the maximality of $\mathfrak{h}$ we have $\mathfrak{h}' = \mathfrak{g}$ and we are done.

We return now to the representation of $\mathfrak{g}$ on V. We may apply the induction hypothesis to the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ found in the preceding paragraph to conclude that there exists a nonzero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{h}$; let $W \subset V$ be the subspace of all such vectors $v \in V$. Let Y be any element of $\mathfrak{g}$ not in $\mathfrak{h}$; since $\mathfrak{h}$ and Y span $\mathfrak{g}$, it will suffice to show that there exists a (nonzero) vector $v \in W$ such that $Y(v) = 0$. Now for any vector $w \in W$ and any $X \in \mathfrak{h}$, we have

$$X(Y(w)) = Y(X(w)) + [X, Y](w).$$

The first term on the right is zero because by hypothesis $w \in W$, $X \in \mathfrak{h}$ and so $X(w) = 0$; likewise, the second term is zero because $[X, Y] = \operatorname{ad}(X)(Y) \in \mathfrak{h}$. Thus, $X(Y(w)) = 0$ for all $X \in \mathfrak{h}$; we deduce that $Y(w) \in W$. But this means that the action of Y on V carries the subspace W into itself; since Y acts nilpotently on V, it follows that there exists a vector $v \in W$ such that $Y(v) = 0$. □

Exercise 9.10*. Show that a Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\operatorname{ad}(X)$ is a nilpotent endomorphism of $\mathfrak{g}$ for every $X \in \mathfrak{g}$.

Engel's theorem, in turn, allows us to prove the basic statement made above that every representation of a solvable Lie group can be put in upper-triangular form. This is implied by

Theorem 9.11 (Lie's Theorem). *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a complex solvable Lie algebra. Then there exists a nonzero vector $v \in V$ that is an eigenvector of X for all $X \in \mathfrak{g}$.*

Exercise 9.12. Show that this implies the existence of a basis for V in terms of which the matrix representative of each $X \in \mathfrak{g}$ is upper triangular.

Proof of Theorem 9.11. Once more, the first step in the argument is to assert that $\mathfrak{g}$ contains an ideal $\mathfrak{h}$ of codimension one. This time, since $\mathfrak{g}$ is solvable we know that $\mathscr{D}\mathfrak{g} \neq \mathfrak{g}$, so that the quotient $\mathfrak{a} = \mathfrak{g}/\mathscr{D}\mathfrak{g}$ is a nonzero abelian Lie algebra; the inverse image in $\mathfrak{g}$ of any codimension one subspace of $\mathfrak{a}$ will then be a codimension one ideal in $\mathfrak{g}$.

Still following the lines of the previous argument, we may by induction assume that there is a vector $v_0 \in V$ that is an eigenvector for all $X \in \mathfrak{h}$. Denote the eigenvalue of X corresponding to v_0 by $\lambda(X)$. We then consider the subspace $W \subset V$ of all vectors satisfying the same relation, i.e., we set

$$W = \{v \in V: X(v) = \lambda(X) \cdot v \ \forall X \in \mathfrak{h}\}.$$

Let Y now be any element of $\mathfrak{g}$ not in $\mathfrak{h}$. As before, it will suffice to show that Y carries some vector $v \in W$ into a multiple of itself, and for this it is enough to show that Y carries W into itself. We prove this in a general context in the following lemma.

Lemma 9.13. *Let $\mathfrak{h}$ be an ideal in a Lie algebra $\mathfrak{g}$. Let V be a representation of $\mathfrak{g}$, and $\lambda\colon \mathfrak{h} \to \mathbb{C}$ a linear function. Set*

$$W = \{v \in V : X(v) = \lambda(X)\cdot v \ \forall X \in \mathfrak{h}\}.$$

Then $Y(W) \subset W$ for all $Y \in \mathfrak{g}$.

PROOF. Let w be any nonzero element of W; to test whether $Y(w) \in W$ we let X be any element of $\mathfrak{h}$ and write

$$\begin{aligned} X(Y(w)) &= Y(X(w)) + [X, Y](w) \\ &= \lambda(X)\cdot Y(w) + \lambda([X, Y])\cdot w \end{aligned} \tag{9.14}$$

since $[X, Y] \in \mathfrak{h}$. This differs from our previous calculation in that the second term on the right is not immediately seen to be zero; indeed, $Y(w)$ will lie in W if and only if $\lambda([X, Y]) = 0$ for all $X \in \mathfrak{h}$.

To verify this, we introduce another subspace of V, namely, the span U of the images w, $Y(w)$, $Y^2(w)$, ... of w under successive applications of Y. This subspace is clearly preserved by Y; we claim that any $X \in \mathfrak{h}$ carries U into itself as well. It is certainly the case that $\mathfrak{h}$ carries w into a multiple of itself, and hence into U, and (9.14) says that $\mathfrak{h}$ carries $Y(w)$ into a linear combination of $Y(w)$ and w, and so into U. In general, we can see that $\mathfrak{h}$ carries $Y^k(w)$ into U by induction: for any $X \in \mathfrak{h}$ we write

$$X(Y^k(w)) = Y(X(Y^{k-1}(w))) + [X, Y](Y^{k-1}(w)). \tag{9.15}$$

Since $X(Y^{k-1}(w)) \in U$ by induction the first term on the right is in U, and since $[X, Y] \in \mathfrak{h}$ the second term is in U as well.

In fact, we see something more from (9.14) and (9.15): it follows that, in terms of the basis w, $Y(w)$, $Y^2(w)$, ... for U, the action of any $X \in \mathfrak{h}$ is upper triangular, with diagonal entries all equal to $\lambda(X)$. In particular, for any $X \in \mathfrak{h}$ the trace of the restriction of X to U is just the dimension of U times $\lambda(X)$. On the other hand, for any element $X \in \mathfrak{h}$ the commutator $[X, Y]$ acts on U, and being the commutator of two endomorphisms of U the trace of this action is zero. It follows then that $\lambda([X, Y]) = 0$, and we are done. □

Exercise 9.16. Show that any irreducible representation of a solvable Lie algebra $\mathfrak{g}$ is one dimensional, and $\mathscr{D}\mathfrak{g}$ acts trivially.

At least for *irreducible* representations, Lie's theorem implies they will all be known for an arbitrary Lie algebra when they are known for the semisimple case. In fact, we have:

Proposition 9.17. *Let $\mathfrak{g}$ be a complex Lie algebra, $\mathfrak{g}_{ss} = \mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$. Every irreducible representation of $\mathfrak{g}$ is of the form $V = V_0 \otimes L$, where V_0 is an irreducible* *representation of* $\mathfrak{g}_{ss}$ [*i.e., a representation of* $\mathfrak{g}$ *that is trivial on* Rad($\mathfrak{g}$)], *and* L *is a one-dimensional representation.*

PROOF. By Lie's theorem there is a $\lambda \in (\text{Rad}(\mathfrak{g}))^*$ such that

$$W = \{v \in V : X(v) = \lambda(X) \cdot v \ \forall X \in \text{Rad}(\mathfrak{g})\}$$

is not zero. Apply the preceding lemma, with $\mathfrak{h} = \text{Rad}(\mathfrak{g})$. Since V is irreducible, we must have $W = V$. In particular, $\text{Tr}(X) = \dim(V) \cdot \lambda(X)$ for $X \in \text{Rad}(\mathfrak{g})$, so λ vanishes on $\text{Rad}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$. Extend λ to a linear function on $\mathfrak{g}$ that vanishes on $[\mathfrak{g}, \mathfrak{g}]$, and let L be the one-dimensional representation of $\mathfrak{g}$ determined by λ; in other words, $Y(z) = \lambda(Y) \cdot z$ for all $Y \in \mathfrak{g}$ and $z \in L$. Then $V \otimes L^*$ is a representation that is trivial on Rad($\mathfrak{g}$), so it comes from a representation of $\mathfrak{g}_{ss}$, as required. □

Exercise 9.18. Show that if $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ that maps isomorphically onto $\mathfrak{g}/\text{Rad}(\mathfrak{g})$, then any irreducible representation of $\mathfrak{g}$ restricts to an irreducible representation of $\mathfrak{g}'$, and any irreducible representation of $\mathfrak{g}'$ extends to a representation of $\mathfrak{g}$.

§9.3. Semisimple Lie Algebras

As is clear from the above, many of the aspects of the representation theory of finite groups that were essential to our approach are no longer valid in the context of general Lie algebras and Lie groups. Most obvious of these is complete reducibility, which we have seen fails for Lie groups; another is the fact that not only can the action of elements of a Lie group or algebra on a vector space be nondiagonalizable, the action of some element of a Lie algebra may be diagonalizable under one representation and not under another.

That is the bad news. The good news is that, if we just restrict ourselves to semisimple Lie algebras, everything is once more as well behaved as possible. For one thing, we have complete reducibility again:

Theorem 9.19 (Complete Reducibility). *Let V be a representation of the semisimple Lie algebra $\mathfrak{g}$ and $W \subset V$ a subspace invariant under the action of $\mathfrak{g}$. Then there exists a subspace $W' \subset V$ complementary to W and invariant under $\mathfrak{g}$.*

The proof of this basic result will be deferred to Appendix C.

The other question, the diagonalizability of elements of a Lie algebra under a representation, requires a little more discussion. Recall first the statement of *Jordan decomposition*: any endomorphism X of a complex vector space V can be uniquely written in the form

$$X = X_s + X_n$$

where X_s is diagonalizable, X_n is nilpotent, and the two commute. Moreover, X_s and X_n may be expressed as polynomials in X.

Now, suppose that $\mathfrak{g}$ is an arbitrary Lie algebra, $X \in \mathfrak{g}$ any element, and $\rho: \mathfrak{g} \to \mathfrak{gl}_n\mathbb{C}$ any representation. We have seen that the image $\rho(X)$ need not be diagonalizable; we may still ask how $\rho(X)$ behaves with respect to the Jordan decomposition. The answer is that, in general, absolutely nothing need be true. For example, just taking $\mathfrak{g} = \mathbb{C}$, we see that under the representation

$$\rho_1: t \mapsto (t)$$

every element is diagonalizable, i.e., $\rho(X)_s = \rho(X)$; under the representation

$$\rho_2: t \mapsto \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

every element is nilpotent [i.e., $\rho(X)_s = 0$]; whereas under the representation

$$\rho_3: t \mapsto \begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix}$$

not only are the images $\rho(X)$ neither diagonalizable nor nilpotent, the diagonalizable and nilpotent parts of $\rho(X)$ are not even in the image $\rho(\mathfrak{g})$ of the representation.

If we assume the Lie algebra $\mathfrak{g}$ is semisimple, however, the situation is radically different. Specifically, we have

Theorem 9.20 (Preservation of Jordan Decomposition). *Let $\mathfrak{g}$ be a semisimple Lie algebra. For any element $X \in \mathfrak{g}$, there exist X_s and $X_n \in \mathfrak{g}$ such that for any representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ we have*

$$\rho(X)_s = \rho(X_s) \quad \textit{and} \quad \rho(X)_n = \rho(X_n).$$

In other words, if we think of ρ as injective and $\mathfrak{g}$ accordingly as a Lie subalgebra of $\mathfrak{gl}(V)$, *the diagonalizable and nilpotent parts of any element X of $\mathfrak{g}$ are again in $\mathfrak{g}$ and are independent of the particular representation ρ.*

The proofs we will give of the last two theorems both involve introducing objects that are not essential for the rest of this book, and we therefore relegate them to Appendix C. It is worth remarking, however, that another approach was used classically by Hermann Weyl; this is the famous *unitary trick*, which we will describe briefly.

A Digression on "The Unitary Trick"

Basically, the idea is that the statements above (complete reducibility, preservation of Jordan decomposition) can be proved readily for the representations of a compact Lie group. To prove complete reducibility, for example, we can proceed more or less just as in the case of a finite group: if the compact group G acts on a vector space, we see that there is a Hermitian metric on V invariant under the action of G by taking an arbitrary metric on V and averaging its images under the action of G. If G fixes a subspace $W \subset V$, it will then fix as well its orthogonal complement $W^{\perp}$ with respect to this metric. (Alternatively, we can choose an arbitrary complement W' to W, not necessarily fixed by G, and average over G the projection map to $g(W')$ with kernel W; this average will have image invariant under G.)

How does this help us analyze the representation of a semisimple Lie algebra? The key fact here (to be proved in Lecture 26) is that *if* $\mathfrak{g}$ *is any complex semisimple Lie algebra, there exists a* (*unique*) *real Lie algebra* $\mathfrak{g}_0$ *with complexification* $\mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{g}$, *such that the simply connected form of the Lie algebra* $\mathfrak{g}_0$ *is a compact Lie group* G. Thus, restricting a given representation of $\mathfrak{g}$ to $\mathfrak{g}_0$, we can exponentiate to obtain a representation of G, for which complete reducibility holds; and we can deduce from this the complete reducibility of the original representation. For example, while it is certainly not true that any representation ρ of the Lie group $SL_n\mathbb{R}$ on a vector space V admits an invariant Hermitian metric (in fact, it cannot, unless it is the trivial representation), we can

(i) let ρ' be the corresponding (complex) representation of the Lie algebra $\mathfrak{sl}_n\mathbb{R}$;
(ii) by linearity extend the representation ρ' of $\mathfrak{sl}_n\mathbb{R}$ to a representation ρ'' of $\mathfrak{sl}_n\mathbb{C}$;
(iii) restrict to a representation ρ''' of the subalgebra $\mathfrak{su}_n \subset \mathfrak{sl}_n\mathbb{C}$;
(iv) exponentiate to obtain a representation ρ'''' of the unitary group SU_n.

We can now argue that

If a subspace $W \subset V$ is invariant under the action of $SL_n\mathbb{R}$,

it must be invariant under $\mathfrak{sl}_n\mathbb{R}$; and since $\mathfrak{sl}_n\mathbb{C} = \mathfrak{sl}_n\mathbb{R} \otimes \mathbb{C}$, it follows that
it will be invariant under $\mathfrak{sl}_n\mathbb{C}$; so of course
it will be invariant under $\mathfrak{su}_n$; and hence
it will be invariant under SU_n.

Now, since SU_n is compact, there will exist a complementary subspace W' preserved by SU_n; we argue that

W' will then be invariant under $\mathfrak{su}_n$; and since $\mathfrak{sl}_n\mathbb{C} = \mathfrak{su}_n \otimes \mathbb{C}$, it follows that
it will be invariant under $\mathfrak{sl}_n\mathbb{C}$. Restricting, we see that
it will be invariant under $\mathfrak{sl}_n\mathbb{R}$, and exponentiating,
it will be invariant under $SL_n\mathbb{R}$.

Similarly, if one wants to know that the diagonal elements of $SL_n\mathbb{R}$ act semisimply in any representation, or equivalently that the diagonal elements of $\mathfrak{sl}_n\mathbb{R}$ act semisimply, one goes through the same reasoning, coming down to the fact that the group of diagonal elements in $\mathfrak{su}_n$ is abelian and *compact*.

In general, most of the theorems about the finite-dimensional representation of semisimple Lie algebras admit proofs along two different lines: either algebraically, using just the structure of the Lie algebra; or by the unitary trick, that is, by associating to a representation of such a Lie algebra a representation of a compact Lie group and working with that. Which is preferable depends very much on taste and context; in this book we will for the most part go with the algebraic proofs, though in the case of the Weyl character formula in Part IV the proof via compact groups is so much more appealing it has to be mentioned.

The following exercises include a few applications of these two theorems.

Exercise 9.21*. Show that a Lie algebra $\mathfrak{g}$ is semisimple if and only if every finite-dimensional representation is semisimple, i.e., every invariant subspace has a complement.

Exercise 9.22. Use Weyl's unitary trick to show that, for $n > 2$, all representations of $SO_n\mathbb{C}$ are semisimple, so that, in particular, the Lie algebras $\mathfrak{so}_n\mathbb{C}$ are semisimple. Do the same for $Sp_{2s}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$, $n \geq 1$. Where does the argument break down for $SO_2\mathbb{C}$?

Exercise 9.23. Show that a real Lie algebra $\mathfrak{g}$ is solvable if and only if the complex Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is solvable. Similarly for nilpotent and semisimple.

Exercise 9.24*. If $\mathfrak{h}$ is an ideal in a Lie algebra $\mathfrak{g}$, show that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are semisimple. Deduce that every semisimple Lie algebra is a direct sum of simple Lie algebras.

Exercise 9.25*. A Lie algebra is called *reductive* if its radical is equal to its center. A Lie group is reductive if its Lie algebra is reductive. For example, $GL_n\mathbb{C}$ is reductive. Show that the following are true for a reductive Lie algebra $\mathfrak{g}$: (i) $\mathscr{D}\mathfrak{g}$ is semisimple; (ii) the adjoint representation of $\mathfrak{g}$ is semisimple; (iii) $\mathfrak{g}$ is a product of a semisimple and an abelian Lie algebra; (iv) $\mathfrak{g}$ has a finite-dimensional faithful semisimple representation. In fact, each of these conditions is equivalent to $\mathfrak{g}$ being reductive.

§9.4. Simple Lie Algebras

There is one more basic fact about Lie algebras to be stated here; though its proof will have to be considerably deferred, it informs our whole approach to the subject. This is the complete classification of simple Lie algebras:

Theorem 9.26. *With five exceptions, every simple complex Lie algebra is isomorphic to either $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, or $\mathfrak{sp}_{2n}\mathbb{C}$ for some n.*

The five exceptions can all be explicitly described, though none is particularly simple except in name; they are denoted $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$. We will give a construction of each later in the book (§22.3). The algebras $\mathfrak{sl}_n\mathbb{C}$ (for $n > 1$), $\mathfrak{so}_n\mathbb{C}$ (for $n > 2$), and $\mathfrak{sp}_{2n}\mathbb{C}$ are commonly called the *classical Lie algebras* (and the corresponding groups the *classical Lie groups*); the other five algebras are called, naturally enough, the *exceptional Lie algebras.*

The nature of the classification theorem for simple Lie algebras creates a dilemma as to how we approach the subject: many of the theorems about simple Lie algebras can be proved either in the abstract, or by verifying them in turn for each of the particular algebras listed in the classification theorem. Another alternative is to declare that we are concerned with understanding only the representations of the classical algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, and $\mathfrak{sp}_{2n}\mathbb{C}$, and verify any relevant theorems just in these cases.

Of these three approaches, the last is in many ways the least satisfactory; it is, however, the one that we shall for the most part take. Specifically, what we will do, starting in Lecture 11, is the following:

(i) Analyze in Lectures 11–13 a couple of examples, namely, $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$, on what may appear to be an ad hoc basis.
(ii) On the basis of these examples, propose in Lecture 14 a general paradigm for the study of representations of a simple (or semisimple) Lie algebra.
(iii) Proceed in Lectures 15–20 to carry out this analysis for the classical algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, and $\mathfrak{sp}_{2n}\mathbb{C}$.
(iv) Give in Part IV and the appendices proofs for general simple Lie algebras of the facts discovered in the preceding sections for the classical ones (as well as one further important result, the *Weyl character formula*).

We can at least partially justify this seemingly inefficient approach by saying that even if one makes a beeline for the general theorems about the structure and representation theory of a simple Lie algebra, to apply these results in practice we would still need to carry out the sort of explicit analysis of the individual algebras done in Lectures 11–20. This is, however, a fairly bald rationalization: the fact is, the reason we are doing it this way is that this is the only way we have ever been able to understand any of the general results.