LECTURE 8

# Lie Algebras and Lie Groups

In this crucial lecture we introduce the definition of the Lie algebra associated to a Lie group and its relation to that group. All three sections are logically necessary for what follows; §8.1 is essential. We use here a little more manifold theory: specifically, the differential of a map of manifolds is used in a fundamental way in §8.1, the notion of the tangent vector to an arc in a manifold is used in §8.2 and §8.3, and the notion of a vector field is introduced in an auxiliary capacity in §8.3. The Campbell-Hausdorff formula is introduced only to establish the First and Second Principles of §8.1 below; if you are willing to take those on faith the formula (and exercises dealing with it) can be skimmed. Exercises 8.27–8.29 give alternative descriptions of the Lie algebra associated to a Lie group, but can be skipped for now.

§8.1:  Lie algebras: motivation and definition
§8.2:  Examples of Lie algebras
§8.3:  The exponential map

## §8.1.  Lie Algebras: Motivation and Definition

Given that we want to study the representations of a Lie group, how do we go about it? As we have said, the notions of generators and relations is hardly relevant here. The answer, of course, is that we have to use the continuous structure of the group. The first step in doing this is

**Exercise 8.1.** Let $G$ be a connected Lie group, and $U \subset G$ any neighborhood of the identity. Show that $U$ generates $G$.

   This statement implies that any map $\rho: G \to H$ between connected Lie groups will be determined by what it does on any open set containing the

identity in $G$, i.e., $\rho$ is determined by its germ at $e \in G$. In fact, we can extend this idea a good bit further: later in this lecture we will establish the

*First Principle*: Let $G$ and $H$ be Lie groups, with $G$ connected. A map $\rho: G \to H$ is uniquely determined by its differential $d\rho_e: T_eG \to T_eH$ at the identity.

This is, of course, great news: we can completely describe a homomorphism of Lie groups by giving a linear map between two vector spaces. It is not really worth that much, however, unless we can give at least some answer to the next, obvious question: *which maps between these two vector spaces actually arise as differentials of group homomorphisms?* The answer to this is expressed in the *Second Principle* below, but it will take us a few pages to get there. To start, we have to ask ourselves what it means for a map to be a homomorphism, and in what ways this may be reflected in the differential.

To begin with, the definition of a homomorphism is simply a $\mathscr{C}^\infty$ map $\rho$ such that

$$\rho(gh) = \rho(g) \cdot \rho(h)$$

for all $g$ and $h$ in $G$. To express this in a more confusing way, we can say that *a homomorphism respects the action of a group on itself by left or right multiplication*: that is, for any $g \in G$ we denote by $m_g: G \to G$ the differentiable map given by multiplication by $g$, and observe that a $\mathscr{C}^\infty$ map $\rho: G \to H$ of Lie groups will be a homomorphism if it carries $m_g$ to $m_{\rho(g)}$ in the sense that the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\ \rho\ } & H \\
{\scriptstyle m_g}\Big\downarrow & & \Big\downarrow{\scriptstyle m_{\rho(g)}} \\
G & \xrightarrow[\ \rho\ ]{} & H
\end{array}
$$

commutes.

The problem with this characterization is that, since the maps $m_g$ have no fixed points, it is hard to associate to them any operation on the tangent space to $G$ at one point. This suggests looking, not at the diffeomorphisms $m_g$, but at the automorphisms of $G$ given by conjugation. Explicitly, for any $g \in G$ we define the map

$$\Psi_g: G \to G$$

by

$$\Psi_g(h) = g \cdot h \cdot g^{-1}.$$

($\Psi_g$ is actually a Lie group map, but that is not relevant for our present purposes.) It is now equally the case that *a homomorphism $\rho$ respects the action of a group $G$ on itself by conjugation*: that is, it will carry $\Psi_g$ into $\Psi_{\rho(g)}$ in the sense that the diagram

$$\begin{array}{ccc}
G & \xrightarrow{\ \rho\ } & H \\
\Psi_g \downarrow & & \downarrow \Psi_{\rho(g)} \\
G & \xrightarrow[\ \rho\ ]{} & H
\end{array}$$

commutes. We have, in other words, a natural map

$$\Psi \colon G \to \mathrm{Aut}(G).$$

The advantage of working with $\Psi_g$ is that it fixes the identity element $e \in G$; we can therefore extract some of its structure by looking at its differential at $e$: we set

$$\mathrm{Ad}(g) = (d\Psi_g)_e \colon T_e G \to T_e G. \tag{8.2}$$

This is a representation

$$\mathrm{Ad} \colon G \to \mathrm{Aut}(T_e G) \tag{8.3}$$

of the group $G$ on its own tangent space, called the *adjoint representation* of the group. This gives a third characterization[1]: *a homomorphism $\rho$ respects the adjoint action of a group $G$ on its tangent space $T_e G$ at the identity.* In other words, for any $g \in G$ the actions of $\mathrm{Ad}(g)$ on $T_e G$ and $\mathrm{Ad}(\rho(g))$ on $T_e H$ must commute with the differential $(d\rho)_e \colon T_e G \to T_e H$, i.e., the diagram

$$\begin{array}{ccc}
T_e G & \xrightarrow{\ (d\rho)_e\ } & T_e H \\
\mathrm{Ad}(g) \downarrow & & \downarrow \mathrm{Ad}(\rho(g)) \\
T_e G & \xrightarrow[\ (d\rho)_e\ ]{} & T_e H
\end{array}$$

commutes; equivalently, for any tangent vector $v \in T_e G$,

$$d\rho(\mathrm{Ad}(g)(v)) = \mathrm{Ad}(\rho(g))(d\rho(v)). \tag{8.4}$$

This is nice, but does not yet answer our question, for preservation of the adjoint representation $\mathrm{Ad} \colon G \to \mathrm{Aut}(T_e G)$ still involves the map $\rho$ on the group $G$ itself, and so is not purely a condition on the differential $(d\rho)_e$. We have instead to go one step further, and *take the differential of the map* $\mathrm{Ad}$. The group $\mathrm{Aut}(T_e G)$ being just an open subset of the vector space of endomorphisms of $T_e G$, its tangent space at the identity is naturally identified with $\mathrm{End}(T_e G)$; taking the differential of the map $\mathrm{Ad}$ we arrive at a map

$$\mathrm{ad} \colon T_e G \to \mathrm{End}(T_e G). \tag{8.5}$$

This is essentially a trilinear gadget on the tangent space $T_e G$; that is, we can view the image $\mathrm{ad}(X)(Y)$ of a tangent vector $Y$ under the map $\mathrm{ad}(X)$ as a

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[1] "Characterization" is not the right word here (or in the preceding case), since we do not mean an equivalent condition, but rather something implied by the condition that $\rho$ be a homomorphism.

function of the two variables $X$ and $Y$, so that we get a bilinear map

$$T_e G \times T_e G \to T_e G.$$

We use the notation $[\ ,\ ]$ for this bilinear map; that is, for a pair of tangent vectors $X$ and $Y$ to $G$ at $e$, we write

$$[X, Y] \stackrel{\text{def}}{=} \mathrm{ad}(X)(Y). \tag{8.6}$$

As desired, the map ad involves only the tangent space to the group $G$ at $e$, and so gives us our final characterization: *the differential $(d\rho)_e$ of a homomorphism $\rho$ on a Lie group $G$ respects the adjoint action of the tangent space to $G$ on itself.* Explicitly, the fact that $\rho$ and $d\rho_e$ respect the adjoint representation implies in turn that the diagram

$$\begin{array}{ccc}
T_e G & \xrightarrow{\ (d\rho)_e\ } & T_e H \\
{\scriptstyle \mathrm{ad}(v)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{ad}(d\rho(v))} \\
T_e G & \xrightarrow[\ (d\rho)_e\ ]{} & T_e H
\end{array}$$

commutes; i.e., for any pair of tangent vectors $X$ and $Y$ to $G$ at $e$,

$$d\rho_e(\mathrm{ad}(X)(Y)) = \mathrm{ad}(d\rho_e(X))(d\rho_e(Y)). \tag{8.7}$$

or, equivalently,

$$d\rho_e([X, Y]) = [d\rho_e(X), d\rho_e(Y)]. \tag{8.8}$$

All this may be fairly confusing (if it is not, you probably do not need to be reading this book). Two things, however, should be borne in mind. They are:

(i) *It is not so bad*, in the sense that we can make the bracket operation, as defined above, reasonably explicit. We do this first for the general linear group $G = \mathrm{GL}_n \mathbb{R}$. Note that in this case conjugation extends to the ambient linear space $E = \mathrm{End}(\mathbb{R}^n) = M_n \mathbb{R}$ of $\mathrm{GL}_n \mathbb{R}$ by the same formula: $\mathrm{Ad}(g)(M) = gMg^{-1}$, and this ambient space is identified with the tangent space $T_e G$; differentiation in $E$ is usual differentiation of matrices. For any pair of tangent vectors $X$ and $Y$ to $\mathrm{GL}_n \mathbb{R}$ at $e$, let $\gamma: I \to G$ be an arc with $\gamma(0) = e$ and tangent vector $\gamma'(0) = X$. Then our definition of $[X, Y]$ is that

$$[X, Y] = \mathrm{ad}(X)(Y) = \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}(\gamma(t))(Y)).$$

Applying the product rule to $\mathrm{Ad}(\gamma(t))(Y) = \gamma(t) Y \gamma(t)^{-1}$, this is

$$= \gamma'(0) \cdot Y \cdot \gamma(0) + \gamma(0) \cdot Y \cdot (-\gamma(0)^{-1} \cdot \gamma'(0) \cdot \gamma(0)^{-1})$$

$$= X \cdot Y - Y \cdot X,$$

which, of course, explains the bracket notation. In general, any time a Lie group is given as a subgroup of a general linear group $\mathrm{GL}_n \mathbb{R}$, we can view its

tangent space $T_e G$ at the identity as a subspace of the space of endomorphisms of $\mathbb{R}^n$; and since bracket is preserved by (differentials of) maps of Lie groups, the bracket operation on $T_e G$ will coincide with the commutator.

(ii) *Even if it were that bad, it would be worth it.* This is because it turns out that the bracket operation is exactly the answer to the question we raised before. Precisely, later in this lecture we will prove the

*Second Principle*: Let $G$ and $H$ be Lie groups, with $G$ connected and simply connected. A linear map $T_e G \to T_e H$ is the differential of a homomorphism $\rho: G \to H$ if and only if it preserves the bracket operation, in the sense of (8.8) above.

We are now almost done: maps between Lie groups are classified by maps between vector spaces preserving the structure of a bilinear map from the vector space to itself. We have only one more question to answer: when does a vector space with this additional structure actually arise as the tangent space at the identity to a Lie group, with the adjoint or bracket product? Happily, we have the answer to this as well. First, though it is far from clear from our initial definition, it follows from our description of the bracket as a commutator that *the bracket is skew-symmetric*, i.e, $[X, Y] = -[Y, X]$. Second, it likewise follows from the description of $[X, Y]$ as a commutator that it satisfies the *Jacobi identity*: for any three tangent vectors $X$, $Y$, and $Z$,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We thus make the

**Definition 8.9.** A *Lie algebra* $\mathfrak{g}$ is a vector space together with a skew-symmetric bilinear map

$$[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity.

We should take a moment out here to make one important point. Why, you might ask, do we define the bracket operation in terms of the relatively difficult operations Ad and ad, instead of just defining $[X, Y]$ to be the commutator $X \cdot Y - Y \cdot X$? The answer is that the "composition" $X \cdot Y$ of elements of a Lie algebra is not well defined. Specifically, any time we embed a Lie group $G$ in a general linear group $GL(V)$, we get a corresponding embedding of its Lie algebra $\mathfrak{g}$ in the space $\text{End}(V)$, and can talk about the composition $X \cdot Y \in \text{End}(V)$ of elements of $\mathfrak{g}$ in this context; but it must be borne in mind that this composition $X \cdot Y$ will depend on the embedding of $\mathfrak{g}$, and for that matter need not even be an element of $\mathfrak{g}$. *Only* the commutator $X \cdot Y - Y \cdot X$ is always an element of $\mathfrak{g}$, independent of the representation. The terminology sometimes heightens the confusion: for example, when we speak of embedding a Lie algebra in the algebra $\text{End}(V)$ of endomorphisms of $V$, the word *algebra* may mean two very different things. In general, when we want

to refer to the endomorphisms of a vector space $V$ (resp. $\mathbb{R}^n$) as a Lie algebra, we will write $\mathfrak{gl}(V)$ (resp. $\mathfrak{gl}_n\mathbb{R}$) instead of $\mathrm{End}(V)$ (resp. $M_n\mathbb{R}$).

To return to our discussion of Lie algebras, a *map* of Lie algebras is a linear map of vector spaces preserving the bracket, in the sense of (8.8); notions like *Lie subalgebra* are defined accordingly. We note in passing one thing that will turn out to be significant: the definition of Lie algebra does not specify the field. Thus, we have real Lie algebras, complex Lie algebras, etc., all defined in the same way; and in addition, given a real Lie algebra $\mathfrak{g}$ we may associate to it a complex Lie algebra, whose underlying vector space is $\mathfrak{g} \otimes \mathbb{C}$ and whose bracket operation is just the bracket on $\mathfrak{g}$ extended by linearity.

**Exercise 8.10\*.** The skew-commutativity and Jacobi identity also follow from the naturality of the bracket (8.8), without using an embedding in $\mathfrak{gl}(V)$:

(a) Deduce the skew-commutativity $[X, X] = 0$ from that fact that any $X$ can be written the image of a vector by $d\rho_e$ for some homomorphism $\rho\colon \mathbb{R} \to G$. (See §8.3 for the existence of $\rho$.)
(b) Given that the bracket is skew-commutative, verify that the Jacobi identity is equivalent to the assertion that

$$\mathrm{ad} = d(\mathrm{Ad})_e\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

preserves the bracket. In particular, ad is a map of Lie algebras.

To sum up our progress so far: taking for the moment on faith the statements made, we have seen that

(i) the tangent space $\mathfrak{g}$ at the identity to a Lie group $G$ is naturally endowed with the structure of a Lie algebra;
(ii) if $G$ and $H$ are Lie groups with $G$ connected and simply connected, the maps from $G$ to $H$ are in one-to-one correspondence with maps of the associated Lie algebras, by associating to $\rho\colon G \to H$ its differential $(d\rho)_e\colon \mathfrak{g} \to \mathfrak{h}$.

Of course, we make the

**Definition 8.11.** A *representation* of a Lie algebra $\mathfrak{g}$ on a vector space $V$ is simply a map of Lie algebras

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}(V),$$

i.e., a linear map that preserves brackets, or an action of $\mathfrak{g}$ on $V$ such that

$$[X, Y](v) = X(Y(v)) - Y(X(v)).$$

Statement (ii) above implies in particular that *representations of a connected and simply connected Lie group are in one-to-one correspondence with repre-*

*sentations of its Lie algebra.* This is, then, the first step of the series of reductions outlined in the introduction to Part II.

At this point, a few words are in order about the relation between representations of a Lie group and the corresponding representations of its Lie algebra. The first remark to make is about tensors. Recall that if $V$ and $W$ are representations of a Lie group $G$, then we define the representation $V \otimes W$ to be the vector space $V \otimes W$ with the action of $G$ described by

$$g(v \otimes w) = g(v) \otimes g(w).$$

The definition for representations of a Lie algebra, however, is quite different. For one thing, if $\mathfrak{g}$ is the Lie algebra of $G$, so that the representation of $G$ on the vector spaces $V$ and $W$ induces representations of $\mathfrak{g}$ on these spaces, we want the tensor product of the representations $V$ and $W$ of $\mathfrak{g}$ to be the representation induced by the action of $G$ on $V \otimes W$ above. But now suppose that $\{\gamma_t\}$ is an arc in $G$ with $\gamma_0 = e$ and tangent vector $\gamma_0' = X \in \mathfrak{g}$. Then by definition the action of $X$ on $V$ is given by

$$X(v) = \frac{d}{dt}\bigg|_{t=0} \gamma_t(v)$$

and similarly for $w \in W$; it follows that the action of $X$ on the tensor product $v \otimes w$ is

$$X(v \otimes w) = \frac{d}{dt}\bigg|_{t=0} (\gamma_t(v) \otimes \gamma_t(w))$$

$$= \left(\frac{d}{dt}\bigg|_{t=0} \gamma_t(v)\right) \otimes w + v \otimes \left(\frac{d}{dt}\bigg|_{t=0} \gamma_t(w)\right),$$

so

$$X(v \otimes w) = X(v) \otimes w + v \otimes X(w). \tag{8.12}$$

This, then, is how we *define* the action of a Lie algebra $\mathfrak{g}$ on the tensor product of two representations of $\mathfrak{g}$. This describes as well other tensors: for example, if $V$ is a representation of the group $G$, $v \in V$ is any vector and $v^2 \in \mathrm{Sym}^2 V$ its square, then for any $g \in G$,

$$g(v^2) = g(v)^2.$$

On the other hand, if $V$ is a representation of the Lie algebra $\mathfrak{g}$ and $X \in \mathfrak{g}$ is any element, we have

$$X(v^2) = 2 \cdot v \cdot X(v). \tag{8.13}$$

One further example: if $\rho: G \to \mathrm{GL}(V)$ is a representation of the group $G$, the dual representation $\rho': G \to \mathrm{GL}(V^*)$ is defined by setting

$$\rho'(g) = {}^t\rho(g^{-1}): V^* \to V^*.$$

Differentiating this, we find that if $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of a Lie

algebra $\mathfrak{g}$, the dual representation of $\mathfrak{g}$ on $V^*$ will be given by

$$\rho'(X) = {}^t\rho(-X) = -{}^t\rho(X)\colon V^* \to V^*. \tag{8.14}$$

A second and related point to be made concerns terminology. Obviously, when we speak of the action of a group $G$ on a vector space $V$ preserving some extra structure on $V$, we mean that literally: for example, if we have a quadratic form $Q$ on $V$, to say that $G$ preserves $Q$ means just that

$$Q(g(v), g(w)) = Q(v, w), \quad \forall g \in G \text{ and } v, w \in V.$$

Equivalently, we mean that the associated action of $G$ on the vector space $\text{Sym}^2 V^*$ fixes the element $Q \in \text{Sym}^2 V^*$. But by the above calculation, the action of the associated Lie algebra $\mathfrak{g}$ on $V$ satisfies

$$Q(v, X(w)) + Q(X(v), w) = 0, \quad \forall X \in \mathfrak{g} \text{ and } v, w \in V \tag{8.15}$$

or, equivalently, $Q(v, X(v)) = 0$ for all $X \in \mathfrak{g}$ and $v \in V$; in other words, *the induced action on $\text{Sym}^2 V^*$ kills the element $Q$.* By way of terminology, then, *we will in general say that the action of a Lie algebra on a vector space preserves some structure when a corresponding Lie group action does.*

The next section will be spent in giving examples. In §8.3 we will establish the basic relations between Lie groups and their Lie algebras, to the point where we can prove the First and Second Principles above. The further statement that any Lie algebra is the Lie algebra of some Lie group will follow from the statement (see Appendix E) that every Lie algebra may be embedded in $\mathfrak{gl}_n\mathbb{R}$.

**Exercise 8.16\*.** Show that if $G$ is connected the image of $\text{Ad}\colon G \to \text{GL}(\mathfrak{g})$ is the adjoint form of the group $G$ when that exists.

**Exercise 8.17\*.** Let $V$ be a representation of a connected Lie group $G$ and $\rho\colon \mathfrak{g} \to \text{End}(V)$ the corresponding map of Lie algebras. Show that a subspace $W$ of $V$ is invariant by $G$ if and only if it is carried into itself under the action of the Lie algebra $\mathfrak{g}$, i.e., $\rho(X)(W) \subset W$ for all $X$ in $\mathfrak{g}$. Hence, $V$ is irreducible over $G$ if and only if it is irreducible over $\mathfrak{g}$.

## §8.2. Examples of Lie Algebras

We start with the Lie algebras associated to each of the groups mentioned in Lecture 7. Each of these groups is given as a subgroup of $\text{GL}(V) = \text{GL}_n\mathbb{R}$, so their Lie algebras will be subspaces of $\text{End}(V) = \mathfrak{gl}_n\mathbb{R}$.

Consider first the special linear group $\text{SL}_n\mathbb{R}$. If $\{A_t\}$ is an arc in $\text{SL}_n\mathbb{R}$ with $A_0 = I$ and tangent vector $A'_0 = X$ at $t = 0$, then by definition we have for any basis $e_1, \ldots, e_n$ of $V = \mathbb{R}^n$,

$$A_t(e_1) \wedge \cdots \wedge A_t(e_n) \equiv e_1 \wedge \cdots \wedge e_n.$$

Taking the derivative and evaluating at $t = 0$ we have by the product rule

$$0 = \frac{d}{dt}\bigg|_{t=0} (A_t(e_1) \wedge \cdots \wedge A_t(e_n))$$

$$= \sum e_1 \wedge \cdots \wedge X(e_i) \wedge \cdots \wedge e_n$$

$$= \text{Trace}(X) \cdot (e_1 \wedge \cdots \wedge e_n).$$

The tangent vectors to $\text{SL}_n\mathbb{R}$ are thus all endomorphisms of trace 0; comparing dimensions we can see that the Lie algebra $\mathfrak{sl}_n\mathbb{R}$ is exactly the vector space of traceless $n \times n$ matrices.

The orthogonal and symplectic cases are somewhat simpler. For example, the orthogonal group $\text{O}_n\mathbb{R}$ is defined to be the automorphisms $A$ of an $n$-dimensional vector space $V$ preserving a quadratic form $Q$, so that if $\{A_t\}$ is an arc in $\text{O}_n\mathbb{R}$ with $A_0 = I$ and $A'_0 = X$ we have for every pair of vectors $v$, $w \in V$

$$Q(A_t(v), A_t(w)) \equiv Q(v, w).$$

Taking derivatives, we see that

$$Q(X(v), w) + Q(v, X(w)) = 0 \tag{8.18}$$

for all $v, w \in V$; this is exactly the condition that describes the orthogonal Lie algebra $\mathfrak{so}_n\mathbb{R} = \mathfrak{o}_n\mathbb{R}$. In coordinates, if the quadratic form $Q$ is given on $V = \mathbb{R}^n$ as

$$Q(v, w) = {}^t v \cdot M \cdot w \tag{8.19}$$

for some symmetric $n \times n$ matrix $M$, then as we have seen the condition on $A \in \text{GL}_n\mathbb{R}$ to be in $\text{O}_n\mathbb{R}$ is that

$$^t A \cdot M \cdot A = M. \tag{8.20}$$

Differentiating, the condition on an $n \times n$ matrix $X$ to be in the Lie algebra $\mathfrak{so}_n\mathbb{R}$ of the orthogonal group is that

$$^t X \cdot M + M \cdot X = 0. \tag{8.21}$$

Note that if $M$ is the identity matrix—i.e., $Q$ is the "standard" quadratic form $Q(v, w) = {}^t v \cdot w$ on $\mathbb{R}^n$—then this says that $\mathfrak{so}_n\mathbb{R}$ *is the subspace of skew-symmetric $n \times n$ matrices*. To put it intrinsically, in terms of the identification of $V$ with $V^*$ given by the quadratic form $Q$, and the consequent identification $\text{End}(V) = V \otimes V^* = V \otimes V$, the Lie algebra $\mathfrak{so}_n\mathbb{R} \subset \text{End}(V)$ is just the subspace $\wedge^2 V \subset V \otimes V$ of skew-symmetric tensors:

$$\mathfrak{so}_n\mathbb{R} = \wedge^2 V \subset \text{End}(V) = V \otimes V. \tag{8.22}$$

All of the above, with the exception of the last paragraph, works equally well to describe the Lie algebra $\mathfrak{sp}_{2n}\mathbb{R}$ of the Lie group $\text{Sp}_{2n}\mathbb{R}$ of transformations preserving a skew-symmetric bilinear form $Q$; that is, $\mathfrak{sp}_{2n}\mathbb{R}$ is the subspace of endomorphisms of $V$ satisfying (8.18) for every pair of vectors $v$, $w \in V$, or, if $Q$ is given by a skew-symmetric $2n \times 2n$ matrix $M$ as in (8.19), the

space of matrices satisfying (8.21). The one statement that has to be substantially modified is the last one of the last paragraph: because $Q$ is skew-symmetric, condition (8.18) is equivalent to saying that

$$Q(X(v), w) = Q(X(w), v)$$

for all $v, w \in V$; thus, in terms of the identification of $V$ with $V^*$ given by $Q$, the Lie algebra $\mathfrak{sp}_{2n}\mathbb{R} \subset \text{End}(V) = V \otimes V^* = V \otimes V$ is the subspace $\text{Sym}^2 V \subset V \otimes V$:

$$\mathfrak{sp}_{2n}\mathbb{R} = \text{Sym}^2 V \subset \text{End}(V) = V \otimes V. \tag{8.23}$$

**Exercise 8.24\*.** With $Q$ a standard skew form, say of Exercise 7.3, describe $\text{Sp}_{2n}\mathbb{R}$ and its Lie algebra $\mathfrak{sp}_{2n}\mathbb{R}$ (as subgroup of $\text{GL}_{2n}\mathbb{R}$ and subalgebra of $\mathfrak{gl}_{2n}\mathbb{R}$). Do a corresponding calculation for $\text{SO}_{k,l}\mathbb{R}$.

One more similar example is that of the Lie algebra $\mathfrak{u}_n$ of the unitary group $U(n)$; by a similar calculation we find that the Lie algebra of complex linear endomorphisms of $\mathbb{C}^n$ preserving a Hermitian inner product $H$ is just the space of matrices $X$ satisfying

$$H(X(v), w) + H(v, X(w)) = 0, \quad \forall v, w \in V;$$

if $H$ is given by $H(v, w) = {}^t\bar{v} \cdot w$, this amounts to saying that $X$ is conjugate skew-symmetric, i.e., that ${}^t\bar{X} = -X$.

**Exercise 8.25.** Find the Lie algebras of the real Lie groups $\text{SL}_n\mathbb{C}$ and $\text{SL}_n\mathbb{H}$— the elements in $\text{GL}_n\mathbb{H}$ whose real determinant is 1.

**Exercise 8.26.** Show that the Lie algebras of the Lie groups $B_n$ and $N_n$ introduced in §7.2 are the algebra $\mathfrak{b}_n\mathbb{R}$ of upper triangular $n \times n$ matrices and the algebra $\mathfrak{n}_n\mathbb{R}$ of strictly upper triangular $n \times n$ matrices, respectively.

If $G$ is a complex Lie group, its Lie algebra is a complex Lie algebra. Just as in the real case, we have the complex Lie algebras $\mathfrak{gl}_n\mathbb{C}$, $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_m\mathbb{C}$, and $\mathfrak{sp}_{2n}\mathbb{C}$ of the Lie groups $\text{GL}_n\mathbb{C}$, $\text{SL}_n\mathbb{C}$, $\text{SO}_m\mathbb{C}$, and $\text{Sp}_{2n}\mathbb{C}$.

**Exercise 8.27.** Let $A$ be any (real or complex) algebra, not necessarily finite dimensional, or even associative. A *derivation* is a linear map $D: A \to A$ satisfying the Leibnitz rule $D(ab) = aD(b) + D(a)b$.

(a) Show that the derivations $\text{Der}(A)$ form a Lie algebra under the bracket $[D, E] = D \circ E - E \circ D$. If $A$ is finite dimensional, so is $\text{Der}(A)$.
(b) The group of automorphisms of $A$ is a closed subgroup $G$ of the group $\text{GL}(A)$ of linear automorphisms of $A$. Show that the Lie algebra of $G$ is $\text{Der}(A)$.
(c) If the algebra $A$ is a Lie algebra, the map $A \to \text{Der}(A)$, $X \mapsto D_X$, where $D_X(Y) = [X, Y]$, is a map of Lie algebras.

**Exercise 8.28\*.** If g is a Lie algebra, the Lie algebra automorphisms of g form a Lie subgroup Aut(g) of the general linear group GL(g).

(a) Show that the Lie algebra of Aut(g) is Der(g). If $G$ is a simply connected Lie group with Lie algebra g, the map Aut($G$) → Aut(g) by $\varphi \mapsto d\varphi$ is one-to-one and onto, giving Aut($G$) the structure of a Lie group with Lie algebra Der(g).
(b) Show that the automorphism group of any connected Lie group is a Lie subgroup of the automorphism group of its Lie algebra.

**Exercise 8.29\*.** For any manifold $M$, the $C^\infty$ vector fields on $M$ form a Lie algebra $\mathfrak{v}(M)$, as follows: a vector field $v$ can be identified with a derivation of the ring $A$ of $C^\infty$ functions on $M$, with $v(f)$ the function whose value at a point $x$ of $M$ is the value of the tangent vector $v_x$ on $f$ at $x$. Show that the vector fields on $M$ form a Lie algebra, in fact a Lie subalgebra of the Lie algebra Der($A$). If a Lie group $G$ acts on $M$, the $G$-invariant vector fields form a Lie subalgebra $\mathfrak{v}_G M$ of $\mathfrak{v}(M)$. If the action is transitive, the invariant vector fields form a finite-dimensional Lie algebra.

If $G$ is a Lie group, $\mathfrak{v}_G(G) = T_e G$ becomes a Lie algebra by the above process. Show that this bracket agrees with that defined using the adjoint map (8.6). This gives another proof that the bracket is skew-symmetric and satisfies Jacobi's identity.

# §8.3.  The Exponential Map

The essential ingredient in studying the relationship between a Lie group $G$ and its Lie algebra g is the exponential map. This may be defined in very straightforward fashion, using the notion of *one-parameter subgroups*, which we study next. Suppose that $X \in$ g is any element, viewed simply as a tangent vector to $G$ at the identity. For any element $g \in G$, denote by $m_g \colon G \to G$ the map of manifolds given by multiplication on the left by $g$. Then we can define a vector field $v_X$ on all of $G$ simply by setting

$$v_X(g) = (m_g)_*(X).$$

This vector field is clearly invariant under left translation (i.e., it is carried into itself under the diffeomorphism $m_g$ for all $g$); and it is not hard to see that this gives an identification of g with the space of all left-invariant vector fields on $G$. Under this identification, the bracket operation on the Lie algebra g corresponds to Lie bracket of vector fields; indeed, this may be adopted as the definition of the Lie algebra associated to a Lie group (cf. Exercise 8.29). For our present purposes, however, all we need to know is that $v_X$ exists and is left-invariant.

Given any vector field $v$ on a manifold $M$ and a point $p \in M$, a basic theorem from differential equations allows us to integrate the vector field. This

gives a differentiable map $\varphi: I \to M$, defined on some open interval $I$ containing 0, with $\varphi(0) = p$, whose tangent vector at any point is the vector assigned to that point by $v$, i.e., such that

$$\varphi'(t) = v(\varphi(t))$$

for all $t$ in $I$. The map $\varphi$ is uniquely characterized by these properties. Now suppose the manifold in question is a Lie group $G$, the vector field the field $v_X$ associated to an element $X \in \mathfrak{g}$, and $p$ the identity. We arrive then at a map $\varphi: I \to G$; we claim that, at least where $\varphi$ is defined, it is a *homomorphism*, i.e., $\varphi(s + t) = \varphi(s)\varphi(t)$ whenever $s$, $t$, and $s + t$ are in $I$. To prove this, fix $s$ and let $t$ vary; that is, consider the two arcs $\alpha$ and $\beta$ given by $\alpha(t) = \varphi(s) \cdot \varphi(t)$ and $\beta(t) = \varphi(s + t)$. Of course, $\alpha(0) = \beta(0)$; and by the invariance of the vector field $v_X$, we see that the tangent vectors satisfy $\alpha'(t) = v_X(\alpha(t))$ and $\beta'(t) = v_X(\beta(t))$ for all $t$. By the uniqueness of the integral curve of a vector field on a manifold, we deduce that $\alpha(t) = \beta(t)$ for all $t$.

From the fact that $\varphi(s + t) = \varphi(s)\varphi(t)$ for all $s$ and $t$ near 0, it follows that $\varphi$ extends uniquely to all of $\mathbb{R}$, defining a homomorphism

$$\varphi_X: \mathbb{R} \to G$$

with $\varphi_X'(t) = v_X(\varphi)(t) = (m_{\varphi(t)})_*(X)$ for all $t$.

**Exercise 8.30.** Establish the *product rule* for derivatives of arcs in a Lie group $G$: if $\alpha$ and $\beta$ are arcs in $G$ and $\gamma(t) = \alpha(t) \cdot \beta(t)$, then

$$\gamma'(t) = dm_{\alpha(t)}(\beta'(t)) + dn_{\beta(t)}(\alpha'(t)),$$

where for any $g \in G$, the map $m_g$ (resp. $n_g$): $G \to G$ is given by left (resp. right) multiplication by $g$. Use this to give another proof that $\varphi$ is a homomorphism.

**Exercise 8.31.** Show that $\varphi_X$ is uniquely determined by the fact that it is a homomorphism of $\mathbb{R}$ to $G$ with tangent vector $\varphi_X'(0)$ at the identity equal to $X$. Deduce that if $\psi: G \to H$ is a map of Lie groups, then $\varphi_{\psi_* X} = \psi \circ \varphi_X$.

The Lie group map $\varphi_X: \mathbb{R} \to G$ is called the *one-parameter subgroup of G with tangent vector X at the identity*. The construction of these one-parameter subgroups for each $X$ amounts to the verification of the Second Principle of §8.1 for homomorphisms from $\mathbb{R}$ to $G$. The fact that there exists such a one-parameter subgroup of $G$ with any given tangent vector at the identity is crucial. For example, it is not hard to see (we will do this in a moment) that these one-parameter subgroups fill up a neighborhood of the identity in $G$, which immediately implies the First Principle of §8.1. To carry this out, we define the *exponential map*

$$\exp: \mathfrak{g} \to G$$

by

$$\exp(X) = \varphi_X(1). \tag{8.32}$$

Note that by the uniqueness of $\varphi_X$, we have

$$\varphi_{(\lambda X)}(t) = \varphi_X(\lambda t);$$

so that the exponential map restricted to the lines through the origin in $\mathfrak{g}$ gives the one-parameter subgroups of $G$. Indeed, Exercise 8.31 implies the characterization:

**Proposition 8.33.** *The exponential map is the unique map from* $\mathfrak{g}$ *to* $G$ *taking* $0$ *to* $e$ *whose differential at the origin*

$$(\exp_*)_0 \colon T_0\mathfrak{g} = \mathfrak{g} \to T_eG = \mathfrak{g}$$

*is the identity, and whose restrictions to the lines through the origin in* $\mathfrak{g}$ *are one-parameter subgroups of* $G$.

This in particular implies (cf. Exercise 8.31) that the exponential map is natural, in the sense that for any map $\psi\colon G \to H$ of Lie groups the diagram

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\ \psi_*\ } & \mathfrak{h} \\
{\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\
G & \xrightarrow{\ \psi\ } & H
\end{array}
$$

commutes.

Now, since the differential of the exponential map at the origin in $\mathfrak{g}$ is an isomorphism, the image of exp will contain a neighborhood of the identity in $G$. If $G$ is connected, this will generate all of $G$; from this follows the First Principle: *if $G$ is connected, then the map $\psi$ is determined by its differential $(d\psi)_e$ at the identity.*

Using (8.32), we can write down the exponential map very explicitly in the case of $GL_n\mathbb{R}$, and hence for any subgroup of $GL_n\mathbb{R}$. We just use the standard power series for the function $e^x$, and set, for $X \in \mathrm{End}(V)$,

$$\exp(X) = 1 + X + \frac{X^2}{2} + \frac{X^3}{6} + \cdots. \tag{8.34}$$

Observe that this converges and is invertible, with inverse $\exp(-X)$. Clearly, the differential of this map from $\mathfrak{g}$ to $G$ at the origin is the identity; and by the standard power series computation, the restriction of the map to any line through the origin in $\mathfrak{g}$ is a one-parameter subgroup of $G$. Thus, the map coincides with the exponential as defined originally; and by naturality the same is true for any subgroup of $G$. (Note that, as we have pointed out, the individual terms in the expression on the right of (8.34) are very much dependent of the particular embedding of $G$ in a general linear group $GL(V)$ and correspondingly of $\mathfrak{g}$ in $\mathrm{End}(V)$, even though the *sum* on the right in (8.34) is not.)

This explicit form of the exponential map allows us to give substance to

the assertion that "the group structure of $G$ is encoded in the Lie algebra." Explicitly, we claim that not only do the exponentials $\exp(X)$ generate $G$, but for $X$ and $Y$ in a sufficiently small neighborhood of the origin in $\mathfrak{g}$, we can write down the product $\exp(X) \cdot \exp(Y)$ as an exponential. To do this, we introduce first the "inverse" of the exponential map: for $g \in G \subset \mathrm{GL}_n\mathbb{R}$, we set

$$\log(g) = (g - I) - \frac{(g - I)^2}{2} + \frac{(g - I)^3}{3} - \cdots \in \mathfrak{gl}_n\mathbb{R}.$$

Of course, this will be defined only for $g$ sufficiently close to the identity in $G$; but where it is defined it will be an inverse to the exponential map.

Now, we define a new bilinear operation on $\mathfrak{gl}_n\mathbb{R}$: we set

$$X * Y = \log(\exp(X) \cdot \exp(Y)).$$

We have to be careful what we mean by this, of course; we substitute for $g$ in the expression above for $\log(g)$ the quantity

$$\exp(X) \cdot \exp(Y) = \left( I + X + \frac{X^2}{2} + \cdots \right) \cdot \left( I + Y + \frac{Y^2}{2} + \cdots \right)$$

$$= I + (X + Y) + \left( \frac{X^2}{2} + X \cdot Y + \frac{Y^2}{2} \right) + \cdots,$$

being careful, of course, to preserve the order of the factors in each product. Doing this, we arrive at

$$X * Y = (X + Y) + \left( -\frac{(X + Y)^2}{2} + \left( \frac{X^2}{2} + X \cdot Y + \frac{Y^2}{2} \right) \right) + \cdots$$

$$= X + Y + \tfrac{1}{2}[X, Y] + \cdots.$$

Observe in particular that the terms of degree 2 in $X$ and $Y$ do not involve the squares of $X$ and $Y$ or the product $X \cdot Y$ alone, but only the commutator. In fact, this is true of each term in the formula, i.e., the quantity $\log(\exp(X) \cdot \exp(Y))$ can be expressed purely in terms of $X$, $Y$, and the bracket operation; the resulting formula is called the *Campbell–Hausdorff formula* (although the actual formula in closed form was given by Dynkin). To degree three, it is

$$X * Y = X + Y + \tfrac{1}{2}[X, Y] \pm \tfrac{1}{12}[X, [X, Y]] \pm \tfrac{1}{12}[Y, [Y, X]] + \cdots.$$

**Exercise 8.35\*.** Verify (and find the correct signs in) the cubic term of the Campbell–Hausdorff formula.

**Exercise 8.36.** Prove the assertion of the last paragraph that the power series $\log(\exp(X) \cdot \exp(Y))$ can be expressed purely in terms of $X$, $Y$, and the bracket operation.

**Exercise 8.37.** Show that for $X$ and $Y$ sufficiently small, the power series $\log(\exp(X) \cdot \exp(Y))$ converges.

**Exercise 8.38\*.** (a) Show that there is a constant $C$ such that for $X$, $Y \in \mathfrak{gl}_n$, $X * Y = X + Y + [X, Y] + E$, where $\|E\| \leq C(\|X\| + \|Y\|)^3$.
  (b) Show that $\exp(X + Y) = \lim_{n \to \infty} (\exp(X/n) \cdot \exp(Y/n))^n$.
  (c) Show that

$$\exp([X, Y]) = \lim_{n \to \infty} \left( \exp\left(\frac{X}{n}\right) \cdot \exp\left(\frac{Y}{n}\right) \cdot \exp\left(-\frac{X}{n}\right) \cdot \exp\left(-\frac{Y}{n}\right) \right)^{n^2}.$$

**Exercise 8.39.** Show that if $G$ is a subgroup of $\mathrm{GL}_n\mathbb{R}$, the elements of its Lie algebra are the "infinitesimal transformations" of $G$ in the sense of von Neumann, i.e., they are the matrices in $\mathfrak{gl}_n\mathbb{R}$ which can be realized as limits

$$\lim_{t \to 0} \frac{A_t - I}{\varepsilon_t}, \quad A_t \in G, \varepsilon_t > 0, \varepsilon_t \to 0.$$

**Exercise 8.40.** Show that $\exp$ is surjective for $G = \mathrm{GL}_n\mathbb{C}$ but not for $G = \mathrm{GL}_n^+\mathbb{R}$ if $n > 1$, or for $G = \mathrm{SL}_2\mathbb{C}$.

By the Campbell–Hausdorff formula, we can not only identify all the elements of $G$ in a neighborhood of the identity, but we can also say what their pairwise products are, thus making precise the sense in which $\mathfrak{g}$ and its bracket operation determines $G$ and its group law locally. Of course, we have not written a closed-form expression for the Campbell–Hausdorff formula; but, as we will see shortly, its very existence is significant. (For such a closed form, see [Se1, I§4.8].)

We now consider another very natural question, namely, when a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra of (i.e., tangent space at the identity to) an immersed subgroup of $G$. Obviously, a necessary condition is that $\mathfrak{h}$ is closed under the bracket operation; we claim here that this is sufficient as well:

**Proposition 8.41.** *Let $G$ be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then the subgroup of the group $G$ generated by $\exp(\mathfrak{h})$ is an immersed subgroup $H$ with tangent space $T_eH = \mathfrak{h}$.*

PROOF. Note that the subgroup generated by $\exp(\mathfrak{h})$ is the same as the subgroup generated by $\exp(U)$, where $U$ is any neighborhood of the origin in $\mathfrak{h}$. It will suffice, then (see Exercise 8.42), to show that the image of $\mathfrak{h}$ under the exponential map is "locally" closed under multiplication, i.e., that for a sufficiently small disc $\Delta \subset \mathfrak{h}$, the product $\exp(\Delta) \cdot \exp(\Delta)$ (that is, the set of pairwise products $\exp(X) \cdot \exp(Y)$ for $X$, $Y \in \Delta$) is contained in the image of $\mathfrak{h}$ under the exponential map.

We will do this under the hypothesis that $G$ may be realized as a subgroup of a general linear group $\mathrm{GL}_n\mathbb{R}$, so that we can use the formula (8.34) for the exponential map. This is a harmless assumption, given the statement (to be proved in Appendix E) that any finite-dimensional Lie algebra may be

embedded in the Lie algebra $\mathfrak{gl}_n\mathbb{R}$: the subgroup of $GL_n\mathbb{R}$ generated by $\exp(g)$ will be a group isogenous to $G$, and, as the reader can easily check, proving the proposition for a group isogenous to $G$ is equivalent to proving it for $G$.

It thus suffices to prove the assertion in case the group $G$ is $GL_n\mathbb{R}$. But this is exactly the content of the Campbell–Hausdorff formula.            □

When applied to an embedding of a Lie algebra $\mathfrak{g}$ into $\mathfrak{gl}_n$, we see, in particular, that *every finite-dimensional Lie algebra is the Lie algebra of a Lie group*. From what we have seen, this Lie group is unique if we require it to be simply connected, and then all others are obtained by dividing this simply connected model by a discrete subgroup of its center.

**Exercise 8.42\*.** Let $G_0 = \exp(\Delta)$, where $\Delta$ is a disk centered at the origin in $\mathfrak{g}$, and let $H_0 = \exp(\Delta \cap \mathfrak{h})$. Show that $G_0^{-1} = G_0$, $H_0^{-1} = H_0$, and $H_0 \cdot H_0 \cap G_0 = H_0$. Use this to show that the subgroup $H$ of $G$ generated by $H_0$ is an immersed Lie subgroup of $G$.

As a fairly easy consequence of this proposition, we can finally give a proof of the Second Principle stated in §8.1, which we may restate as

**Second Principle.** *Let $G$ and $H$ be Lie groups with $G$ simply connected, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. A linear map $\alpha: \mathfrak{g} \to \mathfrak{h}$ is the differential of a map $A: G \to H$ of Lie groups if and only if $\alpha$ is a map of Lie algebras.*

PROOF. To see this, consider the product $G \times H$. Its Lie algebra is just $\mathfrak{g} \oplus \mathfrak{h}$. Let $\mathfrak{j} \subset \mathfrak{g} \oplus \mathfrak{h}$ be the graph of the map $\alpha$. Then the hypothesis that $\alpha$ is a map of Lie algebras is equivalent to the statement that $\mathfrak{j}$ is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$; and given this, by the proposition there exists an immersed Lie subgroup $J \subset G \times H$ with tangent space $T_e J = \mathfrak{j}$.

Look now at the map $\pi: J \to G$ given by projection on the first factor. By hypothesis, the differential of this map $d\pi_e: \mathfrak{j} \to \mathfrak{g}$ is an isomorphism, so that the map $J \to G$ is an isogeny; but since $G$ is simply connected *it follows that $\pi$ is an isomorphism*. The projection $\eta: G \cong J \to H$ on the second factor is then a Lie group map whose differential at the identity is $\alpha$.            □

**Exercise 8.43\*.** If $\mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie algebras with kernel $\mathfrak{h}$, show that the kernel $H$ of the corresponding map of simply connected Lie groups $G \to G'$ is a closed subgroup of $G$ with Lie group $\mathfrak{h}$. This does not extend to non-normal subgroups, i.e., to the situation when $\mathfrak{h}$ is not the kernel of a homomorphism: give an example of an immersed subgroup of a simply connected Lie group $G$ whose image in $G$ is not closed.

**Exercise 8.44.** Use the ideas of this lecture to prove the assertion that a compact complex connected Lie group $G$ must be abelian:

(a) Verify that the map $\mathrm{Ad} \colon G \to \mathrm{Aut}(T_e G) \subset \mathrm{End}(T_e G)$ is holomorphic, and, therefore (by the maximum principle), constant.

(b) Deduce that if $\Psi_g$ is conjugation by $g$, then $d\Psi_g$ is the identity, so $\Psi_g(\exp(X)) = \exp(d\Psi_g(X)) = \exp(X)$ for all $X \in T_e G$, which implies that $G$ is abelian.

(c) Show that the exponential map from $T_e G$ to $G$ is surjective, with the kernel a lattice $\Lambda$, so $G = T_e G / \Lambda$ is a complex torus.