LECTURE 10

Lie Algebras in Dimensions One, Two, and Three

Just to get a sense of what a Lie algebra is and what groups might be associated to it, we will classify here all Lie algebras of dimension three or less. We will work primarily with complex Lie algebras and Lie groups, but will mention the real case as well. Needless to say, this lecture is logically superfluous; but it is easy, fun, and serves a didactic purpose, so why not read it anyway. The analyses of both the Lie algebras and the Lie groups are completely elementary, with one exception: the classification of the complex Lie groups associated to abelian Lie algebras involves the theory of complex tori, and should probably be skipped by anyone not familiar with this subject.

§10.1. Dimensions One and Two

To begin with, any one-dimensional Lie algebra $\mathfrak{g}$ is clearly abelian, that is, $\mathbb{C}$ with all brackets zero.

The simply connected Lie group with this Lie algebra is just the group $\mathbb{C}$ under addition; and other connected Lie groups that have $\mathfrak{g}$ as their Lie algebra must all be quotients of $\mathbb{C}$ by discrete subgroups $\Lambda \subset \mathbb{C}$. If Λ has rank one, then the quotient is just $\mathbb{C}^*$ under multiplication. If Λ has rank two, however, G may be any one of a continuously varying family of *complex tori of dimension one* (or *Riemann surfaces of genus one*, or *elliptic curves over* $\mathbb{C}$). The set of isomorphism classes of such tori is parametrized by the complex plane with coordinate j, where the function j on the set of lattices $\Lambda \subset \mathbb{C}$ is as described in, e.g., [Ahl].

Over the real numbers, the situation is completely straightforward: the only real Lie algebra of dimension one is again $\mathbb{R}$ with trivial bracket; the simply connected Lie group associated to it is $\mathbb{R}$ under addition; and the only other connected real Lie group with this Lie algebra is $\mathbb{R}/\mathbb{Z} \cong S^1$.

Dimension Two

Here we have to consider two cases, depending on whether $\mathfrak{g}$ is abelian or not.

Case 1: $\mathfrak{g}$ *abelian.* This is very much like the previous case; the simply connected two-dimensional abelian complex Lie group is just $\mathbb{C}^2$ under addition, and the remaining connected Lie groups with Lie algebra $\mathfrak{g}$ are just quotients of $\mathbb{C}^2$ by discrete subgroups. Such a subgroup $\Lambda \subset \mathbb{C}^2$ can have rank 1, 2, 3, or 4, and we analyze these possibilities in turn (the reader who has seen enough complex tori in the preceding example may wish to skip directly to Case 2 at this point).

If the rank of Λ is 1, we can complete the generator of Λ to a basis for $\mathbb{C}^2$, so that $\Lambda = \mathbb{Z}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $G \cong \mathbb{C}^* \times \mathbb{C}$. If the rank of Λ is 2, there are two possibilities: either Λ lies in a one-dimensional complex subspace of $\mathbb{C}^2$ or it does not. If it does not, a pair of generators for Λ will also be a basis for $\mathbb{C}^2$ over $\mathbb{C}$, so that $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$, and $G \cong \mathbb{C}^* \times \mathbb{C}^*$. If on the other hand Λ does lie in a complex line in $\mathbb{C}^2$, so that we have $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}\tau e_1$ for some $\tau \in \mathbb{C}\setminus\mathbb{R}$, then $G = E \times \mathbb{C}$ will be the product of the torus $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ and $\mathbb{C}$; the remarks above apply to the classification of these (see Exercise 10.1).

The cases where Λ has rank 3 or 4 are a little less clear. To begin with, if the rank of Λ is 3, the main question to ask is whether any rank 2 sublattice Λ' of Λ lies in a complex line. If it does, then we can assume this sublattice is saturated (i.e., a pair of generators for Λ' can be completed to a set of generators for Λ) and write $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}\tau e_1 \oplus \mathbb{Z}e_2$, so that we will have $G = E \times C^*$, where E is a torus as above.

Exercise 10.1*. For two one-dimensional complex tori E and E', show that the complex Lie groups $G = E \times \mathbb{C}$ and $G' = E' \times \mathbb{C}$ are isomorphic if and only if $E \cong E'$. Similarly for $E \times \mathbb{C}^*$ and $E' \times \mathbb{C}^*$.

If, on the other hand, no such sublattice of Λ exists, the situation is much more mysterious. One way we can try to represent G is by choosing a generator for Λ and considering the projection of $\mathbb{C}^2$ onto the quotient of $\mathbb{C}^2$ by the line spanned by this generator; thus, if we write $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}(\alpha e_1 + \beta e_2)$ then (assuming β is not real) we have maps

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2/\mathbb{C}e_1 = \mathbb{C} \\ \big\downarrow & & \big\downarrow \\ G = \mathbb{C}^2/\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}(\alpha e_1 + \beta e_2) & \longrightarrow & \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\beta) \end{array}$$

expressing G as a bundle over a torus $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\beta)$, with fibers isomorphic to $\mathbb{C}^*$. This expression of G does not, however, help us very much to describe the family of all such groups. For one thing, the elliptic curve E is surely not determined by the data of G: if we just exchange e_1 and e_2, for example, we replace E by $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\alpha)$, which, of course, need not even be isogenous to E. Indeed, this yields an example of different algebraic groups isomorphic as complex Lie groups: expressing G as a $\mathbb{C}^*$ bundle in this way gives it the structure of an algebraic variety, which, in turn, determines the elliptic curve E (for example, the field of rational functions on G will be the field of rational functions on E with one variable adjoined). Thus, different expressions of the complex Lie group G as a $\mathbb{C}^*$ bundle yield nonisomorphic algebraic groups.

Finally, the case where Λ has rank 4 remains completely mysterious. Among such two-dimensional complex tori are the *abelian varieties*; these are just the tori that may be embedded in complex projective space (and hence may be realized as algebraic varieties). For polarized abelian varieties (that is, abelian varieties with equivalence class of embedding in projective space) there exists a reasonable moduli theory; but the set of abelian varieties forms only a countable dense union in the set of all complex tori (indeed, the general complex torus possesses no nonconstant meromorphic functions whatsoever). No satisfactory theory of moduli is known for these objects.

Needless to say, the foregoing discussion of the various abelian complex Lie groups in dimension two is completely orthogonal to our present purposes. We hope to make the point, however, that even in this seemingly trivial case there lurk some fairly mysterious phenomena. Of course, none of this occurs in the real case, where the two-dimensional abelian simply connected real Lie group is just $\mathbb{R} \times \mathbb{R}$ and any other connected two-dimensional abelian real Lie group is the quotient of this by a sublattice $\Lambda \subset \mathbb{R} \times \mathbb{R}$ of rank 1 or 2, which is to say either $\mathbb{R} \times S^1$ or $S^1 \times S^1$.

Case 2: $\mathfrak{g}$ *not abelian*. Viewing the Lie bracket as a linear map $[\ ,\ ]: \wedge^2\mathfrak{g} \to \mathfrak{g}$, we see that if it is not zero, it must have one-dimensional image. We can thus choose a basis $\{X, Y\}$ for $\mathfrak{g}$ as vector space with X spanning the image of $[\ ,\ ]$; after multiplying Y by an appropriate scalar we will have $[X, Y] = X$, which of course determines $\mathfrak{g}$ completely. There is thus a unique nonabelian two-dimensional Lie algebra $\mathfrak{g}$ over either $\mathbb{R}$ or $\mathbb{C}$.

What are the complex Lie groups with Lie algebra $\mathfrak{g}$? To find one, we start with the adjoint representation of $\mathfrak{g}$, which is faithful: we have

$$\begin{aligned} \operatorname{ad}(X)\colon X &\mapsto 0, & \qquad \operatorname{ad}(Y)\colon X &\mapsto -X, \\ Y &\mapsto X, & Y &\mapsto 0 \end{aligned}$$

or in matrix notation, in terms of the basis $\{X, Y\}$ for $\mathfrak{g}$,

$$\operatorname{ad}(X) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \operatorname{ad}(Y) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

These generate the algebra $\mathfrak{g} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \subset \mathfrak{gl}_2\mathbb{C}$; we may exponentiate to arrive at the adjoint form

$$G_0 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\} \subset \mathrm{GL}_2\mathbb{C}.$$

Topologically this group is homeomorphic to $\mathbb{C} \times \mathbb{C}^*$. To take its universal cover, we write a general member of G_0 as

$$\begin{pmatrix} e^t & s \\ 0 & 1 \end{pmatrix}.$$

The product of two such matrices is given by

$$\begin{pmatrix} e^t & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{t'} & s' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{t+t'} & s + e^t s' \\ 0 & 1 \end{pmatrix},$$

so we may realize the universal cover G of G_0 as the group of pairs $(t, s) \in \mathbb{C} \times \mathbb{C}$ with group law

$$(t, s) \cdot (t', s') = (t + t', s + e^t s').$$

The center of G is just the subgroup

$$Z(G) = \{(2\pi i n, 0)\} \cong \mathbb{Z},$$

so that the connected groups with Lie algebra $\mathfrak{g}$ form a partially ordered tower

$$\begin{array}{l} G \\ \downarrow \\ \vdots \\ \downarrow \\ G_n = G/n\mathbb{Z} = \{(a, b) \in \mathbb{C}^* \times \mathbb{C}; (a, b) \cdot (a', b') = (aa', b + a^n b')\}. \\ \downarrow \\ \vdots \\ \downarrow \\ G_0 \end{array}$$

Exercise 10.2*. Show that for $n \neq m$ the two groups G_n and G_m are not isomorphic.

Finally, in the real case things are simpler: when we exponentiate the adjoint representation as above, the Lie group we arrive at is already simply connected, and so is the unique connected real Lie group with this Lie algebra.

§10.2. Dimension Three, Rank 1

As in the case of dimension two, we look at the Lie bracket as a linear map from $\Lambda^2\mathfrak{g}$ to $\mathfrak{g}$ and begin our classification by considering the rank of this map (that is, the dimension of $\mathscr{D}\mathfrak{g}$), which may be either 0, 1, 2, or 3. For the case of rank 0, we refer back to the discussion of abelian Lie groups above. We begin with the case of rank 1.

Here the kernel of the map $[\ ,\ ]: \Lambda^2\mathfrak{g} \to \mathfrak{g}$ is two dimensional, which means that for some $X \in \mathfrak{g}$ it consists of all vectors of the form $X \wedge Y$ with Y ranging over all of $\mathfrak{g}$ (X here will just be the vector corresponding to the hyperplane $\ker([\ ,\ ]) \subset \Lambda^2\mathfrak{g}$ under the natural (up to scalars) duality between a three-dimensional vector space and its exterior square). Completing X to a basis $\{X, Y, Z\}$ of $\mathfrak{g}$, we can write $\mathfrak{g}$ in the form

$$[X, Y] = [X, Z] = 0,$$
$$[Y, Z] = \alpha X + \beta Y + \gamma Z$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. If either β or γ is nonzero, we may now rechoose our basis, replacing Y by a multiple of the linear combination $\alpha X + \beta Y + \gamma Z$ and either leaving Z alone (if $\beta \neq 0$) or replacing Z by Y (if $\gamma \neq 0$). We will then have

$$[X, Y] = [X, Z] = 0,$$
$$[Y, Z] = Y$$

from which we see that $\mathfrak{g}$ is just the product of the one-dimensional abelian Lie algebra $\mathbb{C}X$ with the non-abelian two-dimensional Lie algebra $\mathbb{C}Y \oplus \mathbb{C}Z$ described in the preceding discussion. We may thus ignore this case and assume that in fact we have $\beta = \gamma = 0$; replacing X by αX we then have the Lie algebra

$$[X, Y] = [X, Z] = 0,$$
$$[Y, Z] = X.$$

How do we find the Lie groups with this Lie algebra? As before, we need to start with a faithful representation of $\mathfrak{g}$, but here the adjoint representation is useless, since X is in its kernel. We can, however, arrive at a representation of $\mathfrak{g}$ by considering the equations defining $\mathfrak{g}$: we want to find a pair of endomorphisms Y and Z on some vector space that do not commute, but that do commute with their commutator $X = [Y, Z]$; thus,

$$Y(YZ - ZY) - (YZ - ZY)Y = Y^2Z - 2YZY + ZY^2 = 0$$

and similarly for $[Z, [Y, Z]]$. One simple way to find such a pair of endomorphisms is make all three terms Y^2Z, YZY, and Z^2Y in the above equation zero, e.g., by making Y and Z both have square zero, and to have $YZ = 0$ while $ZY \neq 0$. For example, on a three-dimensional vector space with basis e_1, e_2, and e_3 we could take Y to be the map carrying e_3 to e_2 and killing e_1 and e_2, and Z the map carrying e_2 to e_1 and killing e_1 and e_3; we then have $YZ = 0$ while ZY sends e_3 to e_1. We see then that $\mathfrak{g}$ *is just the Lie algebra* $\mathfrak{n}_3$ *of strictly upper-triangular* 3×3 *matrices.* When we exponentiate we arrive at the group

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{C} \right\}$$

which is simply connected. Now the center of G is the subgroup

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b \in \mathbb{C} \right\} \cong \mathbb{C},$$

so the discrete subgroups of $Z(G)$ are just lattices Λ of rank 1 or 2; thus any connected group with Lie algebra $\mathfrak{g}$ is either G, $G/\mathbb{Z}$, or $G/(\mathbb{Z} \times \mathbb{Z})$—that is, an extension of $\mathbb{C} \times \mathbb{C}$ by either $\mathbb{C}$, $\mathbb{C}^*$, or a torus E.

Exercise 10.3. Show that G/Λ is determined up to isomorphism by the one-dimensional $Z(G)/\Lambda$.

A similar analysis holds in the real case: just as before, $\mathfrak{n}_3$ is the unique real Lie algebra of dimension three with commutator subalgebra of dimension one; its simply connected form is the group G of unipotent 3×3 matrices and (the center of this group being $\mathbb{R}$) the only other group with this Lie algebra is the quotient $H = G/\mathbb{Z}$.

Incidentally, the group H represents an interesting example of a group that cannot be realized as a matrix group, i.e., that admits no faithful finite-dimensional representations. One way to see this is to argue that in any irreducible finite-dimensional representation V the center S^1 of H, being compact and abelian, must be diagonalizable; and so under the corresponding representation of the Lie algebra $\mathfrak{g}$ the element X must be carried to a diagonalizable endomorphism of V. But now if $v \in V$ is any eigenvector for X with eigenvalue λ, we also have, arguing as in §9.2,

$$X(Y(v)) = Y(X(v)) = Y(\lambda v) = \lambda Y(v)$$

and similarly $X(Z(v)) = \lambda Z(v)$, i.e., both $Y(v)$ and $Z(v)$ are also eigenvectors for X with eigenvalue λ. Since Y and Z generate $\mathfrak{g}$ and the representation V is irreducible, it follows that X must act as a scalar multiple $\lambda \cdot I$ of the identity; but since $X = [Y, Z]$ is a commutator and so has trace 0, it follows that $\lambda = 0$.

Exercise 10.4*. Show that if G is a simply connected Lie group, and its Lie algebra is solvable, then G cannot contain any nontrivial compact subgroup (in particular, it contains no elements of finite order).

The group H does, however, have an important infinite-dimensional representation. This arises from the representation of the Lie algebra $\mathfrak{g}$ on the space V of $\mathscr{C}^\infty$ functions on the real line $\mathbb{R}$ with coordinate x, in which Y, Z, and X are the operators

$$Y: f \mapsto \pi i x \cdot f,$$

$$Z: f \mapsto \frac{df}{dx}$$

and $X = [Y, Z]$ is $-\pi i$ times the identity. Exponentiating, we see that e^{tY} acts on a function f by multiplying it by the function $(\cos tx + i \cdot \sin tx)$; e^{tZ} sends f to the function F_t where $F_t(x) = f(t + x)$, and e^{tX} sends f to the scalar multiple $e^{-\pi i t} \cdot f$.

§10.3. Dimension Three, Rank 2

In this case, write the commutator subalgebra $\mathscr{D}\mathfrak{g} \subset \mathfrak{g}$ as the span of two elements Y and Z. The commutator of Y and Z can then be written

$$[Y, Z] = \alpha Y + \beta Z.$$

But now the endomorphism $\mathrm{ad}(Y)$ of $\mathfrak{g}$ carries $\mathfrak{g}$ into $\mathscr{D}\mathfrak{g}$, kills Y, and sends Z to $\alpha Y + \beta Z$, and so has trace β; on the other hand, since $\mathrm{ad}(Y)$ is a commutator in $\mathrm{End}(\mathfrak{g})$, it must have trace 0. Thus, β, and similarly α, must be zero; i.e., the subalgebra $\mathscr{D}\mathfrak{g}$ must be abelian. It follows from this that for any element $X \in \mathfrak{g}$ not in $\mathscr{D}\mathfrak{g}$, the map

$$\mathrm{ad}(X): \mathscr{D}\mathfrak{g} \to \mathscr{D}\mathfrak{g}$$

must be an isomorphism. We may now distinguish two possibilities: either $\mathrm{ad}(X)$ is diagonalizible or it is not.

(Note that for the first time we see a case where the classification of the real Lie algebra will be more complicated than that of the complex: in the real case we will have to deal with the third possibility that $\mathrm{ad}(X)$ is diagonalizible over $\mathbb{C}$ but not over $\mathbb{R}$, i.e., that it has two complex conjugate eigenvalues. Though we have not seen it much in these low-dimensional examples, in fact it is generally the case that the real picture is substantially more complicated than the complex one, for essentially just this reason.)

Possibility A: $\mathrm{ad}(X)$ *is diagonalizable.* In this case it is natural to use as a basis for $\mathscr{D}\mathfrak{g}$ a pair of eigenvectors Y, Z for $\mathrm{ad}(X)$; and by multiplying X by a suitable scalar we can assume that one of the eigenvalues (both of which are nonzero) is 1. We thus have the equations for $\mathfrak{g}$

$$[X, Y] = Y, \qquad [X, Z] = \alpha Z, \qquad [Y, Z] = 0 \tag{10.5}$$

for some $\alpha \in \mathbb{C}^*$.

Exercise 10.6. Show that two Lie algebras $\mathfrak{g}_\alpha$, $\mathfrak{g}_{\alpha'}$ corresponding to two different scalars in the structure equations (10.5) are isomorphic if and only if $\alpha = \alpha'$ or

$\alpha = 1/\alpha'$. Observe that we have for the first time a continuously varying family of nonisomorphic complex Lie algebras.

To find the groups with these Lie algebras we go to the adjoint representation, which here is faithful. Explicitly, $\mathrm{ad}(Y)$ carries X to $-Y$ and kills Y and Z; $\mathrm{ad}(Z)$ carries X to $-\alpha Z$ and also kills Y and Z; and $\mathrm{ad}(X)$ carries Y to itself, Z to αZ, and kills X. A general member $aX - bY - cZ$ of the Lie algebra is thus represented (with respect to the basis $\{Y, Z, X\}$ for $\mathfrak{g}$) by the matrix

$$\begin{pmatrix} a & 0 & b \\ 0 & \alpha a & \alpha c \\ 0 & 0 & 0 \end{pmatrix}.$$

Exponentiating, we find that a group with Lie algebra $\mathfrak{g}$ is

$$G = \left\{ \begin{pmatrix} e^t & 0 & u \\ 0 & e^{\alpha t} & v \\ 0 & 0 & 1 \end{pmatrix}, t, u, v \in \mathbb{C} \right\} \subset \mathrm{GL}_3\mathbb{C}.$$

Here we run across a very interesting circumstance. If the complex number α is not rational, then the exponential map from $\mathfrak{g}$ to G is one-to-one, and hence a homeomorphism; thus, in particular, G is simply connected. If, on the other hand, α is rational, G will have nontrivial fundamental group. To see this, observe that we always have an exact sequence of groups

$$1 \to B \to G \to A \to 1,$$

where

$$A = \left\{ \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{\alpha t} & 0 \\ 0 & 0 & 1 \end{pmatrix}, t \in \mathbb{C} \right\}$$

and

$$B = \left\{ \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}, u, v \in \mathbb{C} \right\} \cong \mathbb{C} \times \mathbb{C}.$$

Now when $\alpha \notin \mathbb{Q}$, the group $A \cong \mathbb{C}$ is simply connected; but when $\alpha \in \mathbb{Q}$—whatever its denominator—we have $A \cong \mathbb{C}^*$ and correspondingly $\pi_1(G) = \mathbb{Z}$.

Exercise 10.7. Show that G has no center, and hence when $\alpha \neq \mathbb{Q}$, it is the unique connected group with Lie algebra $\mathfrak{g}$. For $\alpha \in \mathbb{Q}$, describe the universal covering of G and classify all groups with Lie algebra $\mathfrak{g}$.

Observe that in this case, even though we have a continuously varying family of Lie algebras $\mathfrak{g}_\alpha$, we have no corresponding continuously varying family of the adjoint (linear) Lie groups; the simply-connected forms do form a family, however.

Possibility B: ad(X) *is not diagonalizable.* In this case the natural thing to do is to choose a basis $\{Y, Z\}$ of $\mathscr{D}\mathfrak{g}$ with respect to which ad(X) is in Jordan normal form; replacing X by a multiple, we may assume both its eigenvalues are 1 so that we will have the Lie algebra

$$[X, Y] = Y, \qquad [X, Z] = Y + Z, \qquad [Y, Z] = 0. \tag{10.8}$$

With respect to the basis $\{Y, Z, X\}$ for $\mathfrak{g}$, then, the adjoint action of the general element $aX - bY - cZ$ of the Lie algebra is represented by the matrix

$$\begin{pmatrix} a & a & b+c \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix}$$

and exponentiating we find that the corresponding group is

$$G = \left\{ \begin{pmatrix} e^t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix}, t, u, v \in \mathbb{C} \right\}.$$

Exercise 10.9. Show that this group has no center, and hence is the unique connected complex Lie group with its Lie algebra.

Note that the real Lie groups obtained by exponentiating the adjoint action of the Lie algebras given by (10.5) and (10.8) are all homeomorphic to $\mathbb{R}^3$ and have no center, and so are the only connected real Lie groups with these Lie algebras.

Exercise 10.10. Complete the analysis of real Lie groups in Case 2 by considering the third possibility mentioned above: that ad(X) acts on $\mathscr{D}\mathfrak{g}$ with distinct complex conjugate eigenvalues. Observe that in this way we arrive at our first example of two nonisomorphic real Lie algebras whose tensor products with $\mathbb{C}$ are isomorphic.

§10.4. Dimension Three, Rank 3

Our analysis of this final case begins, as in the preceding one, by looking for eigenvectors of the adjoint action of a suitable element $X \in \mathfrak{g}$. Specifically, we claim that we can find an element $H \in \mathfrak{g}$ such that $\mathrm{ad}(H)\colon \mathfrak{g} \to \mathfrak{g}$ has an eigenvector with nonzero eigenvalue. To see this, observe first that for any nonzero $X \in \mathfrak{g}$, the rank of ad(X) must be 2; in particular, we must have $\mathrm{Ker}(\mathrm{ad}(X)) = \mathbb{C}X$. Now start with any $X \in \mathfrak{g}$. Either ad(X) has an eigenvector with nonzero eigenvalue or it is nilpotent; if it is nilpotent, then there exists a vector $Y \in \mathfrak{g}$, not in the kernel of $\mathrm{ad}(X)$ but in the kernel of $\mathrm{ad}(X)^2$—that is, such that $\mathrm{ad}(X)(Y) = \alpha X$ for some nonzero $\alpha \in \mathbb{C}$. But then of course $\mathrm{ad}(Y)(X) = -\alpha X$, so that X is an eigenvector for $\mathrm{ad}(Y)$ with nonzero eigenvalue.

So: choose H and $X \in \mathfrak{g}$ so that X is an eigenvector with nonzero eigenvalue for $\mathrm{ad}(H)$, and write $[H, X] = \alpha X$. Since $H \in \mathscr{D}\mathfrak{g}$, $\mathrm{ad}(H)$ is a commutator in $\mathrm{End}(\mathfrak{g})$, and so has trace 0; it follows that $\mathrm{ad}(H)$ must have a third eigenvector Y with eigenvalue $-\alpha$. To describe the structure of $\mathfrak{g}$ completely it now remains to find the commutator of X and Y; but this follows from the Jacobi identity. We have

$$\begin{aligned}[H, [X, Y]] &= -[X, [Y, H]] - [Y, [H, X]] \\ &= -[X, \alpha Y] - [Y, \alpha X] \\ &= 0,\end{aligned}$$

from which we deduce that $[X, Y]$ must be a multiple of H; since it must be a nonzero multiple, we can multiply X or Y by a scalar to make it 1. Similarly multiplying H by a scalar we can assume α is 1 or any other nonzero scalar. Thus, there is only one possible complex Lie algebra $\mathfrak{g}$ of this type. One could look for endomorphisms H, X, and Y whose commutators satisfy these relations, as we did before. Or we may simply realize that the three-dimensional Lie algebra $\mathfrak{sl}_2\mathbb{C}$ has not yet been seen, so it must be this last possibility. In fact, a natural basis for $\mathfrak{sl}_2\mathbb{C}$ is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

whose Lie algebra is given by

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{10.11}$$

What groups other than $\mathrm{SL}_2\mathbb{C}$ have Lie algebra $\mathfrak{sl}_2\mathbb{C}$? To begin with, the group $\mathrm{SL}_2\mathbb{C}$ is simply connected: for example, the map $\mathrm{SL}_2\mathbb{C} \to \mathbb{C}^2 - \{(0, 0)\}$ sending a matrix to its first row expresses the topological space $\mathrm{SL}_2\mathbb{C}$ as a bundle with fiber $\mathbb{C}$ over $\mathbb{C}^2 - \{(0, 0)\}$. Also, it is not hard to see that the center of $\mathrm{SL}_2\mathbb{C}$ is just the subgroup $\{\pm I\}$ of scalar matrices, so that the only other connected group with Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is the quotient $\mathrm{PSL}_2\mathbb{C} = \mathrm{Sl}_2\mathbb{C}/\{\pm I\}$.

As in the preceding case, the analysis of real three-dimensional Lie algebras $\mathfrak{g}$ with $\mathscr{D}\mathfrak{g} = \mathfrak{g}$ involves one additional possibility. At the outset of the argument above, we started with an arbitrary $H \in \mathfrak{g}$ and said that if $\mathrm{ad}(H)$ had no eigenvector other than H itself, then it would have to be nilpotent. Of course, in the real case it is also possible that $\mathrm{ad}(H)$ has two distinct complex conjugate eigenvalues λ and $\bar{\lambda}$. Since $\mathrm{ad}(H)$ is a commutator in $\mathrm{End}(\mathfrak{g})$ and so has trace 0, λ will have to be purely imaginary in this case; and so multiplying H by a real scalar we can assume that its eigenvalues are i and $-i$. It follows then that we can find $X, Y \in \mathfrak{g}$ with

$$[H, X] = Y \qquad \text{and} \qquad [H, Y] = -X.$$

Using the Jacobi identity as before we may conclude that the commutator of X and Y is a multiple of H; after multiplying each of X and Y by a real scalar we can assume that it is either H or $-H$. Finally, if $[X, Y] = -H$, then we observe that we are in the case we considered before: $\text{ad}(Y)$ will have $X + H$ as an eigenvector with nonzero eigenvalue, and following our previous analysis we may conclude that $\mathfrak{g} \cong \mathfrak{sl}_2\mathbb{R}$. Thus, we are left with the sole additional possibility that $\mathfrak{g}$ has structure equations

$$[H, X] = Y, \qquad [H, Y] = -X, \qquad [X, Y] = H. \tag{10.12}$$

This, finally, we may recognize as the Lie algebra $\mathfrak{su}_2$ of the real Lie group SU(2) (as you may recall, the isomorphism $\mathfrak{su}_2 \otimes \mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$ was used in the last lecture).

What are the real Lie groups with Lie algebras $\mathfrak{sl}_2\mathbb{R}$ and $\mathfrak{su}_2$? To start, the center of the group $SL_2\mathbb{R}$ is again just the scalar matrices $\{\pm I\}$, so the only group dominated by $SL_2\mathbb{R}$ is the quotient $PSL_2\mathbb{R}$. On the other hand, unlike the complex case $SL_2\mathbb{R}$ is not simply connected: now the map associating to a 2×2 matrix its first row expresses $SL_2\mathbb{R}$ as a bundle with fiber $\mathbb{R}$ over $\mathbb{R}^2 - \{(0, 0)\}$, so that $\pi_1(SL_2\mathbb{R}) = \mathbb{Z}$. More precisely $PSL_2\mathbb{R}$ maps to the real projective line $\mathbb{P}^1\mathbb{R}$, which is homeomorphic to the circle, with fiber homeomorphic to $\mathbb{R}^2$, so $\pi_1(PSL_2\mathbb{R}) = \mathbb{Z}$. We thus have a tower of covering spaces of $PSL_2\mathbb{R}$, consisting of the simply-connected group $\tilde{S}$ with center $\mathbb{Z}$ and its quotients $\tilde{S}_n = \tilde{S}/n\mathbb{Z}$ (not all of these are covers of $SL_2\mathbb{R}$, despite the diagram below).

A note: In §10.2 we encountered a real Lie group with no faithful finite-dimensional representations; only its universal cover could be represented as a matrix group. Here we find in some sense the opposite phenomenon: the groups $\tilde{S}$ and $\tilde{S}_n$ have no faithful finite-dimensional representations, all finite-dimensional representations factoring through $SL_2\mathbb{R}$ or $PSL_2\mathbb{R}$. This fact will be proved as a consequence of our discussion of the representations of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ in the next lecture.

What about groups with Lie algebra $\mathfrak{su}_2$? To begin with, there is SU(2), which (again via the map sending a matrix to its first row vector) is homeomorphic to S^3 and thus simply connected. The center of this group is again $\{\pm I\}$, so that the quotient PSU(2) is the only other group with Lie algebra $\mathfrak{su}_2$. (Alternatively, we may realize SU(2) as the group of unit quaternions, cf. Exercise 7.15.)

Finally, we remark that there are other representations of the real and complex Lie groups discussed above. As we will see, the Lie algebra $\mathfrak{so}_3\mathbb{C}$ is isomorphic to $\mathfrak{sl}_2\mathbb{C}$, which induces an isomorphism between the corresponding adjoint forms $PSL_2\mathbb{C}$ and $SO_3\mathbb{C}$ (and between the simply-connected forms $SL_2\mathbb{C}$ and the spin group $\text{Spin}_3\mathbb{C}$). This in turn suggests two more real forms of this group: $SO_3\mathbb{R}$ and $SO^+(2, 1)$. In fact, it is not hard to see that $SO_3\mathbb{R} \cong$ PSU(2), while $SO^+(2, 1) \cong PSL_2\mathbb{R}$. Lastly the isomorphism $\mathfrak{su}_{1,1} \otimes \mathbb{C} \cong$

$\mathfrak{su}_2 \otimes \mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$ implies that the real Lie algebra $\mathfrak{su}_{1,1}$ is isomorphic to either $\mathfrak{su}_2$ or $\mathfrak{sl}_2\mathbb{R}$; in fact, the latter is the case and this induces an isomorphism of groups $SU_{1,1} \cong SL_2\mathbb{R}$. We summarize the isomorphisms mentioned in the diagram below:

$$\begin{array}{ccccc}
\tilde{S} & & & & \\
\downarrow & & & & \\
\tilde{S}_n & & \mathrm{Spin}_3\mathbb{C} & & \mathrm{Spin}_3\mathbb{R} \\
\downarrow & & \| & & \| \\
SU(1,1) = SL_2\mathbb{R} & \hookrightarrow & SL_2\mathbb{C} & \hookleftarrow & SU(2) = \{\text{unit quaternions}\} \\
\downarrow & & \downarrow & & \downarrow \\
PSL_2\mathbb{R} & \hookrightarrow & PSL_2\mathbb{C} & \hookleftarrow & PSU(2) = \{\text{unit quaternions}\}/\pm 1 \\
\| & & \| & & \| \\
SO^+(2,1) & \hookrightarrow & SO_3\mathbb{C} & \hookleftarrow & SO_3\mathbb{R}
\end{array} \qquad (10.13)$$

Note also the coincidences:

$$Sp_2(\mathbb{C}) = SL_2(\mathbb{C}), \qquad Sp_2(\mathbb{R}) = SL_2(\mathbb{R}), \qquad (10.14)$$

which follow from the fact that Sp refers to preserving a skew-symmetric bilinear form, and for 2×2 matrices the determinant is such a form.

Exercise 10.15. Identify the Lie algebras $\mathfrak{so}_3$, $\mathfrak{su}_2$, $\mathfrak{su}_{1,1}$, $\mathfrak{so}_{2,1}$, and verify the assertions made about the corresponding Lie groups in the diagram.

Exercise 10.16. For each of the Lie algebras encountered in this lecture, compute the lower central series and the derived series, and say whether the algebra is nilpotent, solvable, simple, or semisimple.

Exercise 10.17. The following are Lie groups of dimension two or three, so must appear on our list. Find them: (i) the group of affine transformations of the line ($x \mapsto ax + b$, under composition); (ii) the group of upper-triangular 2×2 matrices; (iii) the group of orientation preserving Euclidean transformations of the plane (compositions of translations and rotations).

Exercise 10.18. Locate $\mathbb{R}^3$ with the usual cross-product on our list of Lie algebras. More generally, consider the family of Lie algebras parametrized by real quadruples (a, b, c, d), each with basis X, Y, Z with bracket given by

$$[X, Y] = aZ + dY, \qquad [Y, Z] = bX, \qquad [Z, X] = cY - dZ.$$

Classify this Lie algebra as (a, b, c, d) varies in $\mathbb{R}^4$, showing in particular that every three-dimensional Lie algebra can be written in this way.

Exercise 10.19. Realize the isomorphism of $SU(1, 1)$ with $SL_2\mathbb{R}$ by identifying them with the groups of complex automorphisms of the unit disk and the upper half-plane, respectively.

Exercise 10.20. Classify all Lie algebras of dimension four and rank 1; in particular, show that they are all direct sums of Lie algebras described above.

Exercise 10.21. Show more generally that there exists a Lie algebra of dimension m and rank 1 that is not a direct sum of smaller Lie algebras if and only if m is odd; in case m is odd show that this Lie algebra is unique and realize it as a Lie subalgebra of $\mathfrak{sl}_n\mathbb{C}$.