LECTURE 7

Lie Groups

In this lecture we introduce the definitions and basic examples of Lie groups and Lie algebras. We assume here familiarity with the definition of differentiable manifolds and maps between them, but no more; in particular, we do not mention vector fields, differential forms, Riemannian metrics, or any other tensors. Section 7.3, which discusses maps of Lie groups that are covering space maps of the underlying manifolds, may be skimmed and referred back to as needed, though working through it will help promote familiarity with basic examples of Lie groups.

§7.1: Lie groups: definitions
§7.2: Examples of Lie groups
§7.3: Two constructions

§7.1. Lie Groups: Definitions

You probably already know what a Lie group is; it is just a set endowed simultaneously with the compatible structures of a group and a $\mathscr{C}^\infty$ manifold. "Compatible" here means that the multiplication and inverse operations in the group structure

$$\times : G \times G \to G$$

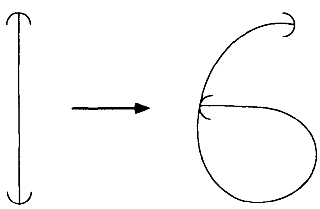
and

$$\iota : G \to G$$

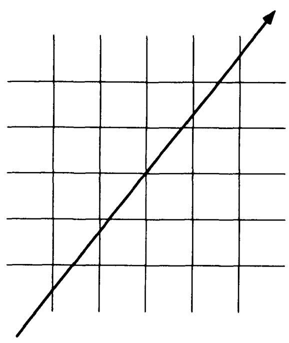
are actually differentiable maps (logically, this is equivalent to the single requirement that the map $G \times G \to G$ sending (x, y) to $x \cdot y^{-1}$ is $\mathscr{C}^\infty$).

A *map*, or *morphism*, between two Lie groups G and H is just a map $\rho: G \to H$ that is both differentiable and a group homomorphism. In general, qualifiers applied to Lie groups refer to one or another of the two structures, usually without much ambiguity; thus, *abelian* refers to the group structure, *n-dimensional* or *connected* refers to the manifold structure. Sometimes a condition on one structure turns out to be equivalent to a condition on the other; for example, we will see below that to say that a map of connected Lie groups $\varphi: G \to H$ is a surjective map of groups is equivalent to saying that the differential $d\varphi$ is surjective at every point.

One area where there is some potential confusion is in the definition of a Lie subgroup. This is essentially a difficulty inherited directly from manifold theory, where we have to make a distinction between a *closed submanifold* of a manifold M, by which we mean a subset $X \subset M$ that inherits a manifold structure from M (i.e., that may be given, locally in M, by setting a subset of the local coordinates equal to zero), and an *immersed submanifold*, by which we mean the image of a manifold X under a one-to-one map with injective differential everywhere—that is, a map that is an embedding *locally in X*. The distinction is necessary simply because the underlying topological space structure of an immersed submanifold may not agree with the topological structure induced by the inclusion of X in M. For example, the map from X to M could be the immersion of an open interval in $\mathbb{R}$ into the plane $\mathbb{R}^2$ as a figure "6":



Another standard example of this, which is also an example in the category of groups, would be to take M to be the two-dimensional real torus $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$, and X the image in M of a line $V \subset \mathbb{R}^2$ having irrational slope:



The upshot of this is that we define a *Lie subgroup* (or *closed Lie subgroup*, if we want to emphasize the point) of a Lie group G to be a subset that is simultaneously a subgroup and a *closed* submanifold; and we define an *immersed subgroup* to be the image of a Lie group H under an injective morphism to G. (That a one-to-one morphism of Lie groups has everywhere injective differential will follow from discussions later in this lecture.)

The definition of a *complex Lie group* is exactly analogous, the words "differentiable manifold" being replaced by "complex manifold" and all related notions revised accordingly. Similarly, to define an *algebraic group* one replaces "differentiable manifold" by "algebraic variety" and "differentiable map" by "regular morphism." As we will see, the category of complex Lie groups is in many ways markedly different from that of real Lie groups (for example, there are many fewer complex Lie groups than real ones). Of course, the study of algebraic groups in general is quite different from either of these since an algebraic group comes with a field of definition that may or may not be a subfield of $\mathbb{C}$ (it may, for that matter, have positive characteristic). In practice, though, while the two are not the same (we will see examples of this in Lecture 10, for example), the category of algebraic groups over $\mathbb{C}$ behaves very much like the category of complex Lie groups.

§7.2. Examples of Lie Groups

The basic example of a Lie group is of course the *general linear group* $\mathrm{GL}_n\mathbb{R}$ of invertible $n \times n$ real matrices; this is an open subset of the vector space of all $n \times n$ matrices, and gets its manifold structure accordingly (so that the entries of the matrix are coordinates on $\mathrm{GL}_n\mathbb{R}$). That the multiplication map $\mathrm{GL}_n\mathbb{R} \times \mathrm{GL}_n\mathbb{R} \to \mathrm{GL}_n\mathbb{R}$ is differentiable is clear; that the inverse map $\mathrm{GL}_n\mathbb{R} \to \mathrm{GL}_n\mathbb{R}$ is follows from Cramer's formula for the inverse. Occasionally $\mathrm{GL}_n\mathbb{R}$ will come to us as the group of automorphisms of an n-dimensional real vector space V; when we want to think of $\mathrm{GL}_n\mathbb{R}$ in this way (e.g., without choosing a basis for V and thereby identifying G with the group of matrices), we will write it as $\mathrm{GL}(V)$ or $\mathrm{Aut}(V)$. A *representation* of a Lie group G, of course, is a morphism from G to $\mathrm{GL}(V)$.

Most other Lie groups are defined initially as subgroups of GL_n (though they may appear in other contexts as subgroups of other general linear groups, which is, of course, the subject matter of these lectures). For the most part, such subgroups may be described either by equations on the entries of an $n \times n$ matrix, or as the subgroup of automorphisms of $V \cong \mathbb{R}^n$ preserving some structure on $\mathbb{R}^n$. For example, we have:

the *special linear group* $\mathrm{SL}_n\mathbb{R}$ of automorphisms of $\mathbb{R}^n$ preserving the volume element; equivalently, $n \times n$ matrices A with determinant 1.

the group B_n of *upper-triangular* matrices; equivalently, the subgroup of automorphisms of $\mathbb{R}^n$ preserving the flag[1]

[1] In general, a *flag* is a sequence of subspaces of a fixed vector space, each properly contained in the next; it is a *complete* flag if each has one dimension larger than the preceding, and *partial* otherwise.

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots V_{n-1} \subset V_n = \mathbb{R}^n,$$

where V_i is the span of the standard basis vectors $e_1, \ldots, e_i$. Note that choosing a different basis and correspondingly a different flag yields a different subgroup of $\mathrm{GL}_n\mathbb{R}$, but one isomorphic to (indeed, conjugate to) B_n. Somewhat more generally, for any sequence of positive integers $a_1, \ldots, a_k$ with sum n we can look at the group of block-upper-triangular matrices; this is the subgroup of automorphisms of $\mathbb{R}^n$ preserving a partial flag

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k = \mathbb{R}^n,$$

where the dimension of V_i is $a_1 + \cdots + a_i$. If the subspace V_i is spanned by the first $a_1 + \cdots + a_i$ basis vectors, the group will be the set of matrices of the form

$$\left(\begin{array}{c|c|c|c} * & * & * & * \\ \hline 0 & * & * & * \\ \hline 0 & 0 & * & * \\ \hline 0 & 0 & 0 & * \end{array}\right) \begin{array}{l} \}a_1 \\ \}a_2 \\ \\ \}a_k \end{array}$$

The group N_n of *upper-triangular unipotent* matrices (that is, upper triangular with 1's on the diagonal); equivalently, the subgroup of automorphisms of $\mathbb{R}^n$ preserving the complete flag $\{V_i\}$ where V_i is the span of the standard basis vectors $e_1, \ldots, e_i$, and acting as the identity on the successive quotients V_{i+1}/V_i. As before, we can, for any sequence of positive integers $a_1, \ldots, a_k$ with sum n, look at the group of block-upper-triangular unipotent matrices; this is the subgroup of automorphisms of $\mathbb{R}^n$ preserving a partial flag and acting as the identity on successive quotients, i.e., matrices of the form

$$\left(\begin{array}{c|c|c|c} I & * & * & * \\ \hline 0 & I & * & * \\ \hline 0 & 0 & I & * \\ \hline 0 & 0 & 0 & I \end{array}\right) \begin{array}{l} \}a_1 \\ \}a_2 \\ \\ \}a_k \end{array}$$

Next, there are the subgroups of $\mathrm{GL}_n\mathbb{R}$ defined as the group of transformations of $V = \mathbb{R}^n$ of determinant 1 preserving some bilinear form $Q\colon V \times V \to \mathbb{R}$. If the bilinear form Q is symmetric and positive definite, the group we get is called the (*special*) *orthogonal group* $\mathrm{SO}_n\mathbb{R}$ (sometimes written $\mathrm{SO}(n)$; see p. 100). If Q is symmetric and nondegenerate but not definite—e.g., if it has k positive eigenvalues and l negative—the group is denoted $\mathrm{SO}_{k,l}\mathbb{R}$ or $\mathrm{SO}(k, l)$; note that $\mathrm{SO}(k, l) \cong \mathrm{SO}(l, k)$. If Q is skew-symmetric and nondegenerate, the group is called the *symplectic group* and denoted $\mathrm{Sp}_n\mathbb{R}$; note that in this case n must be even.

The equations that define the subgroup of $\mathrm{GL}_n\mathbb{R}$ preserving a bilinear form Q are easy to write down. If we represent Q by a matrix M—that is, we write

$$Q(v, w) = {}^t v \cdot M \cdot w$$

for all $v, w \in \mathbb{R}^n$—then the condition

$$Q(Av, Aw) = Q(v, w)$$

translates into the condition that

$${}^t v \cdot {}^t A \cdot M \cdot A \cdot w = {}^t v \cdot M \cdot w$$

for all v and w; this is equivalent to saying that

$${}^t A \cdot M \cdot A = M.$$

Thus, for example, if Q is the symmetric form $Q(v, w) = {}^t v \cdot w$ given by the identity matrix $M = I_n$, the group $\mathrm{SO}_n\mathbb{R}$ is just the group of $n \times n$ real matrices A of determinant 1 such that ${}^t A = A^{-1}$.

Exercise 7.1*. Show that in the case of $\mathrm{Sp}_{2n}\mathbb{R}$ the requirement that the transformations have determinant 1 is redundant; whereas in the case of $\mathrm{SO}_n\mathbb{R}$ if we do not require the transformations to have determinant 1 the group we get (denoted $\mathrm{O}_n\mathbb{R}$, or sometimes $\mathrm{O}(n)$) is disconnected.

Exercise 7.2*. Show that $\mathrm{SO}(k, l)$ has two connected components if k and l are both positive. The connected component containing the identity is often denoted $\mathrm{SO}^+(k, l)$. (Composing with a projection onto $\mathbb{R}^k$ or $\mathbb{R}^l$, we may associate to an automorphism $A \in \mathrm{SO}(k, l)$ automorphisms of $\mathbb{R}^k$ and $\mathbb{R}^l$; $\mathrm{SO}^+(k, l)$ will consist of those $A \in \mathrm{SO}(k, l)$ whose associated automorphisms preserve the orientations of $\mathbb{R}^k$ and $\mathbb{R}^l$.)

Note that if the form Q is degenerate, a transformation preserving Q will carry its kernel

$$\mathrm{Ker}(Q) = \{v \in V: Q(v, w) = 0 \ \forall w \in V\}$$

into itself; so that the group we get is simply the group of matrices preserving the subspace $\mathrm{Ker}(Q)$ and preserving the induced nondegenerate form $\tilde{Q}$ on the quotient $V/\mathrm{Ker}(Q)$. Likewise, if Q is a general bilinear form, that is, neither symmetric nor skew-symmetric, a linear transformation preserving Q will preserve the symmetric and skew-symmetric parts of Q individually, so we just get an intersection of the subgroups encountered already. At any rate, we usually limit our attention to nondegenerate forms that are either symmetric or skew-symmetric.

Of course, the group $\mathrm{GL}_n\mathbb{C}$ of complex linear automorphisms of a complex vector space $V = \mathbb{C}^n$ can be viewed as subgroup of the general linear group $\mathrm{GL}_{2n}\mathbb{R}$; it is, thus, a real Lie group as well, as is the subgroup $\mathrm{SL}_n\mathbb{C}$ of $n \times n$ complex matrices of determinant 1. Similarly, the subgroups $\mathrm{SO}_n\mathbb{C} \subset \mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C} \subset \mathrm{SL}_{2n}\mathbb{C}$ of transformations of a complex vector space preserving a symmetric and skew-symmetric nondegenerate *bilinear* form, respectively, are real as well as complex Lie subgroups. Note that since all nondegenerate bilinear symmetric forms on a complex vector space are isomorphic (in particular, there is no such thing as a signature), there is only one complex orthogonal subgroup $SO_n\mathbb{C} \subset SL_n\mathbb{C}$ up to conjugation; there are no analogs of the groups $SO_{k,l}\mathbb{R}$.

Another example we can come up with here is the *unitary group* U_n or $U(n)$, defined to be the group of complex linear automorphisms of an n-dimensional complex vector space V preserving a positive definite *Hermitian* inner product H on V. (A *Hermitian* form H is required to be conjugate linear in the first[2] factor, and linear in the second: $H(\lambda v, \mu w) = \bar{\lambda} H(v, w)\mu$, and $H(w, v) = \overline{H(v, w)}$; it is *positive definite* if $H(v, v) > 0$ for $v \neq 0$.)

Just as in the case of the subgroups SO and Sp, it is easy to write down the equations for $U(n)$: for some $n \times n$ matrix M we can write the form H as

$$H(v, w) = {}^t\bar{v} \cdot M \cdot w, \quad \forall v, w \in \mathbb{C}^n$$

(note that for H to be conjugate symmetric, M must be conjugate symmetric, i.e., ${}^tM = \bar{M}$); then the group $U(n)$ is just the group of $n \times n$ complex matrices A satisfying

$${}^t\bar{A} \cdot M \cdot A = M.$$

In particular, if H is the "standard" Hermitian inner product $H(v, w) = {}^t\bar{v} \cdot w$ given by the identity matrix, $U(n)$ will be the group of $n \times n$ complex matrices A such that ${}^t\bar{A} = A^{-1}$.

Exercise 7.3. Show that if H is a Hermitian form on a complex vector space V, then the real part $R = \operatorname{Re}(H)$ of H is a symmetric form on the underlying real space, and the imaginary part $C = \operatorname{Im}(H)$ is a skew-symmetric real form; these are related by $C(v, w) = R(iv, w)$. Both R and C are invariant by multiplication by i: $R(iv, iw) = R(v, w)$. Show conversely that any such real symmetric R is the real part of a unique Hermitian H. Show that if H is standard, so is R, and C corresponds to the matrix $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Deduce that

$$U(n) = O(2n) \cap Sp_{2n}\mathbb{R}.$$

Note that the determinant of a unitary matrix can be any complex number of modulus 1; the *special unitary group*, $SU(n)$, is the subgroup of $U(n)$ of automorphisms with determinant 1. The subgroup of $GL_n\mathbb{C}$ preserving an indefinite Hermitian inner product with k positive eigenvalues and l negative ones is denoted $U_{k,l}$ or $U(k, l)$; the subgroup of those of determinant 1 is denoted $SU_{k,l}$ or $SU(k, l)$.

In a similar vein, the group $GL_n\mathbb{H}$ of quaternionic linear automorphisms of an n-dimensional vector space V over the ring $\mathbb{H}$ of quaternions is a real

[2] This choice of which factor is linear and which conjugate linear is less common than the other. It makes little difference in what follows, but it does have the small advantage of being compatible with the natural choice for quaternions.

Lie subgroup of the group $GL_{4n}\mathbb{R}$, as are the further subgroups of $\mathbb{H}$-linear transformations of V preserving a bilinear form. Since $\mathbb{H}$ is not commutative, care must be taken with the conventions here, and it may be worth a little digression to go through this now. We take the vector spaces V to be right $\mathbb{H}$-modules; $\mathbb{H}^n$ is the space of column vectors with *right* multiplication by scalars. In this way the $n \times n$ matrices with entries in $\mathbb{H}$ act in the usual way on $\mathbb{H}^n$ on the left. Scalar multiplication on the left (only) is $\mathbb{H}$-linear.

View $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C} \cong \mathbb{C}^2$. Then left multiplication by elements of $\mathbb{H}$ give $\mathbb{C}$-linear endomorphisms of $\mathbb{C}^2$, which determines a mapping $\mathbb{H} \to M_2\mathbb{C}$ to the 2×2 complex matrices. In particular, $\mathbb{H}^* = GL_1\mathbb{H} \hookrightarrow GL_2\mathbb{C}$. Similarly $\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n = \mathbb{C}^{2n}$, so we have an embedding $GL_n\mathbb{H} \hookrightarrow GL_{2n}\mathbb{C}$. Note that a $\mathbb{C}$-linear mapping $\varphi\colon \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$-linear exactly when it commutes with j: $\varphi(vj) = \varphi(v)j$. If $v = v_1 + jv_2$, then $v \cdot j = -\bar{v}_2 + j\bar{v}_1$, so multiplication by j takes $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ to $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$. It follows that if J is the matrix of the preceding exercise, then

$$GL_n\mathbb{H} = \{A \in GL_{2n}\mathbb{C}\colon AJ = J\bar{A}\}.$$

Those matrices with real determinant 1 form a subgroup $SL_n\mathbb{H}$.

A *Hermitian form* (or "symplectic scalar product") on a quaternionic vector space V is an $\mathbb{R}$-bilinear form $K\colon V \times V \to \mathbb{H}$ that is conjugate $\mathbb{H}$-linear in the first factor and $\mathbb{H}$-linear in the second: $K(v\lambda, w\mu) = \bar{\lambda}K(v, w)\mu$, and satisfies $K(w, v) = \overline{K(v, w)}$. It is *positive definite* if $K(v, v) > 0$ for $v \neq 0$. (The conjugate $\bar{\lambda}$ of a quaternion $\lambda = a + bi + cj + dk$ is defined to be $a - bi - cj - dk$.) The *standard* Hermitian form on $\mathbb{H}^n$ is $\Sigma\bar{v}_i w_i$. The group of automorphisms of an n-dimensional quaternionic space preserving such a form is called the *compact symplectic* group and denoted $Sp(n)$ or $U_{\mathbb{H}}(n)$. The *standard* Hermitian form on $\mathbb{H}^n$ is $\Sigma\bar{v}_i w_i$.

Exercise 7.4. Regarding V as a complex vector space, show that every quaternionic Hermitian form K has the form

$$K(v, w) = H(v, w) + jQ(v, w),$$

where H is a complex Hermitian form and Q is a skew-symmetric complex linear form on V, with H and Q related by $Q(v, w) = H(vj, w)$, and H satisfying the condition $H(vj, wj) = \overline{H(v, w)}$. Conversely, any such Hermitian H is the complex part of a unique K. If K is standard, so is H, and Q is given by the same matrix as in Exercise 7.3. Deduce that

$$Sp(n) = U(2n) \cap Sp_{2n}\mathbb{C}.$$

This shows that the two notions of "symplectic" are compatible.

More generally, if K is not positive definite, but has signature (p, q), say the standard $\sum_{i=1}^{p} \bar{v}_i w_i - \sum_{i=p+1}^{p+q} \bar{v}_i w_i$, the automorphisms preserving it form a group $U_{p,q}\mathbb{H}$. Or if the form is a skew Hermitian form (satisfying the same linearity conditions, but with $K(w, v) = -\overline{K(v, w)}$), the group is denoted $U_n^*\mathbb{H}$.

Exercise 7.5. Identify, among all the real Lie groups described above, which ones are compact.

Complex Lie Groups

So far, all of our examples have been examples of real Lie groups. As for *complex* Lie groups, these are fewer in number. The general linear group $GL_n\mathbb{C}$ is one, and again, all the elementary examples come to us as subgroups of the general linear group $GL_n\mathbb{C}$. There is, for example, the subgroup $SO_n\mathbb{C}$ of automorphisms of an n-dimensional complex vector space V having determinant 1 and preserving a nondegenerate symmetric bilinear form Q (note that Q no longer has a signature); and likewise the subgroup $Sp_n\mathbb{C}$ of transformations of determinant 1 preserving a skew-symmetric bilinear form.

Exercise 7.6. Show that the subgroup $SU(n) \subset SL_n\mathbb{C}$ is *not* a complex Lie subgroup. (It is not enough to observe that the defining equations given above are not holomorphic.)

Exercise 7.7. Show that none of the complex Lie groups described above is compact.

We should remark here that both of these exercises are immediate consequences of the general fact that *any compact complex Lie group is abelian*; we will prove this in the next lecture. A *representation* of a complex Lie group G is a map of complex Lie groups from G to $GL(V) = GL_n\mathbb{C}$ for an n-dimensional complex vector space V; note that such a map is required to be complex analytic.

Remarks on Notation

A common convention is to use a notation without subscripts or mention of ground field to denote the *real groups*:

$$O(n), \quad SO(n), \quad SO(p, q), \quad U(n), \quad SU(n), \quad SU(p, q), \quad Sp(n)$$

and to use subscripts for the algebraic groups GL_n, SL_n, SO_n, and Sp_n. This, of course, introduces some anomalies: for example, $SO_n\mathbb{R}$ is $SO(n)$, but $Sp_n\mathbb{R}$ is not $Sp(n)$; but some violation of symmetry seems inevitable in any notation. The notations $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ are often used in place of our $GL_n\mathbb{R}$ or $GL_n\mathbb{C}$, and similarly for SL, SO, and Sp.

Also, where we have written Sp_{2n}, some write Sp_n. In practice, it seems that those most interested in algebraic groups or Lie algebras use the former notation, and those interested in compact groups the latter. Other common notations are $U^*(2n)$ in place of our $GL_n\mathbb{H}$, $Sp(p, q)$ for our $U_{p,q}\mathbb{H}$, and $O^*(2n)$ for our $U_n^*\mathbb{H}$.

Exercise 7.8. Find the dimensions of the various real Lie groups $\mathrm{GL}_n\mathbb{R}$, $\mathrm{SL}_n\mathbb{R}$, B_n, N_n, $\mathrm{SO}_n\mathbb{R}$, $\mathrm{SO}_{k,l}\mathbb{R}$, $\mathrm{Sp}_{2n}\mathbb{R}$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{GL}_n\mathbb{C}$, $\mathrm{SL}_n\mathbb{C}$, $\mathrm{GL}_n\mathbb{H}$, and $\mathrm{Sp}(n)$ introduced above.

§7.3. Two Constructions

There are two constructions, in some sense inverse to one another, that arise frequently in dealing with Lie groups (and that also provide us with further examples of Lie groups). They are expressed in the following two statements.

Proposition 7.9. *Let G be a Lie group, H a connected manifold, and $\varphi: H \to G$ a covering space map.*[3] *Let e' be an element lying over the identity e of G. Then there is a unique Lie group structure on H such that e' is the identity and φ is a map of Lie groups; and the kernel of φ is in the center of H.*

Proposition 7.10. *Let H be a Lie group, and $\Gamma \subset Z(H)$ a discrete subgroup of its center. Then there is a unique Lie group structure on the quotient group $G = H/\Gamma$ such that the quotient map $H \to G$ is a Lie group map.*

The proof of the second proposition is straightforward. To prove the first, one shows that the multiplication on G lifts uniquely to a map $H \times H \to H$ which takes (e', e') to e', and verifies that this product satisfies the group axioms. In fact, it suffices to do this when H is the universal covering of G, for one can then apply the second proposition to intermediate coverings. □

Exercise 7.11*. (a) Show that any discrete normal subgroup of a connected Lie group G is in the center $Z(G)$.

(b) If $Z(G)$ is discrete, show that $G/Z(G)$ has trivial center.

These two propositions motivate a definition: we say that a Lie group map between two Lie groups G and H is an *isogeny* if it is a covering space map of the underlying manifolds; and we say two Lie groups G and H are *isogenous* if there is an isogeny between them (in either direction). Isogeny is not an equivalence relation, but generates one; observe that every isogeny equivalence class has an initial member (that is, one that maps to every other one by an isogeny)—that is, just the universal covering space $\tilde{G}$ of any one—and, if the center of this universal cover is discrete, as will be the case for all our semisimple groups, a final object $\tilde{G}/Z(\tilde{G})$ as well. For any group G in such an equivalence class, we will call $\tilde{G}$ the *simply connected form* of the group G, and $\tilde{G}/Z(\tilde{G})$ (if it exists) the *adjoint form* (we will see later a more general definition of adjoint form).

[3] This means that φ is a continuous map with the property that every point of G has a neighborhood U such that $\varphi^{-1}(U)$ is a disjoint union of open sets each mapping homeomorphically to U.

Exercise 7.12. If $H \to G$ is a covering of connected Lie groups, show that $Z(G)$ is discrete if and only if $Z(H)$ is discrete, and then $H/Z(H) = G/Z(G)$. Therefore, if $Z(G)$ is discrete, the adjoint form of G exists and is $G/Z(G)$.

To apply these ideas to some of the examples discussed, note that the center of SL_n (over $\mathbb{R}$ or $\mathbb{C}$) is just the subgroup of multiples of the identity by an nth root of unity; the quotient may be denoted $\mathrm{PSL}_n\mathbb{R}$ or $\mathrm{PSL}_n\mathbb{C}$. In the complex case, $\mathrm{PSL}_n\mathbb{C}$ is isomorphic to the quotient of $\mathrm{GL}_n\mathbb{C}$ by its center $\mathbb{C}^*$ of scalar matrices, and so one often writes $\mathrm{PGL}_n\mathbb{C}$ instead of $\mathrm{PSL}_n\mathbb{C}$. The center of the group SO_n is the subgroup $\{\pm I\}$ when n is even, and trivial when n is odd; in the former case the quotient will be denoted $\mathrm{PSO}_n\mathbb{R}$ or $\mathrm{PSO}_n\mathbb{C}$. Finally the center of the group Sp_{2n} is similarly the subgroup $\{\pm I\}$, and the quotient is denoted $\mathrm{PSp}_{2n}\mathbb{R}$ or $\mathrm{PSp}_{2n}\mathbb{C}$.

Exercise 7.13*. Realize $\mathrm{PGL}_n\mathbb{C}$ as a matrix group, i.e., find an embedding (faithful representation) $\mathrm{PGL}_n\mathbb{C} \hookrightarrow \mathrm{GL}_N\mathbb{C}$ for some N. Do the same for the other quotients above.

In the other direction, whenever we have a Lie group that is not simply connected, we can ask what its universal covering space is. This is, for example, how the famous *spin groups* arise: as we will see, the orthogonal groups $\mathrm{SO}_n\mathbb{R}$ and $\mathrm{SO}_n\mathbb{C}$ have fundamental group $\mathbb{Z}/2$, and so by the above there exist connected, two-sheeted covers of these groups. These are denoted $\mathrm{Spin}_n\mathbb{R}$ and $\mathrm{Spin}_n\mathbb{C}$, and will be discussed in Lecture 20; for the time being, the reader may find it worthwhile (if frustrating) to try to realize these as matrix groups. The last exercises of this section sketch a few steps in this direction which can be done now by hand.

Exercise 7.14. Show that the universal covering of $U(n)$ can be identified with the subgroup of the product $U(n) \times \mathbb{R}$ consisting of pairs (g, t) with $\det(g) = e^{\pi i t}$.

Exercise 7.15. We have seen in Exercise 7.4 that

$$\mathrm{SU}(2) = \mathrm{Sp}(2) = \{q \in \mathbb{H}: q\bar{q} = 1\}.$$

Identifying $\mathbb{R}^3$ with the imaginary quaternions (with basis i, j, k), show that, for $q\bar{q} = 1$, the map $v \mapsto qv\bar{q}$ maps $\mathbb{R}^3$ to itself, and is an isometry. Verify that the resulting map

$$\mathrm{SU}(2) = \mathrm{Sp}(2) \to \mathrm{SO}(3)$$

is a $2:1$ covering map. Since the equation $q\bar{q} = 1$ describes a 3-sphere, $\mathrm{SU}(2)$ is the universal covering of $\mathrm{SO}(3)$; and $\mathrm{SO}(3)$ is the adjoint form of $\mathrm{SU}(2)$.

Exercise 7.16. Let $M_2\mathbb{C} = \mathbb{C}^4$ be the space of 2×2 matrices, with symmetric form $Q(A, B) = \frac{1}{2}\,\mathrm{Trace}(AB^\natural)$, where $B^\natural$ is the adjoint of the matrix B; the quadratic form associated to Q is the determinant. For g and h in $\mathrm{SL}_2\mathbb{C}$, the mapping $A \mapsto gAh^{-1}$ is in $\mathrm{SO}_4\mathbb{C}$. Show that this gives a 2 : 1 covering

$$\mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C} \to \mathrm{SO}_4\mathbb{C},$$

which, since $\mathrm{SL}_2\mathbb{C}$ is simply connected, realizes the universal covering of $\mathrm{SO}_4\mathbb{C}$.

Exercise 7.17. Identify $\mathbb{C}^3$ with the space of traceless matrices in $M_2\mathbb{C}$, so $g \in \mathrm{SL}_2\mathbb{C}$ acts by $A \mapsto gAg^{-1}$. Show that this gives a 2 : 1 covering

$$\mathrm{SL}_2\mathbb{C} \to \mathrm{SO}_3\mathbb{C},$$

which realizes the universal covering of $\mathrm{SO}_3\mathbb{C}$.