LECTURE 25

More Character Formulas

In this lecture we give two more formulas for the multiplicities of an irreducible representation of a semisimple Lie algebra or group. First, Freudenthal's formula (§25.1) gives a straightforward way of calculating the multiplicity of a given weight once we know the multiplicity of all higher ones. This in turn allows us to prove in §25.2 the Weyl character formula, as well as another multiplicity formula due to Kostant. Finally, in §25.3 we give Steinberg's formula for the decomposition of the tensor product of two arbitrary irreducible representations of a semisimple Lie algebra, and also give formulas for some pairs $\mathfrak{h} \subset \mathfrak{g}$ for the decomposition of the restriction to $\mathfrak{h}$ of irreducible representations of $\mathfrak{g}$.

§25.1. Freudenthal's Multiplicity Formula

Freudenthal's formula gives a general way of computing the multiplicities of a representation, i.e., the dimensions of its weight spaces, by working down successively from the highest weight. The result is similar to (but more complicated than) what we did for $\mathfrak{sl}_3\mathbb{C}$ in Lecture 13, where we found the multiplicities along successive concentric hexagons in the weight diagram.

Let Γ_λ be the irreducible representation with highest weight λ, which will be fixed throughout this discussion. Let $n_\mu = n_\mu(\Gamma_\lambda)$ be the dimension of the weight space[1] of weight μ in Γ_λ, i.e., $\mathrm{Char}(\Gamma_\lambda) = \sum n_\mu e(\mu)$. Freudenthal gives a formula for n_μ in terms of multiplicities of weights that are higher than μ.

[1] In the literature, these multiplicities n_μ are often referred to as "inner multiplicities."

W. Fulton et al., *Representation Theory*
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Proposition 25.1 (Freudenthal's Multiplicity Formula). *With the above notation,*

$$c(\mu)\cdot n_\mu(\Gamma_\lambda) = 2 \sum_{\alpha\in R^+} \sum_{k\geq 1} (\mu + k\alpha, \alpha) n_{\mu+k\alpha},$$

where $c(\mu) = \|\lambda + \rho\|^2 - \|\mu + \rho\|^2$.

Here $\|\beta\|^2 = (\beta, \beta)$, $(\ ,\ )$ is the Killing form, and ρ is half the sum of the positive roots.

Exercise 25.2*. Verify that $c(\mu)$ is positive if $\mu \neq \lambda$ and $n_\mu > 0$.

The proof of Freudenthal's formula uses a *Casimir operator*, denoted C. This is an endomorphism of any representation V of the semisimple Lie algebra $\mathfrak{g}$, and is constructed as follows. Take any basis $U_1, \ldots, U_r$ for $\mathfrak{g}$, and let $U_1', \ldots, U_r'$ be the dual basis with respect to the Killing form on $\mathfrak{g}$. Set

$$C = U_1 U_1' + \cdots + U_r U_r',$$

i.e., for any $v \in V$, $C(v) = \sum U_i \cdot (U_i' \cdot v)$.

Exercise 25.3. Verify that C is independent of the choice of basis[2].

The key fact is

Exercise 25.4*. Show that C commutes with every operation in $\mathfrak{g}$, i.e.,

$$C(X \cdot v) = X \cdot C(v) \quad \text{for all } X \in \mathfrak{g},\ v \in V.$$

The idea is to use a special basis for the construction of C, so that each term $U_i U_i'$ will act as multiplication by a constant on any weight space, and this constant can be calculated in terms of multiplicities. Then Schur's lemma can be applied to know that, in case V is irreducible, C itself is multiplication by a scalar. Taking traces will lead to a relation among multiplicities, and a little algebraic manipulation will give Freudenthal's formula.

The basis for $\mathfrak{g}$ to use is a natural one: Choose the basis $H_1, \ldots, H_n$ for the Cartan subalgebra $\mathfrak{h}$, where $H_i = H_{\alpha_i}$ corresponds to the simple root α_i, and let H_i' be the dual basis for the restriction of the Killing form to $\mathfrak{h}$. For each root α, choose a nonzero $X_\alpha \in \mathfrak{g}_\alpha$. The dual basis will then have X_α' in $\mathfrak{g}_{-\alpha}$. In fact, if we let $Y_\alpha \in \mathfrak{g}_{-\alpha}$ be the usual element so that X_α, Y_α, and $H_\alpha = [X_\alpha, Y_\alpha]$ are the canonical basis for the subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$ that they span, then

$$X_\alpha' = ((\alpha, \alpha)/2)\, Y_\alpha. \tag{25.5}$$

Exercise 25.6*. Verify (25.5) by showing that $(X_\alpha, Y_\alpha) = 2/(\alpha, \alpha)$.

[2] In fancy language, C is an element of the universal enveloping algebra of $\mathfrak{g}$, but we do not need this.

Now we have the Casimir operator

$$C = \sum H_i H_i' + \sum_{\alpha \in R} X_\alpha X_\alpha',$$

and we analyze the action of C on the weight space V_μ corresponding to weight μ for any representation V. Let $n_\mu = \dim(V_\mu)$. First we have

$$\sum H_i H_i' \text{ acts on } V_\mu \text{ by multiplication by } (\mu, \mu) = \|\mu\|^2. \tag{25.7}$$

Indeed, $H_i H_i'$ acts by multiplication by $\mu(H_i)\mu(H_i')$. If we write $\mu = \sum r_i \omega_i$, where the ω_i are the fundamental weights, then $\mu(H_i) = r_i$, and if $\mu = \sum r_i' \omega_i'$, with ω_i' the dual basis to ω_i, then similarly $\mu(H_i') = r_i'$. Hence $\sum \mu(H_i)\mu(H_i') = \sum r_i r_i' = (\mu, \mu)$, as asserted.

Now consider the action of $X_\alpha X_\alpha' = ((\alpha, \alpha)/2) X_\alpha Y_\alpha$ on V_μ. Restricting to the subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ and to the subrepresentation $\bigoplus_i V_{\mu + i\alpha}$ corresponding to the α-string through μ, we are in a situation which we know very well. Suppose this string is

$$V_\beta \oplus V_{\beta-\alpha} \oplus \cdots \oplus V_{\beta - m\alpha},$$

so $m = \beta(H_\alpha)$ [cf. (14.10)], and let k be the integer such that $\mu = \beta - k\alpha$. We assume for now that $k \leq m/2$.

On the first term V_β, $X_\alpha Y_\alpha$ acts by multiplication by $m = \beta(H_\alpha) = 2(\beta, \alpha)/(\alpha, \alpha)$, so $X_\alpha X_\alpha'$ acts by multiplication by (β, α). In general, on the part of $V_{\beta - k\alpha}$ which is the image of V_β by multiplication by $(Y_\alpha)^k$, we know [cf. (11.5)] that $X_\alpha Y_\alpha$ acts by multiplication by $(k+1)(m-k)$. This gives us a subspace of V_μ of dimension n_β on which $X_\alpha X_\alpha'$ acts by multiplication by

$$(k+1)((\beta, \alpha) - k(\alpha, \alpha)/2) = (k+1)((\mu, \alpha) + k(\alpha, \alpha)/2).$$

Now peel off the subrepresentation (over $\mathfrak{s}_\alpha$) of V spanned by V_β, and apply the same reasoning to what is left. We have a subspace of $V_{\beta-\alpha}$ of dimension $n_{\beta-\alpha} - n_\beta$ to which the same analysis can be made. From this we get a subspace of V_μ of dimension $n_{\beta-\alpha} - n_\beta$ on which $X_\alpha X_\alpha'$ acts by multiplication by

$$(k)((\mu, \alpha) + (k-1)(\alpha, \alpha)/2).$$

Continuing to peel off subrepresentations, the space V_μ is decomposed into pieces on which $X_\alpha X_\alpha'$ acts by multiplication by a scalar. The trace of $X_\alpha X_\alpha'$ on V_μ is therefore the sum

$$\begin{aligned} & n_\beta \cdot (k+1)((\mu, \alpha) + k(\alpha, \alpha)/2) + (n_{\beta-\alpha} - n_\beta) \cdot (k)((\mu, \alpha) + (k-1)(\alpha, \alpha)/2) \\ & \quad + \cdots + ((n_{\beta - k\alpha} - n_{\beta-(k-1)\alpha}) \cdot (1)((\mu, \alpha) + (0)(\alpha, \alpha)/2). \end{aligned}$$

Canceling in successive terms, this simplifies to

$$\operatorname{Trace}(X_\alpha X_\alpha'|_{V_\mu}) = \sum_{i=0}^{k} (\mu + i\alpha, \alpha) n_{\mu + i\alpha}. \tag{25.8}$$

One pleasant fact about this sum is that it may be extended to all $i \geq 0$, since $n_{\mu + i\alpha} = 0$ for $i > k$.

In case $k \geq m/2$, the computation is similar, peeling off representations from the other end, starting with $V_{\beta-m\alpha}$. The only difference is that the action of $X_\alpha Y_\alpha$ on $V_{\beta-m\alpha}$ is zero. The result is

$$\operatorname{Trace}(X_\alpha X'_\alpha|_{V_\mu}) = -\sum_{i=1}^{\infty} (\mu - i\alpha, \alpha) n_{\mu-i\alpha}. \tag{25.9}$$

Exercise 25.10. Show that $X_\alpha X'_\alpha = X_{-\alpha} X'_{-\alpha} + ((\alpha, \alpha)/2) H_\alpha$, and deduce (25.9) directly from (25.8) by replacing α by $-\alpha$.

In fact, (25.8) is valid for all μ and α, as we see from the identity

$$\sum_{i=-\infty}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} = 0. \tag{25.11}$$

Exercise 25.12*. Verify (25.11) by using the symmetry of the α-string through β.

Now we add the assumption that V is irreducible, so C is multiplication by some scalar c. Taking the trace of C on V_μ and adding, we get

$$c n_\mu = (\mu, \mu) n_\mu + \sum_{\alpha \in R} \sum_{i \geq 0} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}. \tag{25.13}$$

Note that when $i = 0$ the two terms for α and $-\alpha$ cancel each other, so the summation can begin at $i = 1$ instead. Rewriting this in terms of the positive weights, and using (25.11) the sums become

$$\sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha} + \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu - i\alpha, \alpha) n_{\mu-i\alpha}$$

$$= n_\mu \sum_{\alpha \in R^+} (\mu, \alpha) + 2 \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}.$$

Summarizing, and observing that $\sum_{\alpha \in R^+} (\mu, \alpha) = (\mu, 2\rho)$, we have

$$c n_\mu = ((\mu, \mu) + (\mu, 2\rho)) n_\mu + 2 \sum_{\alpha \in R^+} \sum_{i=1}^{\infty} (\mu + i\alpha, \alpha) n_{\mu+i\alpha}.$$

Note that $(\mu, \mu) + (\mu, 2\rho) = (\mu + \rho, \mu + \rho) - (\rho, \rho) = \|\mu + \rho\|^2 - \|\rho\|^2$. To evaluate the constant we evaluate on the highest weight space V_λ, where $n_\lambda = 1$ and $n_{\lambda+i\alpha} = 0$ for $i > 0$. Hence,

$$c = (\lambda, \lambda) + (\lambda, 2\rho) = \|\lambda + \rho\|^2 - \|\rho\|^2. \tag{25.14}$$

Combining the preceding two equations yields Freudenthal's formula. □

Exercise 25.15. Apply Freudenthal's formula to the representations of $\mathfrak{sl}_3\mathbb{C}$ considered in §13.2, verifying again that the multiplicities are as prescribed on the hexagons and triangles.

Exercise 25.16. Use Freudenthal's formula to calculate multiplicities for the representations $\Gamma_{1,0}$, $\Gamma_{0,1}$, and $\Gamma_{2,0}$ of $(\mathfrak{g}_2)$.

§25.2. Proof of (WCF); the Kostant Multiplicity Formula

It is not unreasonable to anticipate that Weyl's character formula can be deduced from Freudenthal's inductive formula, but some algebraic manipulation is certainly required. Let

$$\chi_\lambda = \operatorname{Char}(\Gamma_\lambda) = \sum n_\mu e(\mu)$$

be the character of the irreducible representation with highest weight λ. Freudenthal's formula, in form (25.13), reads[3]

$$c \cdot \chi_\lambda = \sum_\mu (\mu, \mu) n_\mu e(\mu) + \sum_\mu \sum_{\alpha \in R} \sum_{i=0}^{\infty} (\mu + i\alpha, \alpha) n_{\mu + i\alpha} e(\mu),$$

where $c = \|\lambda + \rho\|^2 - \|\rho\|^2$. To get this to look anything like Weyl's formula, we must get rid of the inside sums over i. If α is fixed, they will disappear if we multiply by $e(\alpha) - 1$, as successive terms cancel:

$$(e(\alpha) - 1) \cdot \sum_\mu \sum_{i=0}^{\infty} (\mu + i\alpha, \alpha) n_{\mu + i\alpha} e(\mu) = \sum_\mu (\mu, \alpha) n_\mu e(\mu + \alpha).$$

Let $P = \prod_{\alpha \in R} (e(\alpha) - 1) = (e(\alpha) - 1) \cdot P_\alpha$, where $P_\alpha = \prod_{\beta \neq \alpha} (e(\beta) - 1)$. The preceding two formulas give

$$c \cdot P \cdot \chi_\lambda = P \cdot \sum_\mu (\mu, \mu) n_\mu e(\mu) + \sum_{\mu, \alpha} (\mu, \alpha) P_\alpha n_\mu e(\mu + \alpha). \qquad (25.17)$$

Note also that

$$P = (-1)^r A_\rho \cdot A_\rho,$$

where r is the number of positive roots, so at least the formula now involves the ingredients that go into (WCF).

We want to prove (WCF): $A_\rho \cdot \chi_\lambda = A_{\lambda + \rho}$. We have seen in §24.1 that both sides of this equation are alternating, and that both have highest weight term $e(\lambda + \rho)$, with coefficient 1. On the right-hand side the only terms that appear are those of the form $\pm e(W(\lambda + \rho))$, for W in the Weyl group. To prove (WCF), it suffices to prove that the only terms appearing with nonzero coefficients in $A_\rho \cdot \chi_\lambda$ are these same $e(W(\lambda + \rho))$, for then the alternating property and the knowledge of the coefficient of $e(\lambda + \rho)$ determine all the coefficients. This can be expressed as:

[3] In this section we work in the ring $\mathbb{C}[\Lambda]$ of finite sums $\sum m_\mu e(\mu)$ with complex coefficients m_μ.

Claim. *The only terms $e(\nu)$ occurring in $A_\rho \cdot \chi_\lambda$ with nonzero coefficient are those with $\|\nu\| = \|\lambda + \rho\|$.*

To see that this is equivalent, note that by definition of A_ρ and χ_λ, the terms in $A_\rho \cdot \chi_\lambda$ are all of the form $\pm e(\nu)$, where $\nu = \mu + W(\rho)$, for μ a weight of Γ_λ and W in the Weyl group. But if $\|\mu + W(\rho)\| = \|\lambda + \rho\|$, since the metric is invariant by the Weyl group, this gives $\|W^{-1}(\mu) + \rho\| = \|\lambda + \rho\|$. But we saw in Exercise 25.2 that this cannot happen unless $\mu = W(\lambda)$, as required.

We are thus reduced to proving the claim. This suggests looking at the "Laplacian" operator that maps $e(\mu)$ to $\|\mu\|^2 e(\mu)$, that is, the map

$$\Delta\colon \mathbb{C}[\Lambda] \to \mathbb{C}[\Lambda]$$

defined by

$$\Delta\left(\sum m_\mu e(\mu)\right) = \sum (\mu, \mu) m_\mu e(\mu).$$

The claim is equivalent to the assertion that $F = A_\rho \cdot \chi_\lambda$ satisfies the "differential equation"

$$\Delta(F) = \|\lambda + \rho\|^2 F.$$

From the definition $\Delta(\chi_\lambda) = \sum (\mu, \mu) n_\mu e(\mu)$. And $\Delta(A_\rho) = \|\rho\|^2 A_\rho$. In general, since $\|W(\alpha)\| = \|\alpha\|$ for all $W \in \mathfrak{W}$,

$$\Delta(A_\alpha) = \sum (-1)^W \|W(\alpha)\|^2 e(W(\alpha)) = \|\alpha\|^2 A_\alpha.$$

So we would be in good shape if we had a formula for Δ of a product of two functions. One expects such a formula to take the form

$$\Delta(fg) = \Delta(f)g + 2(\nabla f, \nabla g) + f\Delta(g), \tag{25.18}$$

where ∇ is a "gradient," and (,) is an "inner product." Taking $f = e(\mu)$, $g = e(\nu)$, we see that we need to have $(\nabla e(\mu), \nabla e(\nu)) = (\mu, \nu) e(\mu + \nu)$. There is indeed such a gradient and inner product. Define a homomorphism

$$\nabla\colon \mathbb{C}[\Lambda] \to \mathfrak{h}^* \otimes \mathbb{C}[\Lambda] = \operatorname{Hom}(\mathfrak{h}, \mathbb{C}[\Lambda])$$

by the formula $\nabla(e(\mu)) = \mu \cdot e(\mu)$, and define the bilinear form (,) on $\mathfrak{h}^* \otimes \mathbb{C}[\Lambda]$ by the formula $(\alpha e(\mu), \beta e(\nu)) = (\alpha, \beta) e(\mu + \nu)$, where (α, β) is the Killing form on $\mathfrak{h}^*$.

Exercise 25.19. With these definitions, verify that (25.18) is satisfied, as well as the Leibnitz rule

$$\nabla(fg) = \nabla(f)g + f\nabla(g).$$

For example, $\nabla(\chi_\lambda) = \sum_\mu n_\mu \mu \cdot e(\mu)$, and, by the Leibnitz rule,

$$\nabla(P) = \sum_{\alpha \in R} P_\alpha \alpha \cdot e(\alpha).$$

But now look at formula (25.17). This reads

$$c \cdot P\chi_\lambda = P\Delta(\chi_\lambda) + (\nabla P, \nabla \chi_\lambda).$$

Since, also by the exercise, $\nabla(P) = 2(-1)^r A_\rho \nabla(A_\rho)$, we may cancel $(-1)^r A_\rho$ from each term in the equation, getting

$$c \cdot A_\rho \chi_\lambda = A_\rho \Delta(\chi_\lambda) + 2(\nabla A_\rho, \nabla \chi_\lambda).$$

By the identity (25.18), the right-hand side of this equation is

$$\Delta(A_\rho \chi_\lambda) - \Delta(A_\rho)\chi_\lambda = \Delta(A_\rho \chi_\lambda) - \|\rho\|^2 A_\rho \chi_\lambda.$$

Since $c = \|\lambda + \rho\|^2 - \|\rho\|^2$, this gives $\|\lambda + \rho\|^2 A_\rho \chi_\lambda = \Delta(A_\rho \chi_\lambda)$, which finishes the proof. □

We conclude this section with a proof of another general multiplicity formula, discovered by Kostant. It gives an elegant closed formula for the multiplicities, but at the expense of summing over the entire Weyl group (although as we will indicate below, there are many interesting cases where all but a few terms of the sum vanish). It also involves a kind of partition counting function. For each weight μ, let $P(\mu)$ be the number of ways to write μ as a sum of positive roots; set $P(0) = 1$. Equivalently,

$$\prod_{\alpha \in R^+} \frac{1}{1 - e(\alpha)} = \sum_\mu P(\mu) e(\mu). \tag{25.20}$$

Proposition 25.21. (Kostant's Multiplicity Formula). *The multiplicity $n_\mu(\Gamma_\lambda)$ of weight μ in the irreducible representation Γ_λ is given by*

$$n_\mu(\Gamma_\lambda) = \sum_{W \in \mathfrak{W}} (-1)^W P(W(\lambda + \rho) - (\mu + \rho)),$$

where ρ is half the sum of the positive roots.

Proof. Write $(A_\rho)^{-1} = e(-\rho)/\prod(1 - e(-\alpha)) = \sum_\nu P(\nu)e(-\nu - \rho)$. By (WCF),

$$\begin{aligned} \chi_\lambda = A_{\lambda+\rho}(A_\rho)^{-1} &= \sum_{W,\nu} (-1)^W e(W(\lambda + \rho)) P(\nu) e(-\nu - \rho) \\ &= \sum_{W,\nu} (-1)^W P(\nu) e(W(\lambda + \rho) - (\nu + \rho)) \\ &= \sum_{W,\mu} (-1)^W P(W(\lambda + \rho) - (\mu + \rho)) e(\mu), \end{aligned}$$

as seen by writing $\mu = W(\lambda + \rho) - (\nu + \rho)$. □

In fact, the proof shows that Kostant's formula is equivalent to Weyl's formula, cf. [Cart].

One way to interpret Kostant's formula, at least for weights μ close to the highest weight λ of Γ_λ, is as a sort of converse to Proposition 14.13(ii). Recall that this says that Γ_λ will be generated by the images of its highest weight vector v under successive applications of the generators of the negative root spaces; in practice, we used this fact to bound from above the multiplicities of various weights μ close to λ by counting the number of ways of getting from λ to μ by adding negative roots. The problem in making this precise was always that we did not know how many relations there were among these images, if any. Kostant's formula gives an answer: for example, if the difference $\lambda - \mu$ is small relative to λ, we see that the only nonzero term in the sum is the principle term, corresponding to $W = 1$; in this case the answer is that there are no relations other than the trivial ones $X(Y(v)) - Y(X(v)) = [X, Y](v)$. When μ gets somewhat smaller, other terms appear corresponding to single reflections W in the walls of the Weyl chamber for which $W(\lambda + \rho)$ is higher than $\mu + \rho$; we can think of these terms, which all appear with sign -1, as correction terms indicating the presence of relations. As μ gets smaller still, of course, more terms appear of both signs, and this viewpoint breaks down.

To see how this works in practice, the reader can for example carry out the analysis of the example at the end of §13.1.

Exercise 25.22* (Kostant). Prove the following formula for the function P, which can be used to calculate it inductively: $P(0) = 1$, and, for $\mu \neq 0$,

$$P(\mu) = -\sum_{W \neq 1} (-1)^W P(\mu + W(\rho) - \rho).$$

Exercise 25.23* (Racah). Deduce from Kostant's formula and the preceding exercise the following inductive formula for the multiplicities n_μ of μ in Γ_λ: $n_\mu = 1$ if $\mu = \lambda$, and if μ is any other weight of Γ_λ, then

$$n_\mu = -\sum_{W \neq 1} (-1)^W n_{\mu+\rho-W(\rho)}.$$

Show, in fact, that for any weight μ

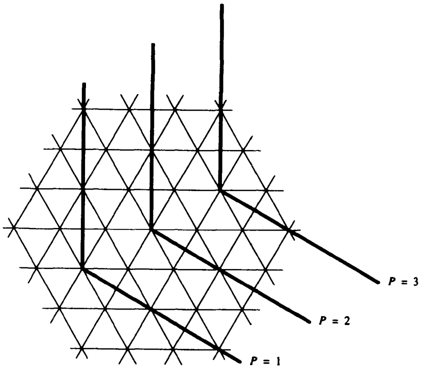
$$\sum_{W \in \mathfrak{W}} (-1)^W n_{\mu+\rho-W(\rho)} = \sum_{W'} (-1)^{W'},$$

where the second sum is over those $W' \in \mathfrak{W}$ such that $W'(\lambda + \rho) = \mu + \rho$.

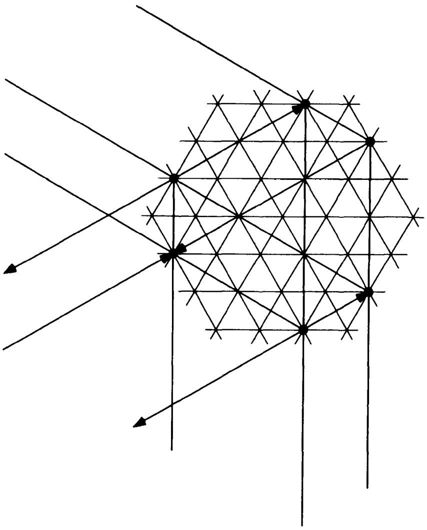
Note that Kostant's formula, more than any of the others, shows us directly the pattern of multiplicities in the irreducible representations of $\mathfrak{sl}_3\mathbb{C}$. For one thing, it is easy to represent the function P diagrammatically: in the weight lattice of $\mathfrak{sl}_3\mathbb{C}$, the function $P(\mu)$ will be a constant 1 on the rays $\{aL_2 - aL_1\}_{a \geq 0}$ and $\{aL_3 - aL_2\}_{a \geq 0}$ through the origin in the direction of the two simple positive roots $L_2 - L_1$ and $L_3 - L_2$. It will have value 2 on the translates $\{aL_2 - (a + 3)L_1\}_{a \geq -1}$ and $\{aL_3 - (a - 3)L_2\}_{a \geq 2}$ of these two rays by the third positive root $L_3 - L_1$: for example, the first of these can be written as

$$\begin{aligned} aL_2 - (a + 3)L_1 &= (a + 1)\cdot(L_2 - L_1) + L_3 - L_1 \\ &= (a + 2)\cdot(L_2 - L_1) + L_3 - L_2; \end{aligned}$$

and correspondingly its value will increase by 1 on each successive translate of these rays by $L_3 - L_1$. The picture is thus



Now, the prescription given in the Kostant formula for the multiplicities is to take six copies of this function flipped about the origin, translated so that the vertex of the outer shell lies at the points $w(\lambda + \rho) - \rho$ and take their alternating sum. Superimposing the six pictures we arrive at



which shows us clearly the hexagonal pattern of the multiplicities.

Exercise 25.24*. A nonzero dominant weight λ of a simple Lie algebra is called *minuscule* if $\lambda(H_\alpha) = 0$ or 1 for each positive root α.

(a) Show that if λ is minuscule, then every weight space of Γ_λ is one dimensional.

(b) Show that λ is minuscule if and only if all the weights of Γ_λ are conjugate under the Weyl group.
(c) Show that a minuscule weight must be one of the fundamental weights. Find the minuscule weights for each simple Lie algebra.

§25.3. Tensor Products and Restrictions To Subgroups

In the case of the general or special linear groups, we saw general formulas for describing how the tensor product $\Gamma_\lambda \otimes \Gamma_\mu$ of two irreducible representations decomposes:

$$\Gamma_\lambda \otimes \Gamma_\mu = \bigoplus_\nu N_{\lambda\mu\nu}\Gamma_\nu.$$

In these cases the multiplicities $N_{\lambda\mu\nu}$ can be described by a combinatorial formula: the Littlewood–Richardson rule. In general, such a decomposition is equivalent to writing

$$\chi_\lambda \chi_\mu = \sum_\nu N_{\lambda\mu\nu}\chi_\nu \tag{25.25}$$

in $\mathbb{Z}[\Lambda]$, where $\chi_\lambda = \mathrm{Char}(\Gamma_\lambda)$ denotes the character.[4] By Weyl's character formula, these multiplicities $N_{\lambda\mu\nu}$ are determined by the identity

$$A_{\lambda+\rho} \cdot A_{\mu+\rho} = \sum_\nu N_{\lambda\mu\nu} A_\rho \cdot A_{\nu+\rho}. \tag{25.26}$$

This formula gives an effective procedure for calculating the coefficients $N_{\lambda\mu\nu}$, if one that is tedious in practice: we can peel off highest weights, i.e., successively subtract from $A_{\lambda+\rho} \cdot A_{\mu+\rho}$ multiples of $A_\rho \cdot A_{\nu+\rho}$ for the highest ν that appears.

There are some explicit formulas for the other classical groups. R. C. King [Ki2] has showed that for both the symplectic or orthogonal groups, the multiplicities $N_{\lambda\mu\nu}$ are given by the formula

$$N_{\lambda\mu\nu} = \sum_{\zeta,\sigma,\tau} M_{\zeta\sigma\lambda} \cdot M_{\zeta\tau\mu} \cdot M_{\sigma\tau\nu}, \tag{25.27}$$

where the M's denote the Littlewood–Richardson multiplicities, i.e., the corresponding numbers for the general linear group, and the sum is over all partitions ζ, σ, τ. For other formulas for the classical groups, see [Mur1], [Wel, p. 230].

Exercise 25.28*. For $\mathfrak{so}_4\mathbb{C}$, show that all the nonzero multiplicities $N_{\lambda\mu\nu}$ are 1's, and these occur for ν in a rectangle with sides making 45° angles to the axes. Describe this rectangle.

[4] In the literature these multiplicities $N_{\lambda\mu\nu}$ are often called "outer multiplicities," and the problem of finding them, or decomposing the tensor product, the "Clebsch–Gordan" problem.

Steinberg has also given a general formula for the multiplicities $N_{\lambda\mu\nu}$. Since it involves a double summation over the Weyl group, using it in a concrete situation may be a challenge.

Proposition 25.29 (Steinberg's Formula). *The multiplicity of Γ_ν in $\Gamma_\lambda \otimes \Gamma_\mu$ is*

$$N_{\lambda\mu\nu} = \sum_{W,W'} (-1)^{WW'} P(W(\lambda + \rho) + W'(\mu + \rho) - \nu - 2\rho),$$

where the sum is over pairs $W, W' \in \mathfrak{W}$, and P is the counting function appearing in Kostant's multiplicity formula.

Exercise 25.30*. Prove Steinberg's formula by multiplying (25.25) by A_ρ, using (WCF) to get $\chi_\lambda A_{\mu+\rho} = \sum N_{\lambda\mu\nu} A_{\nu+\rho}$. Write out both sides, using Kostant's formula for χ_λ, and compute the coefficient of the term $e(\beta + \rho)$ on each side, for any β. This gives

$$\sum_{W,W'} (-1)^{WW'} P(W(\lambda + \rho) + W'(\mu + \rho) - \beta - 2\rho) = \sum_{W} (-1)^{W} N_{\lambda,\mu,W(\beta+\rho)-\rho}.$$

Show that for $\beta = \nu$ all the terms on the right are zero but $N_{\lambda\mu\nu}$.

Exercise 25.31 (Racah). Use the Steinberg and Kostant formulas to show that

$$N_{\lambda\mu\nu} = \sum_{W} (-1)^{W} n_{\nu+\rho-W(\mu+\rho)}(\Gamma_\lambda).$$

The following is the generalization of something we have seen several times:

Exercise 25.32. If λ and μ are dominant weights, and α is a simple root with $\lambda(H_\alpha)$ and $\mu(H_\alpha)$ not zero, show that $\lambda + \mu - \alpha$ is a dominant weight and $\Gamma_\lambda \otimes \Gamma_\mu$ contains the irreducible representation $\Gamma_{\lambda+\mu-\alpha}$ with multiplicity one. So

$$\Gamma_\lambda \otimes \Gamma_\mu = \Gamma_{\lambda+\mu} \oplus \Gamma_{\lambda+\mu-\alpha} \oplus \text{others}.$$

In case $\mu = \lambda$, with $\lambda(H_\alpha) \neq 0$, $\mathrm{Sym}^2(\Gamma_\lambda)$ contains $\Gamma_{\lambda+\mu}$, while $\wedge^2(\Gamma_\lambda)$ contains $\Gamma_{\lambda+\mu-\alpha}$.

Exercise 25.33. If $\lambda + \zeta$ is a dominant weight for each weight ζ of Γ_μ, show that the irreducible representations appearing in $\Gamma_\lambda \otimes \Gamma_\mu$ are exactly the $\Gamma_{\lambda+\zeta}$. In fact, with no assumptions, every component of $\Gamma_\lambda \otimes \Gamma_\mu$ always has this form. One can show that $N_{\lambda\mu\nu}$ is the dimension of

$$\{v \in (\Gamma_\lambda)_{\nu-\mu} \colon H_i^{l_i+1}(v) = 0, 1 \leq i \leq n, l_i = \mu(H_i)\}.$$

For this, see [Žel, §131].

For other general formulas for the multiplicities $N_{\lambda\mu\nu}$ see [Kem], [K-N], [Li], and [Kum1], [Kum2].

We have seen in Exercise 6.12 a formula for decomposing the representation Γ_λ of $GL_m\mathbb{C}$ when restricted to the subgroup $GL_{m-1}\mathbb{C}$. In this case the multiplicities of the irreducible components again have a simple combinatorial description. There are similar formulas for other classical groups. In the literature, such formulas are often called "branching formulas," or "modification rules." We will just state the analogues of this formula for the symplectic and orthogonal cases:

For $\mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}$, and Γ_λ the irreducible representation of $\mathfrak{so}_{2n+1}\mathbb{C}$ given by $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, the restriction is

$$\mathrm{Res}^{\mathfrak{so}_{2n+1}\mathbb{C}}_{\mathfrak{so}_{2n}\mathbb{C}}(\Gamma_\lambda) = \bigoplus \Gamma_{\bar{\lambda}}, \tag{25.34}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n)$ with

$$\lambda_1 \geq \bar{\lambda}_1 \geq \lambda_2 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{n-1} \geq \lambda_n \geq |\bar{\lambda}_n|,$$

with the $\bar{\lambda}_i$ and λ_i simultaneously all integers or all half integers.

For $\mathfrak{so}_{2n-1}\mathbb{C} \subset \mathfrak{so}_{2n}\mathbb{C}$, and Γ_λ the irreducible representation of $\mathfrak{so}_{2n}\mathbb{C}$ given by $\lambda = (\lambda_1 \geq \cdots \geq |\lambda_n|)$,

$$\mathrm{Res}^{\mathfrak{so}_{2n}\mathbb{C}}_{\mathfrak{so}_{2n-1}\mathbb{C}}(\Gamma_\lambda) = \bigoplus \Gamma_{\bar{\lambda}}, \tag{25.35}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1})$ with

$$\lambda_1 \geq \bar{\lambda}_1 \geq \lambda_2 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{n-1} \geq |\lambda_n|,$$

with the $\bar{\lambda}_i$ and λ_i simultaneously all integers or all half integers.

For $\mathfrak{sp}_{2n-2}\mathbb{C} \subset \mathfrak{sp}_{2n}\mathbb{C}$, and Γ_λ the irreducible representation of $\mathfrak{sp}_{2n}\mathbb{C}$ given by $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, the restriction is

$$\mathrm{Res}^{\mathfrak{sp}_{2n}\mathbb{C}}_{\mathfrak{sp}_{2n-2}\mathbb{C}}(\Gamma_\lambda) = \bigoplus N_{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}}, \tag{25.36}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1})$ with $\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_{n-1} \geq 0$, and the multiplicity $N_{\lambda\bar{\lambda}}$ is the number of sequences $p_1, \ldots, p_n$ of integers satisfying

$$\lambda_1 \geq p_1 \geq \lambda_2 \geq p_2 \geq \cdots \geq \lambda_n \geq p_n \geq 0$$

and

$$p_1 \geq \bar{\lambda}_1 \geq p_2 \cdots \geq p_{n-1} \geq \bar{\lambda}_{n-1} \geq p_n.$$

As in the case of $GL_n\mathbb{C}$, these formulas are equivalent to identities among symmetric polynomials. The reader may enjoy trying to work them out from this point of view, cf. Exercise 23.43 and [Boe]. A less computational approach is given in [Žel].

As we saw in the case of the general linear group, these branching rules can be used inductively to compute the dimensions of the weight spaces. For example, for $\mathfrak{so}_m\mathbb{C}$ consider the chain

$$\mathfrak{so}_m\mathbb{C} \supset \mathfrak{so}_{m-1}\mathbb{C} \supset \mathfrak{so}_{m-2}\mathbb{C} \supset \cdots \supset \mathfrak{so}_3\mathbb{C}.$$

Decomposing a representation successively from one layer to the next will finally write it as a sum of one-dimensional weight spaces, and the dimension can be read off from the number of "partitions" in chains that start with the given λ. The representations can be constructed from these chains, as described by Gelfand and Zetlin, cf. [Žel, §10].

Similarly, one can ask for formulas for decomposing restrictions for other inclusions, such as the natural embeddings: $Sp_{2n}\mathbb{C} \subset SL_{2n}\mathbb{C}$, $SO_m\mathbb{C} \subset SL_m\mathbb{C}$, $GL_m\mathbb{C} \times GL_n\mathbb{C} \subset GL_{m+n}\mathbb{C}$, $GL_m\mathbb{C} \times GL_n\mathbb{C} \subset GL_{mn}\mathbb{C}$, $SL_n\mathbb{C} \subset Sp_{2n}\mathbb{C}$, $SL_n\mathbb{C} \subset SO_{2n+1}\mathbb{C}$, $SL_n\mathbb{C} \subset SO_{2n}\mathbb{C}$, to mention just a few. Such formulas are determined in principle by computing what happens to generators of the representation rings, which is not hard: one need only decompose exterior or symmetric products of standard representations, cf. Exercise 23.31. A few closed formulas for decomposing more general representations can also be found in the literature. We state what happens when the irreducible representations of $GL_m\mathbb{C}$ are restricted to the orthogonal or symplectic subgroups, referring to [Lit3] for the proofs:

For $O_m\mathbb{C} \subset GL_m\mathbb{C}$, with $m = 2n$ or $2n+1$, given $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$,

$$\mathrm{Res}^{GL_m\mathbb{C}}_{O_m\mathbb{C}}(\Gamma_\lambda) = \bigoplus N_{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}}, \tag{25.37}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_n \geq 0)$, where

$$N_{\lambda\bar{\lambda}} = \sum_\delta N_{\delta\bar{\lambda}\lambda},$$

with $N_{\delta\bar{\lambda}\lambda}$ the Littlewood–Richardson coefficient, and the sum over all $\delta = (\delta_1 \geq \delta_2 \geq \cdots)$ with all δ_i even.

Exercise 23.38. Show that the representation $\Gamma_{(2,2)}$ of $GL_m\mathbb{C}$ restricts to the direct sum

$$\Gamma_{(2,2)} \oplus \Gamma_{(2)} \oplus \Gamma_{(0)}$$

over $O_m\mathbb{C}$. (This decomposition is important in differential geometry: the *Riemann–Christoffel* tensor has type (2, 2), and the above three components of its decomposition are the *conformal curvature* tensor, the *Ricci* tensor, and the *scalar* curvature, respectively.)

Similarly for $Sp_{2n}\mathbb{C} \subset GL_{2n}\mathbb{C}$,

$$\mathrm{Res}^{GL_{2n}\mathbb{C}}_{Sp_{2n}\mathbb{C}}(\Gamma_\lambda) = \bigoplus N_{\lambda\bar{\lambda}}\Gamma_{\bar{\lambda}}, \tag{25.39}$$

the sum over all $\bar{\lambda} = (\bar{\lambda}_1 \geq \cdots \geq \bar{\lambda}_n \geq 0)$, where

$$N_{\lambda\bar{\lambda}} = \sum_\eta N_{\eta\bar{\lambda}\lambda},$$

$N_{\eta\bar{\lambda}\lambda}$ is the Littlewood–Richardson coefficient, and the sum is over all $\eta = (\eta_1 = \eta_2 \geq \eta_3 = \eta_4 \geq \cdots)$ with each part occurring an even number of times.

It is perhaps worth pointing out the the decomposition of tensor products is a special case of the decomposition of restrictions: the exterior tensor product $\Gamma_\lambda \boxtimes \Gamma_\mu$ of two irreducible representations of G is an irreducible representation of $G \times G$, and the restriction of this to the diagonal embedding of G in $G \times G$ is the usual tensor product $\Gamma_\lambda \otimes \Gamma_\mu$.

There are also some general formulas, valid whenever $\bar{\mathfrak{g}}$ is a semisimple Lie subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Assume that the Cartan subalgebra $\bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{h}$, so we have a restriction from $\mathfrak{h}^*$ to $\bar{\mathfrak{h}}^*$, and we assume the half-spaces determining positive roots are compatible. We write $\bar{\mu}$ for weights of $\bar{\mathfrak{g}}$, and we write $\mu \downarrow \bar{\mu}$ to mean that a weight μ of $\mathfrak{g}$ restricts to $\bar{\mu}$. Similarly write $\bar{W}$ for a typical element of the Weyl group of $\bar{\mathfrak{g}}$, and $\bar{\rho}$ for half the sum of its positive weights. If λ (resp. $\bar{\lambda}$) is a dominant weight for $\mathfrak{g}$ (resp. $\bar{\mathfrak{g}}$), let $N_{\lambda\bar{\lambda}}$ denote the multiplicity with which $\Gamma_{\bar{\lambda}}$ appears in the restriction of Γ_λ to $\bar{\mathfrak{g}}$, i.e.,

$$\mathrm{Res}(\Gamma_\lambda) = \bigoplus_{\bar{\lambda}} N_{\lambda\bar{\lambda}} \Gamma_{\bar{\lambda}}.$$

Exercise 25.40*. Show that, for any dominant weight λ of $\mathfrak{g}$ and any weight $\bar{\mu}$ of $\bar{\mathfrak{g}}$,

$$\sum_{\mu \downarrow \bar{\mu}} n_\mu(\Gamma_\lambda) = \sum_{\bar{\lambda}} N_{\lambda\bar{\lambda}} n_{\bar{\mu}}(\Gamma_{\bar{\lambda}}).$$

Exercise 25.41* (Klimyk). Show that

$$N_{\lambda\bar{\lambda}} = \sum_{\bar{W}} (-1)^{\bar{W}} \sum_{\mu \downarrow \bar{\lambda} + \bar{\rho} - \bar{W}(\bar{\rho})} n_\mu(\Gamma_\lambda).$$

Exercise 25.42. Show that if the formula of the preceding exercise is applied to the diagonal embedding of $\mathfrak{g}$ in $\mathfrak{g} \times \mathfrak{g}$, then the Racah formula of Exercise 25.31 results.

For additional formulas of a similar vein, as well as discussions of how they can be implemented on a computer, there are several articles in *SIAM J. Appl. Math.* **25**, 1973.

Finally, we note that it is possible, for any semisimple Lie algebra $\mathfrak{g}$, to make the direct sum of all its irreducible representations into a commutative algebra, generalizing constructions we saw in Lectures 15, §17, and §19. Let $\Gamma_{\omega_1}, \ldots, \Gamma_{\omega_n}$ be the irreducible representations corresponding to the fundamental weights $\omega_1, \ldots, \omega_n$. Let

$$A^\bullet = \mathrm{Sym}^\bullet(\Gamma_{\omega_1} \oplus \cdots \oplus \Gamma_{\omega_n}).$$

This is a commutative graded algebra, the direct sum of pieces

$$A^{\mathbf{a}} = \bigoplus_{a_1, \ldots, a_n} \mathrm{Sym}^{a_1}(\Gamma_{\omega_1}) \otimes \cdots \otimes \mathrm{Sym}^{a_n}(\Gamma_{\omega_n}),$$

where $\mathbf{a} = (a_1, \ldots, a_n)$ is an n-tuple of non-negative integers. Then $A^{\mathbf{a}}$ is the direct sum of the irreducible representation Γ_λ whose highest weight is $\lambda = \sum a_i \omega_i$, and a sum $J^{\mathbf{a}}$ of representations whose highest weight is strictly smaller. As before, weight considerations show that $J^\bullet = \bigoplus_{\mathbf{a}} J^{\mathbf{a}}$ is an ideal in $A^\bullet$, so the quotient

$$A^\bullet / J^\bullet = \bigoplus_{\lambda} \Gamma_\lambda$$

is the direct sum of all the irreducible representations. The product

$$\Gamma_\lambda \otimes \Gamma_\mu \to \Gamma_{\lambda+\mu}$$

in this ring is often called *Cartan multiplication*; note that the fact that $\Gamma_{\lambda+\mu}$ occurs once in the tensor product determines such a projection, but only up to multiplication by a scalar.

Using ideas of §25.1, it is possible to give generators for the ideal $J^\bullet$. If C is the Casimir operator, we know that C acts on all representations and is multiplication by the constant $c_\lambda = (\lambda, \lambda) + (2\lambda, \rho)$ on the irreducible representation with highest weight λ. Therefore, if $\lambda = \sum a_i \omega_i$, the endomorphism $C - c_\lambda I$ of $A^\bullet$ vanishes on the factor Γ_λ, and on each of the representations Γ_μ of lower weight μ it is multiplication by $c_\mu - c_\lambda \neq 0$ [cf. (25.2)]. It follows that

$$J^\bullet = \text{Image}(C - c_\lambda I: A^\bullet \to A^\bullet).$$

Exercise 25.43*. Write $C = \sum U_i U_i'$ as in §25.1. Show that for $v_1, \ldots, v_m$ vectors in the fundamental weight spaces, with $v_j \in \Gamma_{\alpha_j}$ and $\sum \alpha_j = \sum a_i \omega_i$, the element $(C - c_\lambda I)(v_1 \cdot v_2 \cdot \ldots \cdot v_m)$ is the sum over all pairs j, k, with $1 \leq j < k \leq m$, of the terms

$$\left(\sum_i (U_i(v_j) \cdot U_i'(v_k) + U_i'(v_j) \cdot U_i(v_k)) - 2(\alpha_j, \alpha_k) v_j \cdot v_k\right) \cdot \prod_{l \neq j,k} v_l.$$

From this exercise follows a theorem of Kostant: $J^\bullet$ is generated by the elements

$$\sum_i (U_i(v) \cdot U_i'(w) + U_i'(v) \cdot U_i(w)) - 2(\alpha, \beta) v \cdot w$$

for $v \in \Gamma_\alpha$, $w \in \Gamma_\beta$, with α and β fundamental roots. For the classical Lie algebras, this formula can be used to find concrete realizations of the ring. If one wants a similar ring for a semisimple Lie group, one has the same ring, of course, when the group is simply connected; this leads to the ring described in Lectures 15 and 17 for $SL_n\mathbb{C}$ and $Sp_{2n}\mathbb{C}$. For $SO_m\mathbb{C}$, little change is needed when m is odd, but there is more work for m even. Details can be found in [L-T].