

## LECTURE 18

# Orthogonal Lie Algebras

In this and the following two lectures we carry out for the orthogonal Lie algebras what we have already done in the special linear and symplectic cases. As in those cases, we start by working out in general the structure of the orthogonal Lie algebras, describing the roots, root spaces, Weyl group, etc., and then go to work on low-dimensional examples. There is one new phenomenon here: as it turns out, all three of the Lie algebras we deal with in §18.2 are isomorphic to symplectic or special linear Lie algebras we have already analyzed (this will be true of  $\mathfrak{so}_6\mathbb{C}$  as well, but of no other orthogonal Lie algebra). As in the previous cases, the analysis of the Lie algebras and their representation theory will be completely elementary. Algebraic geometry does intrude into the discussion, however: we have described the isomorphisms between the orthogonal Lie algebras discussed and special linear and symplectic ones in terms of projective geometry, since that is what seems to us most natural. This should not be a problem; there are many other ways of describing these isomorphisms, and readers who disagree with our choice can substitute their own.

§18.1:  $\mathrm{SO}_m\mathbb{C}$  and  $\mathfrak{so}_m\mathbb{C}$

§18.2: Representations of  $\mathfrak{so}_3\mathbb{C}$ ,  $\mathfrak{so}_4\mathbb{C}$ , and  $\mathfrak{so}_5\mathbb{C}$

### §18.1. $\mathrm{SO}_m\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$

We will take up now the analysis of the Lie algebras of orthogonal groups. Here there is, as we will see very shortly, a very big difference in behavior between the so-called “even” orthogonal Lie algebras  $\mathfrak{so}_{2n}\mathbb{C}$  and the “odd” orthogonal Lie algebras  $\mathfrak{so}_{2n+1}\mathbb{C}$ . Interestingly enough, the latter seem at first glance to be more complicated, especially in terms of notation; but when we analyze their representations we see that in fact they behave more regularly than the even ones. In any event, we will try to carry out the analysis in parallel

fashion for as long as is feasible; when it becomes necessary to split up into cases, we will usually look at the even orthogonal Lie algebras first and then consider the odd.

Let  $V$  be a  $m$ -dimensional complex vector space, and

$$Q: V \times V \rightarrow \mathbb{C}$$

a nondegenerate, symmetric bilinear form on  $V$ . The orthogonal group  $\text{SO}_m\mathbb{C}$  is then defined to be the group of automorphisms  $A$  of  $V$  of determinant 1 preserving  $Q$ —that is, such that  $Q(Av, Aw) = Q(v, w)$  for all  $v, w \in V$ —and the orthogonal Lie algebra  $\mathfrak{so}_m\mathbb{C}$  correspondingly consists of endomorphisms  $A: V \rightarrow V$  satisfying

$$Q(Av, w) + Q(v, Aw) = 0 \tag{18.1}$$

for all  $v$  and  $w \in V$ . As in the case of the symplectic Lie algebras, to carry out our analysis we want to write  $Q$  explicitly in terms of a basis for  $V$ , and here is where the cases of even and odd  $m$  first separate. In case  $m = 2n$  is even, we will choose a basis for  $V$  in terms of which the quadratic form  $Q$  is given by

$$Q(e_i, e_{i+n}) = Q(e_{i+n}, e_i) = 1$$

and

$$Q(e_i, e_j) = 0 \quad \text{if } j \neq i \pm n.$$

The bilinear form  $Q$  may be expressed as

$$Q(x, y) = {}^t x \cdot M \cdot y,$$

where  $M$  is the  $2n \times 2n$  matrix given in block form as

$$M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix};$$

the group  $\text{SO}_{2n}\mathbb{C}$  is thus the group of  $2n \times 2n$  matrices  $A$  with  $\det(A) = 1$  and

$$M = {}^t A \cdot M \cdot A,$$

and the Lie algebra  $\mathfrak{so}_{2n}\mathbb{C}$  correspondingly the space of matrices  $X$  satisfying the relation

$${}^t X \cdot M + M \cdot X = 0.$$

Writing a  $2n \times 2n$  matrix  $X$  in block form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we have

$${}^t X \cdot M = \begin{pmatrix} {}^t C & {}^t A \\ {}^t D & {}^t B \end{pmatrix}$$

and

$$M \cdot X = \begin{pmatrix} C & D \\ A & B \end{pmatrix}$$

so that this relation is equivalent to saying that *the off-diagonal blocks B and C of X are skew-symmetric, and the diagonal blocks A and D of X are negative transposes of each other.*

**Exercise 18.2.** Show that with this choice of basis,

$$SO_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \cong \mathbb{C}^*,$$

and  $so_2 \mathbb{C} = \mathbb{C}$ .

The situation in case the dimension  $m$  of  $V$  is odd is similar, if a little messier. To begin with, we will take  $Q$  to be expressible, in terms of a basis  $e_1, \dots, e_{2n+1}$  for  $V$ , by

$$Q(e_i, e_{i+n}) = Q(e_{i+n}, e_i) = 1 \quad \text{for } 1 \leq i \leq n;$$

$$Q(e_{2n+1}, e_{2n+1}) = 1;$$

and

$$Q(e_i, e_j) = 0 \quad \text{for all other pairs } i, j.$$

The bilinear form  $Q$  may be expressed as

$$Q(x, y) = {}^t x \cdot M \cdot y,$$

where  $M$  is the  $(2n + 1) \times (2n + 1)$  matrix

$$M = \left( \begin{array}{c|c|c} 0 & I_n & 0 \\ \hline I_n & 0 & 0 \\ \hline 0 & 0 & 1 \end{array} \right)$$

(the diagonal blocks here having widths  $n, n,$  and  $1$ ). The Lie algebra  $so_{2n+1} \mathbb{C}$  is correspondingly the space of matrices  $X$  satisfying the relation  ${}^t X \cdot M + M \cdot X = 0$ ; if we write  $X$  in block form as

$$X = \left( \begin{array}{c|c|c} A & B & E \\ \hline C & D & F \\ \hline G & H & J \end{array} \right),$$

then this is equivalent to saying that, as in the previous case, *B and C are skew-symmetric and A and D negative transposes of each other; and in addition  $E = -{}^t H, F = -{}^t G,$  and  $J = 0.$*

With these choices, we may take as Cartan subalgebra—in both the even and odd cases—the subalgebra of matrices diagonal in this representation.<sup>1</sup>

<sup>1</sup> Note that if we had taken the simpler choice of  $Q$ , with  $M$  the identity matrix, the Lie algebra would have consisted of skew-symmetric matrices, and there would have been no nonzero diagonal matrices in the Lie algebra.

The subalgebra  $\mathfrak{h}$  is thus generated by the  $n \ 2n \times 2n$  matrices  $H_i = E_{i,i} - E_{n+i,n+i}$  whose action on  $V$  is to fix  $e_i$ , send  $e_{n+i}$  to its negative, and kill all the remaining basis vectors; note that this is the same whether  $m = 2n$  or  $2n + 1$ . We will correspondingly take as basis for the dual vector space  $\mathfrak{h}^*$  the dual basis  $L_j$ , where  $\langle L_j, H_i \rangle = \delta_{i,j}$ .

Given that the Cartan subalgebra of  $\mathfrak{so}_{2n}\mathbb{C}$  coincides, as a subspace of  $\mathfrak{sl}_{2n}\mathbb{C}$ , with the Cartan subalgebra of  $\mathfrak{sp}_{2n}\mathbb{C}$ , we can use much of the description of the roots of  $\mathfrak{sp}_{2n}\mathbb{C}$  to help locate the roots and root spaces of  $\mathfrak{so}_{2n}\mathbb{C}$ . For example, we saw in Lecture 16 that the endomorphism

$$X_{i,j} = E_{i,j} - E_{n+j,n+i} \in \mathfrak{sp}_{2n}\mathbb{C}$$

is an eigenvector for the action of  $\mathfrak{h}$  with eigenvalue  $L_i - L_j$ . Since  $X_{i,j}$  is also an element of  $\mathfrak{so}_{2n}\mathbb{C}$ , we see that  $L_i - L_j$  is likewise a root of  $\mathfrak{so}_{2n}\mathbb{C}$ , with root space generated by  $X_{i,j}$ . Less directly but using the same analysis, we find that the endomorphisms

$$Y_{i,j} = E_{i,n+j} - E_{j,n+i}$$

and

$$Z_{i,j} = E_{n+i,j} - E_{n+j,i}$$

are eigenvectors for the action of  $\mathfrak{h}$ , with eigenvalues  $L_i + L_j$  and  $-L_i - L_j$ , respectively (note that  $Y_{i,j}$  and  $Z_{i,j}$  do not coincide with their definitions in Lecture 16). In sum, then, *the roots of the Lie algebra  $\mathfrak{so}_{2n}\mathbb{C}$  are the vectors  $\{\pm L_i \pm L_j\}_{i \neq j} \subset \mathfrak{h}^*$ .*

The case of the algebra  $\mathfrak{so}_{2n+1}\mathbb{C}$  is similar; indeed, all the eigenvectors for the action of  $\mathfrak{h}$  found above in  $\mathfrak{so}_{2n}\mathbb{C}$ , viewed as endomorphisms of  $\mathbb{C}^{2n+1}$ , are likewise eigenvectors for the action of  $\mathfrak{h}$  on  $\mathfrak{so}_{2n+1}\mathbb{C}$ . In addition, we have the endomorphisms

$$U_i = E_{i,2n+1} - E_{2n+1,n+i}$$

and

$$V_i = E_{n+i,2n+1} - E_{2n+1,i}$$

which are eigenvectors with eigenvalues  $+L_i$  and  $-L_i$ , respectively. The roots of  $\mathfrak{so}_{2n+1}\mathbb{C}$  are thus the roots  $\pm L_i \pm L_j$  of  $\mathfrak{so}_{2n}\mathbb{C}$ , together with additional roots  $\pm L_i$ .

We note that we could have arrived at these statements without decomposing the Lie algebras  $\mathfrak{so}_m\mathbb{C}$ : the description (18.1) of the orthogonal Lie algebra may be interpreted as saying that, in terms of the identification of  $V$  with  $V^*$  given by the form  $Q$ ,  $\mathfrak{so}_m\mathbb{C}$  is just the Lie algebra of skew-symmetric endomorphisms of  $V$  (an endomorphism being skew-symmetric if it is equal to minus its transpose). That is, the adjoint representation of  $\mathfrak{so}_m\mathbb{C}$  is isomorphic to the wedge product  $\wedge^2 V$ . In the even case  $m = 2n$ , since the weights of  $V$  are  $\pm L_i$  (inasmuch as the subalgebras  $\mathfrak{h} \subset \text{End}(V)$  coincide, the weights of  $V$  must likewise be the same for  $\mathfrak{so}_{2n}\mathbb{C}$  as for  $\mathfrak{sp}_{2n}\mathbb{C}$ ), it follows that the roots of  $\mathfrak{so}_{2n}\mathbb{C}$

are just the pairwise distinct sums  $\pm L_i \pm L_j$ . In the odd case  $m = 2n + 1$ , we see that  $e_{2n+1} \in V$  is an eigenvector for the action of  $\mathfrak{h}$  with eigenvalue 0, so that the weights of the standard representation  $V$  are  $\{\pm L_i\} \cup \{0\}$  and the weights of the adjoint representation correspondingly  $\{\pm L_i \pm L_j\} \cup \{\pm L_i\}$ .

**Exercise 18.3.** Use a similar analysis to find the roots of  $\mathfrak{sp}_{2n}\mathbb{C}$  without explicit calculation.

To make a comparison with the Lie algebra  $\mathfrak{sp}_{2n}\mathbb{C}$ , we can say that the root diagram of  $\mathfrak{so}_{2n}\mathbb{C}$  looks like that of  $\mathfrak{sp}_{2n}\mathbb{C}$  with the roots  $\pm 2L_i$  removed, whereas the root diagram of  $\mathfrak{so}_{2n+1}\mathbb{C}$  looks like that of  $\mathfrak{sp}_{2n}\mathbb{C}$  with the roots  $\pm 2L_i$  replaced by  $\pm L_i$ . Note that this immediately tells us what the Weyl groups are: first, in the case of  $\mathfrak{so}_{2n+1}\mathbb{C}$ , the Weyl group is the same as that of  $\mathfrak{sp}_{2n}\mathbb{C}$ :

$$1 \rightarrow (\mathbb{Z}/2)^n \rightarrow \mathfrak{W}_{\mathfrak{so}_{2n+1}\mathbb{C}} \rightarrow \mathfrak{S}_n \rightarrow 1.$$

In the case of  $\mathfrak{so}_{2n}\mathbb{C}$ , the Weyl group is the subgroup of the Weyl group of  $\mathfrak{sp}_{2n}\mathbb{C}$  generated by reflection in the hyperplanes perpendicular to the roots  $\pm L_i \pm L_j$ , without the additional generator given by reflection in the roots  $\pm L_i$ . This subgroup still acts as the full symmetric group on the set of coordinate axes in  $\mathfrak{h}^*$ ; but the kernel of this action, instead of acting as  $\pm I$  on each of the coordinate axes independently, will consist of transformations of determinant 1; i.e., will act as  $-1$  on an even number of axes. (That every such transformation is indeed in the Weyl group is easy to see: for example, reflection in the plane perpendicular to  $L_i + L_j$  followed by reflection in the plane perpendicular to  $L_i - L_j$  will send  $L_i$  to  $-L_i$ ,  $L_j$  to  $-L_j$ , and  $L_k$  to  $L_k$  for  $k \neq i, j$ .) Another way to say this is that the Weyl group is the subgroup of the Weyl group of  $\mathfrak{sp}_{2n}\mathbb{C}$  consisting of transformations whose determinant agrees with the sign of the induced permutation of the coordinate axes; so that while the Weyl group of  $\mathfrak{sp}_{2n}\mathbb{C}$  fits into the exact sequence

$$1 \rightarrow (\mathbb{Z}/2)^n \rightarrow \mathfrak{W}_{\mathfrak{sp}_{2n}\mathbb{C}} \rightarrow \mathfrak{S}_n \rightarrow 1,$$

the Weyl group of  $\mathfrak{so}_{2n}\mathbb{C}$  has instead the sequence

$$1 \rightarrow (\mathbb{Z}/2)^{n-1} \rightarrow \mathfrak{W}_{\mathfrak{so}_{2n}\mathbb{C}} \rightarrow \mathfrak{S}_n \rightarrow 1.$$

We can likewise describe the Weyl chambers of  $\mathfrak{so}_{2n}\mathbb{C}$  and  $\mathfrak{so}_{2n+1}\mathbb{C}$  by direct comparison with  $\mathfrak{sp}_{2n}\mathbb{C}$ . To start, to choose an ordering of the roots we take as linear functional on  $\mathfrak{h}^*$  a form  $l = c_1 H_1 + \cdots + c_n H_n$ , where  $c_1 > c_2 > \cdots > c_n > 0$ . The positive roots in the case of  $\mathfrak{so}_{2n+1}\mathbb{C}$  are then

$$R^+ = \{L_i + L_j\}_{i < j} \cup \{L_i - L_j\}_{i < j} \cup \{L_i\}_i,$$

whereas in the case of  $\mathfrak{so}_{2n}\mathbb{C}$  we have

$$R^+ = \{L_i + L_j\}_{i < j} \cup \{L_i - L_j\}_{i < j}.$$

The primitive positive roots are

$$\begin{aligned}
 &L_1 - L_2, L_2 - L_3, \dots, L_{n-1} - L_n, L_n && \text{for } \mathfrak{so}_{2n+1}\mathbb{C}; \\
 &L_1 - L_2, L_2 - L_3, \dots, L_{n-1} - L_n, L_{n-1} + L_n && \text{for } \mathfrak{so}_{2n}\mathbb{C}.
 \end{aligned}$$

In the first case, the Weyl chamber is exactly the same as for  $\mathfrak{sp}_{2n}\mathbb{C}$ , namely, for  $m = 2n + 1$ ,

$$\mathcal{W} = \{ \sum a_i L_i : a_1 \geq a_2 \geq \dots \geq a_n \geq 0 \}$$

since the roots are the same except for the factor of 2 on some. In the case of  $\mathfrak{so}_{2n}\mathbb{C}$ , since there is no root along the line spanned by the  $L_i$ , the equality  $a_n = 0$  does not describe a face of the Weyl chamber; however, since  $L_{n-1} + L_n$  is still a root (and a positive one) we still have the inequality  $a_{n-1} + a_n \geq 0$  in  $\mathcal{W}$ , so that we can write, for  $m = 2n$ ,

$$\mathcal{W} = \{ \sum a_i L_i : a_1 \geq a_2 \geq \dots \geq a_{n-1} \geq |a_n| \}.$$

(Note that in the case of  $\mathfrak{so}_{2n}\mathbb{C}$  we could have chosen our linear functional  $l = c_1 H_1 + \dots + c_n H_n$  with  $c_1 > c_2 > \dots > -c_n > 0$ ; the ordering of the roots, and consequently the Weyl chamber, would still be the same.)

As for the Killing form, the same considerations as for the symplectic case show that it must be, up to scalars, the standard quadratic form:  $B(H_i, H_j) = \delta_{i,j}$ . (This was implicit in the above description of the Weyl group.) The explicit calculation is no more difficult, and we leave it as an exercise:

$$B\left(\sum a_i H_i, \sum b_i H_i\right) = \begin{cases} (4n - 2) \sum a_i b_i & \text{if } m = 2n + 1 \\ (4n - 4) \sum a_i b_i & \text{if } m = 2n. \end{cases}$$

Next, to describe the representations of the orthogonal Lie algebras we have to determine the weight lattice in  $\mathfrak{h}^*$ ; and to do this we must, as before, locate the copies  $\mathfrak{s}_\alpha$  of  $\mathfrak{sl}_2\mathbb{C}$  corresponding to the root pairs  $\pm\alpha$ , and the corresponding distinguished elements  $H_\alpha$  of  $\mathfrak{h}$ . This is so similar to the case of  $\mathfrak{sp}_{2n}\mathbb{C}$  that we will leave the actual calculations as an exercise; we will simply state here the results that in  $\mathfrak{so}_m\mathbb{C}$  for any  $m$ ,

(i) the distinguished copy  $\mathfrak{s}_{L_i - L_j}$  of  $\mathfrak{sl}_2\mathbb{C}$  associated to the root  $L_i - L_j$  is the span of the root spaces  $\mathfrak{g}_{L_i - L_j} = \mathbb{C} \cdot X_{i,j}$ ,  $\mathfrak{g}_{-L_i - L_j} = \mathbb{C} \cdot X_{j,i}$  and their commutator  $[X_{i,j}, X_{j,i}] = E_{i,i} - E_{j,j} + E_{n+j,n+j} - E_{n+i,n+i}$ , with distinguished element  $H_{L_i - L_j} = H_i - H_j$  (this is exactly as in the case of  $\mathfrak{sp}_{2n}\mathbb{C}$ );

(ii) the distinguished copy  $\mathfrak{s}_{L_i + L_j}$  of  $\mathfrak{sl}_2\mathbb{C}$  associated to the root  $L_i + L_j$  is the span of the root spaces  $\mathfrak{g}_{L_i + L_j} = \mathbb{C} \cdot Y_{i,j}$ ,  $\mathfrak{g}_{-L_i - L_j} = \mathbb{C} \cdot Z_{i,j}$  and their commutator  $[Y_{i,j}, Z_{i,j}] = -E_{i,i} + E_{j,j} - E_{n+j,n+j} + E_{n+i,n+i} = -H_i - H_j$ , with distinguished element  $H_{L_i + L_j} = H_i + H_j$  (so that we have also  $H_{-L_i - L_j} = -H_i - H_j$ ); and in the case of  $\mathfrak{so}_{2n+1}\mathbb{C}$ ,

(iii) the distinguished copy  $\mathfrak{s}_{L_i}$  of  $\mathfrak{sl}_2\mathbb{C}$  associated to the root  $L_i$  is the span of the root spaces  $\mathfrak{g}_{L_i} = \mathbb{C} \cdot U_i$ ,  $\mathfrak{g}_{-L_i} = \mathbb{C} \cdot V_i$  and their commutator  $[U_i, V_i] = [E_{i,2n+1} - E_{2n+1,n+i}, E_{n+i,2n+1} - E_{2n+1,i}] = -H_i$ , with distinguished element  $H_{L_i} = 2H_i$  (so that  $H_{-L_i} = -2H_i$  as well).

**Exercise 18.4.** Verify the computations made here.

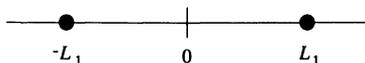
Again, the configuration of distinguished elements resembles that of  $\mathfrak{sp}_{2n}\mathbb{C}$  closely; that of  $\mathfrak{so}_{2n+1}\mathbb{C}$  differs from it by the substitution of  $\pm 2H_i$  for  $\pm H_i$ , whereas that of  $\mathfrak{so}_{2n}\mathbb{C}$  differs by the removal of the  $\pm H_i$ . The effect on the weight lattice is the same in either case: *for both even and odd orthogonal Lie algebras, the weight lattice  $\Lambda_W$  is the lattice generated by the  $L_i$  together with the element  $(L_1 + \cdots + L_n)/2$ .*

**Exercise 18.5.** Show that

$$\Lambda_W/\Lambda_R = \begin{cases} \mathbb{Z}/2 & \text{if } m = 2n + 1 \\ \mathbb{Z}/4 & \text{if } m = 2n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } m = 2n \text{ and } n \text{ is even.} \end{cases}$$

## §18.2. Representations of $\mathfrak{so}_3\mathbb{C}$ , $\mathfrak{so}_4\mathbb{C}$ , and $\mathfrak{so}_5\mathbb{C}$

To give some examples, start with the case  $n = 1$ . Of course,  $\mathfrak{so}_2\mathbb{C} \cong \mathbb{C}$  is not semisimple. The root system of  $\mathfrak{so}_3\mathbb{C}$ , on the other hand, looks like that of  $\mathfrak{sl}_2\mathbb{C}$ :

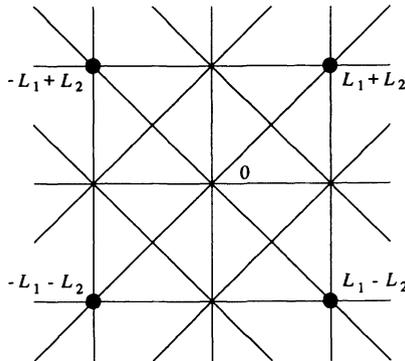


This is because, in fact, the two Lie algebras are isomorphic. Indeed, like the symplectic group, the quotient  $\text{PSO}_m\mathbb{C}$  of the orthogonal group by its center can be realized as the motions of the projective space  $\mathbb{P}V$  preserving isotropic subspaces for the quadratic form  $Q$ ; in particular, this means we can realize  $\text{PSO}_m\mathbb{C}$  as the group of motions of  $\mathbb{P}V = \mathbb{P}^{m-1}$  carrying the quadric hypersurface

$$\bar{Q} = \{[v]: Q(v, v) = 0\}$$

into itself. In the first case of this, we see that the group  $\text{PSO}_3\mathbb{C}$  is the group of motions of the projective plane  $\mathbb{P}^2$  carrying a conic curve  $C \subset \mathbb{P}^2$  into itself. But we have seen before that this group is also  $\text{PGL}_2\mathbb{C}$  (the conic curve is itself isomorphic to  $\mathbb{P}^1$ , and the group acts as its full group of automorphisms), giving us the isomorphism  $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$ . One thing to note here is that the “standard” representation of  $\mathfrak{so}_3\mathbb{C}$  is not the standard representation of  $\mathfrak{sl}_2\mathbb{C}$ , but rather its symmetric square. In fact, the irreducible representation with highest weight  $\frac{1}{2}L_1$  is not contained in tensor powers of the standard representation of  $\mathfrak{so}_3\mathbb{C}$ . This will turn out to be significant: the standard representation of  $\mathfrak{sl}_2\mathbb{C}$ , viewed as a representation of  $\mathfrak{so}_3\mathbb{C}$ , is the first example of a *spin* representation of an orthogonal Lie algebra.

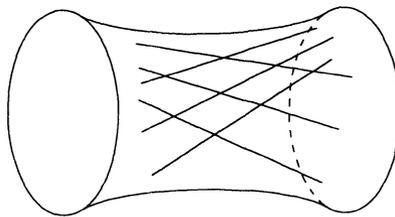
The next examples involve two-dimensional Cartan algebras. First we have  $\mathfrak{so}_4\mathbb{C}$ , whose root diagram looks like



Note one thing about this diagram: the roots are located on the union of two complementary lines. This says, by Exercise 14.33, that the Lie algebra  $\mathfrak{so}_4 \mathbb{C}$  is decomposable, and in fact should be the sum of two algebras each of whose root diagrams looks like that of  $\mathfrak{sl}_2 \mathbb{C}$ ; explicitly,  $\mathfrak{so}_4 \mathbb{C}$  is the direct sum of the two algebras  $\mathfrak{s}_\alpha$ , for  $\alpha = L_1 + L_2$  and  $\alpha = L_1 - L_2$ . In fact, we can see this isomorphism

$$\mathfrak{so}_4 \mathbb{C} \cong \mathfrak{sl}_2 \mathbb{C} \times \mathfrak{sl}_2 \mathbb{C}, \tag{18.6}$$

as in the previous example, geometrically. Precisely, we may realize the group  $\text{PSO}_4 \mathbb{C} = \text{SO}_4 \mathbb{C} / \{ \pm I \}$  as the connected component of the identity in the group of motions of projective three-space  $\mathbb{P}^3$  carrying a quadric hypersurface  $\bar{Q}$  into itself. But a quadric hypersurface in  $\mathbb{P}^3$  has two rulings by lines, and these two rulings give an isomorphism of  $\bar{Q}$  with a product  $\mathbb{P}^1 \times \mathbb{P}^1$



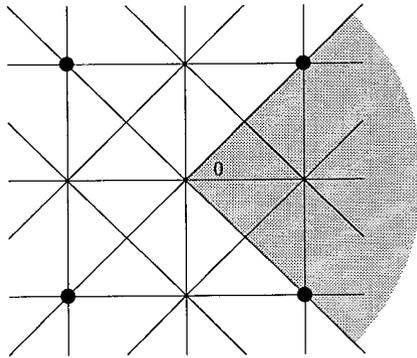
$\text{PSO}_4 \mathbb{C}$  thus acts on the product  $\mathbb{P}^1 \times \mathbb{P}^1$ ; and since the connected component of the identity in the automorphism group of this variety is just the product  $\text{PGL}_2 \mathbb{C} \times \text{PGL}_2 \mathbb{C}$ , we get an inclusion

$$\text{PSO}_4 \mathbb{C} \rightarrow \text{PGL}_2 \mathbb{C} \times \text{PGL}_2 \mathbb{C}.$$

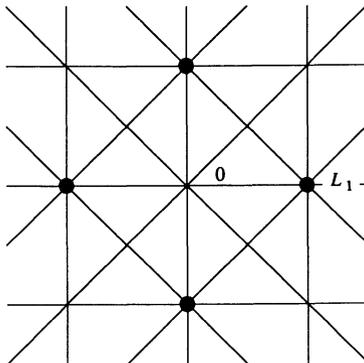
Another way of saying this is to remark that  $\text{PSO}_4 \mathbb{C}$  acts on the variety of isotropic 2-planes for the quadratic form  $Q$  on  $V$ ; and this variety is just the disjoint union of two copies of  $\mathbb{P}^1$ . To see in this case that the map is an

isomorphism, consider the tensor product  $V = U \otimes W$  of the pullbacks to  $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$  of the standard representations of the two factors. Clearly the action on  $\mathbb{P}(U \otimes W)$  will preserve the points corresponding to decomposable tensors (that is, points of the form  $[u \otimes w]$ ); but the locus of such points is just a quadric hypersurface, giving us the inverse inclusion of  $\mathrm{PGL}_2\mathbb{C} \times \mathrm{PGL}_2\mathbb{C}$  in  $\mathrm{PSO}_4\mathbb{C}$ .

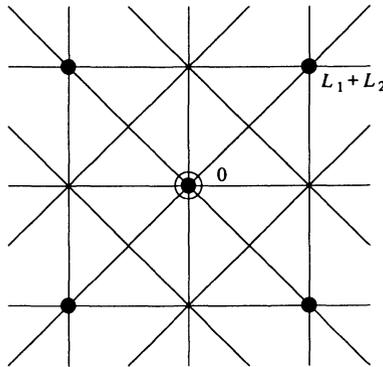
In fact, all of this will fall out of the analysis of the representations of  $\mathfrak{so}_4\mathbb{C}$ , if we just pursue it as usual. To begin with, the Weyl chamber we have selected looks like



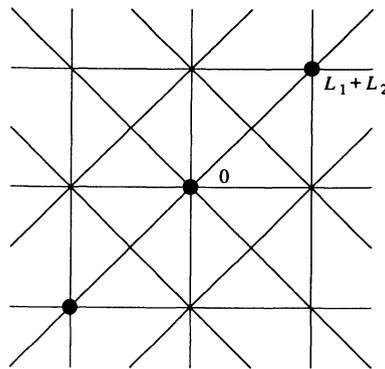
Now, the standard representation has, as noted above, weight diagram



with highest weight  $L_1$  (note that the highest weight of the standard representation lies in this case in the interior of the Weyl chamber, something of an anomaly). Its second exterior power will have weights  $\pm L_1 \pm L_2$  and 0 (occurring with multiplicity 2), i.e., diagram



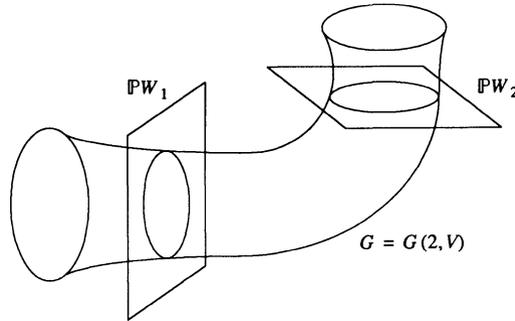
We see one thing about this representation right away, namely, that it cannot be irreducible. Indeed, the images of the highest weight  $L_1 + L_2$  under the Weyl group consist just of  $\pm(L_1 + L_2)$ , so that the diagram of the irreducible representation with this highest weight is



We see from this that the second exterior power  $\wedge^2 V$  of the standard representation of  $\mathfrak{so}_4 \mathbb{C}$  must be the direct sum of the irreducible representations  $W_1 = \Gamma_{L_1 + L_2}$  and  $W_2 = \Gamma_{L_1 - L_2}$  with highest weights  $L_1 + L_2$  and  $L_1 - L_2$ . Since  $\wedge^2 V$  is at the same time the adjoint representation, this says that  $\mathfrak{so}_4 \mathbb{C}$  itself must be a product of Lie algebras with adjoint representations  $\Gamma_{L_1 + L_2}$  and  $\Gamma_{L_1 - L_2}$ .

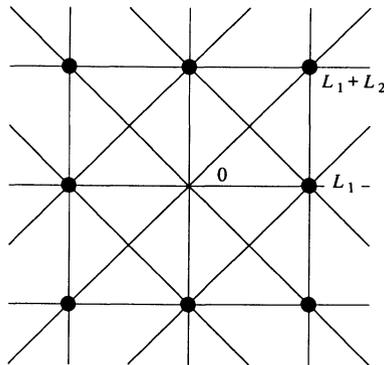
One way to derive the picture of the ruling of the quadric in  $\mathbb{P}^3$  from this decomposition is to view  $\mathfrak{so}_4 \mathbb{C}$  as a subalgebra of  $\mathfrak{sl}_4 \mathbb{C}$ , and the action of  $\text{PSO}_4 \mathbb{C}$  on  $\mathbb{P}(\wedge^2 V)$  as a subgroup of the group of motions of  $\mathbb{P}^2(\wedge^2 V) = \mathbb{P}^5$  preserving the Grassmannian  $G = G(2, V)$  of lines in  $\mathbb{P}^3$ . In fact, we see from the above that the action of  $\text{PSO}_4$  on  $\mathbb{P}^5$  will preserve a pair of complementary 2-planes  $\mathbb{P}W_1$  and  $\mathbb{P}W_2$ ; it follows that this action must carry into themselves

the intersections of these 2-planes with the Grassmannian. These intersections are conic curves, corresponding to one-parameter families of lines sweeping out a quadric surface (necessarily the same quadric, since the action of  $SO_4\mathbb{C}$  on  $V$  preserves a unique quadratic form); thus, the two rulings of the quadric.



Note one more aspect of this example: as in the case of  $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$ , the weights of the standard representation of  $\mathfrak{so}_4\mathbb{C}$  do not generate the weight lattice, but rather a sublattice  $\mathbb{Z}\{L_1, L_2\}$  of index 2 in  $\Lambda_W$ . Thus, there is no way of constructing all the representations of  $\mathfrak{so}_4\mathbb{C}$  by applying linear- or multilinear-algebraic constructions to the standard representation; it is only after we are aware of the isomorphism  $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$  that we can construct, for example, the representation  $\Gamma_{(L_1+L_2)/2}$  with highest weight  $(L_1 + L_2)/2$  (of course, this is just the pullback from the first factor of  $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$  of the standard representation of  $\mathfrak{sl}_2\mathbb{C}$ ).

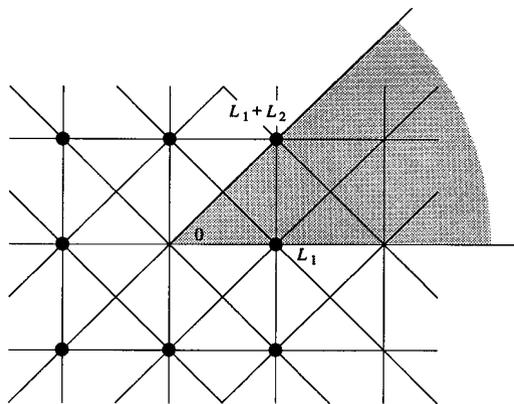
We come now to the case of  $\mathfrak{so}_5\mathbb{C}$ , which is more interesting. The root diagram in this case looks like



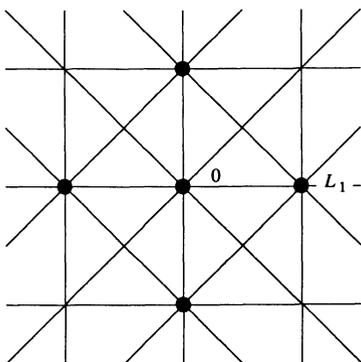
(as in the preceding example, the weight lattice is the lattice of intersections of all the lines drawn). The first thing we should notice about this diagram is that it is isomorphic to the weight diagram of the Lie algebra  $\mathfrak{sp}_4\mathbb{C}$ ; the diagram just appears here rotated through an angle of  $\pi/4$ . Indeed, this is not accidental; the two Lie algebras  $\mathfrak{sp}_4\mathbb{C}$  and  $\mathfrak{so}_5\mathbb{C}$  are isomorphic, and it is not hard to construct this isomorphism explicitly. To see the isomorphism geometrically, we simply have to recall the identification, made in Lecture 14, of the group  $\mathrm{PSp}_4\mathbb{C}$  with a group of motions of  $\mathbb{P}^4$ . There, we saw that the larger group  $\mathrm{PGL}_4\mathbb{C}$  could be identified with the automorphisms of the projective space  $\mathbb{P}(\wedge^2 V) = \mathbb{P}^5$  preserving the Grassmannian  $G = G(2, 4) \subset \mathbb{P}(\wedge^2 V)$ . The subgroup  $\mathrm{PSP}_4\mathbb{C} \subset \mathrm{PGL}_4\mathbb{C}$  thus preserves both the Grassmannian  $G$ , which is a quadric hypersurface in  $\mathbb{P}^5$ , and the decomposition of  $\wedge^2 V$  into the span  $\mathbb{C} \cdot Q$  of the skew form  $Q \in \wedge^2 V^* \cong \wedge^2 V$  and its complement  $W$ , and so acts on  $\mathbb{P}W$  carrying the intersection  $G_L = G \cap \mathbb{P}W$  into itself. We thus saw that  $\mathrm{PSP}_4\mathbb{C}$  was a subgroup of the group of motions of projective space  $\mathbb{P}^4$  preserving a quadric hypersurface, and asserted that in fact it was the whole group.

(To see the reverse inclusion directly, we can invoke a little algebraic geometry, which tells us that the locus of isotropic lines for a quadric in  $\mathbb{P}^4$  is isomorphic to  $\mathbb{P}^3$ , so that  $\mathrm{PSO}_5\mathbb{C}$  acts on  $\mathbb{P}^3$ . Moreover, this action preserves the subset of pairs of points in  $\mathbb{P}^3$  whose corresponding lines in  $\mathbb{P}^4$  intersect, which, for a suitably defined skew-symmetric bilinear form  $\tilde{Q}$ , is exactly the set of pairs  $([v], [w])$  such that  $\tilde{Q}(v, w) = 0$ , so that we have an inclusion of  $\mathrm{PSO}_5\mathbb{C}$  in  $\mathrm{PSP}_4\mathbb{C}$ .)

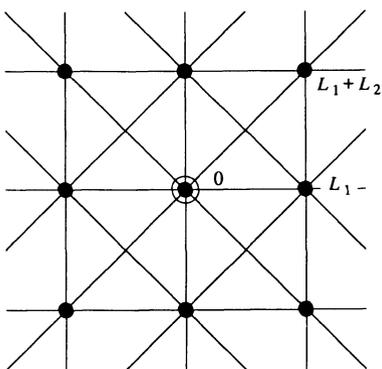
Let us proceed to analyze the representations of  $\mathfrak{so}_5\mathbb{C}$  as we would ordinarily, bearing in mind the isomorphism with  $\mathfrak{sp}_4\mathbb{C}$ . To begin with, we draw the Weyl chamber picked out above in  $\mathfrak{h}^*$ :



As for the representations of  $\mathfrak{so}_5\mathbb{C}$ , we have to begin with the standard, which has weight diagram



This we see corresponds to the representation  $W = \wedge^2 V/\mathbb{C} \cdot Q$  of  $\mathfrak{sp}_4\mathbb{C}$ . Next, the second exterior power of the standard representation of  $\mathfrak{so}_5\mathbb{C}$  has weights



This is of course the adjoint representation of  $\mathfrak{so}_5\mathbb{C}$ ; it is the irreducible representation with highest weight  $L_1 + L_2$ . Note that it corresponds to the symmetric square  $\text{Sym}^2 V$  of the standard representation of  $\mathfrak{sp}_4\mathbb{C}$  (see Exercise 16.8).

**Exercise 18.7.** Show that contraction with the quadratic form  $Q \in \text{Sym}^2 V^*$  preserved by the action of  $\mathfrak{so}_5\mathbb{C}$  induces maps

$$\varphi: \text{Sym}^a V \rightarrow \text{Sym}^{a-2} V.$$

Show that the kernel of this contraction is exactly the irreducible representation with highest weight  $a \cdot L_1$ . Compare this with the analysis in Exercise 16.11.

**Exercise 18.8.** Examine the symmetric power  $\text{Sym}^a(\wedge^2 V)$  of the representation  $\wedge^2 V$ . This will contain a copy of the irreducible representation  $\Gamma_{a(L_1+L_2)}$ ; what else will it contain? Interpret these other factors in light of the isomorphism  $\mathfrak{so}_5 \mathbb{C} \cong \mathfrak{sp}_4 \mathbb{C}$ .

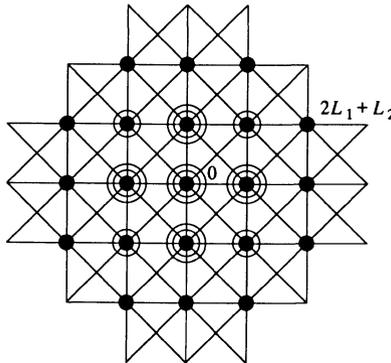
**Exercise 18.9.** For an example of a “mixed” tensor, consider the irreducible representation  $\Gamma_{2L_1+L_2}$ . Show that this is contained in the kernels of the wedge product map

$$\varphi: V \otimes \wedge^2 V \rightarrow \wedge^3 V$$

and the composition

$$\varphi': V \otimes \wedge^2 V \rightarrow V^* \otimes \wedge^2 V \rightarrow V,$$

where the first map is induced by the isomorphism  $\tilde{Q}: V \rightarrow V^*$  and the second is the contraction  $V^* \otimes \wedge^2 V \rightarrow V$ . Is it equal to the intersection of these kernels? Show that the weight diagram of this representation is



After you are done with this analysis, compare with the analysis given of the corresponding representation in Lecture 16.

Note that, as in the case of the other orthogonal Lie algebras studied so far (and as is the case for all  $\mathfrak{so}_m \mathbb{C}$ ), the weights of the standard representation do not generate the weight lattice, but only the sublattice of index two generated by the  $L_i$ . Thus, the tensor algebra of the standard representation will contain only one-half of all the irreducible representations of  $\mathfrak{so}_5 \mathbb{C}$ . Now, we do know that there are others, and even something about them—for example, we see in the following exercise that the irreducible representation of  $\mathfrak{so}_5 \mathbb{C}$  with highest weight  $(L_1 + L_2)/2$  is a sort of “symmetric square root” of the adjoint representation:

**Exercise 18.10.** Show, using only root and weight diagrams for  $\mathfrak{so}_5 \mathbb{C}$ , that the exterior square  $\wedge^2 V$  of the standard representation of  $\mathfrak{so}_5 \mathbb{C}$  is actually the symmetric square of an irreducible representation.

We can also describe this irreducible representation via the isomorphism of  $\mathfrak{so}_5\mathbb{C}$  with  $\mathfrak{sp}_4\mathbb{C}$ : it is just the standard representation of  $\mathfrak{sp}_4\mathbb{C}$  on  $\mathbb{C}^4$ . We do not at this point have, however, a way of constructing this representation without invoking the isomorphism. This representation, the representation of  $\mathfrak{so}_3\mathbb{C}$  with highest weight  $L_1/2$ , and the representation of  $\mathfrak{so}_4\mathbb{C}$  with highest weight  $(L_1 + L_2)/2$  discussed above are called *spin* representations of the corresponding Lie algebras and will be the subject matter of Lecture 20.