

LECTURE 26

Real Lie Algebras and Lie Groups

In this lecture we indicate how to complete the last step in the process outlined at the beginning of Part II: to take our knowledge of the classification and representation theory of complex algebras and groups and deduce the corresponding statements in the real case. We do this in the first section, giving a list of the simple classical real Lie algebras and saying a few words about the corresponding groups and their (complex) representations. The existence of a compact group whose Lie algebra has as complexification a given semisimple complex Lie algebra makes it possible to give another (indeed, the original) way to prove the Weyl character formula; we sketch this in §26.2. Finally, we can ask in regard to real Lie groups G a question analogous to one asked for the representations of finite groups in §3.5: which of the complex representations V of G actually come from real ones. We answer this in the most commonly encountered cases in §26.3. In this final lecture, proofs, when we attempt them, are generally only sketched and may require more than the usual fortitude from the reader.

§26.1: Classification of real simple Lie algebras and groups

§26.2: Second proof of Weyl's character formula

§26.3: Real, complex, and quaternionic representations

§26.1. Classification of Real Simple Lie Algebras and Groups

Having described the semisimple complex Lie algebras, we now address the analogous problem for real Lie algebras. Since the complexification $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ of a semisimple real Lie algebra \mathfrak{g}_0 is a semisimple complex Lie algebra and we have classified those, we are reduced to the problem of describing the *real forms* of the complex semisimple Lie algebras: that is, for a given complex Lie algebra \mathfrak{g} , finding all real Lie algebras \mathfrak{g}_0 with

$$\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}.$$

We saw many of the real forms of the classical complex Lie groups and algebras back in Lectures 7 and 8. In this section we will indicate one way to approach the question systematically, but we will only include sketches of proofs.

To get the idea of what to expect, let us work out real forms of $\mathfrak{sl}_2\mathbb{C}$ in detail. To do this, suppose \mathfrak{g}_0 is any real Lie subalgebra of $\mathfrak{sl}_2\mathbb{C}$, with $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_2\mathbb{C}$. The natural thing to do is to try to carry out our analysis of semisimple Lie algebras for the real Lie algebra \mathfrak{g}_0 : that is, find an element $H \in \mathfrak{g}_0$ such that $\text{ad}(H)$ acts semisimply on \mathfrak{g}_0 , decompose \mathfrak{g}_0 into eigenspaces, and so on. The first part of this presents no problem: since the subset of $\mathfrak{sl}_2\mathbb{C}$ of non-semisimple matrices is a proper algebraic subvariety, it cannot contain the real subspace $\mathfrak{g}_0 \subset \mathfrak{sl}_2\mathbb{C}$, so that we can certainly find a semisimple $H \in \mathfrak{g}_0$.

The next thing is to consider the eigenspaces of $\text{ad}(H)$ acting on \mathfrak{g} . Of course, $\text{ad}(H)$ has one eigenvalue 0, corresponding to the eigenspace $\mathfrak{h}_0 = \mathbb{R} \cdot H$ spanned by H . The remaining two eigenvalues must then sum to zero, which leaves just two possibilities:

(i) $\text{ad}(H)$ has eigenvalues λ and $-\lambda$, for λ a nonzero real number; multiplying H by a real scalar, we can take $\lambda = 2$. In this case we obtain a decomposition of the vector space \mathfrak{g}_0 into one-dimensional eigenspaces

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2}.$$

We can then choose $X \in \mathfrak{g}_2$ and $Y \in \mathfrak{g}_{-2}$; the standard argument then shows that the bracket $[X, Y]$ is a nonzero multiple of H , which we may take to be 1 by rechoosing X and Y . We thus have the real form $\mathfrak{sl}_2\mathbb{R}$, with the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(ii) $\text{ad}(H)$ has eigenvalues $i\lambda$ and $-i\lambda$ for λ some nonzero real number; again, adjusting H by a real scalar we may take $\lambda = 1$. In this case, of course, there are no real eigenvectors for the action of $\text{ad}(H)$ on \mathfrak{g}_0 ; but we can decompose \mathfrak{g}_0 into the direct sum of \mathfrak{h}_0 and the two-dimensional subspace $\mathfrak{g}_{\{i, -i\}}$ corresponding to the pair of eigenvalues i and $-i$. We may then choose a basis B and C for $\mathfrak{g}_{\{i, -i\}}$ with

$$[H, B] = C \quad \text{and} \quad [H, C] = -B.$$

The commutator $[B, C]$ will then be a nonzero multiple of H , which we may take to be either H or $-H$ (we can multiply B and C simultaneously by a scalar μ , which multiplies the commutator $[B, C]$ by μ^2). In the latter case, we see that \mathfrak{g}_0 is isomorphic to $\mathfrak{sl}_2\mathbb{R}$ again: these are the relations we get if we take as basis for $\mathfrak{sl}_2\mathbb{C}$ the three vectors

$$H = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally, if the commutator $[B, C] = H$, we do get a new example: \mathfrak{g}_0 is in this case isomorphic to the algebra

$$\mathfrak{su}_2 = \{A: {}^t\bar{A} = -A \text{ and } \text{trace}(A) = 0\} \subset \mathfrak{sl}_2\mathbb{C},$$

which has as basis

$$H = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}.$$

Exercise 26.1. Carry out this analysis for the real Lie algebras $\mathfrak{so}_3\mathbb{R}$ and $\mathfrak{so}_{2,1}\mathbb{R}$. In particular, give an isomorphism of each with either $\mathfrak{sl}_2\mathbb{R}$ or \mathfrak{su}_2 .

This completes our analysis of the real forms of $\mathfrak{sl}_2\mathbb{C}$. In the general case, we can try to apply a similar analysis, and indeed at least one aspect generalizes: given a real form $\mathfrak{g}_0 \subset \mathfrak{g}$ of the complex semisimple Lie algebra \mathfrak{g} , we can find a real subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ such that $\mathfrak{h}_0 \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$; this is called a *Cartan subalgebra* of \mathfrak{g}_0 . There is a further complication in the case of Lie algebras of rank 2 or more: the values on \mathfrak{h}_0 of a root $\alpha \in R$ of \mathfrak{g} need not be either all real or all purely imaginary. We, thus, need to consider the root spaces \mathfrak{g}_α , $\mathfrak{g}_{\bar{\alpha}}$, $\mathfrak{g}_{-\alpha}$, and $\mathfrak{g}_{-\bar{\alpha}}$, and the subalgebra they generate, at the same time. Moreover, as we saw in the above example, whether the values of the roots $\alpha \in R$ of \mathfrak{g} on the real subspace \mathfrak{h}_0 are real, purely imaginary, or neither will in general depend on the choice of \mathfrak{h}_0 .

Exercise 26.2*. In the case of $\mathfrak{g}_0 = \mathfrak{sl}_3\mathbb{R} \subset \mathfrak{g} = \mathfrak{sl}_3\mathbb{C}$, suppose we choose as Cartan subalgebra \mathfrak{h}_0 the space spanned over \mathbb{R} by the elements

$$H_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Show that this is indeed a Cartan subalgebra, and find the decomposition of \mathfrak{g} into eigenspaces for the action of $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$. In particular, find the roots of \mathfrak{g} as linear functions on \mathfrak{h} , and describe the corresponding decomposition of \mathfrak{g}_0 .

Judging from these examples, it is probably prudent to resist the temptation to try to carry out an analysis of real semisimple Lie algebras via an analogue of the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$ in this case. Rather, in the present book, we will do two things. First, we will give the statement of the classification theorem for the real forms of the classical algebras—that is, we will list all the simple real Lie algebras whose complexifications are classical algebras. Second, we will focus on two distinguished real forms possessed by any real semisimple Lie algebra, the *split form* and the *compact form*. These are the two forms that you see most often; and the existence of the latter in particular will be essential in the following section.

For the first, it turns out to be enough to work out the complexifications $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_0 \oplus i \cdot \mathfrak{g}_0$ of the real Lie algebras \mathfrak{g}_0 we know. The list is:

<u>Real Lie algebra</u>	<u>Complexification</u>
$\mathfrak{sl}_n \mathbb{R}$	$\mathfrak{sl}_n \mathbb{C}$
$\mathfrak{sl}_n \mathbb{C}$	$\mathfrak{sl}_n \mathbb{C} \times \mathfrak{sl}_n \mathbb{C}$
$\mathfrak{sl}_n \mathbb{H} = \mathfrak{gl}_n \mathbb{H} / \mathbb{R}$	$\mathfrak{sl}_{2n} \mathbb{C}$
$\mathfrak{so}_{p,q} \mathbb{R}$	$\mathfrak{so}_{p+q} \mathbb{C}$
$\mathfrak{so}_n \mathbb{C}$	$\mathfrak{so}_n \mathbb{C} \times \mathfrak{so}_n \mathbb{C}$
$\mathfrak{sp}_{2n} \mathbb{R}$	$\mathfrak{sp}_{2n} \mathbb{C}$
$\mathfrak{sp}_{2n} \mathbb{C}$	$\mathfrak{sp}_{2n} \mathbb{C} \times \mathfrak{sp}_{2n} \mathbb{C}$
$\mathfrak{su}_{p,q}$	$\mathfrak{sl}_{p+q} \mathbb{C}$
$\mathfrak{u}_{p,q} \mathbb{H}$	$\mathfrak{sp}_{2(p+q)} \mathbb{C}$
$\mathfrak{u}_n^* \mathbb{H}$	$\mathfrak{so}_{2n} \mathbb{C}$

The last two in the left-hand column are the Lie algebras of the groups $U_{p,q} \mathbb{H}$ and $U_n^* \mathbb{H}$ of automorphisms of a quaternionic vector space preserving a Hermitian form with signature (p, q) , and a skew-symmetric Hermitian form, respectively.

We should first verify that the algebras on the right are indeed the complexifications of those on the left. Some are obvious, such as the complexification

$$(\mathfrak{sl}_n \mathbb{R})_{\mathbb{C}} = \mathfrak{sl}_n \mathbb{R} \oplus i \cdot \mathfrak{sl}_n \mathbb{R} = \mathfrak{sl}_n \mathbb{C}.$$

The same goes for $\mathfrak{so}_{p,q} \mathbb{R}$ and $\mathfrak{sp}_{2n} \mathbb{R}$.

Next, consider the complexification of

$$\mathfrak{su}_n = \{A \in \mathfrak{sl}_n \mathbb{C} : {}^t \bar{A} = -A\}.$$

To see that $\mathfrak{sl}_n \mathbb{C} = \mathfrak{su}_n \oplus i \cdot \mathfrak{su}_n$, let $M \in \mathfrak{sl}_n \mathbb{C}$, and write

$$M = \frac{1}{2}(M - {}^t \bar{M}) + \frac{1}{2}(M + {}^t \bar{M}) = \frac{1}{2}A + \frac{1}{2}B;$$

then $A \in \mathfrak{su}_n$, $iB \in \mathfrak{su}_n$, and $M = \frac{1}{2}A - i(i/2)B$.

The general case of $\mathfrak{su}_{p,q} \subset \mathfrak{sl}_{p+q} \mathbb{C}$ is similar: if the form is given by $(x, y) = {}^t \bar{x} Q y$, then $\mathfrak{su}_{p,q} = \{A : {}^t A Q = -Q A\}$. Writing $M \in \mathfrak{sl}_{p+q} \mathbb{C}$ in the form

$$M = \frac{1}{2}(M - Q \cdot {}^t \bar{M} \cdot Q) - i \cdot (\frac{1}{2}(iM + iQ \cdot {}^t \bar{M} \cdot Q))$$

and using $\bar{Q} = {}^t Q = Q^{-1} = Q$, one sees that $M \in \mathfrak{su}_{p,q} \oplus i \cdot \mathfrak{su}_{p,q}$.

For the complexification of $\mathfrak{sl}_m \mathbb{C}$, embed $\mathfrak{sl}_m \mathbb{C}$ in $\mathfrak{sl}_m \mathbb{C} \times \mathfrak{sl}_m \mathbb{C}$ by $A \mapsto (A, \bar{A})$. Given any pair (B, C) , write

$$\begin{aligned} (B, C) &= \frac{1}{2}(B + \bar{C}, \bar{B} + C) + \frac{1}{2}(B - \bar{C}, -\bar{B} + C) \\ &= \frac{1}{2}(B + \bar{C}, \bar{B} + C) - i \cdot (\frac{1}{2}(iB + i\bar{C}, i\bar{B} + iC)). \end{aligned}$$

For the quaternionic Lie algebra, from the description of $\mathfrak{GL}_n \mathbb{H}$ we saw in Lecture 7, we have

$$\mathfrak{gl}_n \mathbb{H} = \{A \in \mathfrak{gl}_{2n} \mathbb{C} : AJ = J\bar{A}\},$$

with $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. As before, for $M \in \mathfrak{gl}_{2n}\mathbb{C}$, we can write

$$M = \frac{1}{2}(M - J \cdot \overline{M} \cdot J) - i \cdot \left(\frac{1}{2}(iM + iJ \cdot \overline{M} \cdot J)\right)$$

to see that $\mathfrak{gl}_n\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{gl}_{2n}\mathbb{C}$.

Exercise 26.3. Verify the rest of the list.

The theorem, which also goes back to Cartan, is that *this includes the complete list of simple real Lie algebras associated to the classical complex types (A_n) – (D_n)* . In fact, there are an additional 17 simple real Lie algebras associated with the five exceptional Lie algebras. The proof of this theorem is rather long, and we refer to the literature (cf. [H-S], [Hel], [Ar]) for it.

Split Forms and Compact Forms

Rather than try to classify in general the real forms \mathfrak{g}_0 of a semisimple Lie algebra \mathfrak{g} , we would like to focus here on two particular forms that are possessed by every semisimple Lie algebra and that are by far the most commonly dealt with in practice: the *split form* and the *compact form*.

These represent the two extremes of behavior of the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$ with respect to the real subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$. To begin with, the *split form* of \mathfrak{g} is a form \mathfrak{g}_0 such that there exists a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ (that is, a subalgebra whose complexification $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{g}$ is a Cartan subalgebra of \mathfrak{g}) whose action on \mathfrak{g}_0 has all real eigenvalues—i.e., such that all the roots $\alpha \in R \subset \mathfrak{h}^*$ of \mathfrak{g} (with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}$) assume all real values on the subspace \mathfrak{h}_0 . In this case we have a direct sum decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \left(\bigoplus \mathfrak{i}_\alpha\right)$$

of \mathfrak{g}_0 into \mathfrak{h}_0 and one-dimensional eigenspaces \mathfrak{i}_α for the action of \mathfrak{h}_0 (each \mathfrak{i}_α will just be the intersection of the root space $\mathfrak{g}_\alpha \subset \mathfrak{g}$ with \mathfrak{g}_0); each pair \mathfrak{i}_α and $\mathfrak{i}_{-\alpha}$ will generate a subalgebra isomorphic to $\mathfrak{sl}_2\mathbb{R}$. As we will see momentarily, this uniquely characterizes the real form \mathfrak{g}_0 of \mathfrak{g} .

By contrast, in the *compact form* all the roots $\alpha \in R \subset \mathfrak{h}^*$ of \mathfrak{g} (with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C} \subset \mathfrak{g}$) assume all purely imaginary values on the subspace \mathfrak{h}_0 . We accordingly have a direct sum decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \left(\bigoplus \mathfrak{l}_\alpha\right)$$

of \mathfrak{g}_0 into \mathfrak{h}_0 and two-dimensional spaces on which \mathfrak{h}_0 acts by rotation (each \mathfrak{l}_α will just be the intersection of the root space $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ with \mathfrak{g}_0); each \mathfrak{l}_α will generate a subalgebra isomorphic to \mathfrak{su}_2 .

The existence of the split form of a semisimple complex Lie algebra was already established in Lecture 21: one way to construct a real—even rational

—form \mathfrak{g}_0 of a semisimple Lie algebra \mathfrak{g} is by starting with any generator X_{α_i} for the root space for each positive simple root α_i , completing it to standard basis $X_{\alpha_i}, Y_{\alpha_i}$, and $H_i = [X_{\alpha_i}, Y_{\alpha_i}]$ for the corresponding $\mathfrak{s}_{\alpha_i} = \mathfrak{sl}_2\mathbb{C}$, and taking \mathfrak{g}_0 to be the real subalgebra generated by these elements. Choosing a way to write each positive root as a sum of simple roots even determined a basis $\{H_i \in \mathfrak{h}, X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha \in \mathfrak{g}_{-\alpha}\}$ for \mathfrak{g}_0 , as in (21.20). The Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 is the real span of these H_i . Note that once \mathfrak{h} is fixed for \mathfrak{g} , the real subalgebra \mathfrak{h}_0 is uniquely determined as the span of the H_α for all roots α . The algebra \mathfrak{g}_0 is determined up to isomorphism; it is sometimes called the *natural* real form of \mathfrak{g} . Note that this also demonstrates the uniqueness of the split form: it is the only real form \mathfrak{g}_0 of \mathfrak{g} that has a Cartan subalgebra \mathfrak{h}_0 acting on \mathfrak{g}_0 with all real eigenvalues.

As for the compact form of a semisimple Lie algebra, it owes much of its significance (as well as its name) to the last condition in

Proposition 26.4. *Suppose \mathfrak{g} is any complex semisimple Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{g}$ a real form of \mathfrak{g} . Let \mathfrak{h}_0 be a Cartan subalgebra of \mathfrak{g}_0 , $\mathfrak{h} = \mathfrak{h}_0 \otimes \mathbb{C}$ the corresponding Cartan subalgebra of \mathfrak{g} . The following are equivalent:*

- (i) *Each root $\alpha \in R \subset \mathfrak{h}^*$ of \mathfrak{g} assumes purely imaginary values on \mathfrak{h}_0 , and for each root α the subalgebra of \mathfrak{g}_0 generated by the intersection I_α of $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ with \mathfrak{g}_0 is isomorphic to \mathfrak{su}_2 ;*
- (ii) *The restriction to \mathfrak{g}_0 of the Killing form of \mathfrak{g} is negative definite;*
- (iii) *The real Lie group G_0 with Lie algebra \mathfrak{g}_0 is compact.*

In (iii), G_0 can be taken to be the adjoint form of \mathfrak{g}_0 . However, a theorem of Weyl ensures that the fundamental group of any such G_0 is finite, so the condition is independent of the choice of G_0 . Note also that, by the equivalence with (ii) and (iii), the condition (i) must be independent of the choice of Cartan subalgebra \mathfrak{h}_0 . This is in contrast with the split case, where we require only that there exist a Cartan subalgebra whose action on \mathfrak{g} has all real eigenvalues; as we saw in the case of $\mathfrak{sl}_2\mathbb{R}$, in the split case a different \mathfrak{h}_0 may have imaginary eigenvalues.

PROOF. We start by showing that the first condition implies the second; this will follow from direct observation. To begin with, the value of the Killing form on $H \in \mathfrak{h}_0$ is visibly

$$B(H, H) = \sum (\alpha(H))^2 < 0.$$

Next, the subspaces I_α are orthogonal to one another with respect to B , so it remains only to verify $B(Z, Z) < 0$ for a general member $Z \in I_\alpha$. To do this, let X and Y be generators of \mathfrak{g}_α and $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}$ respectively, chosen so as to form, together with their commutator $H = [X, Y]$ a standard basis for $\mathfrak{sl}_2\mathbb{C}$. By the analysis of real forms of $\mathfrak{sl}_2\mathbb{C}$ above, we may take as generators of the algebra generated by I_α the elements $iH, U = X - Y$ and $V = iX + iY$. If we set

$$Z = aU + bV = (a + ib) \cdot X + (-a + ib) \cdot Y,$$

then we have

$$\begin{aligned} \operatorname{ad}(Z) \circ \operatorname{ad}(Z) &= (a + ib)^2 \operatorname{ad}(X) \circ \operatorname{ad}(X) \\ &\quad - (a^2 + b^2)(\operatorname{ad}(X) \circ \operatorname{ad}(Y) + \operatorname{ad}(Y) \circ \operatorname{ad}(X)) \\ &\quad + (a - ib)^2 \operatorname{ad}(Y) \circ \operatorname{ad}(Y). \end{aligned}$$

Now, $\operatorname{ad}(X) \circ \operatorname{ad}(X)$ and $\operatorname{ad}(Y) \circ \operatorname{ad}(Y)$ have no trace, so we can write

$$\operatorname{trace}(\operatorname{ad}(Z) \circ \operatorname{ad}(Z)) = -2 \cdot (a^2 + b^2) \cdot \operatorname{trace}(\operatorname{ad}(X) \circ \operatorname{ad}(Y)). \quad (26.5)$$

By direct examination, in the representation $\operatorname{Sym}^n V$ of $\mathfrak{sl}_2 \mathbb{C}$, $\operatorname{ad}(X) \circ \operatorname{ad}(Y)$ acts by multiplication by $(n - \lambda)(n + \lambda - 2)/4 \geq 0$ on the λ -eigenspace for H , from which we deduce that the right-hand side of (26.5) is negative.

Next, we show that the second condition implies the third. This is immediate: the adjoint form G_0 is the connected component of the identity of the group $\operatorname{Aut}(\mathfrak{g}_0)$. In particular, it is a closed subgroup of the adjoint group of \mathfrak{g} , and it acts faithfully on the real vector space \mathfrak{g}_0 , preserving the bilinear form B . If B is negative definite it follows that G_0 is a closed subgroup of the orthogonal group $\operatorname{SO}_m \mathbb{R}$, which is compact.

Finally, if we know that G_0 is compact, by averaging we can construct a positive definite inner product on \mathfrak{g}_0 invariant under the action of G_0 . For any X in \mathfrak{g}_0 , $\operatorname{ad}(X)$ is represented by a skew-symmetric matrix $A = (a_{i,j})$ with respect to an orthonormal basis of \mathfrak{g}_0 (cf. (14.23)), so $B(X, X) = \operatorname{Tr}(A \circ A) = \sum_{i,j} a_{i,j} a_{j,i} = -\sum a_{i,j}^2 \leq 0$. In particular, the eigenvalues of $\operatorname{ad}(X)$ must be purely imaginary. Therefore $\alpha(\mathfrak{h}_0) \subset i\mathbb{R}$ and $\bar{\alpha} = -\alpha$ for any root α , from which (i) follows. \square

We now claim that *every semisimple complex Lie algebra has a unique compact form*. To see this we need an algebraic notion which is, in fact, crucial to the classification theorem mentioned above: that of *conjugate linear involution*. If $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of a real Lie algebra \mathfrak{g}_0 , there is a map $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ which takes $x \otimes z$ to $x \otimes \bar{z}$ for $x \in \mathfrak{g}_0$ and $z \in \mathbb{C}$; it is conjugate linear, preserves Lie brackets, and σ^2 is the identity. The real algebra \mathfrak{g}_0 is the fixed subalgebra of σ , and conversely, given such a conjugate linear involution σ of a complex Lie algebra \mathfrak{g} , its fixed algebra \mathfrak{g}^σ is a real form of \mathfrak{g} . To prove the claim, we start with the split, or natural form, as constructed in Lecture 21 and referred to above. With a basis for \mathfrak{g} chosen as in this construction, it is not hard to show that there is a unique Lie algebra automorphism φ of \mathfrak{g} that takes each element of \mathfrak{h} to its negative and takes each X_α to Y_α (this follows from Claim 21.25). This automorphism φ is a complex linear involution which preserves the real subalgebra \mathfrak{g}_0 . This automorphism commutes with the associated conjugate linear σ . The composite $\sigma\varphi = \varphi\sigma$ is a conjugate linear involution, from which it follows that its fixed part $\mathfrak{g}_c = \mathfrak{g}^{\sigma\varphi}$ is another real form of \mathfrak{g} . This has Cartan subalgebra $\mathfrak{h}_c = \mathfrak{h}^{\sigma\varphi} = i \cdot \mathfrak{h}_0$. We have seen that the restriction of the Killing form to \mathfrak{h}_0 is positive definite. It follows that its restriction to \mathfrak{h}_c is negative definite, and hence that \mathfrak{g}_c is a compact form of \mathfrak{g} . Finally, this construction of \mathfrak{g}_c from \mathfrak{g}_0 is reversible, and from this one can deduce the uniqueness of the compact form.

We may see directly from this construction that

$$\mathfrak{g}_c = \mathfrak{h}_c \oplus \bigoplus_{\alpha \in R^+} \mathfrak{l}_\alpha,$$

where $\mathfrak{l}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})^{\sigma\varphi}$ is a real plane with $\mathfrak{l}_\alpha \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ and $[\mathfrak{h}_c, \mathfrak{l}_\alpha] \subset \mathfrak{l}_\alpha$.

Exercise 26.6. Verify that $\{A_j = i \cdot H_j; 1 \leq j \leq n\}$ is a basis for \mathfrak{h}_c , $\{B_\alpha = X_\alpha - Y_\alpha, C_\alpha = i \cdot (X_\alpha + Y_\alpha)\}$ is a basis for \mathfrak{l}_α , and the action is given by

$$[A_j, B_\alpha] = p \cdot C_\alpha \quad \text{and} \quad [A_j, C_\alpha] = -p \cdot B_\alpha,$$

where p is the integer $\alpha(H_j)$. In particular, \mathfrak{h}_c acts by rotations on the planes \mathfrak{l}_α .

Our classical Lie algebras \mathfrak{g} all came equipped with a natural real form \mathfrak{g}_0 , and with a basis of the above type. These split forms are:

Complex simple Lie algebra	Split form
$\mathfrak{sl}_{n+1} \mathbb{C}$	$\mathfrak{sl}_{n+1} \mathbb{R}$
$\mathfrak{so}_{2n+1} \mathbb{C}$	$\mathfrak{so}_{n+1, n}$
$\mathfrak{sp}_{2n} \mathbb{C}$	$\mathfrak{sp}_{2n} \mathbb{R}$
$\mathfrak{so}_{2n} \mathbb{C}$	$\mathfrak{so}_{n, n}$

Exercise 26.7. For each of these split forms, find the corresponding compact form \mathfrak{g}_c .

Exercise 26.8. Let \mathfrak{g}_0 be a real semisimple Lie algebra. Show that a subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 is a Cartan subalgebra if and only if it is a maximal abelian subalgebra and the adjoint action on \mathfrak{g}_0 is semisimple.

Exercise 26.9*. Starting with a real form \mathfrak{g}_0 of \mathfrak{g} with associated conjugation σ , show that one can always find a compact form \mathfrak{g}_c of \mathfrak{g} such that $\sigma(\mathfrak{g}_c) = \mathfrak{g}_c$, and such that

$$\mathfrak{g}_0 = \mathfrak{t} \oplus \mathfrak{p},$$

where $\mathfrak{t} = \mathfrak{h}_0 = \mathfrak{g}_0 \cap \mathfrak{g}_c$, and $\mathfrak{p} = \mathfrak{g}_0 \cap (i \cdot \mathfrak{g}_c)$. Such a decomposition is called a *Cartan decomposition* of \mathfrak{g}_0 . It is unique up to inner automorphism.

Exercise 26.10*. For any real form \mathfrak{g}_0 of \mathfrak{g} , given by a conjugation σ , show that there is a Cartan subalgebra \mathfrak{h} of \mathfrak{g} that is preserved by σ , so $\mathfrak{g}_0 \cap \mathfrak{h}$ is a Cartan subalgebra of \mathfrak{g}_0 .

Naturally, the various special isomorphisms between complex Lie algebras ($\mathfrak{sl}_2 \mathbb{C} \cong \mathfrak{so}_3 \mathbb{C} \cong \mathfrak{sp}_2 \mathbb{C}$, etc.) give rise to special isomorphisms among their real forms. For example, we have already seen that

$$\mathfrak{sl}_2 \mathbb{R} \cong \mathfrak{su}_{1,1} \cong \mathfrak{so}_{2,1} \cong \mathfrak{sp}_2 \mathbb{R},$$

while

$$\mathfrak{su}_2 \cong \mathfrak{so}_3 \mathbb{R} \cong \mathfrak{sl}_1 \mathbb{H} \cong \mathfrak{u}_1 \mathbb{H}$$

(cf. Exercise 26.1). Similarly, each of the remaining three special isomorphisms of complex semisimple Lie algebras gives rise to isomorphisms between their real forms, as follows:

- (i) $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$
 compact forms: $\mathfrak{so}_4\mathbb{R} \cong \mathfrak{su}_2 \times \mathfrak{su}_2$
 split forms: $\mathfrak{so}_{2,2} \cong \mathfrak{sl}_2\mathbb{R} \times \mathfrak{sl}_2\mathbb{R}$
 others: $\mathfrak{so}_{3,1} \cong \mathfrak{sl}_2\mathbb{C}$, $\mathfrak{u}_2^*\mathbb{H} \cong \mathfrak{su}_2 \times \mathfrak{sl}_2\mathbb{R}$.
- (ii) $\mathfrak{sp}_4\mathbb{C} \cong \mathfrak{so}_5\mathbb{C}$
 compact forms: $\mathfrak{u}_2\mathbb{H} \cong \mathfrak{so}_5\mathbb{R}$
 split forms: $\mathfrak{sp}_4\mathbb{R} \cong \mathfrak{so}_{3,2}$
 other: $\mathfrak{u}_{1,1}\mathbb{H} \cong \mathfrak{so}_{4,1}$.
- (iii) $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$
 compact forms: $\mathfrak{su}_4 \cong \mathfrak{so}_6\mathbb{R}$
 split forms: $\mathfrak{sl}_4\mathbb{R} \cong \mathfrak{so}_{3,3}$
 others: $\mathfrak{su}_{2,2} \cong \mathfrak{so}_{4,2}$; $\mathfrak{su}_{3,1} \cong \mathfrak{u}_3^*\mathbb{H}$; $\mathfrak{sl}_2\mathbb{H} \cong \mathfrak{so}_{5,1}$.

In addition, the extra automorphism of $\mathfrak{so}_8\mathbb{C}$ coming from triality gives rise to an isomorphism $\mathfrak{u}_4^*\mathbb{H} \cong \mathfrak{so}_{6,2}$.

Exercise 26.11. Verify some of the isomorphisms above. (Of course, in the case of compact and split forms, these are implied by the corresponding isomorphisms of complex Lie algebras, but it is worthwhile to see them directly in any case.)

Real Groups

We turn now to problem of describing the real Lie groups with these Lie algebras. Let G be the adjoint form of the semisimple complex Lie algebra \mathfrak{g} . If \mathfrak{g}_0 is a real form of \mathfrak{g} , the associated conjugate linear involution σ of \mathfrak{g} that fixes \mathfrak{g}_0 lifts to an involution $\tilde{\sigma}$ of G . (This follows from the functorial nature of the adjoint form, noting that G is regarded now as a real Lie group.) The fixed points $G^{\tilde{\sigma}}$ of this involution then form a closed subgroup of G ; its connected component of the identity G_0 is a real Lie group whose Lie algebra is \mathfrak{g}_0 . G is called the *complexification* of G_0 .

We have seen in §23.1 that if $\Gamma = \Gamma_{\mathfrak{w}}$ is the lattice of those elements in \mathfrak{h} on which all roots take integral values, then $2\pi i\Gamma$ is the kernel of the exponential mapping $\exp: \mathfrak{h} \rightarrow G$ to the adjoint form. If \mathfrak{h}_0 is a Cartan subalgebra of \mathfrak{g}_0 , $T = \exp(\mathfrak{h}_0)$ will be compact precisely when the intersection of \mathfrak{h}_0 with the kernel $2\pi i\Gamma$ is a lattice of maximal rank. In this case, T will be a product of n copies of the circle S^1 , $n = \dim(\mathfrak{h})$, and, since the Killing form on \mathfrak{h}_0 is negative definite, the corresponding real group G_0 will also be compact. Such a G_0 will be a maximal compact subgroup of G .

When $G_0 \subset G$ is a maximal compact subgroup, they have the same irreducible complex representations. Indeed, for any complex group G' , each complex

homomorphism from G to G' is the extension of a unique real homomorphism from G_0 to G' . This follows from the corresponding fact for Lie algebras and the fact that G_0 and G have the same fundamental group. This is another general fact, which implies the finiteness of the fundamental group of G_0 ; we omit the proof, noting only that it can be seen directly in the classical cases:

Exercise 26.12*. Prove that $\pi_1(G_0) \rightarrow \pi_1(G)$ is an isomorphism for each of the classical adjoint groups.

Exercise 26.13*. The special isomorphisms of real Lie algebras listed above give rise to special isomorphisms of real Lie groups. Can you find these?

It is another general fact that any compact (connected) Lie group is a quotient

$$(G_1 \times G_2 \times \cdots \times G_r \times T)/Z,$$

where the G_i are simple compact Lie groups, $T \cong (S^1)^k$ is a torus, and Z is a discrete subgroup of the center. In particular, its Lie algebra is the direct sum of a semisimple compact Lie algebra and an abelian Lie algebra. This provides another reason why the classification of irreducible representations in the real compact case and the semisimple complex case are essentially the same.

Representations of Real Lie Algebras

Finally, we should say a word here about the irreducible representations (always here in complex vector spaces!) of simple real Lie algebras. In some cases these are easily described in terms of the complex case: for example, the irreducible representations of \mathfrak{su}_m or $\mathfrak{sl}_m \mathbb{R}$ are the same as those for $\mathfrak{sl}_m \mathbb{C}$, i.e., they are the restrictions of the irreducible representations $\Gamma_\lambda = \mathbb{S}_\lambda \mathbb{C}^m$ corresponding to partitions or Young diagrams λ . This is the situation in general whenever the complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ of the real Lie algebra \mathfrak{g}_0 is still simple: the representations of \mathfrak{g}_0 on complex vector spaces are exactly the representations of \mathfrak{g} . The situation is slightly different when we have a simple real Lie algebra whose complexification is not simple: for example, the irreducible representations of $\mathfrak{sl}_m \mathbb{C}$, regarded as a real Lie algebra, are of the form $\Gamma_\lambda \otimes \bar{\Gamma}_\mu$, where $\bar{\Gamma}_\mu$ is the conjugate representation of Γ_μ . The situation in general is expressed in the following

Exercise 26.14. Show that if \mathfrak{g}_0 is a simple real Lie algebra whose complexification \mathfrak{g} is simple, its irreducible representations are the restrictions of (uniquely determined) irreducible representations of \mathfrak{g} . If \mathfrak{g}_0 is the underlying real algebra of a simple complex Lie algebra, show that the irreducible representations of \mathfrak{g}_0 are of the form $V \otimes \bar{W}$, where V and W are (uniquely determined) irreducible representations of the complex Lie algebra.

§26.2. Second Proof of Weyl's Character Formula

The title of this section is perhaps inaccurate: what we will give here is actually a sketch of the first proof of the Weyl character formula. Weyl, in his original proof, used what he called the “unitarian trick,” which is to say he introduces the compact form of a given semisimple Lie algebra and uses integration on the corresponding compact group G . (This trick was already described in §9.3, in the context of proving complete reducibility of representations of a semisimple algebra.)

Indeed, the main reason for including this section (which is, after all, logically unnecessary) is to acquaint the reader with the “classical” treatment of Lie groups via their compact forms. This treatment follows very much the same lines as the representation theory of finite groups. To begin with, we replace the average $(1/|G|)\sum_{g \in G} f(g)$ by the integral $\int_G f(g) d\mu$, the volume element $d\mu$ chosen to be translation invariant and such that $\int_G d\mu = 1$. If $\rho: G \rightarrow \text{Aut}(V)$ is a finite-dimensional representation, with character

$$\chi_V(g) = \text{Trace}(\rho(g)),$$

then $\int_G \rho(g) d\mu \in \text{Hom}(V, V)$ is idempotent, and it is the projection onto the invariant subspace V^G . So $\int_G \chi_V(g) d\mu = \dim(V^G)$. Applied to $\text{Hom}(V, W)$ as before, since $\chi_{\text{Hom}(V, W)} = \bar{\chi}_V \chi_W$, it follows that

$$\int_G \bar{\chi}_V \chi_W d\mu = \dim(\text{Hom}_G(V, W)).$$

So if V and W are irreducible,

$$\int_G \bar{\chi}_V \chi_W d\mu = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$$

Up to now, everything is completely analogous to the case of finite groups, and is proved in exactly the same way. The last general fact, analogous to the basic Proposition 2.30, is harder in the compact case:

Peter–Weyl Theorem. The characters of irreducible representations span a dense subspace of the space of continuous class functions.

It is, moreover, the case that the coordinate functions of the irreducible matrix representations span a dense subspace of all continuous (or L^2) functions on G . For the proof of these statements we refer to [Ad] or [B-tD]. Given the fundamental role that (2.30) played in the analysis of representations of finite groups, it is not surprising that the Peter–Weyl theorem is the cornerstone of most treatments of compact groups, even though it has played no role so far in this book.

We now proceed to indicate how the original proof of the Weyl character

formula went in this setting. In this section, G will denote a fixed compact group, whose Lie algebra \mathfrak{g} is a real form of the semisimple complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We have seen that

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{l}_{\alpha},$$

compatible with the usual decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus (\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha})$ when complexified. The real Cartan algebra \mathfrak{h} acts by rotations on the planes \mathfrak{l}_{α} .

Now let $T = \exp(\mathfrak{h}) \subset G$. As before we have chosen \mathfrak{h} so that it contains the lattice $2\pi i\Gamma$ which is the kernel of the exponential map from $\mathfrak{h}_{\mathbb{C}}$ to the simply-connected form of $\mathfrak{g}_{\mathbb{C}}$, so $T \cong (S^1)^n$ is a compact torus.

In this compact case we can realize the Weyl group on the group level again:

Claim 26.15. $N(T)/T \cong \mathfrak{W}$.

PROOF. For each pair of roots $\alpha, -\alpha$, we have a subalgebra $\mathfrak{s}_{\alpha} \cong \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}_{\mathbb{C}}$, with a corresponding $\mathfrak{su}_2 \subset \mathfrak{g}$. Exponentiating gives a subgroup $SU(2) \subset G$.

The element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts by Ad, taking H to $-H$, X to Y , and Y to X . It is in $N(T)$, and, with B as in the preceding section, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp\left(\frac{1}{2}\pi iB\right)$.

Then $\exp\left(\frac{1}{2}\pi iB\right) \in \mathfrak{g}$ acts by reflection in the hyperplane $\alpha^{\perp} \subset \mathfrak{h}$. □

Note that \mathfrak{W} acting on \mathfrak{h} takes the lattice $2\pi i\Gamma$ to itself, so \mathfrak{W} acts on $T = \mathfrak{h}/2\pi i\Gamma$ by conjugation.

Theorem 26.16. *Every element of G is conjugate to an element of T . A general element is conjugate to $|\mathfrak{W}|$ such elements of T .*

Sketch of a proof: Note that G acts by left multiplication on the left coset space $X = G/T$. For any $z \in G$, consider the map $f_z: X \rightarrow X$ which takes yT to zyT . The claim is that f_z must have a fixed point, i.e., there is a y such that $y^{-1}zy \in T$. Since all f_z are homotopic, and X is compact, the Lefschetz number of f_z is the topological Euler characteristic of X . The first statement follows from the claim that this Euler characteristic is not zero. This is a good exercise for the classical groups; see [Bor2] for a general proof. For another proof see Remark 26.20 below.

For the second assertion, check first that any element that commutes with every element of T is in T . Take an ‘‘irrational’’ element x in T so that its multiples are dense in T . Then for any $y \in G$, $yx y^{-1} \in T \Leftrightarrow yTy^{-1} = T$, and $yx y^{-1} = x \Leftrightarrow y \in T$. This gives precisely $|\mathfrak{W}|$ conjugates of x that are in T .

Corollary 26.17. *The class functions on G are the \mathfrak{W} -invariant functions on T .*

Suppose G is a real form of the complex semisimple group $G_{\mathbb{C}}$, i.e., G is a real analytic closed subgroup of $G_{\mathbb{C}}$, and the Lie algebra of $G_{\mathbb{C}}$ is $\mathfrak{g}_{\mathbb{C}}$. The characters on $G_{\mathbb{C}}$ can be written $\sum n_{\mu} e^{2\pi i \mu}$, the sum over μ in the weight lattice Λ ; they are invariant under the Weyl group. From what we have seen, they can be identified with \mathfrak{B} -invariant functions on the torus T . Let us work this out for the classical groups:

Case (A_n): $G = \text{SU}(n + 1)$. The Lie algebra \mathfrak{su}_{n+1} consists of skew-Hermitian matrices,

$$\mathfrak{h} = \mathfrak{su}_{n+1} \cap \mathfrak{sl}_{n+1} \mathbb{R} = \{\text{imaginary diagonal matrices of trace } 0\},$$

and $T = \{\text{diag}(e^{2\pi i \vartheta_1}, \dots, e^{2\pi i \vartheta_{n+1}}) : \sum \vartheta_j = 0\}$. In this case, the Weyl group \mathfrak{B} is the symmetric group \mathfrak{S}_{n+1} , represented by permutation matrices (with one entry ± 1 on each row and column, other entries 0) modulo T . Let $z_i: T \rightarrow S^1$ correspond to the i th diagonal entry $e^{2\pi i \vartheta_i}$. So characters on T are symmetric polynomials in z_1, \dots, z_{n+1} modulo the relation $z_1 \cdots z_{n+1} = 1$. Therefore, characters on $\text{SU}(n + 1)$ are symmetric polynomials in z_1, \dots, z_{n+1} .

Case (B_n): $G = \text{SO}(2n + 1)$. \mathfrak{h} consists of matrices with n 2×2 blocks of the form

$$\begin{pmatrix} \cos(2\pi \vartheta_i) & -\sin(2\pi \vartheta_i) \\ \sin(2\pi \vartheta_i) & \cos(2\pi \vartheta_i) \end{pmatrix}$$

along the diagonal, and one 1 in the lower right corner. Again we see that $T = (S^1)^n$. This time $N(T)$ will have block permutations to interchange the blocks, and also matrices with some blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the squares along the diagonal, with the other blocks 2×2 identity matrices, with a ± 1 in the corner to make the determinant positive; these take ϑ_i to $-\vartheta_i$ for each i where a block is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This again realizes the Weyl group as a semidirect product of \mathfrak{S}_n and $(\mathbb{Z}/2)^n$. With z_i identified with $e^{2\pi i \vartheta_i}$ again, we see that the characters are the symmetric polynomials in the variables $z_i + z_i^{-1}$, i.e., in $\cos(2\pi \vartheta_1), \dots, \cos(2\pi \vartheta_n)$.

Case (D_n): $G = \text{SO}(2n)$. \mathfrak{h} is as in the preceding case, but with no lower corner. Since we have no corner to put a -1 in, there can be only an even number of blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, reflecting the fact that \mathfrak{B} is a semidirect product of $(\mathbb{Z}/2)^{n-1}$ and \mathfrak{S}_n . This time the invariants are symmetric polynomials in the $z_i + z_i^{-1}$, and one additional $\prod_i (z_i - z_i^{-1})$.

Case (C_n): $G = \text{Sp}(2n)$. \mathfrak{h} consists of imaginary diagonal matrices, T consists of diagonal matrices with entries $e^{2\pi i \vartheta_i}$. The Weyl group is generated by

permutation matrices and diagonal matrices with entries which are 1's and quaternionic j 's: \mathfrak{B} is a semidirect product of $(\mathbb{Z}/2)^n$ and \mathfrak{S}_n . The invariants are symmetric polynomials in the $z_i + z_i^{-1}$.

The key to Weyl's analysis is to calculate the integral of a class function f on G as a suitable integral over the torus T . For this, consider the map

$$\pi: G/T \times T \rightarrow G, \quad \pi(xT, y) = xyx^{-1}.$$

By what we said earlier, π is a generically finite-sheeted covering, with $|\mathfrak{B}|$ sheets. It follows that

$$\int_G f d\mu = \frac{1}{|\mathfrak{B}|} \int_{G/T \times T} \pi^*(f) \pi^* d\mu.$$

Now $\pi^*(f)(xT, y) = f(y)$ since f is a class function. To calculate $\pi^* d\mu$, consider the induced map on tangent spaces

$$\pi_* = d\pi: \mathfrak{g}/\mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{g}.$$

At the point $(x_0 T, y_0) \in G/T \times T$,

$$(x_0 e^{tx} T, y_0 e^{ty}) \mapsto x_0 e^{tx} y_0 e^{ty} e^{-tx} x_0^{-1}.$$

We want to calculate

$$\frac{d}{dt} (x_0 e^{tx} y_0 e^{ty} e^{-tx} x_0^{-1})|_{t=0} (x_0 y_0 x_0^{-1})^{-1},$$

which is

$$x_0(x y_0 + y_0 y - y_0 x) x_0^{-1} (x_0 y_0^{-1} x_0^{-1}) = x_0(x + y_0 y y_0^{-1} - y_0 x y_0^{-1}) x_0^{-1}.$$

Now $y_0 y y_0^{-1} = y$ since $y_0 \in T$ and $y \in \mathfrak{h}$. To calculate the determinant of π_* we can ignore the volume-preserving transformation $x_0(\) x_0^{-1}$. If we identify \mathfrak{g} with $\mathfrak{g}/\mathfrak{h} \times \mathfrak{h}$, the matrix becomes

$$\begin{pmatrix} I - \text{Ad}(y_0) & 0 \\ 0 & I \end{pmatrix}.$$

So the determinant of π_* is $\det(I - \text{Ad}(y_0))$. Now $(\mathfrak{g}/\mathfrak{h})_{\mathbb{C}} = \bigoplus \mathfrak{g}_{\alpha}$, and $\text{Ad}(y_0)$ acts as $e^{2\pi i \alpha(y_0)}$ on \mathfrak{g}_{α} . Hence

$$\det(\pi_*) = \prod_{\alpha \in R} (1 - e^{2\pi i \alpha}), \tag{26.18}$$

as a function on T alone, independent of the factor G/T . This gives *Weyl's integration formula*:

$$\int_G f d\mu_G = \frac{1}{|\mathfrak{B}|} \int_T f(y) \prod_{\alpha \in R} (1 - e^{2\pi i \alpha(y)}) d\mu_T. \tag{26.19}$$

Remark 26.20. The same argument gives another proof of the theorem that G is covered by conjugates of T . This amounts to the assertion that the map

$\pi: G/T \times T \rightarrow G$ of compact manifolds is surjective. By what we saw above, for a generic point $y_0 \in T$ there are exactly $|\mathfrak{B}|$ points in $\pi^{-1}(y_0)$, and at each of these the Jacobian determinant is the same (nonzero) number. It follows that the topological degree of the map π is $|\mathfrak{B}|$, so the map must be surjective.

Now $(1 - e^{2\pi i \alpha})(1 - e^{-2\pi i \alpha}) = (e^{\pi i \alpha} - e^{-\pi i \alpha})(\overline{e^{\pi i \alpha} - e^{-\pi i \alpha}})$, so if we set

$$\Delta = \prod_{\alpha \in \bar{R}^+} (e^{\pi i \alpha} - e^{-\pi i \alpha}),$$

then $\det(\pi_*) = \Delta \bar{\Delta}$. As we saw in Lemma 24.3, $\Delta = A_\rho$, where ρ is half the sum of the positive roots and, for any weight μ ,

$$A_\mu = \sum_{W \in \mathfrak{B}} (-1)^W e^{2\pi i W(\mu)}.$$

Now we can complete the second proof of Weyl’s character formula: the character of the representation with highest weight λ is $A_{\lambda+\rho}/A_\rho$. Since we saw in §24.1 that $A_{\lambda+\rho}/A_\rho$ has highest weight λ and (see Corollary 24.6) its value at the identity is positive, it suffices to show that the integral of $\int_G \chi \bar{\chi} = 1$, where $\chi = A_{\lambda+\rho}/A_\rho$. By Weyl’s integration formula,

$$\begin{aligned} \int_G \chi \bar{\chi} &= \frac{1}{|\mathfrak{B}|} \int_T \chi \bar{\chi} \Delta \bar{\Delta} = \frac{1}{|\mathfrak{B}|} \int_T A_{\lambda+\rho} \overline{A_{\lambda+\rho}} \\ &= \frac{1}{|\mathfrak{B}|} \int_T \sum_{W \in \mathfrak{B}} (-1)^W e^{2\pi i W(\lambda+\rho)} \cdot \sum_{W \in \mathfrak{B}} (-1)^W e^{-2\pi i W(\lambda+\rho)} = 1, \end{aligned}$$

which concludes the proof.

§26.3. Real, Complex, and Quaternionic Representations

The final topic we want to take up is the classification of irreducible complex representations of semisimple Lie groups or algebras into those of real, quaternionic, or complex type. To define our terms, given a real semisimple Lie group G_0 or its Lie algebra \mathfrak{g}_0 and a representation of G_0 or \mathfrak{g}_0 on a complex vector space V we say that the representation V is *real*, or of *real type*, if it comes from a representation of G_0 or \mathfrak{g}_0 on a real vector space V_0 by extension of scalars ($V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$); this is equivalent to saying that it has a conjugate linear endomorphism whose square is the identity. It is *quaternionic* if it comes from a quaternionic representation by restriction of scalars, or equivalently if it has a conjugate linear endomorphism whose square is minus the identity. Finally, we say that the representation is *complex* if it is neither of these. (Compare with Theorem 3.37 for finite groups.)

Having completely classified the irreducible representations of the classical complex Lie algebras, and having described all the real forms of these Lie

algebras, we have a clear-cut problem: to determine the type of the restriction of each representation to each real form. Rather than try to answer this in every case, however, we will instead mention some of the ideas that allow us to answer this question, and then focus on the cases of the split forms (where the answer is easy) and the compact forms (where the answer is more interesting, and where we have more tools to play with). We assume the complexification \mathfrak{g} of \mathfrak{g}_0 is simple, so irreducible representations of \mathfrak{g}_0 are restrictions of unique irreducible representations of \mathfrak{g} (cf. (26.14)); in particular, we have the classification of irreducible representations by dominant weights.

To begin with, the tensor products of two real, or two quaternionic, or of a pair of complex conjugate representations is always real; and exterior powers of real and quaternionic representations are equally easy to analyze, as for finite groups (see Exercise 3.43). Such tensor and exterior powers may not be irreducible, but the following criterion can often be used to describe an irreducible component of highest weight that occurs inside them:

Exercise 26.21*. Suppose W is a representation of a semisimple group G that is real or quaternionic, and suppose W has a highest weight λ that occurs with multiplicity 1. Show that the irreducible representations Γ_λ with highest weight λ has the same type as W .

We may apply this in particular to the tensor product $\Gamma_\lambda \otimes \Gamma_\mu$ of the irreducible representations of \mathfrak{g} with highest weights λ and μ ; since the irreducible representation $\Gamma_{\lambda+\mu}$ with highest weight $\lambda + \mu$ appears once in this tensor product, we deduce

Exercise 26.22*. (i) If Γ_λ and Γ_μ are both real or both quaternionic, then $\Gamma_{\lambda+\mu}$ is real. (ii) If Γ_λ is real and Γ_μ is quaternionic, then $\Gamma_{\lambda+\mu}$ is quaternionic. (iii) If Γ_λ and Γ_μ are complex and conjugate, then $\Gamma_{\lambda+\mu}$ is real.

The last two exercises almost completely answer the question of the representations of the split forms of the classical groups: we have

Proposition 26.23. *Every irreducible representation of the split forms $\mathfrak{sl}_{n+1}\mathbb{R}$, $\mathfrak{so}_{n+1,n}\mathbb{R}$, $\mathfrak{sp}_{2n}\mathbb{R}$, and $\mathfrak{so}_{n,n}\mathbb{R}$ of the classical Lie algebras is real.*

PROOF. In each of these cases, the standard representation V is real, from which it follows that the exterior powers $\wedge^k V$ are real, from which it follows that the symmetric powers $\text{Sym}^{a_k}(\wedge^k V)$ are real. Now, in the cases of $\mathfrak{sl}_{n+1}\mathbb{R}$ and $\mathfrak{sp}_{2n}\mathbb{R}$, we have seen that the highest weights ω_k of the representations $\wedge^k V$ for $k = 1, \dots, n$ form a set of fundamental weights: that is, every irreducible representation Γ has highest weight $\sum a_k \cdot \omega_k$ for some non-negative integers a_1, \dots, a_n . It follows that Γ appears once in the tensor product

$$\text{Sym}^{a_1} V \otimes \text{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \text{Sym}^{a_n}(\wedge^n V)$$

and so is real. (Alternatively, Weyl's construction produces real representations when applied to real vector spaces.)

The only difference in the orthogonal case is that some of the exterior powers $\wedge^k V$ of the standard representation must be replaced in this description by the spin representation(s). That the spin representations are real follows from the construction in Lecture 20, cf. Exercise 20.23; the result in this case then follows as before. \square

The Compact Case

We turn now to the compact forms of the classical Lie algebras. In this case, the theory behaves very much like that of finite groups, discussed in Lecture 5. Specifically, any action of a compact group G_0 on a complex vector space V preserves a nondegenerate Hermitian inner product (obtained, for example, by choosing one arbitrarily and averaging its translates under the action of G_0). It follows that the dual of V is isomorphic to its conjugate, so that V will be either real or quaternionic exactly when it is isomorphic to its dual V^* . (In terms of characters, this says that the character $\text{Char}(V)$ is invariant under the automorphism of $\mathbb{Z}[\Lambda]$ which takes $e(\mu)$ to $e(-\mu)$; for groups, this says the character is real.) More precisely, an irreducible representation of a compact group/Lie algebra will be real (resp. quaternionic) if and only if it has an invariant nondegenerate symmetric (resp. skew-symmetric) bilinear form. In other words, the classification of an irreducible V is determined by whether

$$V \otimes V = \text{Sym}^2 V \oplus \wedge^2 V$$

contains the trivial representation, and, if so, in which factor. So determining which type a representation belongs to is a very special case of the general plethysm problem of decomposing such representations.

With this said, we consider in turn the algebras \mathfrak{su}_n , \mathfrak{u}_n , \mathfrak{h} , and \mathfrak{so}_m .

Let Γ_λ be the irreducible representation of $\mathfrak{sl}_n \mathbb{C}$ with highest weight $\lambda = \sum a_i \cdot \omega_i$, where $\omega_i = L_1 + \dots + L_i, i = 1, \dots, n - 1$ are the fundamental weights of $\mathfrak{sl}_n \mathbb{C}$. The dual of Γ will have highest weight $\sum a_{n-i} \cdot \omega_i$, so that Γ will be real or quaternionic if and only if $a_i = a_{n-i}$ for all i . We now distinguish three cases:

(i) If n is odd, then the sublattice of weights $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i is freely generated by the sums $\omega_i + \omega_{n-i}$ for $i = 1, \dots, (n - 1)/2$. Now, ω_i is the highest weight of the exterior power $\wedge^i V$, so that the irreducible representation with highest weight $\omega_i + \omega_{n-i}$ will appear once in the tensor product

$$\wedge^i V \otimes \wedge^{n-i} V = (\wedge^i V) \otimes (\wedge^i V)^*,$$

which by Exercise 26.21 above is real. It follows that for any weight $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i , the irreducible representation Γ_λ is real.

(ii) If $n = 2k$ is even, then the sublattice of weights $\lambda = \sum a_i \cdot \omega_i$ with

$a_i = a_{n-i}$ for all i is freely generated by the sums $\omega_i + \omega_{n-i}$ for $i = 1, \dots, k-1$, together with the weight ω_k . As before, the irreducible representations with highest weight $\omega_i + \omega_{n-i}$ are all real. Moreover, in case n is divisible by 4 the representation $\wedge^k V$ is real as well, since $\wedge^k V$ admits a symmetric bilinear form

$$\wedge^k V \otimes \wedge^k V \rightarrow \wedge^{2k} V = \mathbb{C}$$

given by wedge product. It follows then as before that for any weight $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i , the irreducible representation Γ_λ is real.

(iib) In case n is congruent to 2 mod 4, the analysis is similar to the last case except that wedge product gives a skew-symmetric bilinear pairing on $\wedge^k V$. The representation $\wedge^k V$ is thus quaternionic, and it follows that for any weight $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i , the irreducible representation Γ_λ is real if a_k is even, quaternionic if a_k is odd. In sum, then, we have

Proposition 26.24. *For any weight $\lambda = \sum a_i \cdot \omega_i$ of \mathfrak{su}_n , the irreducible representation Γ_λ with highest weight λ is: complex if $a_i \neq a_{n-i}$ for any i ; real if $a_i = a_{n-i}$ for all i and n is odd, or $n = 4k$, or $n = 4k + 2$ and a_{2k+1} is even; and quaternionic if $a_i = a_{n-i}$ for all i and $n = 4k + 2$ and a_{2k+1} is odd.*

Next, we consider the case of the compact form $\mathfrak{u}_n \mathbb{H}$ of $\mathfrak{sp}_{2n} \mathbb{C}$. To begin with, we note that since the restriction to $\mathfrak{u}_n \mathbb{H}$ of the standard representation of $\mathfrak{sp}_{2n} \mathbb{C}$ on $V \cong \mathbb{C}^{2n}$ is quaternionic, the exterior power $\wedge^k V$ is real for k even and quaternionic for k odd. Since the highest weights ω_k of $\wedge^k V$ for $k = 1, \dots, n$ form a set of fundamental weights, this completely determines the type of the irreducible representations of $\mathfrak{u}_n \mathbb{H}$: we have

Proposition 26.25. *For any weight $\lambda = \sum a_i \cdot \omega_i$ of $\mathfrak{u}_n \mathbb{H}$, the irreducible representation Γ_λ with highest weight λ is real if a_i is even for all odd i , and quaternionic if a_i is odd for any odd i .*

Next, we consider the odd orthogonal algebras. Part of this is easy: since the restriction to $\mathfrak{so}_{2n+1} \mathbb{R}$ of the standard representation V of $\mathfrak{so}_{2n+1} \mathbb{C}$ is real, so are all its exterior powers; and it follows that any representation of $\mathfrak{so}_{2n+1} \mathbb{R}$ whose highest weight lies in the sublattice of index two generated by the highest weights of these exterior powers is real. It remains, then, to describe the type of the spin representation; the answer, whose verification we leave as Exercise 26.28 below, is that the spin representation Γ_α of $\mathfrak{so}_{2n+1} \mathbb{C}$ (that is, the irreducible representation whose highest weight is one-half the highest weight of $\wedge^n V$) is real when $n \equiv 0$ or 3 mod 4, and quaternionic if $n \equiv 1$ or 2 mod 4. This yields

Proposition 26.26. *Let ω_i be the highest weight of the representation $\wedge^i V$ of $\mathfrak{so}_{2n+1} \mathbb{C}$. For any weight $\lambda = a_1 \omega_1 + \dots + a_{n-1} \omega_{n-1} + a_n \omega_n / 2$ of $\mathfrak{so}_{2n+1} \mathbb{R}$, the irreducible representation Γ_λ with highest weight λ is real if a_n is even, or if n is*

congruent to 0 or 3 mod 4; if a_n is odd and $n \equiv 1$ or 2 mod 4, then Γ_λ is quaternionic.

(Note that, in each of the last two cases, the fact that every representation is either real or quaternionic follows from the observation that the Weyl group action on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ includes multiplication by -1 .)

Finally, we have the even orthogonal Lie algebras. As before, the exterior powers of the standard representation V are all real, but we now have two spin representations to deal with, with highest vectors (in the notation of Lecture 19) $\alpha = (L_1 + \cdots + L_n)/2$ and $\beta = (L_1 + \cdots + L_{n-1} - L_n)/2$. The first question is whether these two are self-conjugate or conjugate to each other. In case n is even, as in the case of the symplectic and odd orthogonal algebras, the Weyl group action on the Cartan subalgebra contains multiplication by -1 (the Weyl group contains the automorphism of \mathfrak{h}^* reversing the sign of any even number of the basis elements L_i), so that Γ_α and Γ_β will be isomorphic to their duals; if n is odd, on the other hand, we see that Γ_α will have $-\beta$ as a weight, so that Γ_α and Γ_β will be complex representations dual to each other. We consider these cases in turn.

(i) Suppose first that n is odd, and say λ is any weight, written as

$$\lambda = a_1\omega_1 + \cdots + a_{n-2}\omega_{n-2} + a_{n-1}\beta + a_n\alpha.$$

If $a_{n-1} \neq a_n$, the representation Γ_λ with highest weight λ will not be isomorphic to its dual, and so will be complex. On the other hand, $\Gamma_{\alpha+\beta}$ appears once in $\Gamma_\alpha \otimes \Gamma_\beta = \text{End}(\Gamma_\alpha)$, and so is real; thus, if $a_{n-1} = a_n$, the representation Γ_λ will be real.

(ii) If, by contrast, n is even then all representations of $\mathfrak{so}_{2n}\mathbb{R}$ will be either real or quaternionic. The half-spin representations Γ_α and Γ_β are real if $n \equiv 0 \pmod{4}$, quaternionic if $n \equiv 2 \pmod{4}$, a fact that we leave as Exercise 26.28. It follows that, with λ as above, Γ_λ will be real if either n is divisible by 4, or if $a_{n-1} + a_n$ is even; if $n \equiv 2 \pmod{4}$ and $a_{n-1} + a_n$ is odd, Γ_λ will be quaternionic. In sum, then, we have

Proposition 26.27. *The representation Γ_λ of $\mathfrak{so}_{2n}\mathbb{R}$ with highest weight $\lambda = a_1\omega_1 + \cdots + a_{n-2}\omega_{n-2} + a_{n-1}\beta + a_n\alpha$ will be complex if n is odd and $a_{n-1} \neq a_n$; it will be quaternionic if $n \equiv 2 \pmod{4}$ and $a_{n-1} + a_n$ is odd; and it will be real otherwise.*

Exercise 26.28*. Verify the statements made above about the types of the spin representation Γ_α of the orthogonal Lie algebras, i.e., that the spin representation Γ_α of $\mathfrak{so}_{2n+1}\mathbb{R}$ is real when $n \equiv 0$ or 3 (mod 4); and quaternionic if $n \equiv 1$ or 2 (mod 4), and that the half-spin representations of $\mathfrak{so}_{2n}\mathbb{R}$ are real if $n \equiv 0 \pmod{4}$ and quaternionic if $n \equiv 2 \pmod{4}$. Show, in fact, that the even Clifford algebras $C_m^{\text{even}} \subset C_m = C(0, m)$ are products of one or two copies of matrix algebras over \mathbb{R} , \mathbb{C} , or \mathbb{H} , with \mathbb{R} occurring for $m \equiv 0$ or $\pm 1 \pmod{8}$, \mathbb{C} occurring for $m \equiv \pm 2 \pmod{8}$, and \mathbb{H} for $m \equiv \pm 3$ or 4 mod 8.

Exercise 26.29. Show that for a representation V of a compact group G ,

$$\int_G \chi_V(g^2) = \begin{cases} 0 & \text{if } V \text{ is complex} \\ 1 & \text{if } V \text{ is real} \\ -1 & \text{if } V \text{ is quaternionic.} \end{cases}$$

Exercise 26.30*. Show that for a representation V of a compact group, the number of irreducible real components it contains, minus the number of quaternionic representations, is the number of times the trivial representation occurs in $\psi^2 V$ in the representation ring, where ψ^2 is the Adams operation (cf. Exercise 23.39).