

LECTURE 1

Representations of Finite Groups

In this lecture we give the basic definitions of representation theory, and prove two of the basic results, showing that every representation is a (unique) direct sum of irreducible ones. We work out as examples the case of abelian groups, and the simplest nonabelian group, the symmetric group on 3 letters. In the latter case we give an analysis that will turn out not to be useful for the study of finite groups, but whose main idea is central to the study of the representations of Lie groups.

§1.1: Definitions

§1.2: Complete reducibility; Schur's lemma

§1.3: Examples: Abelian groups; \mathfrak{S}_3

§1.1. Definitions

A *representation* of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho: G \rightarrow GL(V)$ of G to the group of automorphisms of V ; we say that such a map *gives V the structure of a G -module*. When there is little ambiguity about the map ρ (and, we're afraid, even sometimes when there is) we sometimes call V itself a representation of G ; in this vein we will often suppress the symbol ρ and write $g \cdot v$ or gv for $\rho(g)(v)$. The dimension of V is sometimes called the *degree* of ρ .

A *map* φ between two representations V and W of G is a vector space map $\varphi: V \rightarrow W$ such that

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ g \downarrow & & \downarrow g \\ V & \xrightarrow{\varphi} & W \end{array}$$

commutes for every $g \in G$. (We will call this a G -linear map when we want to distinguish it from an arbitrary linear map between the vector spaces V and W .) We can then define $\text{Ker } \varphi$, $\text{Im } \varphi$, and $\text{Coker } \varphi$, which are also G -modules.

A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under G . A representation V is called *irreducible* if there is no proper nonzero invariant subspace W of V .

If V and W are representations, the *direct sum* $V \oplus W$ and the *tensor product* $V \otimes W$ are also representations, the latter via

$$g(v \otimes w) = gv \otimes gw.$$

For a representation V , the n th tensor power $V^{\otimes n}$ is again a representation of G by this rule, and the *exterior powers* $\wedge^n(V)$ and *symmetric powers* $\text{Sym}^n(V)$ are subrepresentations¹ of it. The *dual* $V^* = \text{Hom}(V, \mathbb{C})$ of V is also a representation, though not in the most obvious way: we want the two representations of G to respect the natural pairing (denoted $\langle \cdot, \cdot \rangle$) between V^* and V , so that if $\rho: G \rightarrow \text{GL}(V)$ is a representation and $\rho^*: G \rightarrow \text{GL}(V^*)$ is the dual, we should have

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$$

for all $g \in G$, $v \in V$, and $v^* \in V^*$. This in turn forces us to define the dual representation by

$$\rho^*(g) = {}^t\rho(g^{-1}): V^* \rightarrow V^*$$

for all $g \in G$.

Exercise 1.1. Verify that with this definition of ρ^* , the relation above is satisfied.

Having defined the dual of a representation and the tensor product of two representations, it is likewise the case that if V and W are representations, then $\text{Hom}(V, W)$ is also a representation, via the identification $\text{Hom}(V, W) = V^* \otimes W$. Unraveling this, if we view an element of $\text{Hom}(V, W)$ as a linear map φ from V to W , we have

$$(g\varphi)(v) = g\varphi(g^{-1}v)$$

for all $v \in V$. In other words, the definition is such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow g & & \downarrow g \\ V & \xrightarrow{g\varphi} & W \end{array}$$

commutes. Note that the dual representation is, in turn, a special case of this:

¹ For more on exterior and symmetric powers, including descriptions as quotient spaces of tensor powers, see Appendix B.

when $W = \mathbb{C}$ is the *trivial* representation, i.e., $gw = w$ for all $w \in \mathbb{C}$, this makes V^* into a G -module, with $g\varphi(v) = \varphi(g^{-1}v)$, i.e., $g\varphi = {}^t(g^{-1})\varphi$.

Exercise 1.2. Verify that in general the vector space of G -linear maps between two representations V and W of G is just the subspace $\text{Hom}(V, W)^G$ of elements of $\text{Hom}(V, W)$ fixed under the action of G . This subspace is often denoted $\text{Hom}_G(V, W)$.

We have, in effect, taken the identification $\text{Hom}(V, W) = V^* \otimes W$ as the definition of the representation $\text{Hom}(V, W)$. More generally, the usual identities for vector spaces are also true for representations, e.g.,

$$V \otimes (U \oplus W) = (V \otimes U) \oplus (V \otimes W),$$

$$\wedge^k(V \oplus W) = \bigoplus_{a+b=k} \wedge^a V \otimes \wedge^b W,$$

$$\wedge^k(V^*) = \wedge^k(V)^*,$$

and so on.

Exercise 1.3*. Let $\rho: G \rightarrow \text{GL}(V)$ be any representation of the finite group G on an n -dimensional vector space V and suppose that for any $g \in G$, the determinant of $\rho(g)$ is 1. Show that the spaces $\wedge^k V$ and $\wedge^{n-k} V^*$ are isomorphic as representations of G .

If X is any finite set and G acts on the left on X , i.e., $G \rightarrow \text{Aut}(X)$ is a homomorphism to the permutation group of X , there is an associated *permutation representation*: let V be the vector space with basis $\{e_x: x \in X\}$, and let G act on V by

$$g \cdot \sum a_x e_x = \sum a_x e_{gx}.$$

The *regular representation*, denoted R_G or R , corresponds to the left action of G on itself. Alternatively, R is the space of complex-valued functions on G , where an element $g \in G$ acts on a function α by $(g\alpha)(h) = \alpha(g^{-1}h)$.

Exercise 1.4*. (a) Verify that these two descriptions of R agree, by identifying the element e_x with the characteristic function which takes the value 1 on x , 0 on other elements of G .

(b) The space of functions on G can also be made into a G -module by the rule $(g\alpha)(h) = \alpha(hg)$. Show that this is an isomorphic representation.

§1.2. Complete Reducibility; Schur's Lemma

As in any study, before we begin our attempt to classify the representations of a finite group G in earnest we should try to simplify life by restricting our search somewhat. Specifically, we have seen that representations of G can be

built up out of other representations by linear algebraic operations, most simply by taking the direct sum. We should focus, then, on representations that are “atomic” with respect to this operation, i.e., that cannot be expressed as a direct sum of others; the usual term for such a representation is *indecomposable*. Happily, the situation is as nice as it could possibly be: a representation is atomic in this sense if and only if it is irreducible (i.e., contains no proper subrepresentations); and every representation is the direct sum of irreducibles, in a suitable sense uniquely so. The key to all this is

Proposition 1.5. *If W is a subrepresentation of a representation V of a finite group G , then there is a complementary invariant subspace W' of V , so that $V = W \oplus W'$.*

PROOF. There are two ways of doing this. One can introduce a (positive definite) Hermitian inner product H on V which is preserved by each $g \in G$ (i.e., such that $H(gv, gw) = H(v, w)$ for all $v, w \in V$ and $g \in G$). Indeed, if H_0 is any Hermitian product on V , one gets such an H by averaging over G :

$$H(v, w) = \sum_{g \in G} H_0(gv, gw).$$

Then the perpendicular subspace W^\perp is complementary to W in V . Alternatively (but similarly), we can simply choose an arbitrary subspace U complementary to W , let $\pi_0: V \rightarrow W$ be the projection given by the direct sum decomposition $V = W \oplus U$, and average the map π_0 over G : that is, take

$$\pi(v) = \sum_{g \in G} g(\pi_0(g^{-1}v)).$$

This will then be a G -linear map from V onto W , which is multiplication by $|G|$ on W ; its kernel will, therefore, be a subspace of V invariant under G and complementary to W . \square

Corollary 1.6. *Any representation is a direct sum of irreducible representations.*

This property is called *complete reducibility*, or *semisimplicity*. We will see that, for continuous representations, the circle S^1 , or any compact group, has this property; integration over the group (with respect to an invariant measure on the group) plays the role of averaging in the above proof. The (additive) group \mathbb{R} does not have this property: the representation

$$a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

leaves the x axis fixed, but there is no complementary subspace. We will see other Lie groups such as $SL_n(\mathbb{C})$ that are semisimple in this sense. Note also that this argument would fail if the vector space V was over a field of finite characteristic since it might then be the case that $\pi(v) = 0$ for $v \in W$. The failure

of complete reducibility is one of the things that makes the subject of *modular representations*, or representations on vector spaces over finite fields, so tricky.

The extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

Schur's Lemma 1.7. *If V and W are irreducible representations of G and $\varphi: V \rightarrow W$ is a G -module homomorphism, then*

- (1) *Either φ is an isomorphism, or $\varphi = 0$.*
- (2) *If $V = W$, then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$, I the identity.*

PROOF. The first claim follows from the fact that $\text{Ker } \varphi$ and $\text{Im } \varphi$ are invariant subspaces. For the second, since \mathbb{C} is algebraically closed, φ must have an eigenvalue λ , i.e., for some $\lambda \in \mathbb{C}$, $\varphi - \lambda I$ has a nonzero kernel. By (1), then, we must have $\varphi - \lambda I = 0$, so $\varphi = \lambda I$. \square

We can summarize what we have shown so far in

Proposition 1.8. *For any representation V of a finite group G , there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i .

PROOF. It follows from Schur's lemma that if W is another representation of G , with a decomposition $W = \bigoplus W_j^{\oplus b_j}$, and $\varphi: V \rightarrow W$ is a map of representations, then φ must map the factor $V_i^{\oplus a_i}$ into that factor $W_j^{\oplus b_j}$ for which $W_j \cong V_i$; when applied to the identity map of V to V , the stated uniqueness follows. \square

In the next lecture we will give a formula for the projection of V onto $V_i^{\oplus a_i}$. The decomposition of the i th summand into a direct sum of a_i copies of V_i is not unique if $a_i > 1$, however.

Occasionally the decomposition is written

$$V = a_1 V_1 \oplus \cdots \oplus a_k V_k = a_1 V_1 + \cdots + a_k V_k, \quad (1.9)$$

especially when one is concerned only about the isomorphism classes and multiplicities of the V_i .

One more fact that will be established in the following lecture is that a finite group G admits only finitely many irreducible representations V_i up to isomorphism (in fact, we will say how many). This, then, is the framework of the classification of all representations of G : by the above, once we have described

the irreducible representations of G , we will be able to describe an arbitrary representation as a linear combination of these. Our first goal, in analyzing the representations of any group, will therefore be:

(i) *Describe all the irreducible representations of G .*

Once we have done this, there remains the problem of carrying out in practice the description of a given representation in these terms. Thus, our second goal will be:

(ii) *Find techniques for giving the direct sum decomposition (1.9), and in particular determining the multiplicities a_i of an arbitrary representation V .*

Finally, it is the case that the representations we will most often be concerned with are those arising from simpler ones by the sort of linear- or multilinear-algebraic operations described above. We would like, therefore, to be able to describe, in the terms above, the representation we get when we perform these operations on a known representation. This is known generally as

(iii) *Plethysm: Describe the decompositions, with multiplicities, of representations derived from a given representation V , such as $V \otimes V$, V^* , $\wedge^k(V)$, $\text{Sym}^k(V)$, and $\wedge^k(\wedge^l V)$. Note that if V decomposes into a sum of two representations, these representations decompose accordingly; e.g., if $V = U \oplus W$, then*

$$\wedge^k V = \bigoplus_{i+j=k} \wedge^i U \otimes \wedge^j W,$$

so it is enough to work out this plethysm for irreducible representations. Similarly, if V and W are two irreducible representations, we want to decompose $V \otimes W$; this is usually known as the *Clebsch–Gordan* problem.

§1.3. Examples: Abelian Groups; \mathfrak{S}_3

One obvious place to look for examples is with abelian groups. It does not take long, however, to deal with this case. Basically, we may observe in general that if V is a representation of the finite group G , abelian or not, each $g \in G$ gives a map $\rho(g): V \rightarrow V$; but *this map is not generally a G -module homomorphism*: for general $h \in G$ we will have

$$g(h(v)) \neq h(g(v)).$$

Indeed, $\rho(g): V \rightarrow V$ will be G -linear for every ρ if (and only if) g is in the center $Z(G)$ of G . In particular if G is abelian, and V is an irreducible representation, then by Schur's lemma every element $g \in G$ acts on V by a scalar multiple of the identity. Every subspace of V is thus invariant; so that V must be one dimensional. The irreducible representations of an abelian group G are thus simply elements of the dual group, that is, homomorphisms

$$\rho: G \rightarrow \mathbb{C}^*.$$

We consider next the simplest nonabelian group, $G = \mathfrak{S}_3$. To begin with, we have (as with any nontrivial symmetric group) two one-dimensional representations: we have the trivial representation, which we will denote U , and the *alternating representation* U' , defined by setting

$$gv = \text{sgn}(g)v$$

for $g \in G$, $v \in \mathbb{C}$. Next, since G comes to us as a permutation group, we have a natural permutation representation, in which G acts on \mathbb{C}^3 by permuting the coordinates. Explicitly, if $\{e_1, e_2, e_3\}$ is the standard basis, then $g \cdot e_i = e_{g(i)}$, or, equivalently,

$$g \cdot (z_1, z_2, z_3) = (z_{g^{-1}(1)}, z_{g^{-1}(2)}, z_{g^{-1}(3)}).$$

This representation, like any permutation representation, is not irreducible: the line spanned by the sum $(1, 1, 1)$ of the basis vectors is invariant, with complementary subspace

$$V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_2 + z_3 = 0\}.$$

This two-dimensional representation V is easily seen to be irreducible; we call it the *standard representation* of \mathfrak{S}_3 .

Let us now turn to the problem of describing an arbitrary representation of \mathfrak{S}_3 . We will see in the next lecture a wonderful tool for doing this, called *character theory*; but, as inefficient as this may be, we would like here to adopt a more ad hoc approach. This has some virtues as a didactic technique in the present context (admittedly dubious ones, consisting mainly of making the point that there are other and far worse ways of doing things than character theory). The real reason we are doing it is that it will serve to introduce an idea that, while superfluous for analyzing the representations of finite groups in general, will prove to be the key to understanding representations of Lie groups.

The idea is a very simple one: since we have just seen that the representation theory of a finite abelian group is virtually trivial, we will start our analysis of an arbitrary representation W of \mathfrak{S}_3 by looking just at the action of the abelian subgroup $\mathfrak{A}_3 = \mathbb{Z}/3 \subset \mathfrak{S}_3$ on W . This yields a very simple decomposition: if we take τ to be any generator of \mathfrak{A}_3 (that is, any three-cycle), the space W is spanned by eigenvectors v_i for the action of τ , whose eigenvalues are of course all powers of a cube root of unity $\omega = e^{2\pi i/3}$. Thus,

$$W = \bigoplus V_i,$$

where

$$V_i = \mathbb{C}v_i \quad \text{and} \quad \tau v_i = \omega^{a_i} v_i.$$

Next, we ask how the remaining elements of \mathfrak{S}_3 act on W in terms of this decomposition. To see how this goes, let σ be any transposition, so that τ and σ together generate \mathfrak{S}_3 , with the relation $\sigma\tau\sigma = \tau^2$. We want to know where σ sends an eigenvector v for the action of τ , say with eigenvalue ω^i ; to answer

this, we look at how τ acts on $\sigma(v)$. We use the basic relation above to write

$$\begin{aligned}\tau(\sigma(v)) &= \sigma(\tau^2(v)) \\ &= \sigma(\omega^{2i} \cdot v) \\ &= \omega^{2i} \cdot \sigma(v).\end{aligned}$$

The conclusion, then, is that if v is an eigenvector for τ with eigenvalue ω^i , then $\sigma(v)$ is again an eigenvector for τ , with eigenvalue ω^{2i} .

Exercise 1.10. Verify that with $\sigma = (12)$, $\tau = (123)$, the standard representation has a basis $\alpha = (\omega, 1, \omega^2)$, $\beta = (1, \omega, \omega^2)$, with

$$\tau\alpha = \omega\alpha, \quad \tau\beta = \omega^2\beta, \quad \sigma\alpha = \beta, \quad \sigma\beta = \alpha.$$

Suppose now that we start with such an eigenvector v for τ . If the eigenvalue of v is $\omega^i \neq 1$, then $\sigma(v)$ is an eigenvector with eigenvalue $\omega^{2i} \neq \omega^i$, and so is independent of v ; and v and $\sigma(v)$ together span a two-dimensional subspace V' of W invariant under \mathfrak{S}_3 . In fact, V' is isomorphic to the standard representation, which follows from Exercise 1.10. If, on the other hand, the eigenvalue of v is 1, then $\sigma(v)$ may or may not be independent of v . If it is not, then v spans a one-dimensional subrepresentation of W , isomorphic to the trivial representation if $\sigma(v) = v$ and to the alternating representation if $\sigma(v) = -v$. If $\sigma(v)$ and v are independent, then $v + \sigma(v)$ and $v - \sigma(v)$ span one-dimensional representations of W isomorphic to the trivial and alternating representations, respectively.

We have thus accomplished the first two of the goals we have set for ourselves above in the case of the group $G = \mathfrak{S}_3$. First, we see from the above that *the only three irreducible representations of \mathfrak{S}_3 are the trivial, alternating, and standard representations U , U' and V* . Moreover, for an arbitrary representation W of \mathfrak{S}_3 we can write

$$W = U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c};$$

and we have a way to determine the multiplicities a , b , and c : c , for example, is the number of independent eigenvectors for τ with eigenvalue ω , whereas $a + c$ is the multiplicity of 1 as an eigenvalue of σ , and $b + c$ is the multiplicity of -1 as an eigenvalue of σ .

In fact, this approach gives us as well the answer to our third problem, finding the decomposition of the symmetric, alternating, or tensor powers of a given representation W , since if we know the eigenvalues of τ on such a representation, we know the eigenvalues of τ on the various tensor powers of W . For example, we can use this method to decompose $V \otimes V$, where V is the standard two-dimensional representation. For $V \otimes V$ is spanned by the vectors $\alpha \otimes \alpha$, $\alpha \otimes \beta$, $\beta \otimes \alpha$, and $\beta \otimes \beta$; these are eigenvectors for τ with eigenvalues ω^2 , 1, 1, and ω , respectively, and σ interchanges $\alpha \otimes \alpha$ with $\beta \otimes \beta$, and $\alpha \otimes \beta$ with $\beta \otimes \alpha$. Thus $\alpha \otimes \alpha$ and $\beta \otimes \beta$ span a subrepresentation

isomorphic to V , $\alpha \otimes \beta + \beta \otimes \alpha$ spans a trivial representation U , and $\alpha \otimes \beta - \beta \otimes \alpha$ spans U' , so

$$V \otimes V \cong U \oplus U' \oplus V.$$

Exercise 1.11. Use this approach to find the decomposition of the representations $\text{Sym}^2 V$ and $\text{Sym}^3 V$.

Exercise 1.12. (a) Decompose the regular representation R of \mathfrak{S}_3 .

(b) Show that $\text{Sym}^{k+6} V$ is isomorphic to $\text{Sym}^k V \oplus R$, and compute $\text{Sym}^k V$ for all k .

Exercise 1.13*. Show that $\text{Sym}^2(\text{Sym}^3 V) \cong \text{Sym}^3(\text{Sym}^2 V)$. Is $\text{Sym}^m(\text{Sym}^n V)$ isomorphic to $\text{Sym}^n(\text{Sym}^m V)$?

As we have indicated, the idea of studying a representation V of a group G by first restricting the action to an abelian subgroup, getting a decomposition of V into one-dimensional invariant subspaces, and then asking how the remaining generators of the group act on these subspaces, does not work well for finite G in general; for one thing, there will not in general be a convenient abelian subgroup to use. This idea will turn out, however, to be the key to understanding the representations of Lie groups, with a torus subgroup playing the role of the cyclic subgroup in this example.

Exercise 1.14*. Let V be an irreducible representation of the finite group G . Show that, up to scalars, there is a *unique* Hermitian inner product on V preserved by G .